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Séminaire de Probabilités

Catherine Donati-Martin
Antoine Lejay
Alain Rouault *Editors*

In Memoriam Marc Yor

Séminaire de Probabilités XLVII

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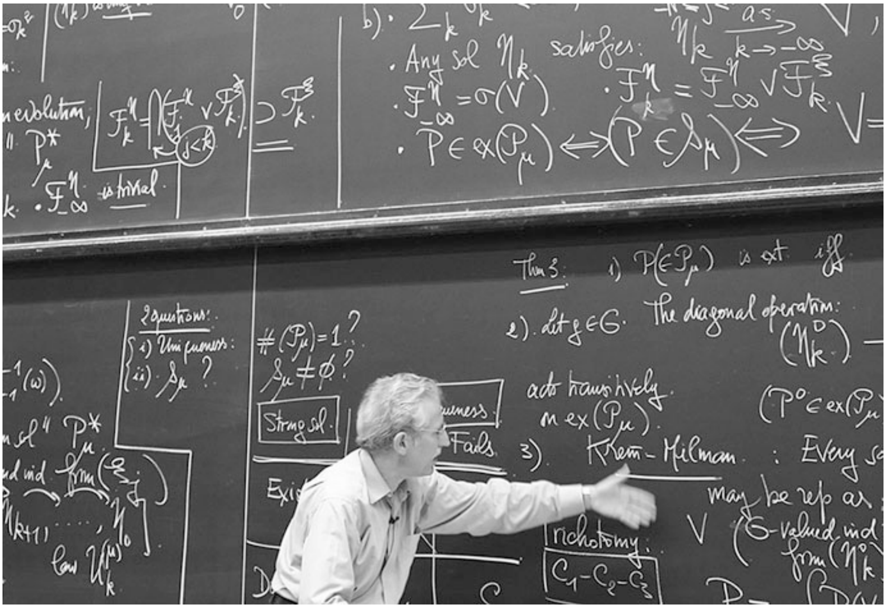

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Preface

Right after Marc Yor's death which deeply distressed us in January 2014, devoting a volume of the Séminaire de Probabilités to his memory appeared natural and essential to us. This Séminaire, created by Paul-André Meyer, was quite successfully continued, owing notably to many years of Marc's untiring activity. We have called for contributions from his friends, collaborators, and former students (with apology for possible omissions). This special volume gathers precious and moving testimonies, as well as many scientific articles.

Beyond this homage to Marc as a man and a mathematician, we wish that this volume will incite young researchers to become acquainted with his work and to draw from it inspiration towards new openings.

We want to thank all authors and referees; they kept the fixed deadlines. We also want to thank our publisher Springer who made this volume possible, and especially Ms. McCrory for her valuable help.

We also draw attention to the special issue "Marc Yor, La passion du mouvement brownien", numéro spécial Gazette des Mathématiciens-Matapli, 2015, and to the links on Zhan Shi's homepage <http://www.proba.jussieu.fr/~zhan/>.

Versailles, France — Catherine Donati-Martin
Vandœuvre-lès-Nancy, France — Antoine Lejay
Versailles, France — Alain Rouault

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Premier contact

Comment as-tu connu Marc ?

Je lui ai fait passer un DEA, il avait dix ans de moins que moi. Je ne sais pas comment il a pu me prendre comme directeur de mémoire. Je lui ai proposé un sujet, comme souvent, un sujet que je ne connaissais absolument pas, comme test. Il y avait des choses compliquées, des ensembles de capacité nulle. Pour la thèse il est parti avec Priouret. Un jour où il exposait une histoire de champ, quelqu'un a dit : « C'est de la bouillie pour les chats ! ». Yor était furieux. Moi, comme je n'avais rien dit, il m'a gardé une profonde admiration. Je crois que le fait qu'il m'ait bien aimé, ça vient de son DEA.

Collaboration scientifique

Quand tu écrivais des articles avec lui, ça se passait comment ?

C'était bien. C'est étonnant qu'on ait travaillé si longtemps ensemble parce que nous étions tellement différents, voire à l'opposé. D'abord comme culture : moi, c'était les Markov, lui c'était le calcul stochastique. Devant un problème donné, typiquement les temps locaux ou le problème de Skorokhod, j'insistais sur l'idée markovienne et Marc disait qu'avec le calcul stochastique ça prenait une demi-ligne.

Du coup vous mainteniez les deux explications, ou bien il n'y en avait qu'une qui sortait?

Non, on prenait celle de Yor, ne serait-ce que pour des questions de rédaction. D'abord, personne ne connaissait plus les Markov... Yor n'aimait pas l'infini dimensionnel. Moi, je n'aimais ni n'aimais pas, je ne comprenais pas. Lui, il comprenait un peu mais il savait que ce n'était pas son truc. Un jour, à propos de je ne sais quoi, il me dit : « Peut-être que Malliavin ou ses disciples s'occuperont de ça quand ils seront revenus sur terre ».

J'ai conscience d'avoir participé à une espèce d'aventure scientifique que plus personne ne connaît, à part nous, et sans doute infiniment plus importante que ce que tout le monde connaît. On a de la chance, moi j'ai eu de la chance. Par hasard...

C'est le cas de le dire !

Le séminaire

Comment êtes-vous, Marc et toi, devenus rédacteurs du Séminaire de Probabilités ?

Quand Meyer nous a confié le Séminaire c'était pour une raison : il en avait assez de refuser des articles. Il espérait que Yor allait pouvoir refuser des articles. En fait, Yor les réécrivait. Un peu plus tôt, Walsh était venu en France et nous avions fait un groupe de travail sur les temps locaux. Walsh est génial ; il a une intuition probabiliste incroyable. Il avait plein d'idées, de petites idées très simples alors que Yor, ce qui l'intéressait, c'était la plus grande généralité. Il y a eu de nombreuses contributions, très techniques par les gens du labo, que nous avons décidé de publier. Il fallait donc relire tout ça. Je me sentais incapable de le faire, alors j'ai dit à Yor : « Je me charge de l'introduction ». C'était une bonne idée parce que j'ai repris les idées de Walsh dans le cadre des martingales continues. Ça m'a passionné parce que je me suis dit qu'à partir des idées de Walsh on pouvait tout dérouler y compris la formule d'Itô. Yor a commencé par râler et puis je suis arrivé à dégonfler BDG, au moins dans un sens. On a donc fait cette introduction et le volume est paru dans la collection Astérisque.[1] Meyer a vu ce volume et a été séduit, peut-être par cette introduction qui remettait tout en place, très différente de ce qu'il avait fait, lui, au Séminaire. Il a donc vu que nous étions capables d'éditer un volume, il nous a refilé le bébé.[2]

Et avant vous, ça se passait comment ?

Les tout premiers volumes étaient composés à partir des exposés faits au Séminaire de Strasbourg, par des Strasbourgeois ou des Parisiens. Au sein de l'école de Strasbourg, l'ambiance était ainsi : on ne se piquait pas les idées, on se les donnait. Meyer invitait des étrangers de passage en Europe, des « Américains-Français », comme les appelait Stroock, c'est-à-dire ceux qui avaient intégré un peu

[1] Volume 52–53, 1978.

[2] N.D.L.R. : les volumes XIV à XX ont été édités par Azéma et Yor et les volumes XXI à XXXVII par Azéma, Yor et Meyer, Émery, Ledoux.

ce qu'avait fait Meyer, comme Chung... Au début, personne ne pensait à envoyer à Meyer un papier en lui demandant de le publier dans le Séminaire de Strasbourg. Meyer a développé sa publication à partir de feuilles ronéotées au début et puis c'est devenu petit à petit un endroit où ça devenait pas mal d'être publié, y compris par des gens qui n'étaient jamais venus à Strasbourg, des Américains qui envoyaient leurs papiers. Meyer publiait tout ce qu'il faisait, plus tout ce qu'il réécrivait. Ça faisait déjà la moitié du Séminaire. Moi, j'ai continué à y publier.

Quels étaient les rapports de Yor et Meyer ?

Ils ont toujours été bons. Yor m'aimait bien, mais la personne qu'il admirait c'était Meyer.

Et concrètement, la réalisation d'un volume, ça se passait comment ?

Quand je m'en occcupais avec Yor, les contributions de Strasbourg arrivaient tout empaquetées, prêtes à être publiées, on ne savait pas comment était faite la chirurgie interne. Nous, on ne s'occupait que des Parisiens et des autres provinciaux.

Dans ce qui n'était pas le paquet de Strasbourg, c'est vous qui relisiez les articles ?

Meyer ne réintervenait pas. Yor faisait la plus grande partie du travail. Il réécrivait, c'est plutôt moi qui refusais des articles. Après, Yor venait me voir, on discutait il me disait qu'il y avait quand même des choses intéressantes. Yor passait tous les deux jours dans mon bureau pour le Séminaire ; ça me permettait de le voir et de discuter de maths, parfois de maths dans le Séminaire. Il prenait tout le paquet et à la fin de l'entretien, il partait sans un mot de reproche. Il savait où étaient rangés les papiers pour le Séminaire, dans un coin de mon bureau. Il passait parfois discuter avec moi quand il pensait que c'était quelque chose qui m'intéressait. On ne pouvait pas dire que je ne faisais strictement rien.

Vous envoyiez quelquefois à des rapporteurs ?

Quand Yor n'était vraiment pas au courant, ce qui arrivait assez peu souvent, on pouvait envoyer à un rapporteur ou discuter avec quelqu'un. Il réécrivait, ou donnait des indications extrêmement précises. Ça arrivait parfois qu'il envoie l'article quand c'était vraiment en dehors de ses domaines de compétence. Mais la vedette des vedettes pour Marc, c'était quand même Pitman. Il était tombé amoureux du Théorème de Pitman. Je me souviens, un jour où j'avais mauvaise conscience, d'avoir demandé à Meyer d'enlever mon nom du Séminaire, et Meyer m'a dit quelque chose qui, vu la suite des événements avec Yor me paraît prophétique, il m'a dit : « Azéma, ne t'en va pas, ou ça va devenir trop sérieux ». Maintenant que j'y repense...

Et pour susciter des articles, contacter des gens, comment vous faisiez ?

Déjà, Yor avait beaucoup d'étudiants... Il y avait une tradition parmi de nombreux probabilistes de se tourner spontanément vers le Séminaire pour publier. Si une soumission arrivait et qu'on ne connaissait pas du tout l'auteur, on faisait un vrai travail de referee, contrairement à ce que certains ont pu dire.

En regardant les volumes, je trouve que le Séminaire de Proba s'est ouvert à plein de sujets différents : systèmes de particules, hypercontractivité, log Sobolev. Ce n'était dans les champs de compétence ni de Meyer, ni de vous ?

Ça doit être dû à Ledoux. Je l'avais rencontré par l'intermédiaire de Fernique, puis je l'avais vu à Saint-Flour. J'avais trouvé qu'il était très fort. Quand il a fallu un peu s'élargir, nous avons pensé à lui pour se joindre au Comité de Rédaction.

Durant notre conversation, les points suivants ont été abordés :

Les relations internationales

Yor a joué un rôle fondamental. Il connaissait David Williams. Un peu avant le congrès de Durham, dans les années 80, au moment du début du Calcul de Malliavin, Williams avait envoyé quelques chercheurs anglais au labo de proba et Yor s'est occupé d'eux comme s'ils avaient été irlandais. Il connaissait la carrière de tous ces jeunes Anglais : Rogers, Barlow… En fait, ce sont les Français, en particulier Yor, qui ont persuadé les Anglais que de nouveaux champs probabilistes existaient et ça s'est conclu en apothéose par le fait que Williams organise ce sommet de Durham avec le calcul de Malliavin. Donc ça a été un peu le début de la révolution anglaise des probabilités animée par tous ces jeunes et ça doit beaucoup à Yor.

Les jeunes

Yor poussait beaucoup ses élèves. Avec ses appréciations, on ne pouvait pas tellement distinguer entre quelqu'un de moyen et quelqu'un de bon. Il disait : « Quelqu'un qui aime les mathématiques, ça ne peut pas être quelqu'un d'inexistant, il ne faut pas le décourager, il faut même le pousser ». En soit c'était quelque chose de formidable. Heureusement qu'il n'avait pas beaucoup de sens tactique, sinon ça aurait pu faire une mafia.

Les mathématiques financières

Les banques avaient conclu un contrat avec le labo, qui devait leur faire un cours pour leur section R & D, salle de marchés et c'est Yor qui s'en était chargé. Il s'était vu suggérer un jour par un correspondant, payé 1 000 dollars par après-midi (alors que lui, était payé 1 000 dollars tous les 5 ans) : « Ce serait bien de mettre tout ça au propre » (sous-entendu plutôt dans le style papier glacé que dans celui de vos torchons habituels).

Plus tard, il a été atteint par la critique contre les maths financières. Il se sentait responsable. Il avait répondu à la tribune de Michel Rocard.[3] Il était persuadé qu'il était en partie responsable. Yor n'avait pas complètement tort. Quand on voit la chance ou la malchance qu'on a eue avec les produits dérivés. C'est 6 ou 7 fois le PIB mondial. . . .

Pourquoi n'as-tu pas souhaité rédiger d'éloge de Marc ?

Faire un éloge de Yor, c'était vraiment très difficile. J'ai essayé vraiment mais quand j'ai relu, j'ai vu que je parlais plus de moi que de Yor. J'étais trop lié avec lui, les choses dont je me souviens, ce sont des choses qu'on a faites ensemble. Tous les ans, on devait faire un rapport pour le CNRS alors bien entendu j'étais en retard pour le rendre, le bibliothécaire n'était pas content. Je lui dis : « Cette année j'ai fait deux articles et je les ai faits tous les deux avec Yor donc allez voir la bibliographie de Yor parce que lui, il a sûrement rendu à temps ». La même année, moi, j'avais deux articles, il en avait peut-être cinquante.

Marc avait organisé un séminaire avec les physiciens. Tout le monde venait voir Marc dès qu'il y avait des problèmes de probabilités, Duplantier par exemple. C'était bien, parce que sa faiblesse, c'est qu'il ne se posait pas toujours des bons problèmes, mais si tu lui posais n'importe quel problème, en tout cas s'il était soluble par le calcul stochastique, trois jours après il avait la solution.

Ce qui m'impressionnait toujours, c'était ses tableaux. Quand il expliquait quelque chose au tableau, il commençait par la première ligne puis la deuxième etc jusqu'en bas du tableau et ensuite pour continuer, il écrivait entre la première et la deuxième ligne, etc. . . Il se ménageait toujours un petit espace en bas à droite pour écrire le résultat final. En fait si on voulait faire un musée Yor, ce n'est pas le musée de tous ses brouillons qu'il faudrait faire mais c'est le musée de tous ses tableaux.

Il écrivait en spirale et il n'y avait pas une faute de calcul. Il faisait le calcul devant toi en direct, alors que moi j'oubliais toujours le ½ dans la formule d'Itô, alors Marc me regardait dans les yeux et disait : « Ah, Azéma ! . . . » Et ça, c'était au bout de trois ans parce que, les trois premières années, à chaque fois, il fallait qu'il me rappelle la formule d'Itô.

Yor, c'était quelqu'un qui était fondamentalement bon, il n'était extraordinairement pas rancunier. Il y a des gens qui lui ont fait des crasses mais je ne l'ai jamais entendu dire du mal de personne.

En guise d'hommage

Au revoir, Marc, au revoir l'infatigable, au revoir le passionné l'inquiet, au revoir le magicien aux yeux bleus, Adieu.

Jacques Azéma

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[3] Marc Yor, « Ebauche de réponse à M. Michel Rocard » — Images des Mathématiques, CNRS, 2009.

Lettre à un Grand Monsieur

Pauline Barrieu

Cher Monsieur,

Vos leçons de mathématiques, vos conseils de chercheur, votre exemple m'ont façonnée, m'ont guidée et éclairée depuis ma thèse jusqu'à aujourd'hui encore.

Votre patience, votre gentillesse, votre disponibilité sans faille, votre droiture, votre loyauté, votre sensibilité, votre humanité… j'ai eu le très grand privilège de connaître ce Grand Monsieur que vous étiez.

Il me reste également en mémoire nos conversations plus récentes sur la poésie, pour laquelle nous partagions ce grand attrait, notamment les Haïkus, allant à l'essentiel, élégants et subtils, comme une très belle preuve mathématique.

Aussi aujourd'hui, pour hononer votre mémoire, je me suis permis de choisir ce Haïku, qui, je l'espère, vous aurait plu :

Sur le sentier de montagne
Le soleil se lève
Au parfum des pruniers.
Bashô (1644–1694)

Chapeaux bas, un Grand Monsieur s'en va.

⋆ ⋆ ⋆ ⋆ ⋆

Souvenirs d'une thèse avec Marc

Jean Bertoin

C'est Jean Giraud, alors responsable du département de mathématiques à l'ENS Saint-Cloud, qui m'avait conseillé sans la moindre hésitation de m'adresser à Marc quand je lui avais fait part de mon intention de m'orienter vers les probabilités. J'ai alors rencontré Marc pour la première fois dans le bureau qu'il occupait dans le couloir 56–66 à Jussieu. Même si les piles de documents soigneusement rangés sur sa table de travail atteignaient déjà une hauteur respectable, ce n'étaient là que les prémices de ce qu'allait devenir plus tard son antre à Chevaleret, où s'entasseraient les cartons et les dossiers.

Marc m'a tout de suite plu par sa gentillesse, sa grande simplicité, et la lueur dans ses yeux quand il expliquait des mathématiques ; et après cette rencontre, j'ai décidé de suivre le DEA de Probabilités du Laboratoire éponyme. Le cours intitulé « Temps locaux browniens et théorie des excursions » que Marc y donnait, était de loin le plus dense et le plus difficile de tous. C'était surtout un cours d'une incroyable richesse. Marc s'attachait à explorer tous les aspects de chaque résultat, à le faire apparaître dans des contextes différents, à l'étendre. Il commençait souvent par dire : « *Cherchons à mieux comprendre…* », puis nous guidait à la découverte d'une variété insoupçonnée d'identités en loi qu'il reliait les unes aux autres. Lorsqu'il entamait un calcul au tableau, sa main restait d'abord en suspens quelques secondes, semblant hésiter un moment pour choisir l'endroit précis où la craie allait se poser. Puis la main dansait, et le tableau se couvrait de formules jusque dans les moindres recoins. À la fin d'une démonstration, il se retournait vers nous, les yeux brillants, et un petit sourire se dessinait sous sa moustache. Après ses cours, j'essayais maladroitement de refaire les calculs de transformées de Laplace de fonctionnelles de processus de Bessel que Marc avait enchaînés comme par magie.

À la fin de l'année du DEA, j'ai demandé à Marc s'il accepterait d'encadrer ma thèse. Marc m'explique qu'il revenait de Strasbourg où Meyer lui avait posé une question à propos des processus de Dirichlet que venait d'introduire Hans Föllmer. Il sort de la poche de son veston une enveloppe pliée en quatre. La question de Meyer était griffonnée au dos de l'enveloppe, il me la tend. J'avais maintenant un sujet de thèse, j'étais officiellement thésard ! Aujourd'hui, où pour entamer une thèse il faut auparavant rédiger un projet détaillé avec un plan de travail précis pour assurer un financement, obtenir une bourse ou une allocation, signer une charte, etc., l'anecdote peut faire sourire. Mais quand même, être encadré par Marc Yor, sur un sujet suggéré par Paul-André Meyer, il y avait de quoi être fier. Et je le suis encore.

Pendant les deux ans où j'ai préparé ma thèse à Paris, Marc a été d'une disponibilité et d'une gentillesse à toute épreuve. Tous ses anciens étudiants peuvent témoigner de même. On pouvait passer le voir à n'importe quel moment pour lui poser des questions, discuter d'un problème, ou lui présenter ses premières trouvailles, sans jamais avoir l'impression de le déranger ou de l'ennuyer. La thèse était assez avancée quand j'ai dû partir pour le Mexique et y effectuer un séjour de deux ans au titre de la coopération. Je suis arrivé à Mexico en septembre 1986, un an après le terrible tremblement de terre qui l'avait frappé ; les séquelles du séisme étaient encore visibles partout dans la ville. Je n'avais jamais quitté l'Europe auparavant, et j'ouvrais les yeux sur un pays fascinant qui continue de m'émerveiller près de trente ans après. J'aurais aisément pu alors me détourner des mathématiques, il y avait tant de choses à découvrir. Découverte d'une autre culture, d'autres modes de vie, d'autres gens, découverte de soi-même au fond bien sûr.

En 1986 au Mexique, le coût d'un appel téléphonique vers la France était prohibitif, et il n'y avait évidemment pas encore d'internet ni de courrier électronique ; la plupart des communications transatlantiques se faisaient par voie postale. La poste mexicaine était alors peu fiable ; il fallait utiliser la valise diplomatique et compter deux bonnes semaines entre l'envoi d'une lettre et sa réception. Marc est la personne avec laquelle j'ai le plus correspondu pendant ces deux années, davantage qu'avec ma famille ou mes amis. J'attendais avec impatience ses courriers qui répondaient à ceux que j'avais envoyés un mois plus tôt, et aurais reconnu sur l'enveloppe son écriture souple entre mille. Il me semble qu'écrire lui procurait un plaisir tout particulier ; peut-être est-ce en partie pour cela qu'il s'est longtemps refusé à utiliser le courrier électronique, préférant correspondre par fax dont il était au Laboratoire de très loin le plus gros utilisateur. Malgré les milliers de kilomètres qui nous séparaient, Marc répondait assidument à mes questions, relisait et commentait les manuscrits que je lui envoyais, posait de nouvelles questions. Je n'aurais probablement pas achevé ma thèse sans son soutien, ses encouragements, et les stimulations constantes qu'il apportait. Il y a quelques mois, en faisant le tri des affaires de Marc dans son bureau de Jussieu, Monique Jeanblanc a trouvé un dossier avec des copies de nos correspondances de cette époque. Elle me les a renvoyées, je les relis avec une profonde émotion.

Tous ceux qui ont bien connu Marc savent combien il était peu au fait des choses administratives. J'en ai fait ma première expérience quelques mois après avoir soutenu ma thèse, lors d'une candidature au CNRS en 1987. Les auditions devaient avoir lieu en décembre ; elles étaient alors purement formelles : il fallait seulement signer une feuille de présence et j'avais entendu dire qu'un candidat dans l'incapacité de se déplacer pouvait demander à quelqu'un d'autre de signer à la place. Je me renseigne auprès de Marc, soucieux d'éviter un voyage en France au moment où ma compagne était en toute fin de grossesse à Mexico. Marc répond qu'il était très important que je sois présent, que cela montrerait ma détermination. Je m'exécute malgré les difficultés que cela représentait. Après avoir signé la feuille d'émargement, j'explique les circonstances de ma présence à Bernard Prum, qui présidait la section pour les mathématiques du Comité National. Bernard s'en étonne, et confirme que Marc aurait pu sans problème venir signer à ma place. Je rentre à Mexico, David naît quelques jours plus tard. Cette année là, j'échoue au concours du CNRS. Je suis évidemment déçu, et suis prêt à prendre un poste d'enseignant dans un lycée à mon retour en France. Marc me réconforte, continue de m'encourager, stimule de nouveaux travaux, et je suis admis au concours suivant grâce à son soutien.

Je n'évoquerai pas ici mes relations avec Marc durant les nombreuses années que j'ai passées au Laboratoire de Probabilités, il me suffira de dire que l'influence considérable qu'il a eu sur mes propres recherches est une évidence. En juillet 2011, lorsque je quitte le Laboratoire, Marc organise une journée spéciale à l'occasion de mon départ. Il prépare une affiche pour l'annonce, elle représente une carte du Mexique et une carte de France reliées par un pont. Lors de cette journée, Marc montre aux participants un charola d'Olinla[4] que je lui avais offert en revenant du Mexique, pour le remercier de son aide tout au long de la préparation de ma thèse.

Marc avait une foi sans faille dans la science. Infatigablement, il a su communiquer sa passion à ses étudiants, à ses collègues, il les a aidés, encouragés, stimulés. J'ai conscience que ma vie professionnelle aurait probablement pris une tournure bien différente si je n'avais rencontré Marc il y a un peu plus de trente ans, s'il ne m'avait transmis une part de son savoir, guidé mes premières recherche, mais surtout s'il ne m'avait soutenu dans des moments déterminants. Je suis particulièrement heureux d'avoir eu la chance de côtoyer pendant toutes ces années non seulement l'un des plus grands probabilistes au monde, mais bien davantage, un homme passionné, profondément généreux, et qui a toujours su rester modeste et pudique. Un homme rare.

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[4]Plateau en bois peint réalisé par des artisans du Guerrero.

Maria Emilia Caballero

I would like to give a brief testimony of my first visit to the "Laboratoire de probabilités (LP)", when it occupied one and a half corridors of the former building of the "Université de Paris VI" and the "Université de Paris VII". The first thing that surprised me was the friendly atmosphere of this very special "laboratoire". Professors were very accessible, casually dressed and you could see many young people, some of them finishing their PhD's, others already appointed to some other University and beginning their careers, all of them strongly attracted by the presence of people like Marc Yor and the feverish activity that he displayed: his course which gave birth to the now famous book "Continuous Martingales and Brownian Motion", the organization of seminars on fine properties of Brownian Motion and the discussions that took place in his entourage played a central role in creating this exciting academic space. Among the young students were : Jean-François Le Gall, Jean Bertoin, Sonia Fourati, Philippe Biane, Nathalie Eisenbaum, Catherine Donati-Martin, and many others. They were all very busy, but they had time to discuss, not only interesting mathematical subjects, but also a great variety of topics. Many more mathematicians came from nearby universities every week to the Tuesday's Probability Seminar. These young students and researchers are all now accomplished mathematicians.

My discovery of the LP determined my future mathematical activity. Before this, I was mainly an analyst and from this point on, I became more and more interested in probability theory and the various activities of the group leaded by Marc Yor. This derived in a fruitful collaboration between the Institute of Mathematics at UNAM (University of Mexico) and the LP from the University of Paris VI, especially with Jean Bertoin who incidently came to Mexico some time after my stay in Paris in the eighties. All this allowed us, later on, to sign an exchange program between the mentioned institutions and the doors of the LP have always been opened to Mexican scholars. Marc Yor was always a wonderful host for these visitors and found the time to discuss mathematics with most of us. Once I asked him if he had time to discuss a certain problem. He answered without hesitation: "this is not a good question, since the answer to it is: no, I do not have time; but I will gladly discuss the question you have".

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Marc et le Séminaire de Probabilités

Catherine Donati-Martin

A l'occasion des soixante ans de Marc, H. Geman et le LPMA (Paris 6) ont organisé un colloque en juin 2009 à l'institut Henri Poincaré. Cet événement a rassemblé de nombreux enseignants-chercheurs et étudiants. Ce furent deux journées riches en exposés de qualité, en convivialité et en émotion. Marc fut très touché par cette initiative et par les témoignages d'amitié de ses collègues et d'anciens étudiants. J'avais eu l'honneur et la joie de prononcer l'allocution de clôture de ce colloque, que je reproduis ici.

Comme beaucoup d'orateurs de ces deux jours, Marc a guidé mes premiers pas dans la recherche et je voudrais souligner sa très grande disponibilité, son enthousiasme, ses grandes qualités humaines : toujours ouvert aux échanges avec les autres, par exemple les étudiants du laboratoire, les invités de passage. Une petite anecdote personnelle à ce sujet : jeune thésarde au laboratoire de probabilités de Paris 6 il y a plus de 20 ans, j'assistais aux exposés du séminaire de probabilités (le grand séminaire du mardi) où je côtoyais les plus grands probabilistes mais où je n'avais jamais vu exposer de thésards. Marc est invité à donner une conférence à ce séminaire, tout naturellement, il me propose de partager l'affiche avec lui en exposant un travail que nous venions de terminer sur les inégalités de Hardy et nous avons donc fait un exposé à deux voix ; marque de confiance qui est restée gravée dans ma mémoire. J'ai pu constater à de nombreuses reprises que Marc a constamment à cœur de mettre en avant ses étudiants.

Marc a beaucoup contribué au développement de l'école probabiliste, je voudrais évoquer ici une des facettes de son implication dans la diffusion des probabilités à travers l'énorme travail accompli au sein de la rédaction du Séminaire.

Un bref historique du Séminaire de Probabilités : le séminaire est né d'une rencontre entre Paul-André Meyer et Klaus Peters, responsable des mathématiques chez Springer. La volonté était de créer une publication qui mèlerait à la fois des articles d'exposition, des articles de débutants et de probabilistes confirmés. Le séminaire de Strasbourg (nom d'origine) publierait les travaux des conférenciers invités à Strasbourg mais la plupart des « exposés » n'auraient pas lieu sous forme orale. Très vite, le séminaire a franchi les frontières avec des publications d'amis d'outre-Manche puis de toutes nationalités. A partir de 1980 (volume 14), le dénveloppement des probabilités a amené le déplacement de la rédaction à Paris où elle a été prise en charge par Jacques Azéma et Marc Yor.

Quelques années plus tard, le comité de rédaction s'est structuré avec l'arrivée de Michel Émery et Michel Ledoux.

Au cours de ses vingt-cinq ans de présence au comité de rédaction, Marc a joué un rôle clé, en développant le séminaire et en lui donnant un positionnement aujourd'hui mondialement reconnu.

Au delà de son apport scientifique propre (pas moins de 80 articles parus dans le Séminaire), il a mis au service du Séminaire sa légendaire capacité de travail, son exigence de qualité et de perfection, re-rédigeant certains articles pour les rendre plus clairs, dans la droite lignée de P.-A. Meyer.

De par sa renommée et son réseau de relations parmi les meilleurs probabilistes du monde, il a attiré vers le Séminaire des scientifiques de grande valeur, dont beaucoup sont aussi des amis, apportant ainsi un esprit d'ouverture et un label de qualité incontestable au Séminaire.

C'est ainsi que la nouvelle équipe qui a pris la succession du comité de rédaction en 2006 et dont j'ai l'honneur de faire partie, a hérité d'une publication reconnue et même indispensable à la communauté probabiliste du monde entier. [. . .]

Le séminaire a bien sûr évolué. Même si aujourd'hui, tous les articles sont expertisés par un referee anonyme, le Séminaire n'est pas un journal scientifique classique. À côté d'articles originaux à la pointe de la recherche, l'on y trouve des cours spécialisés, des articles permettant d'avoir une vision globale ou différente sur un sujet déjà connu et c'est, j'en suis sûre, un atout considérable.

1966–2009 : 43 ans d'existence du Séminaire ! Cette longévité exceptionnelle pour un séminaire est sans aucun doute dû au dynamisme des éditeurs et tout particulièrement de Marc. Les éditeurs de ce volume sont heureux de dédier le volume 42 à Marc pour ses 60 ans. [. . .]

Depuis 2009, Marc a toujours manifesté son intérêt pour le Séminaire et nous a apporté une aide précieuse par la relecture de nombreux articles. J'ai quitté le LPMA en 2011 et Marc a organisé une journée spéciale pour mon départ et celui de Frédérique Petit.

Depuis mon arrivée à Versailles, nous échangions avec Marc par mail et chacun sait qu'envoyer un mail et utiliser un ordinateur n'etait pas très naturel pour Marc. . .

Un de ses derniers courriels en décembre 2013 (il proposait de nous apporter une aide financière pour les Journées de Probabilités 2014 à Luminy, sur ses crédits IUF) :

Chère Catherine,
C'est avec plaisir que je voudrais vous aider pour le montant que vous souhaitez. [. . .]
Alain m'a donné hier le Séminaire 45. Magnifique ! Félicitations. Amitiés, Marc

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Instantanés

Michel Émery

C'était il y a une vingtaine d'années. Au téléphone, je signale incidemment à Marc que Lester Dubins vient d'être hospitalisé à Luxembourg, une jambe cassée. Il m'interrompt : « Quand pouvons-nous y aller ? Es-tu libre jeudi ? » Plus proche de Lester que lui, j'ai un peu honte. Il a fait ses huit heures de train et nous avons passé quelques moments à bavarder avec Lester.

Plus récemment, début XXIe. Dans son bureau à Chevaleret, à peine avons-nous ouvert les dossiers du prochain volume du Séminaire, que le téléphone sonne. C'est le service des archives du Rectorat, à qui les désamianteurs de Jussieu viennent d'envoyer des documents exhumés d'une gaine technique, probablement stockés là par le secrétariat de Maurice Fréchet. Dans le lot, des papiers de Doblin, dont une carte postale de l'armée ; l'expertise de Marc est sollicitée. Nous passons notre après-midi aux archives, dans les combles de la Sorbonne surchauffés par le soleil de juin. Tant pis pour le Séminaire, il attendra !

Printemps 1991, à Haïfa. De notre hôtel, sur la hauteur, Marc et Jacques Azéma descendent un soir se promener dans le centre. Le lendemain, Jacques charrie gentiment Marc sur ses aventures supposées avec les filles du port ; et Marc de piquer un fard comme une jouvencelle.

On l'a dit et répété : Marc poussait à l'extrême la modestie et la disponibilité ; son bureau ne désemplissait pas de collègues et surtout d'étudiants, aux questions desquels il répondait avec patience-tout en tenant au téléphone un co-auteur outre-Atlantique et en rédigeant un fax. Mais je retiens surtout son regard bleu s'illuminant lorsque s'offrait l'occasion d'aider un jeune débutant ; et aussi un miracle toujours renouvelé : auprès de lui, on ne se sentait pas ridiculement petit.

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Souvenirs

Francis Hirsch

Mes souvenirs de Marc Yor remontent à presque 45 ans. C'était en octobre 1969, Marc avait 20 ans, et j'accueillais la nouvelle promotion de l'ENSET (Ecole Normale Supérieure de l'Enseignement Technique, qui allait devenir, des années plus tard, l'École Normale Supérieure de Cachan).

Il m'a raconté, longtemps après, qu'il avait été reçu à l'École Polytechnique, mais que son expérience du Service Militaire au camp du Larzac, par laquelle il avait commencé sa « scolarité », l'avait incité à démissionner de cette prestigieuse école.

Nous étions dans l'effervescence qui suivit Mai 68, et toute l'Université en était bouleversée… Tout semblait possible et les plus belles utopies fleurissaient. A l'ENSET, on multipliait les « assemblées générales » : Marc intervenait rarement, mais, quand il le faisait, c'était avec passion. Il cherchait à faire partager ses convictions profondes, sans être toujours bien compris.

C'est avec une passion encore bien plus forte que, dès cette époque, il s'est véritablement lancé dans les mathématiques et qu'il y a consacré toute son intelligence et son extraordinaire énergie. Il s'est tout de suite intéressé aux Probabilités et est rentré en 1973 au CNRS (en même temps que Claude Kipnis, qui était de la même promotion que lui à l'ENSET). On ne sait pas qu'un de ses premiers sujets d'étude a été la théorie quantique des champs à laquelle, à cette époque, Philippe Courrège cherchait à intéresser les mathématiciens français. Il éprouvait aussi beaucoup d'attirance pour la théorie des nombres, qui a été le sujet de sa « deuxième thèse ».

Après sa sortie de l'ENSET, nous n'avons cessé d'avoir des relations et de nous rencontrer en diverses occasions. A chacune de ces rencontres, il me parlait de questions mathématiques qu'il pensait pouvoir m'intéresser, mais ce n'est qu'à partir de 2005, lorsque j'ai pris ma « retraite », que nous avons vraiment collaboré et que nous sommes devenus véritablement amis. Je me rappelle que nous parlions aussi assez souvent de littérature : il m'a ainsi fait découvrir la littérature irlandaise qu'il aimait profondément.

Ces dernières années ont constitué pour moi et grâce à lui, une expérience intense et tellement enrichissante !

Peu avant sa mort, il me suggérait encore de nouveaux problèmes, et se disait « partant » pour les étudier avec moi…

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Lectures on Infinity

Yueyun Hu and Zhan Shi

Hardly anything distinguishes Saint-Chéron from other villages in the surrounding, in this part of the far suburb of Paris. A church dominates this small town from the hill, whereas a railway lies along the valley in the south and goes towards Paris. We are 38 km away from the capital. By train, it takes an hour; you had better arrive at the station in time because if you miss the train, you need to wait another half of an hour, and the waiting time is doubled in the weekend.

Marc Yor spent 29 years at Saint-Chéron, from 1985 until the last day. The small and quiet village is not very far from Marc Yor's birth place, Brétigny-sur-Orge, where he grew up for the entire childhood. Marc had some elderly relatives living at Saint-Chéron when he moved here with wife and children. The family was pleased with this well preserved small village, and installed itself in a house only a few meters away from the church. The house partly sat beside an ancient laundry, which was immediately served as a wonderful playground for children. For quite a long period, Marc was committed to the local junior football club. He trained the kids on Wednesday evening, and went for matches in the weekend. At Saint-Chéron, seldom anyone was aware that Marc Yor was an eminent mathematician whose name was printed in the dictionary, but everyone knew him as the coach of the football team.

Before arriving at Saint-Chéron, Marc Yor had spotted its north part, the *quartier de Baville*, a vast area filled with endless fields, groves and footpaths. *Baville* became Marc's favourite place for walking and jogging. When someone visited him at Saint-Chéron, there was a fair chance that the visitor got invited for a walk there.

We were invited several times for an excursion at *Baville*. It was on such occasions that we came to know Marc Yor better as a person, not just as a mathematician or as our teacher. Our conversation, though invariably starting with mathematics and with the problems we were working on, turned gradually to other topics as the walk went on. We discovered *Baville*, the castle and its surroundings with fascination, but our attention was essentially focused on the conversation. We learnt that in his youth, Marc's dream was to become a sailor; shortly after having obtained his *Baccalauréat*, Marc made, in September 1967, a tour around world on the *Ville de Tananarive*. We also realised that Marc was an enthusiastic admirer of the Russian literature, especially of Dostoyevsky and Solzhenitsyn; during one of the long walks that lasted more than an afternoon, he recited some of the Elder Zosima's lectures. Once, Marc was told that, in our effort of French learning, we were reading a novel which was widely considered as representing the Everest of world literature; with disarming thoroughness, he made a convincing analysis of the main characters, leading to the conclusion that the novel was much overestimated.

Our last visit at Saint-Chéron was also in the open air, but exceptionally, without any walk. In September 2011, we (together with several others) were invited by Marc Yor to spend an afternoon in a private garden, which was reputed as among the most beautiful in the village, and which was usually not available for visits. The garden was marvellous, the discovery of plants pleasant. And in the middle of the garden, Marc was seen to be seriously discussing with one of the junior guests, a soon-to-be-seven-years-old boy, and their topic of discussion was ... infinity! At that time, Marc was preparing an article on the notion of infinity for the *Lettre de l'Académie des sciences*, and he was making a case study of his method with the young boy. The junior was not aware of what infinity meant exactly, but was obviously rather excited at discussing on something of which he had some vague feeling or imagination, with an adult who was more than willing to use a language which was within the boy's capacity of understanding.

Nothing distinguishes Saint-Chéron from other villages. The church dominates from the hill. Marc rests in the nearby, not far from his favourite jogging area. We have learnt the notion of infinity.

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Poznan 2010

Michel Ledoux

Une conférence est organisée à l'occasion du centenaire de la naissance de J. Marcinkiewicz. L'événement est chargé d'émotion, quelques jours après l'accident d'avion du gouvernement polonais à Katyne, lieu même de la disparition de J. Marcinkiewiez soixante-dix ans plus tôt.

Marc Yor donne un exposé sur les pénalisations du mouvement brownien. Comme à son habitude, il entraîne, au tableau noir, un auditoire attentif depuis quelques observations initiales fascinantes, jusqu'aux mécanismes les plus profonds et subtils qu'il met en évidence et souhaite faire partager.

En fin de journée, nous prenons le tramway pour rejoindre l'hôtel en centre-ville. A bâtons rompus, il évoque quelques aspects de sa vie personnelle, et notamment une tante atteinte de la maladie d'Alzheimer qu'il vient voir tous les jours, tôt le matin, pour la lever et l'installer avant l'arrivée des services de santé, de la difficulté de communiquer, de trouver des personnels compétents et bienveillants. La conversation allant, je lui mentionne que ma belle-sœur vient de recueillir son père, atteint de cette maladie, chez elle, et qu'elle est un peu désemparée sur la marche à suivre. Le lendemain, Marc me tend deux pages manuscrites de son écriture claire et serrée (il aimait tant écrire, des mathématiques, des lettres), détaillant conseils et recommandations à l'intention de ma belle-sœur, lui souhaitant bon courage.

La conférence se poursuit.

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Une excursion avec Marc Yor[5]

Roger Mansuy

Ce 9 janvier 2014, Marc Yor s'est éteint. Le grand public (et même certains collègues) ne connaît pas ce mathématicien de premier plan et expert internationalement reconnu en probabilité. Difficile de résumer une si riche carrière en quelques mots : plus de 400 articles de recherche avec un nombre exceptionnel de collaborateurs et d'étudiants encadrés, une descendance scientifique pléthorique, un livre de référence sur le calcul stochastique continu,[6] un rôle éditorial important,[7] un siège à l'Académie des Sciences… Il est aujourd'hui quasiment impossible d'évoquer le moindre résultat fin sur le mouvement brownien ou les processus de la galaxie brownienne sans citer Marc Yor ou l'un de ses élèves.

L'humilité est sûrement une des premières qualités que tous ceux qui l'ont connu mentionnent : il la couplait avec une forme de timidité, la peur de négliger le mérite des autres,[8] la mise en avant de ses co-auteurs. À l'heure de peser l'importance d'une œuvre scientifique ou la persistance des idées, cette humilité apparaît comme un obstacle à une juste reconnaissance. J'espère que d'autres, plus avertis, viendront lui rendre un hommage appuyé détaillant ses premiers travaux en théorie des processus dans la veine de Paul-André Meyer, l'étude du mouvement brownien plan, les fonctionnelles quadratiques du mouvement brownien et les excursions, les fonctionnelles exponentielles et l'utilisation du théorème de Girsanov, les pénalisations… et les mathématiques issues de la finance.[9]

J'ai eu la chance d'être son étudiant, de bénéficier de ses lumières, de son soutien et de son aide. Mais plus que le matheux, c'est un homme que j'ai rencontré, entier, riche, émouvant, inoubliable. J'aimerais reconstituer une journée avec Marc Yor ou pas très loin de lui. Une journée « ordinaire ».

Voyage en 2004 ou 2005 au troisième étage d'un bâtiment de la rue du Chevaleret où les mathématiciens de Jussieu ont trouvé refuge le temps du désamiantage. Plus précisément, nous avons monté trois étages dans l'aile D. Il est tôt, voire très tôt pour les usages de la recherche, disons 7:20, nous avons rendez-vous ; les couloirs sont déserts et tous les bureaux fermés et obscurs. Tous ? Non ! Un bureau résiste encore et toujours à cette nuit : celui de Marc Yor. La lumière est allumée, la porte grande ouverte laisse voir des piles et cartons de documents savamment triés (un jour, il a dit, avec beaucoup d'autodérision, qu'il était sûrement plus facile

[5] Article paru dans *Image des Mathématiques* — CNRS 2014. Avec l'aimable autorisation des Éditeurs.

[6] *Continuous Martingales and Brownian Motion*, avec Daniel Revuz, Springer.

[7] Particulièrement les décennies avec la coordination du Séminaire de Probabilités de Strasbourg aux éditions Springer et le travail aux Comptes-Rendus de l'Académie des Sciences.

[8] Nous avons finalement placé 15 pages de références dans les 150 pages du Lecture Notes *Random Times and Enlargements of Filtrations in a Brownian Setting*.

[9] Voir ses tribunes dans *Matapli* et dans la *Gazette des Mathématiciens* reprises sur *Images des mathématiques* où il explique sa position sur les mathématiques financières comme partie des mathématiques et sur la responsabilité des mathématiciens.

d'expliquer son dernier calcul que le rangement de son bureau), un étroit passage de sa chaise jusqu'au téléphone, l'emplacement d'une seconde chaise (ce matin-là, elle est occupée des remarques obtenues pendant son trajet en RER), un tableau noir qui n'a pu être effacé depuis quelques jours (faute d'accès, on y retrouve un reste de formule d'Itô) mais nulle part la moustache du propriétaire des lieux.

Pour le trouver, il faut avancer d'une dizaine de mètres et aller à la bibliothèque du laboratoire. Avec le passe du labo, il a pu ouvrir la porte et s'adonner à ses tâches matinales : lire les fax puis répondre à ses correspondants et coauteurs (rétif à l'ordinateur, il peut ainsi communiquer rapidement et archiver tous les échanges) et lire les dernières revues (en apprenant les sommaires d'une manière qui lui permet souvent de répondre plus vite qu'un thésard avec MathSciNet ou ZentralBlatt). Si nous sommes entrés discrètement, nous avons de grandes chances de le voir plié sur un brouillon très soigné en train de mener un calcul.

C'est d'ailleurs une constante, tout au long de la journée lorsque nous le croiserons en train de faire des mathématiques seul, il sera en train de faire un « calcul ». Certes, les calculs dont on parle ici sont fort élaborés et, pour Marc, faire un calcul, ce n'est pas simplement obtenir le résultat mais désosser chacune des étapes, en isoler tous les arguments pour les généraliser ou les transposer à d'autres situations. Suivre un calcul à ses côtés réclame souvent une bonne concentration et une culture probabiliste au-delà du sujet du calcul. Marc Yor est un calculateur hors norme qui a transformé la dextérité technique en un art. Cette compétence rare qui réclame discipline et finesse de compréhension lui a souvent permis de voir ce que personne d'autre ne voyait.

Revenons à la bibliothèque et manifestons-nous d'un raclement de gorge : salutations polies et installation rapide, nous entrons dans la danse. Nous dressons le bilan de ce que nous avons fait ou essayé de faire, nous nous intéressons à la liste des blocages rencontrés ; Marc donne quelques réponses rapides (il faudra tout débriefer pendant la journée pour être sûr d'avoir tout suivi à sa vitesse) et des références indispensables auxquelles nous n'avions pas pensé. Il y a une forme d'urgence car dès que le labo se réveillera ou qu'un correspondant lointain téléphonera, notre temps de Marc Yor diminuera rapidement. Sachant qu'il accueille tout le monde avec bienveillance, il est très (trop) sollicité : ce temps du matin est un privilège que nous envient tant d'étudiants ou de chercheurs.

Fin de la séquence bibliothèque, on retourne dans notre bureau de thésards et on essaie de se débrouiller seul, de faire germer toutes les idées ou indications. On plaisante avec d'autres thésards d'une remarque récurrente ou d'une lubie du moment. Quelques portes plus loin, ça bourdonne : discussions expertes, coups de téléphone, visites... Au hasard d'un passage dans le couloir, on peut croiser deux personnes qui attendent leur tour pour discuter avec Marc Yor. Lorsqu'il est à son bureau, la matinée est souvent un marathon et on se surprend de sa facilité à changer de sujet sans perdre le fil des discussions ou sortir un brouillon sur lequel il a écrit quelques pistes pour le problème « neuf » que l'on amène.

Le vendredi déroge à cette routine : tout d'abord car il y a un groupe de travail important à 11 h ; ensuite car Marc Yor a doublé ce rendez-vous d'un groupe de travail pour ses thésards, collaborateurs du moment, visiteurs : le WIP (pour Work

In Progress), lieu où, nerveux et stressé, chacun de ses thésards fera ses premiers exposés, entendra ses premières félicitations, écoutera les suggestions de Marc ou de ses camarades thésards. À ce moment-là , la « famille » travaille ensemble.

Avant de repartir pour une après-midi aussi frénétique que la matinée (voire davantage avec le réveil du continent américain, le travail sur les Comptes rendus de l'Académie des sciences ou le séminaire du labo le mardi), la pause déjeuner offre l'occasion de sortir un peu des mathématiques (mais pas toujours). Devant un filet de poisson, Marc peut intervenir dans les discussions d'actualité, évoquer le football ou l'un de ses enfants ou petits-enfants... Il ne monopolise jamais la parole mais est écouté.

Passons directement à la fin de la journée : nous avons l'occasion de recroiser Marc à la bibliothèque, nous allions reposer un livre avant de partir mais Marc a réfléchi et il a une réponse plus convaincante à une discussion du matin, on s'installe à nouveau : nous avons même droit à un petit regard en coin et un sourire lors d'une remarque pertinente (la journée est donc fructueuse). La fatigue semble avoir davantage de prise sur nous que sur lui, nous ne suivons plus aussi assidûment : à peine voyons-nous le mouvement du bic et l'éclat de son alliance Claddagh, il regarde sa montre et, d'un seul coup, il est pressé, expédie son dernier commentaire et se dépêche d'empaqueter ses affaires pour ne pas rater le train qui le ramène dans l'Essonne.

La journée a été longue, plus encore pour Marc et pourtant il trouvera un instant dans le RER ou chez lui pour refaire un calcul, vérifier une référence, créer une nouvelle connexion entre deux articles...

Evidemment toutes les journées n'avaient pas cette structure ; la richesse des rencontres et les possibilités de la recherche permettent des « extras ». Voici quelques souvenirs personnels.

– Une semaine WIP en Normandie : Marc avait rassemblé ses thésards dans une longère pour organiser un groupe de travail intensif. Chaque jour, nous faisions des exposés (préparés dans la soirée, la nuit...) ; les pauses étant dévolues au football dans le jardin, à aller chercher du pain à vélo, à écouter Joseph Najnudel nous interpréter une gnossienne centrée réduite sur le piano droit de la maison...
– Les très longs remerciements lorsque nous avons pensé à l'anniversaire de sa femme lors de l'École d'été de Saint Flour en 2002.
– Un voyage aux États-Unis pour aller faire un cours à Columbia University avec les évocations matinales du panneau *Sakura Park*.
– La dernière relecture des épreuves d'un livre un samedi matin sous un kiosque du jardin du Luxembourg.
– Une longue discussion sur un article en collaboration qui nous plaisait bien : comment lui donner un titre un peu accrocheur ? Au bout d'une bonne vingtaine de propositions, on finit par s'accorder sur *Harnesses, Lévy bridges and Monsieur Jourdain* comme si l'on faisait une bonne blague aux gens qui liraient ce texte : des sourires à chaque fois qu'on en reparlait.

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Témoignage et hommage

Laurent Miclo

Les deux cours qui ont décidé de mon orientation scientifique ont été ceux dispensés par Marc Yor en magistère lors de l'année universitaire 1986/1987. Pourtant leur début a été assez déroutant. Il fallait d'ailleurs qu'il le soit : nous étions plusieurs étudiants provinciaux fraichement débarqués à Paris avec une idée encore assez vague de ce qu'étaient les mathématiques qui allaient nous occuper, malgré nos classes préparatoires. A l'époque la théorie des probabilités n'était pas enseignée avant la troisième année d'université et nous étions plus disposés à apprécier de belles égalités que des raisonnements presque sûrs... Notre premier contact avec les probabilités fut donc le cours de Marc, qui commença par une définition et des propriétés abstraites des tribus. Ce fut suffisamment aride pour qu'il en résulte un certain brouhaha dans l'amphithéâtre de l'IHP. Marc décida de donner plus de chair à ces concepts... en nous parlant directement du mouvement brownien, ce qui n'arrangea pas vraiment la compréhension (enfin du moins la mienne, car je ne saurais préjuger de celle des mathématiciens célèbres en devenir de l'auditoire).

Mais, peut-être intrigués par un objet mathématique si différent de ce que nous avions pu rencontrer jusqu'alors, nous sommes revenus aux séances suivantes, puis nous avons rempilé pour le cours du second semestre. Pour les plus mordus, nous avons également participé à un groupe de travail où nous travaillions les exercices du livre culte que Marc était en train d'écrire avec Daniel Revuz. Le style de Marc, son accessibilité bienveillante et jusqu'à l'élégance de son écriture au tableau, avaient fini par nous séduire et en un an, nous sommes passés du niveau zéro de la théorie de l'aléa à certaines propriétés fines des solutions d'équations différentielles stochastiques. Surtout il nous avait transmis cette intuition assez particulière aux probabilités. Bien sûr j'ai suivi d'autres cours excellents, dans divers domaines des mathématiques, mais c'est à ceux de Marc que je suis resté le plus attaché, presque sentimentalement.

On m'a rapporté à plusieurs reprises que Marc ne craignait pas d'enseigner en première année d'université, prenant le risque de n'être pas apprécié à sa juste valeur et alors que sa renommée aurait pu lui permettre de ne revendiquer que des « cours prestigieux ». Bien qu'il était un puits de connaissance (quand devenu chercheur, je continuais de lui poser des questions, j'étais sûr de repartir avec plusieurs références), il était doué de cette qualité assez rare de ne jamais prendre ses interlocuteurs du haut de son savoir. Il était académicien, mais rien dans son authentique simplicité et dans sa gentillesse ne le suggérait. Au contraire, à chaque fois que j'avais la chance de le retrouver, notamment aux Journées de Probabilités de Luminy, il était extrêmement modeste, cherchant à relativiser ses propres contributions et portant une attention sincère aux travaux des jeunes.

Je n'ai pas revu Marc ces dernières années, alors qu'enflaient les polémiques sur les mathématiques financières. Mais dans le fait qu'il ait pu en être terriblement affecté, je n'ai perçu qu'une autre manifestation de son irréductible intégrité.

C'est autrement que par cette lettre que j'aurais voulu pouvoir exprimer ma profonde gratitude à Marc, en particulier pour l'héritage mathématique et l'honnêteté exigeante qui continueront de nous servir de modèles.

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Joseph Najnudel

De ce que j'ai vécu avec Marc Yor, il m'est difficile de choisir un unique événement particulier à raconter. Voici quelques souvenirs qui me viennent naturellement : la première fois qu'on s'est parlé, et qu'il m'a donné mon sujet de mémoire de DEA (l'homotopie du mouvement brownien sur la sphère privée de trois points), quelques jours avant de partir pour le Japon, le traditionnel séminaire WIP (work in progress) du vendredi matin, avec ses nombreux étudiants en thèse, les cours que nous avons faits ensemble, en particulier en octobre 2005 à l'Université de Warwick, et en juillet 2006 à Torgnon, un village italien de la Vallée d'Aoste. C'est dans ce village que nous avons également regardé ensemble la finale France-Italie de la Coupe du Monde de football avec les participants de l'école d'été ! À Torgnon, j'ai également fait un concert d'orgue, puis deux autres concerts à l'église de Saint-Chéron, en décembre 2009 et avril 2010 : les trois concerts d'orgue que j'ai faits dans ma vie ont donc été organisés par Marc... Je serai toujours reconnaissant pour toute l'aide que Marc m'a apporté, pour ses qualités humaines et son enthousiasme pour la recherche. Je garderai toujours le souvenir de notre collaboration, en particulier à la fin de ma thèse et au début de mon post-doc, où nous nous appliquions à essayer de percer le mystère des pénalisations browniennes...

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An Appraisal of Marc's Work

David Williams

In 2007, I was asked to write a brief appraisal of Marc's work. I wrote:

"Marc Yor has made an immense contribution to Probability Theory, perhaps the greatest contribution of any European of the post-Meyer generations. Interestingly, he has made most of his contribution in a quite different way from Meyer. Meyer could be said to follow in the magnificent Bourbaki tradition, developing the fundamentals of a huge area of Probability Theory. In the early part of his career, Yor made important contributions in this spirit to Martingale Theory and Stochastic-Integral Theory. But for most of his career, he has preferred to concentrate on concrete problems. The number of papers he has written is astonishing. Yet all are of real interest, most contain surprises, and all develop the theory via the concrete in that results true in much wider contexts (and stated and proved in those wider contexts) may be found throughout his work on very concrete things. He continually asks of a surprising concrete result: 'What are the real reasons why it is true?'. He seems always to believe that each explanation matters, leads somewhere – very much in the spirit in which Gauss (rightly) regarded quadratic reciprocity (although I am not comparing Yor, brilliant though he is, to Gauss!). We all firmly believe that if anything regarding Brownian motion, diffusions, stochastic integrals, martingales, excursion theory, mathematical finance, the Riemann zeta-function, Bessel functions, theta functions, etc, can be calculated, then Marc can calculate it, even if no-one else can. We know that if we read a Yor paper, then, firstly, we shall enjoy it and be surprised, and, secondly and equally importantly, our understanding of the global structure of the subject will be enhanced. He is someone of whom France should be very proud, someone upholding the tradition of the greatest of all probabilists, Paul Lévy."

How much more this is true now. But the greatest truth about Marc is that he was a wonderful person, kind and generous. Our thoughts go out to his family.

Marc et le dossier Doeblin

Bernard Bru

1 Introduction

Nous voudrions parler de la rencontre de Marc Yor avec Wolfgang Doeblin, c'est-à-dire d'une courte période de travaux historiques menés à bien par Marc entre 2000 et 2001, suivie d'innombrables interventions entre 2001 et 2013 partout dans le monde, qui ont fait du dossier Doeblin probablement l'une des manifestations mathématiques les plus largement connues de la période récente. On ne sait pas si ce dossier sera repris par les générations futures, comme le dossier Galois l'est encore périodiquement, mais on est certain au moins d'une chose : il a commencé d'exister par l'action et les travaux de Marc seul. Un cas d'histoire des mathématiques unique à bien peu près par son ampleur, qu'il s'agit de raconter brièvement.

Wolfgang Doeblin est né à Berlin le 17 mars 1915. Son père, le grand écrivain allemand Alfred Doeblin, est un antinazi de la première heure. Exilé à Paris pendant l'été 1933 avec sa famille, Wolf (le diminutif familial de Wolfgang) y fait l'essentiel de ses études universitaires, à l'issue desquelles il entreprend des recherches en calcul des probabilités, commencées au début de l'année 1936 et interrompues au moment de son incorporation en novembre 1938. Doeblin reprend son travail de recherche, quelques heures volées à ses obligations militaires, pendant l'été 1939 puis l'hiver 1939–1940 et le printemps 1940, alors qu'il cantonne dans les Ardennes et en Lorraine dans des conditions extrêmes (1). En ce très court laps de temps, il a cependant réussi à publier un ensemble de mémoires assez exceptionnels pour que Paul Lévy [1955] fasse de lui l'égal d'Abel et de Galois « pour avoir résolu à un si jeune âge et en si peu de temps des problèmes aussi difficiles » et qu'on puisse affirmer, assurément, que W. Doeblin est, avec Khinchin, Kolmogorov et Lévy, un des maîtres et des créateurs de la nouvelle théorie des probabilités de l'entre-deux-guerres (2). C'est en tout cas l'avis de tous les témoins du temps des plus modestes aux plus considérables qui ont eu accès aux travaux de Doeblin. Par exemple, Gnedenko écrit dans son hommage à Khinchin qui mieux que quiconque savait la profondeur des travaux de Doeblin ([1961], p. 9) : «I remember how proud he was that in our science there had appeared such a bright new representative as V. Doblin, while at the same time he mourned Doblin's untimely death at the hands of Hitler's executioners. » Quant à Doob, qui n'était généralement pas complaisant dans ses jugements mathématiques, lorsqu'il apprit en 1991 que Doeblin avait déposé un pli cacheté à l'Académie des sciences avant de mourir, il fit ce commentaire laconique : ce doit être « quelque chose ». C'est Doob qui avait rédigé les premiers comptes rendus des travaux de Doeblin pour les *Math. Reviews* de 1940, et ses *Stochastic Processes* sont d'esprit doeblinien, comme on sait.

Bref, Doeblin, malgré la brièveté de sa carrière mathématique, n'a jamais été un inconnu, un héros solitaire, incompris, rejeté par la médiocrité ambiante. S'il y a un mythe Doeblin, il ne relève pas de ce genre-là, qu'on associe parfois à Galois ou Abel, à tort le plus souvent. Les travaux de Doeblin ont été reconnus comme fondamentaux dès la fin des années 1930 par les plus grands noms de la discipline à laquelle ils étaient consacrés. Cette reconnaissance s'est poursuivie longtemps après la guerre, au fur et à mesure que des savants découvraient que leurs travaux les plus récents étaient énoncés déjà dans des mémoires de Doeblin peu accessibles (3). De sorte qu'il n'est pas très étonnant qu'on ait souhaité célébrer, ici ou là, le cinquantenaire de sa mort tragique et de son œuvre inachevé. Il faut citer en particulier le bel article de synthèse de T. Lindvall [1991] et le colloque de Blaubeuren, Cohn [1993] (4). C'est à l'occasion de ces célébrations, qu'on a tenté une nouvelle recension des manuscrits de Doeblin déposés aux archives littéraires de Marbach en Allemagne avec les archives de son père et en divers fonds d'archives parisiens, d'où il ressortait que Doeblin avait déposé un pli cacheté « sur l'équation de Kolmogoroff », c'est-à-dire la théorie des diffusions en dimension un, thème qu'il avait abordé au printemps 1938 et sur lequel il avait publié plusieurs notes très peu explicites entre 1938 et 1940.

Il serait trop long de décrire les péripéties de cette histoire, d'autant qu'elle est restée relativement confidentielle et marginale au sein de la communauté probabiliste alors en grande activité (5). Des problèmes difficiles sont résolus et plusieurs théories sont développées aux applications multiples, martingales, calcul stochastique, étude fine du mouvement brownien, etc. qui semblent tout à fait absentes de l'œuvre de Doeblin. Ce qui peut expliquer que le dossier Doeblin n'ait guère passionné les mathématiciens, non plus que le public curieux d'histoire et de littérature, intéressé par son père Alfred (6). Finalement, en avril 2000, les ayants droit de W. Doeblin demandèrent officiellement l'ouverture du pli dont il s'agit. Ce qui fut fait par la commission des plis cachetés de l'Académie, le 18 mai 2000, dans l'indifférence quasi générale. Pierre Dugac qui avait suivi toute l'affaire au titre de la commission étant décédé en mars 2000, c'est Jean-Pierre Kahane qui se chargea du dossier et me le remit, étant bien entendu que je ne ferais que la partie technique du travail, transcription et annotations sommaires pour restituer le texte dans son époque, et qu'il appartiendrait à la Commission de décider ce qu'il y avait lieu d'en faire. À la rentrée 2000, le travail préliminaire était achevé et Jean-Pierre Kahane nous apprit que Marc avait accepté d'évaluer le dossier, au nom de l'Académie des sciences.

2 Automne 2000

On ne sait pas quelles raisons ont poussé Marc à se charger d'un dossier a priori sans rapport visible avec ses centres d'intérêts, d'autant qu'il était débordé de travail. La première raison qui vient à l'esprit est son sens du devoir. Marc avait été élu à l'Académie en 1997. Il a pu se sentir obligé d'accepter cette tâche ingrate, qu'il était sans doute un des seuls, quai Conti, à pouvoir mener à bien dans des délais raisonnables. Il existe naturellement mille autres raisons parmi lesquelles il faut placer au premier rang la curiosité insatiable de Marc pour tout ce qui touche aux mathématiques et son immense capacité de travail et d'enthousiasme.

Donc en novembre 2000, Marc m'a téléphoné pour que je lui remette le « dossier Doeblin », c'est-à-dire la photocopie du manuscrit original, sa transcription, les notes et divers documents pouvant lui permettre de se faire une idée de son contenu. Ici commence notre récit, où tout est baigné, on le verra, dans une sorte de poésie singulière. Marc savait que je me déplaçais le moins possible. Dans ces conditions, tout rendez-vous devenait problématique. Marc m'a proposé que nous nous retrouvions un soir, après son travail, dans la rotonde de la station de métro Porte d'Orléans, à deux pas de la maison. Par la ligne 4, il rejoindrait ensuite le RER C et son domicile de Saint-Chéron, en commençant à travailler sur le dossier. On sait que Marc travaillait tous les matins et tous les soirs dans le RER, au cours de ses longs trajets de chez lui au laboratoire et retour. Le RER C était son second bureau où il n'était dérangé par personne. Ainsi fut fait et tous les rendez vous de travail sur le dossier Doeblin ont eu lieu de la même façon à la station Porte d'Orléans, vers 19 heures, entre deux métros.

Il faut rappeler que cette station était alors en travaux, la ligne 4 devant être prolongée jusqu'à Montrouge. De sorte que l'éclairage très ancien avait été tout à fait négligé et qu'il régnait dans la rotonde comme dans toute la station une atmosphère lugubre et un froid glacial. Le métro parisien ne passe pas pour être particulièrement convivial, mais à l'automne 2000, le terminus de la ligne 4 ressemblait assez au Berlin des années d'après-guerre, et tout y paraissait hors du temps, ce qui est sans doute propice à un travail historique de fond. Quelque temps plus tard, mais très rapidement, Marc m'a téléphoné pour me dire que le dossier était extraordinaire, qu'il fallait tout publier et m'a fixé un autre rendez-vous orléanais pour me donner ses corrections et ses ajouts. De nouveau, dans la rotonde du métro Porte d'Orléans, Marc m'a remis le dossier entièrement annoté et complété et m'a raconté l'histoire suivante. Il avait commencé à lire le dossier dans le RER, sans a priori, en souhaitant se débarrasser au plus vite de ce travail sur commande. Un soir, m'a-t- il dit, il en était arrivé au paragraphe XV intitulé « changements de variables » et là il avait eu un choc, une révélation. Sous ses yeux, il voyait l'énoncé et la démonstration d'une formule d'Itô analogue à celle qu'il enseignait depuis dix ans dans son cours de DEA, à ceci près qu'il n'y avait pas d'intégrale par rapport au mouvement brownien, une intégrale d'Itô, mais un mouvement brownien changé de temps. Marc savait naturellement que Kolmogorov dans son mémoire fondamental [1931] avait consacré un paragraphe aux changements de variables (7). Sous les conditions de Kolmogorov, la loi d'une diffusion satisfait à l'équation parabolique de Kolmogorov, et les techniques de changement de variables dans les équations paraboliques sont classiques à l'époque et permettent sous certaines conditions de se ramener à l'équation de la chaleur. Mais là c'était très différent. Les diffusions générales de Doeblin s'expriment (au moins localement) à l'aide d'un mouvement brownien changé de temps. On obtient une formule de changement de variables à l'aide de ce même mouvement brownien changé de temps d'une autre façon. Dans les deux cas, il s'agit d'un calcul brownien, un calcul stochastique, dont Marc était l'un des maîtres incontestés et qu'il appliquait à mille situations tout à fait actuelles. Doeblin faisait du calcul stochastique, c'est-à-dire cette sorte particulière de calcul des schémas stochastiques en deux parties, une partie classique, correspondant à la composante non aléatoire du mouvement, la dérive, et une partie brownienne pour la composante aléatoire.

Donc Doeblin avait fait aussi du calcul stochastique (8) et se trouvait ainsi au cœur de la théorie moderne, celle de Marc et des probabilistes des universités de l'an 2000. De plus sa théorie des diffusions utilisait, et anticipait donc d'une trentaine d'années, les méthodes de martingales qui conduiraient dans les années 1960 aux travaux fondamentaux de Stroock et Varadhan (9). Voilà qui était formidable. L'enthousiasme de Marc à cette lecture fut tel qu'il attira l'attention du voyageur assis en face de lui, dont on ne sait rien, et qui, probablement, rentrait chez lui après sa journée de travail. Ce dernier, intrigué par le comportement très inhabituel de son vis-à-vis, lui demanda ce qu'il lisait avec tant de passion et Marc commença aussitôt à lui raconter l'histoire du soldat Doblin et bien sûr, emporté par son élan, la formule d'Itô et son rôle en calcul stochastique. Peut-être qu'en donnant, dans le RER C, pour cet unique auditeur, sa première conférence sur le pli cacheté, prit-il conscience qu'il y avait là quelque chose d'unique, un trésor qu'il fallait partager ? Toujours est-il, qu'arrivé à Saint-Chéron, il rédigea un rapport enthousiaste pour la commission académique, qui décida aussitôt de publier le pli cacheté dans les *Comptes rendus de l'Académie*, une procédure prévue par les règlements, bien que très exceptionnelle. C'est Marc qui en fut l'éditeur, avec l'efficacité et le sérieux que l'on connaît (10) et le pli cacheté parut dans le dernier numéro des Comptes rendus de l'année 2000. À l'occasion de sa sortie, l'Académie, par l'intermédiaire de son Service de presse, envoya un communiqué à l'AFP résumant toute l'affaire, qui fut aussitôt repris par toute la presse française et étrangère. Il y a là un phénomène classique d'emballement médiatique, avec cette particularité, classique pour tout ce qui touche aux sciences, de concerner un texte totalement illisible et des résultats dont l'énoncé même est parfaitement impénétrable, pour qui n'est pas spécialiste du sujet. Ainsi le dossier Doeblin devenait l'affaire Doeblin. Un soldat de vingt ans, mort pour la France en 1940, avait envoyé, quelques heures avant sa mort, un document exceptionnel qui anticipait les calculs financiers les plus sophistiqués. Les journaux rivalisèrent d'imagination sur ce thème, faute de pouvoir pénétrer un sujet irréductiblement technique dont on savait seulement qu'il était lié mystérieusement aux salles de marché des grandes banques. D'ailleurs, on était en 2000 et les mathématiques financières semblaient devoir assurer la paix et la prospérité du monde pour l'éternité.

Dans cette cacophonie, Marc resta imperturbable. Il répondit à toutes les demandes d'interviews, participa à deux documentaires importants où il tenait le premier rôle, sans ostentation, avec une exigence scientifique très remarquable. Un éditeur parisien, voulant surfer sur cette vague stochastique inattendue, demanda à Marc Petit, écrivain, peintre, poète et germaniste, de raconter toute l'affaire, le Berlin d'avant 1933, l'exil à Paris des écrivains allemands anti-nazis, l'IHP à la fin des années trente, la guerre de 1939–1940 en France, etc. La première chose qu'il fit fut de demander un rendez-vous à Marc, lequel eut lieu dans un café proche de Jussieu. Les deux hommes sympathisèrent aussitôt. Marc tint à expliquer à l'écrivain la partie mathématique du dossier, notamment les changements de temps du mouvement brownien qui permettent d'écrire les diffusions de Doeblin. Marc utilisa pour cela la nappe en papier recouvrant leur table, qui fut bientôt entièrement couverte de formules écrites très soigneusement de cette écriture si reconnaissable, avec ses notations très précisément écrites et ses signes intégrales en arabesques andalouses. À la fin de l'entretien, Marc Petit qui n'avait rien compris emporta la nappe. Elle lui servit de support mathématique virtuel pour écrire son livre. Elle est maintenant déposée à l'IMEC, avec toutes les archives de Marc Petit (11).

Un point chagrinait Marc particulièrement. Le communiqué de l'Académie évoquait la formule d'Itô, une locution que les journalistes et les commentateurs du pli ignoraient tout à fait. Mal conseillés sans doute par des personnes mieux informées qu'eux, mais pas suffisamment, ils transformèrent allègrement la locution formule d'Itô en intégrale d'Itô, qui sonnait mieux. De sorte que Doeblin fut proclamé par la presse mondiale et la rumeur publique inventeur de l'intégrale d'Itô. C'était un contresens évident pour qui avait lu le pli et l'analyse qu'en avait faite Marc. Il n'y a pas dans le pli la moindre trace d'intégrale stochastique. Selon certains témoignages, celui de Laurent Schwartz en particulier, Doeblin aurait dit à Lévy, au cours d'un repas dominical, qu'il voulait s'intéresser à l'intégrale stochastique, mais il n'en a rien fait à notre connaissance, et, en tout cas, il n'y a pas d'intégrale par rapport au mouvement brownien dans le pli de 1940 (12). Il était donc parfaitement injuste de priver Itô de son intégrale. Pour que les choses soient dites et redites, Marc multiplia dès lors les rectificatifs, notamment dans [2001], [2002] et dans tous les exposés qu'il fit sur ce sujet de 2001 à 2013. Nous n'avons d'ailleurs pas réussi à recenser toutes les interventions de Marc sur le pli cacheté de Doeblin, plusieurs dizaines en tout cas, partout dans le monde et devant les publics les plus variés.

Peu à peu, dans l'indifférence générale et le scepticisme du plus grand nombre, à force de détermination, Marc réussit à ébranler quelque peu le panthéon probabiliste de la fin du 20$^{\text{ème}}$ siècle, dont les plus hautes marches étaient occupées sans conteste par Kolmogorov, pour l'axiomatique moderne des probabilités, Lévy pour l'étude fine du mouvement brownien, Doob pour la théorie des martingales et Itô pour le calcul stochastique. Tout le monde savait bien que c'était simpliste (13), mais ce n'était pas totalement faux et suffisait amplement. Il n'y avait pas lieu de s'y attarder plus longuement. La recherche n'attend pas et elle est loin devant. Marc, l'un des chercheurs les plus actifs et les plus brillants de la théorie, en engageant sa réputation qui n'était pas mince, entreprit au contraire, pendant plus de dix ans, avec une énergie considérable, de faire une place dans l'histoire à Doeblin et à d'autres dont on n'entendait plus parler, et de montrer que ce n'était pas sans intérêt aussi bien pour l'enseignement que pour la recherche, en tout cas pour son enseignement et pour sa recherche (14). Marc a d'abord lu Doeblin parce qu'on le lui avait demandé et qu'il fallait bien tenter de comprendre ce que le soldat téléphoniste avait fait. Mais il a senti, soudain, que ce texte mal écrit sur un cahier de brouillon aux pages déchirées, le concernait personnellement, au moins concernait les mathématiques qu'il pratiquait, et que cela enrichissait son travail, mathématiquement sans doute, mais aussi humainement, par le tragique et la beauté de toute l'histoire et les valeurs qu'elle portait, qui étaient aussi les siennes, rigueur intellectuelle et morale, valeur de l'exemple et du sacrifice, valeur extraordinaire de la vie des gens ordinaires, le soldat Beaujot ou le passager du RER C, (15), … Au paragraphe suivant nous essayons d'analyser sommairement la démarche historique de Marc, sans prétendre y réussir en rien.

3 Questions d'histoire

D'abord une remarque évidente. Si Marc n'avait pas été Marc, il n'aurait vu dans le pli de Doeblin qu'un texte comme les autres appartenant à la préhistoire de la théorie des diffusions, après ceux de Kolmogorov et Feller, et le pli serait retourné dans sa boîte, aux Archives de l'Académie, à la disposition d'un éventuel érudit intéressé par les documents rares et curieux. C'est bien parce que Marc avait enseigné cent fois le calcul stochastique, en avait fait la théorie et l'avait appliquée plus que quiconque, qu'il a vu dans le RER C brusquement que Doeblin faisait ce qu'il faisait cinquante ans après lui. De sorte qu'on peut dire que le Doeblin que Marc a révélé est le Doeblin de Marc plus que le véritable dont on ne saura jamais rien. Mais cette remarque s'applique à toute étude mathématique d'histoire des mathématiques. Soit elle s'en tient au mot à mot sans chercher à aller au-delà et à comprendre ce qui n'est pas encore exprimé, mais qui est là sans doute et qui se dévoilera plus tard. Elle relève alors de l'histoire externe, qui est fort intéressante, mais éloigne généralement des mathématiques vivantes. Soit elle interprète le fond que le mot à mot ne révèle pas et cette interprétation dépend plus ou moins fortement des connaissances et des recherches personnelles de l'historien en question, quelque précaution qu'il prenne pour s'en affranchir et s'en tenir au texte qu'il étudie (16).

Il faut donc chercher plus loin ou ailleurs. Le mieux est encore de prendre un autre exemple et de le comparer à celui du Doeblin de Marc. Pour faire simple, choisissons le dossier Galois que tout le monde connaît, ce qui permet d'aller vite.

Évariste Galois est mort, à 20 ans, des suites d'un duel, le 31 mai 1832. Il avait publié quelques très courts articles aux *Annales de Gergonne* et au *Bulletin de Férussac*, mais son principal mémoire avait été refusé par Poisson pour excès de concision et d'obscurité, ce qui n'était pas faux. À sa mort, tous ses papiers ont été conservés par son ami Michel Chevalier, qui, à plusieurs reprises, a tenté d'y intéresser les plus grands mathématiciens du temps, sans succès, jusqu'à ce que Liouville s'en empare, les étudie longuement et annonce à l'Académie en

1843, qu'ils contiennent une « solution aussi exacte que profonde » d'un « beau problème » de théorie des équations. Il ajoutait qu'il allait publier dans son *Journal* l'ensemble des travaux de ce jeune savant accompagné des explications nécessaires, (17). Ce qu'il fit en 1846, mais sans aucune explication, de sorte qu'on peut penser que, s'il s'est convaincu de la justesse de la démonstration de Galois, il n'en a pas saisi le fond, ce qui maintenant s'appelle la théorie de Galois (18).

La publication des *Œuvres* de Galois ne paraît pas avoir eu de conséquences immédiates. Un mémoire de 1831 résolvant de façon incompréhensible un problème déjà plus ou moins traité par plusieurs savants, Lagrange, Ruffini ou Abel, qui, d'autre part, ne relevait pas des « sciences mathématiques » les plus hautes du moment, analyse, physique mathématique ou mécanique analytique, n'intéressait visiblement pas grand monde. Quant aux circonstances particulières de la mort de Galois, il valait mieux n'en plus parler. Elle mettait en cause l'honneur d'une jeune fille de bonne famille. Le devoir de réserve s'imposait.

Tout ça était bien triste, mais c'était du passé. Le Galois de Liouville disparut donc à son tour, malgré le bel hommage que lui avait rendu le savant toulois en introduction de son édition. Ce n'est pas parce que Galois est mort tragiquement à 20 ans, et qu'il a écrit des mathématiques, qu'il est devenu Galois, (19). Toutefois, grâce à Liouville, on disposait maintenant d'une édition facilement accessible des œuvres de Galois. Il suffisait que quelqu'un les lise ou s'y essaye.

C'est ce qu'a fait, après d'autres, Camille Jordan à la fin des années 1860. Jordan est l'un des premiers mathématiciens français à avoir développé la théorie des groupes. Il connaissait l'œuvre mathématique de Galois, au moins d'après le *Cours d'Algèbre* de Serret (20). Il n'est donc pas surprenant, qu'il ait publié en 1870 un traité sur les équations algébriques, dont la préface commence par un bref historique du sujet et se poursuit ainsi : « Ces beaux résultats n'étaient pourtant que le prélude d'une plus grande découverte. Il était réservé à Galois d'asseoir la théorie des équations sur sa base définitive, en montrant qu'à chaque équation correspond un groupe de substitutions, dans lequel se reflètent ses caractères essentiels, et notamment tous ceux qui ont trait à sa résolution par d'autres équations auxiliaires ... ». En 1870, en France, Jordan était le seul ou l'un des seuls à pouvoir lire Galois de cette façon. Le Galois de Jordan était né, le Galois de la théorie de Galois, dont la vie si courte prenait dès lors une dimension nouvelle et rendait sa mort plus tragique encore. Après le Galois de Jordan, sont nés une suite presque ininterrompue de Galois, dont le plus connu est celui de Bourbaki qui en fait l'initiateur de la vision bourbachique des mathématiques, d'une partie de celle-ci en tout cas, selon laquelle pour résoudre un grand problème, il faut lui associer une structure dans laquelle se reflètent ses caractères essentiels, comme le dit si bien Jordan, dont on réédita sur le champ le traité. Le Jordan de Bourbaki était né par la même occasion.

Il est inutile de développer davantage. Nous avons rappelé cet exemple très connu seulement pour mettre en perspective le travail historique de Marc. Une œuvre aussi riche et aussi visionnaire que celle de Doeblin (ou de Galois, ou de Lévy, ou d'Itô, ou de tant d'autres) ne peut être comprise d'emblée comme un tout cohérent et figé. On ne peut certainement pas la comprendre en totalité en la rapportant aux seules mathématiques de son temps. Elle ne prend sa véritable dimension qu'en étant réinterprétée, si possible, avec l'aide des mathématiques actuelles, en attendant les mathématiques futures qui permettront peut-être de nouvelles réinterprétations. Une œuvre vraiment grande peut se relire sans cesse, s'enrichir des mathématiques nouvelles comme elle peut les enrichir. Le travail historique de Marc s'inscrit dans ce courant et il est remarquable. (21).

Adieu Marc et merci.

Notes

(1) Sur l'œuvre de Doeblin et sa chronologie compliquée, on verra Lévy [1955], Lindvall [1991], Cohn [1993], Doeblin [2000], Charmasson et al. [2005], Mazliak [2007a, b], Doeblin [Œuvres]. Tous les grands traités de probabilité de l'après-guerre comportent une ou plusieurs sections sur les travaux de Doeblin. On verra notamment les ouvrages classiques de Gnedenko-Kolmogorov, Doob, Chung, Loève, Feller, etc. Sur la vie de Wolfgang Doeblin, on se reportera aux références précédentes et bien sûr au beau livre de Marc Petit [2003].

(2) Sur les mathématiques de l'entre-deux-guerres en France et à l'étranger, en particulier sur la théorie des probabilités, on dispose d'un grand nombre de textes très intéressants. On consultera notamment Barbut et al. [2004], Brissaud [2002], Droesbeke [2003], Heyde, Seneta [2001], Kahane [1998], Leloup [2009], Mazliak, Shafer [2009], Pier [1994], Siegmund-Schultze [2001, 2009] ...

(3) Par exemple, Feller [1954a] ajoute en note que le critère d'accessibilité des diffusions à une dimension qu'il vient de découvrir se trouve énoncé dans la note [1939a] de Doeblin, lequel d'ailleurs avait fait le calcul de tête au cours d'une après midi d'août 1938, alors qu'il randonnait dans le Jura, comme son carnet de recherche nous l'apprend, Doeblin [1938b]. De Feller également on verra le commentaire sur les « ensembles de puissances » de Doeblin [1940b], dans son volume 2, [1971], p. 592 : « The technical difficulties presented by the problem at that time were formidable. » Rappelons que Feller était le critique le plus vigilant des à peu près probabilistes de son temps et qu'il n'était pas tendre du tout, Doeblin non plus d'ailleurs. A propos de ce même mémoire de Doeblin [1940b], on peut lire aussi Jain et Orey [1980] qui écrivent « At the time we started working on these problems, we knew Doeblin's famous paper [1940b] only second hand. When we finally turned to the original we were surprised to learn that the main part of our Proposition 1.11 was already in [1940]. ... Actually, our partial ignorance of the contents of [1940b] was fortuitous because without it we might well have been discouraged from attempting further progress. » D'autant plus décourageant que la fin du mémoire [1940b] rédigé à Givet au printemps 1939, alors que son auteur suit un peloton de caporal, n'a jamais été terminé et que son brouillon envoyé à

sa famille en 1940 et déposé aux archives de l'Académie des sciences par sa mère vers 1955, n'a jamais été élucidé. Le texte de Doeblin est codé et incompréhensible. Quant aux résultats principaux, ils sont énoncés dans une note particulièrement énigmatique [1939b] que Lévy [1956] n'a pas réussi à comprendre, et personne après lui.

On peut citer encore les travaux de Chung [1964, 1992], Orey [1971], Duflo [1990], Brémaud [1999], etc. sur la théorie générale des chaînes selon Doeblin [1940a], ou ceux de Lindvall [1992] sur la méthode du couplage de Doeblin, [1938a], etc. Pour des informations actualisées de premier ordre, on se reportera également aux Œuvres de Doeblin à paraître avec les commentaires très intéressants de Iosifescu, Mason, Nummelin et Seneta.

(4) L'article de Torgny Lindvall [1991] a joué un rôle considérable dans la « redécouverte » mathématique de W. Doeblin. Son article présente de façon très claire les principaux résultats publiés de Doeblin et des éléments de biographie importants et originaux. Lindvall a mené une véritable enquête historique pour retrouver les traces du savant disparu en 1940. Il s'est rendu à Housseras, le village vosgien où Wolf est mort et où il est enterré avec ses parents. Il a compris très tôt que l'histoire du soldat Doblin était exceptionnelle et avait une valeur universelle. Il a en particulier écrit pour une revue suédoise grand public un bel article commémoratif [1993]. C'est à notre connaissance la première apparition du dossier Doeblin en dehors du monde mathématique, si l'on excepte l'article ignoré du germaniste Louis Huguet [1984].

Lindvall a prononcé la conférence inaugurale du colloque en l'honneur de Doeblin qui s'est tenu à Blaubeuren en 1991. Ce colloque organisé par K. L. Chung et H. Cohn était présidé par Doob. Il a donné lieu à une intéressante publication, Cohn [1993]. On peut évidemment s'étonner que le colloque ne se soit pas tenu à l'IHP où Doeblin avait fait l'essentiel de ses recherches, mais cela s'est avéré impossible. Après tout, peut-être était-il préférable que le colloque se tînt en Allemagne, où Heinrich Hering de l'université de Göttingen avait trouvé un lieu et un financement ? Blaubeuren est une charmante petite ville du district de Tübingen, connue pour ses remarquables abris sous roche paléolithiques où l'on a retrouvé la plus ancienne flûte connue, vieille de 35 000 ans, et l'on sait que Doeblin était passionné de musique classique, de Mozart en particulier, de sorte que le lieu n'était pas si mal choisi.

(5) Pier [2000] ne cite Doeblin qu'une seule fois et seulement par ricochet, à propos des travaux de Lindvall sur le couplage des diffusions. Doeblin avait mis au point cette méthode dès ses premiers travaux sur les chaînes de Markov homogènes à nombre fini d'états, au premier semestre 1936, ([1938a], n° 2) et on ne savait pas encore en 2000, qu'il l'avait étendue au cas des diffusions dans le pli cacheté alors dans les réserves de l'Académie des sciences. Dans Pier [1994] consacré aux mathématiques des années 1900–1950, Doeblin n'est cité dans aucun des articles, mais seulement dans les « guidelines », p. 24–26, qui proposent une bibliographie sommaire des travaux importants de la période, et seulement parce que son nom a été ajouté in extremis par Pierre Dugac, l'un des éditeurs principaux du volume.

On peut également évoquer à cet égard le peu d'intérêt suscité en France par le colloque de Blaubeuren, supra note 4. Les raisons en sont multiples, mais il n'est guère douteux que la réception eût été différente, si les travaux ou seulement le nom de Doeblin avaient rappelé quelque chose, en 1990, aux probabilistes français les plus en vue. Certains congressistes de Blaubeuren s'étonnèrent publiquement de cette quasi absence française à une conférence en l'honneur d'un Français mort pour la France. S'agissait-il là, une fois encore, d'une manifestation de cette arrogance hexagonale si souvent brocardée par nos amis anglo-saxons ? Fort heureusement, la délégation française, peu nombreuse, mais de qualité, en avait vu d'autres et l'incident n'eut pas de conséquences fâcheuses.

Quoi qu'il en soit, force est de constater qu'en 2000, plus personne ne se passionnait pour la vie et l'œuvre de Wolfgang Doeblin, ou si peu que ce n'est pas la peine d'en parler.

(6) Alfred Doeblin est un des très grands écrivains du 20^e siècle, mais il n'est pas facile à lire, surtout en traduction française. Son roman *Berlin Alexanderplatz* est un grand livre. Nous conseillons également la lecture de *Voyage et destin* [2002] qui fait une description étonnante de la débâcle de juin 1940, et qui, à de certains moments, semble annoncer ou vivre le suicide de Wolf dans les Vosges.

(7) Kolmogorov [1931], § 17. On verra pour des commentaires Shiryaev [1989] [1999]. Serge Bernstein a également une formule de changement de variables dans son cadre [1932], p. 300, [1938], p. 24. Rappelons que c'est Doeblin qui a corrigé et édité ce mémoire important, Bernstein n'ayant pu se rendre au Colloque de Genève de 1937, où Doeblin était présent. On verra à ce sujet Cohn [1993].

La formule d'Itô dans le cadre d'Itô a été publiée dans [1950] et [1951], mais son rôle fondamental en calcul stochastique n'a été compris que quinze ou vingt ans plus tard. On verra notamment Meyer [2000], (et [1966] où la formule d'Itô n'apparaît pas encore), et aussi Yor [2008].

(8) Pour plus de détails, le lecteur se reportera à la superbe introduction rédigée par Marc, [2000b] p. 1033–1035 et aussi à son article [2002], et naturellement au texte de Doeblin lui-même qui est très concis, mais dit l'essentiel. On sait que Doeblin construit ses diffusions de la façon suivante. Ce sont des mouvements continus, dont la loi satisfait à l'équation de Chapman-Kolmogorov, sous les conditions de Kolmogorov-Feller qui définissent ses « coefficients » (la dérive et le coefficient de diffusion) et à une condition supplémentaire de continuité à l'infini qui fait que par un changement de variables qui ramène l'infini en 1, il est continu partout avec probabilité un, (ce qu'il n'est pas en général, les diffusions de Doeblin filant à l'infini en un temps fini avec probabilité positive et se réfléchissant pour redescendre aussitôt sans discontinuité). Si on leur soustrait ensuite la somme cumulée de leur composante non aléatoire, il reste une martingale au sens de Ville, continue, et « donc » un mouvement brownien changé de temps. C'est un des théorèmes de Doeblin, avant les théorèmes analogues des années 1960.

Un nouveau changement de variables ne fait que changer ce changement de temps qui dépend seulement de la composante aléatoire du mouvement, son coefficient de diffusion. C'est donc une véritable formule stochastique.

(9) On peut voir à ce sujet le bel article d'exposition de Varadhan [2001] et les très nombreux ouvrages récents sur ce sujet dont certains (très peu) sont indiqués en bibliographie.

(10) Marc avait lu attentivement non seulement la transcription, mais aussi l'original du pli. Il avait constaté qu'à plusieurs reprises, le transcripteur avait corrigé discrètement les inadvertances diverses d'un texte très technique qui n'avait pas été relu par son auteur. On sait que les historiens les plus consciencieux sont coutumiers de la chose pour éviter de surcharger leurs éditions de manuscrit de notes critiques. Une pratique bien compréhensible, mais sans doute coupable, qu'ils sont les premiers à dénoncer sévèrement chez leurs collègues. Marc accepta les corrections tacites qu'il prit soin cependant de vérifier. C'est également Marc qui précisa à l'éditeur de façon très détaillée la mise en page compliquée, les titres, les photos, etc.

(11) On verra la *Lettre de l'IMEC*, 12 (2013), p. 18.

(12) Doeblin n'a pas besoin d'intégrale stochastique puisqu'il utilise les changements de temps, dans la continuité des travaux de Lévy sur les sommes de variables indépendantes, mais avec des outils entièrement nouveaux, notamment la théorie des martingales continues. Rappelons incidemment que Ville avait développé à l'IHP en 1938 devant Doeblin et Fortet la première théorie des martingales à temps continu [1938, 1939], que Doob a portée aux sommets que l'on sait, Meyer [2000]. Rappelons que les premières intégrales stochastiques par rapport au mouvement brownien remontent aux travaux de Wiener à la fin des années vingt et surtout à Itô pendant la guerre et beaucoup d'autres ensuite. On verra McKean [1969].

(13) L'axiomatique de Kolmogorov est implicite chez un grand nombre d'auteurs notamment chez Borel qui a proposé, en 1909, sa mesure comme modèle du tirage au sort d'un point sur l'intervalle unité, lequel assure de surcroît, « miraculeusement », que son développement en base deux résulte d'un jeu de pile ou face infini et donne à ce dernier une base mathématique qu'il n'avait pas encore. Quant à la définition de Kolmogorov de la probabilité conditionnelle, dans le cadre de son axiomatique, elle est encore loin de rendre compte de la complexité d'une notion que l'on trouve déjà présente au XVIII^{e} siècle dans les travaux de Moivre, Bayes et Laplace. Les travaux de Lévy sur le mouvement brownien commencent en 1939, longtemps après ceux de Bachelier, Wiener, Khinchin et les savants de l'école polonaise. La théorie des martingales de Doob est implicite dans tout le calcul des probabilités classique, et explicite dans la thèse de Ville et les travaux de Lévy et Bernstein sur les variables dépendantes. Le calcul stochastique d'Itô vient après ceux de Wiener, Bernstein, Lévy, etc.

(14) On sait que les documentaires tournés par Marc ont servi et continuent de servir de support à l'enseignement de la théorie des probabilités, aux États-Unis notamment.

(15) Sur le soldat Beaujot, on verra Doeblin [2000], Petit [2003], Handwerk et al [2007], Ellinghaus et al. [2008], etc.

(16) Il ne faut en rien simplifier ces questions (comme nous le faisons ici), et privilégier une approche au détriment des autres. Comme on sait, l'histoire des sciences est une « discipline polymorphe ». Elle est ce regard attentif tourné vers le passé, les cultures, les sociétés, les hommes qui ont accompagné le développement des sciences. C'est aussi une recherche d'origines et de paternité, une enquête sur les méthodes, les problèmes, leur évolution, dont la science moderne a besoin pour se comprendre, s'assumer, s'enseigner et se développer. C'est également une façon de faire revivre le passé, le réinventer pour le transformer en un présent enrichi de culture et de tradition, etc. ... Et chacune de ces approches a sa spécificité et son importance.

(17) Académie des sciences, séance du 4 septembre 1843, C. R., 17 (1843), p. 448–449 : « ... j'espère intéresser l'Académie en lui annonçant que dans les papiers d'Evariste Galois, j'ai trouvé une solution aussi exacte que profonde de ce beau problème : « Etant donnée une équation irréductible de degré premier, décider si elle est ou non résoluble à l'aide de radicaux. » « Le Mémoire de Galois est rédigé peut-être d'une manière un peu trop concise. Je me propose de le compléter par un commentaire qui ne laissera, je crois, aucun doute sur la réalité de la découverte de notre ingénieux et infortuné compatriote. »

Liouville a publié les « Œuvres mathématiques » de Galois à l'automne 1846, Galois [1846], précédé d'un « avertissement » rappelant la vie et la mort du jeune savant, ibid. p. 381–384.

(18) Sur la théorie de Galois, la littérature est immense et se poursuit actuellement de façon intensive. Nous ne la rappelons pas ici. On peut toujours voir le classique Bourbaki [1960], qui a sa propre lecture, cela va sans dire.

(19) Il ne manque pas de jeunes mathématiciens morts tragiquement à la guerre ou en montagne par exemple. Mais peu d'entre eux sont devenus durablement des mythes ou des icônes. Qu'on songe à Heinrich Kornblum, René Gateaux, Robert Jentzsch, Jacques Herbrand, Paul Urysohn, Raymond Paley, etc. La mort tragique, l'extrême jeunesse ne suffisent pas. Il faut que l'œuvre reste vivante, qu'elle soit relue dans une autre perspective, avec des yeux différents. Et ces relectures successives font autant partie de l'histoire des mathématiques que des mathématiques historiques, celles qui se font.

(20) Serret [1866] vol. 2, Section IV, « Les substitutions ». Dans l'introduction de son volume 1, Serret écrit, p. 4, que le « résultat important » de Galois a été « le point de départ des recherches auxquelles se sont livrés depuis sur cette matière MM. Hermite, Kronecker, Betti et plusieurs autres géomètres éminents. » Sur le rôle fondamental de Betti qui a comblé les trous et les non-dits du manuscrit de Galois, on verra par exemple Jordan [1870], et aussi le beau cours de Dugac [1988].

(21) Marc suit la longue tradition des mathématiciens historiens, les Allemands d'avant 1933 notamment, qui, en lisant, dans le texte, Archimède ou Gauss, projettent certes sur ces auteurs illustres, mille fois commentés, les mathématiques qu'ils savent, mais aussi en retour comprennent davantage les

mathématiques qu'ils croient savoir, pour l'honneur de l'esprit humain et la plus grande gloire des mathématiques. On verra notamment les travaux historiques importants de Dieudonné ou de Weil qui procèdent sans doute de la même tradition. On sait bien que cette suite de lectures différentes n'est pas réservée aux seules mathématiques. On la trouve dans les autres sciences, dans les arts, en littérature, en musique etc. Mozart a été entendu de mille façons. Une grande œuvre est vivante. Elle ne meurt jamais tout à fait. Elle renaît sans cesse et devient source d'inspiration nouvelle.

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Integral Representations of Certain Measures in the One-Dimensional Diffusions Excursion Theory

Paavo Salminen, Ju-Yi Yen, and Marc Yor

Abstract In this note we present integral representations of the Itô excursion measure associated with a general one-dimensional diffusion X. These representations and identities are natural extensions of the classical ones for reflected Brownian motion, RBM. As is well known, the three-dimensional Bessel process, BES(3), plays a crucial rôle in the analysis of the Brownian excursions. Our main interest is in showing explicitly how certain excursion theoretical formulae associated with the pair (RBM, BES(3)) generalize to pair $(X, X^{\uparrow})$, where $X^{\uparrow}$ denotes the diffusion obtained from X by conditioning X not to hit 0. We illustrate the results for the pair (R_{-}, R_{+}) consisting of a recurrent Bessel process with dimension $d_{-} = 2(1-\alpha)$, $\alpha \in (0,1)$, and a transient Bessel process with dimension $d_{+} = 2(1+\alpha)$. Pair (RBM, BES(3)) is, clearly, obtained by choosing $\alpha = 1/2$.

1 Introduction and Main Formulae

1.1. Our main aim in this paper is to extend some identities between σ-finite measures associated with the pairs (BM, BES(3)) and (RBM, BES(3)) to a general pair $(X, X^{\uparrow})$, where X is a one dimensional diffusion which satisfies the hypotheses from [12], that is: X is regular, recurrent, and taking values on $\mathbb{R}_{+} := [0,\infty)$ with 0 as an instantaneously reflecting boundary point. We keep the notation from [12];

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in particular, m is the speed measure, S is the scale function of X with $S(0) = 0$, $S(\infty) = \infty$, and

$$\mathscr{G} = \frac{d}{dm}\frac{d}{dS}$$

is the canonical form of the infinitesimal generator of X.

Let $H_0 := \inf\{t : X_t = 0\}$ denote the first hitting time of 0 and introduce a new diffusion $\hat{X}$ via

$$\widehat{X}_t := \begin{cases} X_t, & t < H_0, \\ \partial, & t \geq H_0, \end{cases}$$

where ∂ is a point isolated from $\mathbf{R}_+$ (a "cemetery" point). The semigroup of the process $\hat{X}$ is given by

$$\hat{P}_t(x, dy) := \mathbf{P}_x(\hat{X}_t \in dy) = \mathbf{P}_x(X_t \in dy; t < H_0), \; x > 0.$$

Diffusion $X^{\uparrow}$ is now defined as the h-transform of $\hat{X}$ with $h := S$, that is, the expectation associated with $X^{\uparrow}$, $X_0^{\uparrow} = x > 0$, is given by

$$\mathbf{E}_x^{\uparrow}[F_t] := \frac{\mathbf{E}_x[F_t S(X_{t\wedge H_0})]}{S(x)} = \frac{\mathbf{E}_x[F_t S(X_t)\mathbf{1}_{t<H_0}]}{S(x)}, \tag{1}$$

where F_t is an $\mathscr{F}_t = \sigma\{X_s : s \leq t\}$-measurable positive functional. The notation $\mathbf{P}_x^{\uparrow}$ stands for the probability measure of $X^{\uparrow}$ initiated at $x > 0$. Notice that writing (1) as follows:

$$\mathbf{E}_x[F_t \mathbf{1}_{(t<H_0)}] = S(x)\mathbf{E}_x^{\uparrow}\Big[F_t \frac{1}{S(X_t)}\Big] \tag{1'}$$

presents $\hat{X}$ as an h-transform of $X^{\uparrow}$ with $h(x) := 1/S(x)$. We refer to [8, 13] for pioneering works on h-transforms and excursions; for h-transforms in general, see [4].

Finally, we remark that 0 is an entrance-not-exit boundary point for $X^{\uparrow}$. Consequently, we may extend the semigroup $P_t^{\uparrow}(x, \cdot)$, which a priori is defined on $(0, \infty)$, to $[0, \infty)$ so that the induced process is a diffusion. The diffusion $X^{\uparrow}$ generalizes in our framework the BES(3) process. The discussion to follow revolves around these three Markov processes: X, $X^{\uparrow}$ and $\hat{X}$.

1.2. Let us state first the identities in question for the pair (BM, BES(3)) found, e.g., in [2, p. 79]:

$$\int_0^\infty dt\, \mathbf{W}_0^t = \int_0^\infty dl\, \mathbf{W}_0^{\tau_l} \circ \int_0^\infty du\, \mathbf{w}^u(\cdot\,; u < \zeta), \tag{2}$$

$$\int_0^\infty dl\, \mathbf{W}_0^{\tau_l} = \int_0^\infty \frac{du}{\sqrt{2\pi u}}\, \mathbf{W}_{0,u,0}, \tag{3}$$

$$\mathbf{w}_+(G_t; t < \zeta) = \mathbf{E}_0^{(3)}\Big[G_t\, \frac{1}{2\,\omega_t}\Big], \tag{4}$$

$$\int_0^\infty dt\, \mathbf{w}_+^t(\cdot\,; t < \zeta) = \int_0^\infty da\, (\mathbf{P}_0^{(3)})^{\gamma_a}. \tag{5}$$

In (2)–(5), $\mathbf{W}_0$ denotes Wiener measure associated with standard Brownian motion $\{B_t : t \geq 0\}$ initiated at 0. We may view $\mathbf{W}_0$ as a measure defined in the canonical space C of continuous functions $\omega : \mathbf{R}_+ \mapsto \mathbf{R}$. Let

$$\mathscr{C}_t := \sigma\{\omega(s) : s \leq t\}$$

denote the smallest σ-algebra making the co-ordinate mappings up to time t measurable and take $\mathscr{C}$ to be the smallest σ-algebra including all σ-algebras $\mathscr{C}_t,\ t \geq 0$. The notation $\mathbf{w}$ is used for the Itô excursion measure and $\mathbf{w}_+$ its restriction to positive excursions. Recall that the excursion space for excursions from 0 to 0 associated with B, and also with the diffusion X, is a subset of C, denoted by E, and given by

$$E := \{\varepsilon \in C : \varepsilon(0) = 0,\ \exists\, \zeta(\varepsilon) > 0 \text{ such that } \varepsilon(t) \neq 0\ \forall\, t \in (0, \zeta(\varepsilon)) \\ \text{and } \varepsilon(t) = 0\ \forall\, t \geq \zeta(\varepsilon)\},$$

where ζ is called the lifetime of a generic excursion. The notation $\mathscr{E}_t$ is used for the trace of $\mathscr{C}_t$ on E. The superscript t, e.g., in $\mathbf{W}^t$ and $\mathbf{w}^t$, means the distribution of the path obtained by killing at time t. This operation is used also at random times. In (2) τ_l denotes the inverse local time, that is,

$$\tau_l := \inf\{t : L_t > l\},$$

where $\{L_t : t \geq 0\}$ is the standard Brownian local time, i.e., such that

$$\{|B_t| - L_t :\ t \geq 0\} \tag{6}$$

is a Brownian motion. Clearly, this normalization of the Brownian local time normalizes also the excursion measure $\mathbf{w}$ in (2). Given two trajectories w and w' in C

we define the third one $w^t \circ w'$, where w^t denotes w killed at time t, by concatenation, i.e.,

$$(w^t \circ w')(s) = \begin{cases} w_s, & s \le t, \\ w_t + w'_{s-t} - w'_t, & s \ge t. \end{cases}$$

The symbol $\circ$ in (2), and also in (7), means the image of the product measure under this application of the concatenation of the trajectories with killing at τ_l. In (3), $\mathbf{W}_{0,u,0}$ is the law of Brownian bridge, starting and ending at 0, over the time interval $[0, u]$. In (4), G_t is an arbitrary $\mathscr{C}_t$-measurable positive functional and $\mathbf{E}^{(3)}$ (also $\mathbf{P}^{(3)}$) refer to the three-dimensional Bessel process. Here the quantity $1/(2\,\omega_t)$ may be seen as the Radon-Nikodym density of $\mathbf{w}$ with respect to $\mathbf{P}_0^{(3)}$ when restricted on $\mathscr{C}_t$. Finally, on the RHS of (5), we have the law of the three-dimensional Bessel process $(R_t : t \ge 0)$ stopped at the last passage time of $a > 0$, i.e., at

$$\gamma_a := \sup\{t \ge 0 : R_t = a\}.$$

In [2], identities (2)–(5) proved to be quite useful for the study of a number of Brownian functionals. See also [11], Exercise 4.18 in Chap. XII, as well as the notation preceding Proposition 4.6 in the same reference.

For an illustration of (2), consider the formula:

$$\begin{aligned} \mathbf{W}_0\Big[\int_0^\infty dt\, \mathrm{e}^{-\lambda t} f(B_t)\Big] &= \mathbf{W}_0\Big[\int_0^\infty dl \exp(-\lambda \tau_l)\Big] \int_0^\infty du\, \mathrm{e}^{-\lambda u} \mathbf{w}(f(\varepsilon_u);\, u < \zeta) \\ &= \frac{1}{\sqrt{2\lambda}} \int_0^\infty du\, \mathrm{e}^{-\lambda u}\, \mathbf{w}(f(\varepsilon_u);\, u < \zeta). \end{aligned}$$

Broadly speaking, (2) is a formal manner of writing the compensation formula for Brownian excursions.

1.3. We now state the extensions of identities (2)–(5) to our general diffusions framework. Again, we postpone the definition of some symbols after the statement of the identities:

$$\int_0^\infty dt\, \mathbf{P}_0^t = \int_0^\infty dl\, \mathbf{P}_0^{\tau_l} \circ \int_0^\infty du\, \mathbf{n}^t(\cdot\, ; t < \zeta), \tag{7}$$

$$\int_0^\infty dl\, \mathbf{P}_0^{\tau_l} = \int_0^\infty du\, p_u(0,0) \mathbf{P}_{0,u,0}, \tag{8}$$

$$\mathbf{n}(G_t; t < \zeta) = \mathbf{E}_0^{\uparrow}\Big[G_t \frac{1}{S(\omega_t)}\Big], \tag{9}$$

$$\int_0^{\infty} dt\, \mathbf{n}^t(\cdot\,; t < \zeta) = \int_0^{\infty} m(da)(\mathbf{P}_0^{\uparrow})^{\gamma_a}. \tag{10}$$

In (7)–(10), $\mathbf{P}_0$ denotes the law of X starting at 0, while $\mathbf{n}$ is the excursion measure when $\{L_t : t \geq 0\}$ the local time at 0 is taken to be

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{m(0, \varepsilon)} \int_0^t ds \mathbf{1}_{(0 < X_s < \varepsilon)}$$

and τ_l denotes the right continuous inverse of $t \mapsto L_t$. The measure $\mathbf{P}_0^{\uparrow}$ can be defined as

$$\mathbf{E}_0^{\uparrow}[G_t] = \mathbf{n}(G_t\, S(\omega_t);\ t \leq \zeta), \tag{11}$$

which is equivalent to (9). In (8), $\mathbf{P}_{0,u,0}$ denotes the law of the X-bridge of duration u starting and ending at 0, and $p_t(0, 0)$ is the value of the semigroup density $p_t(x, y)$ of X when $x = y = 0$. This density is taken with respect to the speed measure, i.e.,

$$P_t(x, dy) := \mathbf{P}_x(X_t \in dy) = p_t(x, y)m(dy).$$

In general, we keep the notation introduced in Sect. 1.2. We also refer the reader to a related paper [10] where formulae with some similar flavor as (7)–(10), but based on the decomposition at the maximum, are found.

We remark that (7) and (8) admit quite similar proofs as (2) and (3), whereas (9) and (10) may be considered as different results.

1.4. The organization of the paper is as follows:

- in Sect. 2, we prove (7)–(8), and recall some important facts from [12].
- in Sect. 3, we prove (9)–(10).
- in Sect. 4, we discuss the excursion bridges and the Ito representation of **n**.
- in Sect. 5, we illustrate the discussion with an example and take X to be a recurrent Bessel process, with dimension $d_- = 2(1 - \alpha)$, $\alpha \in (0, 1)$, and $X^{\uparrow}$ the corresponding transient Bessel process with dimension $d_+ = 2(1 + \alpha)$.

2 Proofs of (7) and (8)

2.1. *Proof of (7)*: The result is obtained—as in the Brownian case—using the compensation formula from the excursion theory, see [2, p. 80], also [1, p. 119].

2.2. *Proof of (8)*: For a $\mathscr{C}_t$-measurable positive functional G_t we have

$$\int_0^\infty dl\, \mathbf{E}_0[G_{\tau_l}] = \mathbf{E}_0[\int_0^\infty dL_u\, G_u]$$
$$= \mathbf{E}_0[\int_0^\infty dL_u\, \mathbf{E}_{0,u,0}[G_u]]$$
$$= \int_0^\infty \mathbf{E}_0[dL_u]\, \mathbf{E}_{0,u,0}[G_u],$$

where in the second equality we used conditioning and the fact that the measure induced by $u \mapsto L_u$ is supported by the random set $\{t : X_t = 0\}$. The last equality follows from Fubini's theorem, and the proof is concluded by recalling that $\mathbf{E}_0[dL_u] = p_u(0,0)du$ (see [7, p. 183]).

3 Proofs of (9) and (10)

3.1. *Preliminaries*: Before proving (9) and (10), we recall some key facts and formulae needed in the proofs (see [12]).

Firstly, under $\mathbf{n}$, the process $\{\varepsilon_t : t > 0\}$ is a Markov process with entrance law $\mathbf{n}(\varepsilon_t \in dx) = m(dx)f_{x0}(t)$ and semigroup

$$\hat{P}_t(x, dy) = \mathbf{P}_x(X_t \in dy;\ t < H_0),\ \ x > 0,$$

where H_0 is the first hitting time of 0 and f_{x0} is its $\mathbf{P}_x$- density

$$\mathbf{P}_x(H_0 \in dt) = f_{x0}(t)dt.$$

We use the notation $\hat{p}_t$ and $p_t^{\uparrow}$ for the semigroup densities associated with $\hat{X}$ and $X^{\uparrow}$, respectively, that is

$$\hat{P}_t(x, dy) = \hat{p}_t(x, y)m(dy),$$
$$P_t^{\uparrow}(x, dy) = p_t^{\uparrow}(x, y)m^{\uparrow}(dy).$$

Notice from the h-transform description of $X^{\uparrow}$ that taking the derivative with respect to the speed measure of $X^{\uparrow}$ given by

$$m^{\uparrow}(dy) := (S(y))^2 m(dy) \tag{12}$$

produces the symmetric density. Hence, $S^{\uparrow}(x) = -1/S(x)$ is an appropriate scale function for $X^{\uparrow}$.

Next recall that the law of the last passage time at $x > 0$ for $X^{\uparrow}$ is given by (see, e.g., [9] and [3] p. 27)

$$\frac{\mathbf{P}_0^{\uparrow}(\gamma_x \in dt)}{dt} = \frac{p_t^{\uparrow}(0,x)}{S^{\uparrow}(\infty) - S^{\uparrow}(x)} = -\frac{p_t^{\uparrow}(0,x)}{S^{\uparrow}(x)}, \tag{13}$$

where $S^{\uparrow}(\infty) := \lim_{z\to\infty} S^{\uparrow}(z) = 0$ since it is assumed that X is recurrent implying $S(\infty) = \infty$. The distributions of H_0 for X and γ_x for $X^{\uparrow}$ can be connected via time reversal. Indeed, $\{X_t : 0 < t < H_0\}$ under $\mathbf{P}_x$ when time reversed from H_0 is identical in law with $\{X_t^{\uparrow} : 0 < t < \gamma_x\}$ under $\mathbf{P}_0^{\uparrow}$, see [13] (and [3, p. 35]). Consequently,

$$\mathbf{P}_x(H_0 \in dt) = \mathbf{P}_0^{\uparrow}(\gamma_x \in dt),$$

which yields

$$f_{x0}(t) = p_t^{\uparrow}(0,x)S(x). \tag{14}$$

Thanks to the formula

$$p_t^{\uparrow}(x,y) = \frac{\hat{p}_t(x,y)}{S(x)S(y)} \tag{15}$$

we obtain (see also [7, p. 154]) the useful identity

$$f_{x0}(t) = \lim_{y\downarrow 0} \frac{\hat{p}_t(x,y)}{S(y)}. \tag{16}$$

3.2. *Proof of (9)*: We first show (9) for $G_t = \varphi(\varepsilon_t)$, where φ is a generic function and ε_t the value of the generic excursion at time t. In this case (9) is equivalent with

$$\int m(dx)\varphi(x)f_{x0}(t) = \int m^{\uparrow}(dx)p_t^{\uparrow}(0,x)\frac{1}{S(x)}\varphi(x),$$

and this holds if and only if

$$m(dx)f_{x0}(t) = m^{\uparrow}(dx)\frac{p_t^{\uparrow}(0,x)}{S(x)}. \tag{17}$$

But since $m^{\uparrow}(dx) = (S(x))^2 m(dx)$, (17) is a simple consequence of (14).

To show (9) for a general $\mathscr{C}_t$-measurable positive functional, introduce for $0 < t_1 < t_2 < \cdots < t_k$, $k \in \mathbf{N}$, and $x_i > 0,\ i = 1, 2, \cdots, k$

$$A := \mathbf{n}(\varepsilon_{t_1} \in dx_1, \varepsilon_{t_2} \in dx_2, \ldots, \varepsilon_{t_k} \in dx_k)$$

and

$$B := \mathbf{P}_0^{\uparrow}\big(X_{t_1} \in dx_1, X_{t_2} \in dx_2, \ldots, X_{t_k} \in dx_k\big)\frac{1}{S(x_k)}.$$

We prove that $A = B$ and, then, (9) holds by a standard extension argument. Using the characterization of the law of the excursion process given at the beginning of Sect. 3.1 yields

$$A = m(dx_1)f_{x_1 0}(t_1)\hat{p}_{t_2-t_1}(x_1, x_2)m(dx_2) \cdot \ldots \cdot \hat{p}_{t_k-t_{k-1}}(x_{k-1}, x_k)m(dx_k).$$

On the other hand,

$$B = p_{t_1}^{\uparrow}(0, x_1)m^{\uparrow}(dx_1)p_{t_2-t_1}^{\uparrow}(x_1, x_2)m^{\uparrow}(dx_2) \cdot \ldots \cdot p_{t_k-t_{k-1}}^{\uparrow}(x_{k-1}, x_k)\frac{m^{\uparrow}(dx_k)}{S(x_k)}.$$

From (17), the equality of A and B boils down to the following identity

$$\begin{aligned} &\hat{p}_{t_2-t_1}(x_1, x_2)m(dx_2) \cdot \ldots \cdot \hat{p}_{t_k-t_{k-1}}(x_{k-1}, x_k)m(dx_k) \\ &\quad = S(x_1)p_{t_2-t_1}^{\uparrow}(x_1, x_2)m^{\uparrow}(dx_2) \cdot \ldots \cdot p_{t_k-t_{k-1}}^{\uparrow}(x_{k-1}, x_k)\frac{m^{\uparrow}(dx_k)}{S(x_k)}, \end{aligned} \tag{18}$$

which holds since

$$p_t^{\uparrow}(x, y) = \frac{\hat{p}_t(x, y)}{S(x)S(y)} \quad \text{and} \quad m^{\uparrow}(dx) = S^2(x)m(dx).$$

Notice that (18) can also be proved using

$$\hat{P}_t(x, dy) = P_t^{\uparrow, h}(x, dy), \tag{19}$$

where $h(x) := 1/S(x)$ and

$$P_t^{\uparrow, h}(x, dy) := \frac{1}{h(x)}\, P_t^{\uparrow}(x, dy)h(y),$$

as already observed in (1').

3.3. *Proof of (10)*: With the help of (9), we may write (10) as

$$\int_0^{\infty} dt\, \mathbf{E}_0^{\uparrow}\Big[G_t \frac{1}{S(X_t)}\Big] = \int m(da)\, \mathbf{E}_0^{\uparrow}[G_{\gamma_a}]. \tag{10$'$}$$

Insert here a generic function f to obtain

$$\int_0^\infty dt\, f(t)\, \mathbf{E}_0^\uparrow\Big[G_t \frac{1}{S(X_t)}\Big] = \int m(da)\, \mathbf{E}_0^\uparrow[f(\gamma_a) G_{\gamma_a}]. \tag{10''}$$

Conditioning on γ_a the RHS of (10″) equals

$$\int_0^\infty m(da) \int_0^\infty \mathbf{P}_0^\uparrow(\gamma_a \in dt) f(t) \mathbf{E}_0^\uparrow[G_t|\gamma_a = t].$$

From (13) it follows (see also [9])

$$\mathbf{E}_0^\uparrow[G_t|\gamma_a = t] = \mathbf{E}_0^\uparrow[G_t|X_t = a],$$

and, because f is a generic function, (10″) is equivalent with

$$\mathbf{E}_0^\uparrow\Big[G_t \frac{1}{S(X_t)}\Big] = \int m(da) \frac{\mathbf{P}_0^\uparrow(\gamma_a \in dt)}{dt} \mathbf{E}_0^\uparrow[G_t|X_t = a] \tag{10'''}$$

The LHS of (10‴) can be written as

$$\int p_t^\uparrow(0,a) m^\uparrow(da) \frac{1}{S(a)} \mathbf{E}_0^\uparrow[G_t|X_t = a],$$

which is seen to be equal with the RHS of (10‴) after making therein use of (13) and (12). This completes the proof.

3.4. *Two corollaries of (9)*: Firstly, choosing in (9) $G_t = S(\varepsilon_t)$ gives

$$\mathbf{n}(S(\varepsilon_t); t < \zeta) = 1, \tag{20}$$

since $X^\uparrow$ is conservative.

Secondly, it holds

$$\mathbf{n}\big(f(\varepsilon_t)\, G_t\, \mathbf{1}_{(t<\zeta)}\big) = \mathbf{E}_0^\uparrow\Big[f(X_t)\, \mathbf{E}_0^\uparrow[G_t|X_t]\, \frac{1}{S(X_t)}\Big], \tag{21}$$

which is obtained using the following formulae deduced from (9)

$$\mathbf{n}(\varepsilon_t \in dx; t < \zeta) = \mathbf{P}_0^\uparrow(X_t \in dx) \frac{1}{S(x)} \tag{9'}$$

and

$$\mathbf{n}(G_t|\varepsilon_t = x) = \mathbf{E}_0^{\uparrow}[G_t|X_t = x]. \tag{9''}$$

4 Excursion Bridge

4.1. From the description of $\mathbf{n}$ given in Sect. 3.2 we deduce for $0 < t_1 < t_2 < \cdots < t_k < t$, $k \in \mathbf{N}$, and $x_i > 0,\ i = 1, 2, \cdots, k$ that

$$\begin{aligned}\mathbf{n}(\varepsilon_{t_1} \in dx_1, \varepsilon_{t_2} \in dx_2, \cdots, \varepsilon_{t_k} \in dx_k, \zeta \in dt)\\ = m(dx_1) f_{x_1 0}(t_1) \hat{p}_{t_2 - t_1}(x_1, x_2) m(dx_2) \cdot \ldots\\ \ldots \cdot \hat{p}_{t_k - t_{k-1}}(x_{k-1}, x_k) m(dx_k) f_{x_k,0}(t - t_k) dt.\end{aligned}$$

Recall (see [12, Theorem 2]) that

$$\mathbf{n}(\zeta \in dt) = p_t^{\uparrow}(0, 0) dt = \lim_{x \to 0} \frac{f_{x,0}(t)}{S(x)},$$

and define the probability measure governing the excursion bridge of length t via the finite dimensional distributions defined by

$$\begin{aligned}&\mathbf{n}(\varepsilon_{t_1} \in dx_1, \varepsilon_{t_2} \in dx_2, \cdots, \varepsilon_{t_k} \in dx_k \,|\, \zeta = t)\\ &:= \frac{1}{p_t^{\uparrow}(0, 0)} m(dx_1) f_{x_1 0}(t_1) \hat{p}_{t_2 - t_1}(x_1, x_2) m(dx_2) \cdot \ldots \\ &\qquad \ldots \cdot \hat{p}_{t_k - t_{k-1}}(x_{k-1}, x_k) m(dx_k) f_{x_k,0}(t - t_k).\end{aligned} \tag{22}$$

Using (12), (14), and (15) we may rewrite the RHS of (22) as

$$\begin{aligned}&\frac{1}{p_t^{\uparrow}(0, 0)} p_{t_1}^{\uparrow}(0, x_1)\, m^{\uparrow}(dx_1) p_{t_2 - t_1}^{\uparrow}(x_1, x_2) m^{\uparrow}(dx_2) \cdot \ldots\\ &\qquad \ldots \cdot p_{t_k - t_{k-1}}^{\uparrow}(x_{k-1}, x_k) m^{\uparrow}(dx_k) p_{t - t_k}^{\uparrow}(x_k, 0).\end{aligned} \tag{23}$$

We let also $\mathbf{P}_{0,t,0}^{\uparrow}$ denote the measure induced by the finite dimensional distribution in (23) indicating that this measure may be seen as the law of $X^{\uparrow}$ when started at 0 and conditioned to be at 0 at time t. Using this notation we formulate the Itô representation of $\mathbf{n}$ as:

$$\mathbf{n} = \int_0^{\infty} du\, p_u^{\uparrow}(0, 0) \mathbf{P}_{0,u,0}^{\uparrow}. \tag{24}$$

It is also well motivated to write (24) as follows

$$\mathbf{n} = \int_0^\infty du\, p_u^{\uparrow}(0,0)\hat{\mathbf{P}}_{0,u,0}. \tag{25}$$

Indeed, for all $x, y > 0$ it holds

$$\mathbf{P}^{\uparrow}_{x,u,y} = \hat{\mathbf{P}}_{x,u,y}. \tag{26}$$

The proof of (26) follows from (1'), which implies that $\hat{\mathbf{P}}_x$ and $\mathbf{P}^{\uparrow}_x$ admit the same bridges laws. However, it is perhaps of interest to show directly that the finite dimensional marginals of $\mathbf{P}^{\uparrow}_{x,u,y}$ and $\hat{\mathbf{P}}_{x,u,y}$ coincide. Let $0 < t_1 < \cdots < t_n < u$ and consider

$$\begin{aligned}
&\mathbf{P}^{\uparrow}_{x,u,y}(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n) \\
&\quad := \frac{\mathbf{P}^{\uparrow}_x(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n, X_u \in dy)}{\mathbf{P}^{\uparrow}_x(X_u \in dy)} \\
&\quad = \frac{p^{\uparrow}_{t_1}(x, x_1)m^{\uparrow}(dx_1)\cdot \ldots \cdot p^{\uparrow}_{u-t_n}(x_n, y)m^{\uparrow}(dy)}{p^{\uparrow}_u(x,y)m^{\uparrow}(dy)} \\
&\quad = \frac{\hat{p}_{t_1}(x, x_1)}{S(x)S(x_1)}S(x_1)^2 m(dx_1)\cdot \ldots \cdot \frac{\hat{p}_{u-t_n}(x_n, y)}{S(x_n)S(y)}\frac{S(x)S(y)}{\hat{p}_u(x,y)} \\
&\quad = \hat{p}_{t_1}(x, x_1)m(dx_1)\cdot \ldots \cdot \hat{p}_{t_n-t_{n-1}}(x_{n-1}, x_n)m(dx_n)\frac{\hat{p}_{u-t_n}(x_n, y)}{\hat{p}_u(x,y)} \\
&\quad = \frac{\hat{\mathbf{P}}_x(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n, X_u \in dy)}{\hat{\mathbf{P}}_x(X_u \in dy)} \\
&\quad =: \hat{\mathbf{P}}_{x,u,y}(X_{t_1} \in dx_1, \ldots, X_{t_n} \in dx_n).
\end{aligned}$$

4.2. From (23) it is seen that the following absolute continuity relation holds for $u < t$

$$\mathbf{E}^{\uparrow}_{0,t,0}[F_u] = \mathbf{E}^{\uparrow}_0\Big[F_u \frac{p^{\uparrow}_{t-u}(X_u, 0)}{p^{\uparrow}_t(0,0)}\Big], \tag{27}$$

where F_u is a $\mathscr{C}_u$-measurable positive functional. Our aim is to prove the following extension of this relation:

$$\mathbf{E}^{\uparrow}_{0,t,0}[G_s(F_u \circ \theta_s)] = \mathbf{E}^{\uparrow}_0\Big[G_s \frac{p^{\uparrow}_{t-s}(X_s, 0)}{p^{\uparrow}_t(0,0)}\hat{\mathbf{E}}_{X_s,t-s,0}[F_u]\Big] \tag{28}$$

with F_u and G_s a $\mathscr{C}_u$- and a $\mathscr{C}_s$-measurable, positive variable, respectively, and $u + s < t$.

We analyze the RHS of (28). Therein, $\hat{\mathbf{E}}_{x,t-s,0}$ denotes the law of the $\hat{X}$ bridge starting from x and ending at 0 at time $t-s$ (see also [6]). Hence,

$$\hat{\mathbf{E}}_{x,t-s,0}[F_u] = \hat{\mathbf{E}}_x\Big[F_u \frac{f_{X_u0}(t-s-u)}{f_{x0}(t-s)}\Big]. \tag{29}$$

Since $\hat{X}$ is an h-transform of $X^{\uparrow}$ with $h(x) = 1/S(x)$, see (19), it follows that the RHS of (29) writes

$$S(x)\mathbf{E}_x^{\uparrow}\Big[F_u \frac{f_{X_u0}(t-s-u)}{f_{x0}(t-s)}\frac{1}{S(X_u)}\Big] = \mathbf{E}_x^{\uparrow}\Big[F_u \frac{p^{\uparrow}_{t-s-u}(X_u,0)}{p^{\uparrow}_{t-s}(x,0)}\Big], \tag{30}$$

where we have used (14). Substituting the RHS of (30) into (28) yields

$$\begin{aligned}
&\mathbf{E}_0^{\uparrow}\Big[G_s \frac{p^{\uparrow}_{t-s}(X_s,0)}{p^{\uparrow}_t(0,0)}\mathbf{E}^{\uparrow}_{X_s}\Big[F_u \frac{p^{\uparrow}_{t-s-u}(X_u,0)}{p^{\uparrow}_{t-s}(X_s,0)}\Big]\Big] \\
&= \mathbf{E}_0^{\uparrow}\Big[G_s \mathbf{E}^{\uparrow}_{X_s}\Big[F_u \frac{p^{\uparrow}_{t-s-u}(X_u,0)}{p^{\uparrow}_t(0,0)}\Big]\Big] \\
&= \mathbf{E}_0^{\uparrow}\Big[G_s (F_u \circ \theta_s) \frac{p^{\uparrow}_{t-(s+u)}(X_{u+s},0)}{p^{\uparrow}_t(0,0)}\Big] \\
&= \mathbf{E}^{\uparrow}_{0,t,0}[G_s(F_u \circ \theta_s)],
\end{aligned}$$

where in the second step we have applied the Markov property and in the third formula (27). Formula (28) is now completely proven.

5 The Bessel Case

5.1. In this section, we look at the particular case when X is the Bessel process with dimension $d_- := 2(1-\alpha)$, $\alpha \in (0,1)$, reflected at 0. We let R_- denote this process. Choosing

$$S(a) = a^{2\alpha} \quad \text{and} \quad m(da) = \frac{1}{\alpha}\, a^{1-2\alpha} da \tag{31}$$

as the scale function and the speed measure, respectively, the differential operator associated with R_- takes the form

$$\mathscr{G}_- = \frac{d}{dm}\frac{d}{dS} = \frac{1}{2}\frac{d^2}{da^2} + \frac{1-2\alpha}{2a}\frac{d}{da},$$

and the transition density with respect to m can be found from e.g. [3, p.134] (notice, however, that the normalization therein is different than in the present case). In particular, we have for $a > 0$

$$p_t(0,a) = \frac{\alpha}{2^{-\alpha} t^{1-\alpha} \Gamma(1-\alpha)} \exp\Big(-\frac{a^2}{2t}\Big).$$

For the process $X^{\uparrow}$, i.e., the h-transform of R_- killed at H_0 using $h = S$, we have the scale function and the speed measure given as

$$S^{\uparrow}(a) = -a^{-2\alpha} \quad \text{and} \quad m^{\uparrow}(da) = \frac{1}{\alpha} a^{1+2\alpha} da,$$

respectively. Consequently, $X^{\uparrow}$ is identified as the Bessel process with dimension $d_+ := 2(1+\alpha)$ and we let R_+ denote this process. The operator associated with R_+ is

$$\mathscr{G}_+ = \frac{d}{dm^{\uparrow}} \frac{d}{dS^{\uparrow}} = \frac{1}{2} \frac{d^2}{da^2} + \frac{1+2\alpha}{2a} \frac{d}{da}.$$

The transition density from 0 with respect to $m^{\uparrow}$ is given by

$$p_t^{\uparrow}(0,a) = \frac{1}{2^{\alpha} t^{1+\alpha} \Gamma(\alpha)} \exp\Big(-\frac{a^2}{2t}\Big).$$

Consequently,

$$\mathbf{P}_0^{\uparrow}(\gamma_a \in dt) = -\frac{p_t^{\uparrow}(0,a)}{S^{\uparrow}(a)} dt = \frac{a^{2\alpha}}{2^{\alpha} t^{\alpha+1} \Gamma(\alpha)} \exp\Big(-\frac{a^2}{2t}\Big) dt \tag{32}$$

and it is seen that the common law of H_0 under $\mathbf{P}_a$ and γ_a under $\mathbf{P}_0^{\uparrow}$ is the reciprocal of a gamma variable, precisely $a^2/2g_\alpha$, where g_α denotes a gamma variable with parameter α.

We remark that the same choice of m as in (31) is made in [5], where it is noticed that with this normalization of the local time it holds

$$\{(R_-(t))^{2\alpha} - L_t : t \geq 0\}$$

is a martingale (cf. (6)).

5.2. We conclude by illustrating identities (9) and (10) in finding the law of the excursion length in this particular Bessel case. Putting $G_t \equiv 1$ in (9) yields

$$\begin{aligned}\mathbf{n}(\zeta > t) &= \mathbf{E}_0^{\uparrow}\Big[\frac{1}{S(\omega_t)}\Big] = \int_0^\infty \frac{1}{S(a)}\, p_t^{\uparrow}(0,a)\, m^{\uparrow}(da)\\ &= \int_0^\infty \frac{a}{2^\alpha \Gamma(\alpha+1) t^{\alpha+1}} \exp\Big(-\frac{a^2}{2t}\Big)\, da\\ &= \frac{1}{2^\alpha \Gamma(\alpha+1) t^\alpha}.\end{aligned}$$

The same expression results also from (10) when we use therein a generic function f :

$$\begin{aligned}\int_0^\infty dt f(t)\mathbf{n}(\zeta > t) &= \int_0^\infty m(da)\mathbf{E}_0^{\uparrow}[f(\gamma_a)]\\ &= \int_0^\infty dt f(t) \int_0^\infty \frac{da\, a}{2^\alpha \Gamma(\alpha+1) t^{\alpha+1}} \exp\Big(-\frac{a^2}{2t}\Big),\end{aligned}$$

where formula (32) is applied. We refer to [5] for the above and further results on excursions of Bessel processes; in particular, for a discussion on different normalizations of the local time.

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Epilogue

The work reported in this paper was initiated by Marc Yor together with Ju-Yi Yen in summer 2012. They contacted Paavo Salminen in spring 2013 concerning some open problems in the paper at that time. The trio then started to work together but was unable to finish before Marc Yor's sudden death in January 2014. The present version is written by Paavo Salminen and Ju-Yi Yen during summer and autumn 2014 and is a revision of earlier versions with Marc Yor.

Sticky Particles and Stochastic Flows

Jon Warren

Abstract Gawędzki and Horvai have studied a model for the motion of particles carried in a turbulent fluid and shown that in a limiting regime with low levels of viscosity and molecular diffusivity, pairs of particles exhibit the phenomena of stickiness when they meet. In this paper we characterise the motion of an arbitrary number of particles in a simplified version of their model.

1 Introduction

It was Marc Yor who first explained sticky Brownian motion to me. He was interested in Chitasvili's argument regarding it being a weak but not strong solution to the associated SDE, and Marc showed me his beautifully handwritten notes on the topic. It was part of a wonderful, inspiring summer spent in Paris as a student.

The motivation for this paper comes from a work by Gawędzki and Horvai [4], in which the authors study a model for the motion of particles carried in a turbulent fluid. The trajectories of two distinct particles $\big(X_1(t), t \geq 0\big)$ and $\big(X_2(t), t \geq 0\big)$ are each described by a Brownian motion in $\mathbf{R}^d$ with a covariance of the form

$$\langle X_1, X_2 \rangle(t) = \int_0^t \psi\big(X_1(s) - X_2(s)\big) ds. \tag{1}$$

The $d \times d$ matrix valued function ψ is invariant under the natural action of the orthogonal group and consequently the inter-particle distance $\|X_1(t) - X_2(t)\|$ is a diffusion process on $\mathbf{R}_+$. For different choices of the covariance function ψ, different qualitative behaviours are observed, and these correspond to different boundary conditions at 0 for the diffusion describing the inter-particle distance. See also Le Jan and Raimond [8] for a description of these phases. Gawędzki and Horvai study the case where 0 is both a entrance and exit boundary point, and the function ψ is not smooth at the origin. They then introduce a viscosity effect acting

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at small scales by replacing ψ by a smooth covariance function obtained from ψ by smoothing in a neighbourhood of the origin. Particles moving in this regularized flow never meet, and 0 is now a natural boundary point for the diffusion describing the inter-particle distance. They then further vary the model and consider particles whose motion is affected by molecular diffusivity, modelled by adding, for each particle, a small independent Brownian perturbation to the motion of the flow. If the additional diffusivity and the scale at which viscosity acts both are taken to zero in an appropriate balance then Gawędzki and Horvai show that the inter-particle distance $\|X^{(1)}(t) - X^{(2)}(t)\|$ converges to a diffusion on $\mathbf{R}_+$ with the boundary point being sticky: that is a regular boundary point at which the diffusion spends a strictly positive amount of time.

Sticky boundary behaviour was first identified by Feller, as described in the article [11]. Subsequently the process which is a Brownian motion on $\mathbf{R}_+$ with a sticky boundary at 0 was studied as an example of a stochastic differential equation with no strong solution, see Chitashvili [2] and Warren [13], and recent work by Engelbert and Peskir [3] and Bass [1]. Stochastic flows in which the inter-particle distance evolves as a sticky Brownian motion have been studied by Le Jan and Raimond [10], Le Jan and Lemaire [7], by Howitt and Warren [5, 6], and by Schertzer et al. [12].

In this paper we study a simplification of the Gawędzki-Horvai model. Our goal is to address, in this simplified setting, the question raised by Gawędzki and Horvai of characterizing the behaviour of N particles. We take the dimension of the underlying space to be $d = 1$, and the motion of distinct particles, in the absence of viscosity or molecular diffusivity, to be given by Brownian motions which are independent of one another until the particles meet.

Let ψ be a real-valued, smooth, positive definite function on $\mathbf{R}$, satisfying $\psi(0) = 1$, $|\psi(x)| < 1$ for $x \neq 0$, and $\psi(x) \to 0$ as $|x| \to \infty$. Define the constant a, which we assume is strictly positive, via

$$\frac{1-\psi(x)}{x^2} \to a^2 \text{ as } x \to 0. \tag{2}$$

For each n there exists a smooth flow of Brownian motions associated with the scaled covariance function $\psi(nx)$, the N point motion of which has generator

$$\frac{1}{2}\sum_{i,j} \psi\big(n(x_i - x_j)\big)\frac{\partial^2}{\partial x_i x_j}. \tag{3}$$

As n tends to infinity the covariance functions $\psi(nx)$ converge to the singular covariance $1_0(x)$, and correspondingly, the N-point motions associated with the flows converge to systems of coalescing Brownian motions.

Fix a constant $b > 0$ and for $n \geq 1$, we define generators

$$\mathcal{G}^{N,n} = \frac{1}{2}\sum_{i,j}\psi\big(n(x_i - x_j)\big)\frac{\partial^2}{\partial x_i x_j} + \frac{b^2}{2n^2}\sum_i \frac{\partial^2}{\partial x_i^2} \tag{4}$$

which are perturbations of the generators (3) by addition of the Laplacian with coefficient $b^2/2n^2$. This works against coalescence by giving each particle in the flow a small amount of independent diffusivity. As a consequence paths of particles in the flow can cross and the N-point motions are no longer associated with a flow of maps.

The two effects: approximating a coalescing flow by smooth flows, and adding diffusivity, are in balance as we pass to the limit, as can be seen by the following analysis of the 2-point motion. Let (X_1, X_2) be the two point motion with generator $\mathcal{G}^{2,n}$. It is enough to consider the difference $Z(t) = X_1 - X_2(t)$ which is a diffusion on the real line in natural scale and with speed measure

$$m_n(dz) = \frac{dz}{1 + b^2 n^{-2} - \psi(nz)} \tag{5}$$

As n tends to infinity m_n weakly converges to the measure $m(dz) = dz + \theta^{-1}\delta_0(dz)$ where the constant θ is given by

$$\theta^{-1} = \int_{-\infty}^{\infty} \frac{dz}{b^2 + a^2 z^2} = \frac{\pi}{ab}. \tag{6}$$

Thus the limiting diffusion describing $|X_1 - X_2|$ is a sticky Brownian with the parameter θ describing the degree of stickiness at 0. See, for example, [3] for the construction of sticky Brownian motion via a time change of Brownian motion. The limit of the two point motion is determined by this, together with X_1 and X_2 each being Brownian motions.

This leaves open the limiting behaviour of the perturbed N-point motions for $N \geq 3$. Consistent families of diffusions in $\mathbf{R}^N$ whose components are Brownian motions evolving as independent Brownian motions whenever they are unequal were studied in [5]. For such processes there are times at which many co-ordinates co-incide and it is necessary to describe the sticky behaviour at such times. This is specified by families of of non-negative coefficients $(\theta(k:l); k, l \geq 1)$. Thinking of the N-point motion as a system of N particles $\theta(k:l)$ gives the rate, in an excursion theoretic sense, at which a clump of $k + l$ particles separates into two clumps one consisting of k particles and the other of l particles. The result of this paper is the following identification of these co-efficients for our model.

Theorem 1 *The N-point motions with generators $\mathcal{G}^{N,n}$ converge in law as n tends to infinity to the family of sticky Brownian motions associated to the family of* *parameters* $(\theta(k:l); k, l \geq 1)$ *with* $\theta(k:l)$ *given by*

$$\frac{ab}{2\sqrt{\pi}} \int_{\mathbf{R}} \int_{\mathbf{R}^{k+l}} \frac{e^{-\|x\|^2/2}}{(2\pi)^{(k+l)/2}} \mathbf{1}(x_1, x_2, \ldots x_k < z < x_{k+1}, \ldots, x_{k+l}) dx dz$$

The form of the parameters $\theta(k:l)$ given in this result is highly suggestive of the underlying mechanisms at work. The variables $x_1, \ldots, x_{k+l}$ chosen according to a Gaussian measure can be thought of as the positions of a cluster of $k+l$ particles experiencing independent diffusivity, and the variable z represents a "singularity" in the underlying flow that causes the cluster to separate into two. Of course this is far from being rigorous.

To give Theorem 1 a precise meaning we must specify the law of the family of sticky Brownian motions associated to the family of parameters $(\theta(k:l); k, l \geq 1)$. We do this by means of a well-posed martingale problem, following [5].

Suppose $\big(\theta(k:l); k, l \geq 1)$ is a family of nonnegative parameters satisfying the consistency property

$$\theta(k:l) = \theta(k+1:l) + \theta(k:l+1) \tag{7}$$

For our purposes in this paper we may also assume the symmetry $\theta(k:l) = \theta(l:k)$. We now recall the main result from [5] concerning the characterization of consistent families of sticky Brownian motions.

We begin by partitioning $\mathbf{R}^N$ into cells. A cell $E \subset \mathbf{R}^N$ is determined by some weak total ordering $\preceq$ of the $\{1, 2, \ldots N\}$ via

$$E = \{x \in \mathbf{R}^N : x_i \leq x_j \text{ if and only if } i \preceq j\}. \tag{8}$$

Thus $\{x \in \mathbf{R}^3 : x_1 = x_2 = x_3\}$, $\{x \in \mathbf{R}^3 : x_1 < x_2 = x_3\}$ and $\{x \in \mathbf{R}^3 : x_1 > x_2 > x_3\}$ are three of the thirteen distinct cells into which $\mathbf{R}^3$ is partitioned.

Suppose that I and J are disjoint subsets of $\{1, 2, \ldots, N\}$ with both I and J non-empty. With such a pair we associate a vector $v = v_{I,J}$ belonging to $\mathbf{R}^N$ with components given by

$$v_i = \begin{cases} 0 & \text{if } i \notin I \cup J, \\ +1 & \text{if } i \in I, \\ -1 & \text{if } i \in J. \end{cases} \tag{9}$$

We associate with each point $x \in \mathbf{R}^N$ certain vectors of this form. To this end note that each point $x \in \mathbf{R}^N$ determines a partition $\pi(x)$ of $\{1, 2, \ldots N\}$ such that i and j belong to the same component of $\pi(x)$ if and only if $x_i = x_j$. Then to each point $x \in \mathbf{R}^N$ we associate the set of vectors, denoted by $\mathcal{V}(x)$, which consists of every vector of the form $v = v_{IJ}$ where $I \cup J$ forms one component of the partition $\pi(x)$.

Let L_N be the space of real-valued functions defined on $\mathbf{R}^N$ which are continuous, and whose restriction to each cell is given by a linear function. Given a set of parameters $\big(\theta(k:l); k,l \geq 0\big)$ we define the operator $\mathcal{A}_N^\theta$ from L_N to the space of real valued functions on $\mathbf{R}^N$ which are constant on each cell by

$$\mathcal{A}_N^\theta f(x) = \sum_{v \in \mathcal{V}(x)} \theta(v) \nabla_v f(x). \tag{10}$$

Here on the righthandside $\theta(v) = \theta(k:l)$ where $k = |I|$ is the number of elements in I and $l = |J|$ is the number of elements in J for I and J determined by $v = v_{IJ}$. $\nabla_v f(x)$ denotes the (one-sided) gradient of f in the direction v at the point x, that is

$$\nabla_v f(x) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\big(f(x+\epsilon v) - f(x)\big). \tag{11}$$

We say an $\mathbf{R}^N$-valued stochastic process $\big(X(t); t \geq 0\big)$ solves the $\mathcal{A}_N^\theta$-martingale problem if for each $f \in L_N$,

$$f\big(X(t)\big) - \int_0^t \mathcal{A}_N^\theta f\big(X(s)\big) ds \text{ is a martingale,}$$

relative to some common filtration, and the bracket between co-ordinates X_i and X_j is given by

$$\langle X_i, X_j \rangle(t) = \int_0^t \mathbf{1}(X_i(s) = X_j(s)) ds \qquad \text{for } t \geq 0.$$

In particular $\langle X_i \rangle(t) = t$. According to the main result of [5], for any given starting point $x \in \mathbf{R}^N$, a solution to the $\mathcal{A}_N^\theta$-martingale problem exists and its law is unique. It is a process with this law that we refer to as a family of N sticky Brownian motions associated with the parameters $\big(\theta(k:l); k,l \geq 1\big)$.

2 Heuristic Derivation of Exit Probabilities

Let us write $\big(X(t); t \geq 0\big)$ for the co-ordinate process on N dimensional path space, and we will write $\hat{X}(t)$ for the projection $X(t)$ onto the hyperplane $\mathbf{R}_0^N = \{x \in \mathbf{R}^N : \sum x_i = 0\}$. Suppose that X when governed by a probability measure $\mathbf{P}_x^{N,\theta}$ evolves as the family of N mutually sticky Brownian motions associated with a parameters $\theta = (\theta(k:l); k,l \geq 1)$ started from $x \in \mathbf{R}^N$. Consider, for $\epsilon > 0$, the neighbourhood $D(\epsilon)$ of the origin 0 in $\mathbf{R}_0^N$ given by

$$D(\epsilon) = \{x \in \mathbf{R}_0^N : \max_{i,j}(x_i - x_j) \leq \epsilon\}. \tag{12}$$

We know from [5] that the exit distribution of $\hat{X}$ from $D(\epsilon)$ can, for small ϵ, be described in terms of the $\theta(k:l)$ parameters. In fact if $T(\epsilon)$ denotes the first time that $\hat{X}$ leaves this set, we have

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbf{E}_0^{N,\theta}\big[T(\epsilon)\big]=\frac{1}{2\sum_{k=1}^{N-1}\binom{N}{k}\theta(k:N-k)}, \tag{13}$$

and, for each cell E that corresponds to a (ordered) partition of $\{1,.2,\ldots,N\}$ into two parts having sizes k and $l=N-k$,

$$\lim_{\epsilon\downarrow 0}\mathbf{P}_0^{N,\theta}\big(X(T(\epsilon))\in E\big)=\frac{\theta(k:l)}{\sum_{k=1}^{N-1}\binom{N}{k}\theta(k:N-k)} \tag{14}$$

Notice how this is consistent with the idea that $\theta(k,N-k)$ describes the rate at which a cluster of N particles splits.

In view of these observations on the behaviour of sticky diffusions we can reasonably expect to be able to identify the parameters $\theta(k:l)$ arising in the limiting behaviour of our N point motions with generators (4) by investigating how these processes, for n large, leave neighbourhoods of the origin. Interestingly very close to the origin, at distances of the order $1/n^2$, the N point motions are spherically symmetric, but at larger distances a coalescence effect leads to exit distributions concentrated on points corresponding to the cluster of particles splitting into two subclusters.

We will suppose that X when governed by probability measures $\mathbf{P}_x^{N,n}$ evolves as a diffusion with generator $\mathcal{G}^{N,n}$ starting from $x\in\mathbf{R}^N$. Notice that the generators $\mathcal{G}^{N,n}$ are invariant under shifts $(x_1,x_2,\ldots x_N)\mapsto(x_1+h,x_2+h,\ldots x_N+h)$, and consequently the projection $\hat{X}(t)$ of $X(t)$ is a diffusion also. In view of (13) and (14) it is natural to study the exit time and distribution of $\hat{X}$ from $D(\epsilon)$ under $\mathbf{P}_0^{N,n}$ in order to determine the parameters $\theta(k:l)$ associated with the limiting N point motion. We will estimate the exit distribution (non-rigorously) by approximating the behaviour of $\hat{X}$ on two different scales.

Let $B(r)$ denote the ball of radius r in $\mathbf{R}_0^N$,

$$B(r)=\{x\in\mathbf{R}_0^N:\|x\|\leq r\}.$$

Now, for a fixed small $\epsilon>0$, the map $x\mapsto\psi(x)$ is approximately quadratic for $x\in(-\epsilon,\epsilon)$ and we use this to approximate the covariance matrix of $\hat{X}$ in the ball $B(\epsilon/n)$. Observe that if the matrix A has entries $1-a^2(x_i-x_j)^2$ then for vectors $u,v\in\mathbf{R}_0^N$ we have $(u,Av)=2a^2(u,x)(v,x)$. Consequently we can approximate $\hat{X}$ under $\mathbf{P}^{N,n}$ within the ball $B(\epsilon/n)$ as $(n^{-2}Z(n^2t);t\geq 0)$ where Z is a diffusion with generator $\mathcal{H}^N$ given by, in spherical co-ordinates in $\mathbf{R}_0^N$,

$$\mathcal{H}^N=a^2r^2\frac{\partial^2}{\partial r^2}+\frac{b^2}{2}\nabla^2=\left(\frac{b^2}{2}+a^2r^2\right)\frac{\partial^2}{\partial r^2}+\frac{(N-2)b^2}{2r}\frac{\partial}{\partial r}+\frac{b^2}{2r^2}\Delta_{S^{N-2}}. \tag{15}$$

In particular, the rescaled radial part of $\hat{X}$ is approximated as a diffusion on $(0,\infty)$ with generator

$$\mathcal{H}^N_{\text{rad}} = \left(\frac{b^2}{2} + a^2 r^2\right)\frac{d^2}{dr^2} + \frac{(N-2)b^2}{2r}\frac{d}{dr}. \tag{16}$$

The expected time taken for this diffusion to first reach a level r when started from 0 is equal to $f_0(r)$ where f_0 is the increasing solution to

$$\mathcal{H}^N_{\text{rad}} f_0 = 1, \qquad f_0(0) = 0.$$

The function $f_0(r)$ is asymptotically equal to $r/(\gamma ab)$, see [14], where

$$\gamma = \sqrt{\frac{2}{\pi}}\frac{\Gamma(N/2)}{\Gamma((N-1)/2)} = \frac{1}{\sqrt{\pi}}\int_{\mathbf{R}^{N-1}} \frac{\|x\|e^{-\|x\|^2/2}}{(2\pi)^{(N-1)/2}}\,dx. \tag{17}$$

Thus we have the estimate

$$\mathbf{E}^{N,n}_0[\text{exit time from } B(\epsilon/n)\,] \approx \frac{\epsilon}{n\gamma ab}. \tag{18}$$

Moreover, because of the spherical symmetry of $\mathcal{H}^N$, the exit distribution from this ball is the uniform measure on sphere.

We next consider $\hat{X}$ started from a point x on the sphere of radius ϵ/n which we will assume has distinct co-ordinates. Let σ be the permutation so that $x_{\sigma(1)} > x_{\sigma(2)} > \cdots > x_{\sigma(N)}$, and denote by x^σ the vector $(x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(N)})$. Our second approximation applies to $\hat{X}$ until it first leaves the domain $D(\epsilon)\setminus D(1/(\epsilon n^2))$. If two particles come close to each other, then they have a negligible probability of separating by a significant distance prior to the exit time τ from the domain. Thus we can treat $\hat{X}$ similarly to (the projection to $\mathbf{R}^N_0$) of a system of N coalescing Brownian motions. In particular this means that if $\hat{X}$ exits via the outer part of the boundary then it does so with $\hat{X}^\sigma_1(\tau) - \hat{X}^\sigma_N(\tau) \approx \epsilon$. Consequently applying the optional stopping theorem to the martingale $\hat{X}^\sigma_1(t) - \hat{X}^\sigma_N(t)$ gives rise to the estimate

$$\mathbf{P}^{N,n}_x\big(\hat{X} \text{ exits } D(\epsilon)\setminus D(1/(\epsilon n^2) \text{ via the outer boundary}\big) \approx \frac{x^\sigma_1 - x^\sigma_N}{\epsilon}. \tag{19}$$

Moreover if $\hat{X}$ does exit via the outer boundary then as it does so there are only two clusters of particles (see Lemma 4 for the corresponding statement about coalescing Brownian motion), and applying the optional stopping theorem to $\hat{X}^\sigma_k(t) - \hat{X}^\sigma_{k+1}(t)$

gives

$$\mathbf{P}_x^{N,n}\big(\hat{X}_i^\sigma(\tau)-\hat{X}_{i+1}^\sigma(\tau)\approx 0 \text{ for } i\neq k, \text{ and } \hat{X}_k^\sigma(\tau)-\hat{X}_{k+1}^\sigma(\tau)\approx \epsilon\big) \\ \approx \frac{x_k^\sigma - x_{k+1}^\sigma}{\epsilon}. \tag{20}$$

We now make use of a renewal argument. The diffusion with generator (15) is ergodic, with an invariant measure whose density decays at infinity, see [14]. Consequently, taking account of the scaling by a factor of n^2 we expect that the process $\hat{X}$ spends all but a negligible amount of time at a distance of order $1/n^2$ from the origin prior to exiting $D(\epsilon)$. From this inner region it makes excursions to the sphere of radius ϵ/n and, each time it does, it has a small probability of exiting $D(\epsilon)$ rather than returning to the inner region. When it does return to distances of order $1/n^2$ we can assume by mixing that it is starts afresh and forgets its history. Thus $\hat{X}$ makes approximately a geometrically distributed number of excursions to the sphere of radius ϵ/n before exiting $D(\epsilon)$, and we conclude, neglecting the time spent outside the ball $B(\epsilon/n)$, that the expected time to exit $D(\epsilon)$ is estimated by

$$\frac{\mathbf{E}_0^{N,n}[T_{B(\epsilon/n)}]}{\mathbf{E}_0^{N,n}\big[\mathbf{P}^{N,n}_{X(T_{B(\epsilon/n)})}\big(\hat{X} \text{ exits } D(\epsilon)\setminus D(1/(\epsilon n^2)) \text{ via the outer boundary }\big)\big]},$$

where $T_{B(\epsilon/n)}$ denotes the first time of exiting the ball $B(\epsilon/n)$. Similarly we estimate that the probability of exiting $D(\epsilon)$ at time $T_{D(\epsilon)}$ with $\hat{X}_{i+1}(T_{D(\epsilon)})-\hat{X}_i(T_{D(\epsilon)})\approx 0$ for all $i\neq k$ and $\hat{X}_{k+1}(T_{D(\epsilon)})-\hat{X}_k(T_{D(\epsilon)})\approx\epsilon$ is approximately

$$\frac{\mathbf{E}_0^{N,n}\big[\mathbf{P}^{N,n}_{X(T_{B(\epsilon/n)})}\big(\hat{X}_i(\tau)-\hat{X}_{i+1}(\tau)\approx 0 \text{ for } i\neq k, \hat{X}_k(\tau)-\hat{X}_{k+1}(\tau)\approx\epsilon\big)\big]}{\mathbf{E}_0^{N,n}\big[\mathbf{P}^{N,n}_{X(T_{B(\epsilon/n)})}\big(\hat{X} \text{ exits } D(\epsilon)\setminus D(1/(\epsilon n^2)) \text{ via the outer boundary }\big)\big]},$$

where, as previously, τ is the exit time of $D(\epsilon)\setminus D(1/(\epsilon n^2))$. Thus, in view of (13) and (14), and taking the cell $E=\{x_1=x_2=\ldots=x_k<x_{k+1}=x_{k+2}=\ldots=x_N\}$, we guess that the parameter $\theta(k:N-k)$ associated with a limiting N-point motion should be equal to the limit as n tends to infinity and ϵ tends to zero of

$$\frac{\epsilon\times\mathbf{E}_0^{N,n}\big[\mathbf{P}^{N,n}_{X(T_{B(\epsilon/n)})}\big(\hat{X}_i(\tau)-\hat{X}_{i+1}(\tau)\approx 0 \text{ for } i\neq k, \hat{X}_k(\tau)-\hat{X}_{k+1}(\tau)\approx\epsilon\big)\big]}{2\mathbf{E}_0^{N,n}[T_{B(\epsilon/n)}]}$$

Substituting in our estimates from (18) and (20) and using the fact that the exit distribution from $B(\epsilon/n)$ is uniform we arrive at

$$\frac{\gamma ab}{2}\int_{S^{N-2}}\big(\min_{1\le i\le k} z_i - \max_{k+1\le i\le N} z_i)\big)^+ dz. \tag{21}$$

in which the integral over the unit sphere $S^{N-2} \subset \mathbf{R}_0^N$ is taken with respect to Lebesgue measure on the sphere normalized so $\int_{S^{N-2}} dz = 1$. When we rewrite the spherical integral as a Gaussian integral this agrees the value given in Theorem 1.

3 Proof of Main Result

In view of the characterization of a family of sticky Brownian motions by the $\mathcal{A}_N^\theta$-martingale problem, it is a natural strategy to prove Theorem 1 by considering smooth approximations f_n to a given function $f \in L_N$ and to derive, using weak convergence, from the martingale property, under $\mathbf{P}^{N,n}$, of

$$f_n(X(t)) - \int_0^t \mathcal{G}^{N,n} f_n(X(s))ds \tag{22}$$

that

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))ds,$$

is a martingale under $\mathbf{P}^{N,\theta}$. There are difficulties to be overcome in pursuing this which arise because $\mathcal{A}_N^\theta f$ is not continuous. A key step is to establish the weaker statement described in the following lemma, which gives information about how the limiting process leaves the main diagonal $D = \{x \in \mathbf{R}^N : x_1 = x_2 = \ldots = x_N\}$. Let L_N^0 denote the subspace of L_N containing those functions which are invariant under shifts $(x_1, x_2, \ldots, x_n) \mapsto (x_1 + h, x_2 + h, \ldots, x_n + h)$, and consequently identically equal to 0 on D.

Lemma 1 *Fix $x \in \mathbf{R}^N$, and suppose that $\mathbf{P}_x$ is a subsequential limit of the family of probability measures $\left(\mathbf{P}_x^{N,n}; n \geq 1\right)$. Then for any convex $f \in L_0^N$,*

$$Z^f(t) = f(X(t)) - \mathcal{A}_N^\theta f(0) \int_0^t \mathbf{1}(X(s) \in D)ds$$

is a submartingale under $\mathbf{P}_x$, where the family of parameters θ are specified as in Theorem 1.

We will prove this lemma by applying weak convergence to $\mathbf{P}^{N,n}$ martingales given at (22). But it turns out that we must carefully select suitable smooth approximations f_n. In fact we will choose $f_n(x) = n^2 g(n^{-2}x)$ where the function g is determined according to the next proposition which is adapted from [14].

Recall that the generators $\mathcal{G}^{N,n}$, rescaled and restricted to $\mathbf{R}_0^N$, converge to $\mathcal{H}^N$ given by (15). The constant γ was defined at (17).

Proposition 1 *Let $f : S^{N-2} \to \mathbf{R}$ be a square integral function on the unit sphere $S^{N-2} \subset \mathbf{R}_0^{N-1}$. Let*

$$c = c(f) = \gamma ab \int_{S^{N-2}} f(z)dz$$

where the integral is with respect to normalized Lebesgue measure on the sphere. There exists a unique solution to

$$\mathcal{H}^N g = c$$

satisfying $g(0) = 0$ and

$$\lim_{r\to\infty} g(rz)/r = f(z) \text{ uniformly for } z \in S^{N-2}.$$

Moreover if $y \mapsto \|y\| f(y/\|y\|)$ is a convex function on $\mathbf{R}_0^{N-1}$ then so too is $y \mapsto g(y)$.

Proof (of Lemma 1) Let $f \in L_0^N$ be convex, and consider its restriction to $S^{N-2} \subseteq \mathbf{R}_0^N$. Let $c = c(f) = \gamma ab \int_{S^{N-2}} f(z)dz$ and let g be the corresponding solution to $\mathcal{H}^N g = c$ described in Proposition 1. Extend g to a function on $\mathbf{R}^N$ invariant under shifts $(x_1, x_2, \ldots, x_n) \mapsto (x_1 + h, x_2 + h, \ldots, x_n + h)$, and set $g_n(x) = n^{-2} g(n^2 x)$.

We want to estimate $\mathcal{G}^{N,n} g_n(x)$ in a neighbourhood of the diagonal D. We write

$$\begin{aligned}
\mathcal{G}^{N,n} g_n(x) &= \frac{1}{2}\sum_{i,j} \psi\big(n(x_i - x_j)\big) \frac{\partial^2}{\partial x_i x_j} g_n(x) + \frac{b^2}{2n^2} \sum_i \frac{\partial^2}{\partial x_i^2} g_n(x) \\
&= \left\{ \frac{1}{2}\sum_{i,j} \big(\psi\big(n(x_i - x_j)\big) - 1 + a^2 n^2 (x_i - x_j)^2\big) \frac{\partial^2}{\partial x_i x_j} g_n(x) \right\} \\
&\quad + \left\{ \frac{1}{2}\sum_{i,j} \big(1 - a^2 n^2 (x_i - x_j)^2\big) \frac{\partial^2}{\partial x_i x_j} g_n(x) + \frac{b^2}{2n^2} \sum_i \frac{\partial^2}{\partial x_i^2} g_n(x) \right\}
\end{aligned} \tag{23}$$

The first term in braces appearing here can be controlled as follows. Recall $\hat{x}$ denotes the orthogonal projection of x onto $\mathbf{R}_0^N$ and that $B(r)$ is the ball of radius r in $\mathbf{R}_0^N$. Given $K > 0$ let

$$M(K) = \max_{i,j} \sup_{\hat{x} \in B(K)} \left| \frac{\partial^2}{\partial x_i x_j} g(x) \right| = n^{-2} \max_{i,j} \sup_{\hat{x} \in B(K/n^2)} \left| \frac{\partial^2}{\partial x_i x_j} g_n(x) \right| < \infty.$$

Then given $\epsilon > 0$, we may by (2), choose n_0 so that for all $n \geq n_0$, and x so that $\hat{x} \in B(K/n^2)$,

$$\left|\psi_n(x_i - x_j) - 1 + a^2 n^2 (x_i - x_j)^2\right| \leq \frac{\epsilon}{N^2 K M(K)} n^2 (x_i - x_j)^2 \leq \frac{\epsilon}{N^2 M(K)},$$

and this then entails that the first term in braces is no larger than ϵ in modulus. Because of the shift invariance of g, the second term in braces appearing in Eq. (23) is equal to $\big(\mathcal{H}^N g\big)(n^2 x)$, which in turn is equal to $c(f)$.

Next we claim that

$$c(f) = \mathcal{A}_N^\theta f(0).$$

To verify this it is enough, by linearity, to check it for functions of the form

$$f(x) = \big(\min_{i \in \pi_1} x_i - \max_{i \in \pi_2} x_i\big)^+$$

where $\pi = (\pi_1, \pi_2)$ is an ordered partition of $\{1, \ldots, N\}$ into two non-empty parts. For such f the gradients $\nabla_v f(0)$ appearing in the definition of $\mathcal{A}_N^\theta f(0)$ are all zero except for $\nabla v_{\pi_1, \pi_2} f(0)$ which equals 2. Thus, recalling the values assigned to the parameters $(\theta(k : l)$ in Theorem 1,

$$\mathcal{A}_N^\theta f(0) = 2\theta(|\pi_1|, |\pi_2|) = \frac{ab}{\gamma_N} \int_{S^{N-2}} \big(\min_{i \in \pi_1} z_i - \max_{i \in \pi_2} z_i)\big)^+ dz = c(f).$$

Observe that because g_n is smooth and convex, $\mathcal{G}^{N,n} g_n$ is continuous and non-negative everywhere. This fact, together with the above paragraphs allows us to conclude that given $K > 0$ and $\epsilon > 0$, for all sufficiently large n we have

$$g_n(X(t)) - \big(\mathcal{A}_N^\theta f(0) - \epsilon\big) \int_0^t \mathbf{1}(\hat{X}(s) \in B(K/n^2)) ds \tag{24}$$

is a submartingale under $\mathbf{P}^{N,n}$.

Fix times $s < t$ and let Φ be a bounded, non-negative and continuous function on the path space $\mathbf{C}\big([0, s], \mathbf{R}^N\big)$. Note that the boundary behaviour of g implies that $|g_n(x) - f(x)|/(1 + ||x||) \to 0$ as $n \to \infty$ uniformly for $x \in \mathbf{R}^N$, and that since $\mathbf{E}^{N,n}\big[\|X(s)\|\big]$ and $\mathbf{E}^{N,n}\big[\|X(t)\|\big]$ are bounded uniformly in n, the weak convergence of (a subsequence of) $\mathbf{P}^{N,n}$ to $\mathbf{P}$, implies that (along the subsequence)

$$\mathbf{E}^{N,n}\Big[\Phi(X(r), r \leq s)\big(g_n(X(t)) - g_n(X(s))\big)\Big] \to \\ \mathbf{E}\Big[\Phi(X(r), r \leq s)\big(f(X(t)) - f(X(s))\big)\Big].$$

Let $\phi_K : \mathbf{R}_0^N \to [0,1]$ be a continuous function satisfying $\phi_K(x) = 0$ for $\|x\| \geq 1/K$ and $\phi_K(x) = 1$ for $\|x\| \leq 1/(2K)$ Then we also have by weak convergence (along the subsequence) that

$$\mathbf{E}^{N,n}\left[\Phi(X(r), r \leq s)\int_s^t \phi_K\big(\hat{X}(u)\big)du\right] \to \mathbf{E}\left[\Phi(X(r), r \leq s)\int_s^t \phi_K\big(\hat{X}(u)\big)du\right] \geq \mathbf{E}\left[\Phi(X(r), r \leq s)\int_s^t \mathbf{1}(X(u) \in D)du\right].$$

For a given $\epsilon > 0$, if we choose K large enough, then by virtue of Lemma 2, for all sufficiently large n,

$$\mathbf{E}^{N,n}\left[\Phi(X(r), r \leq s)\int_s^t \mathbf{1}(\hat{X}(u) \in B(K/n^2))du\right] + \epsilon \geq \mathbf{E}^{N,n}\left[\Phi(X(r), r \leq s)\int_s^t \phi_K\big(\hat{X}(u)\big)du\right].$$

From these statements and the fact that the process at (24) is a submartingale for large enough n, it follows that

$$\mathbf{E}\Big[\Phi(X(r), r \leq s)\big(f(X(t)) - f(X(s))\big)\Big] \geq \big(\mathcal{A}_N^\theta f(0) - \epsilon\big)\left(\mathbf{E}\left[\Phi(X(r), r \leq s)\int_s^t \mathbf{1}(X(u) \in D)du\right] - \epsilon\right)$$

Consequently, $s \leq t$, $\Phi \geq 0$ and $\epsilon > 0$ being arbitrary, Z^f is a submartingale under $\mathbf{P}$ as desired. □

We may now give the

Proof (of Theorem 1) Fix $x_0 \in \mathbf{R}^N$. Because the marginal laws of each component $\big(X_i(t); t \geq 0\big)$ converge as $n \to \infty$ it follows that the family of probability measures $\big(\mathbf{P}_{x_0}^{N,n}; n \geq 1\big)$ is tight. Thus it suffices to show that any limit point $\mathbf{P}_{x_0}$ solves the $\mathcal{A}_N^\theta$-martingale problem starting from x_0.

We know, from the analysis of the speed measures in the Introduction, that each pair of components (X_i, X_j) converges in law to a pair of Brownian motions whose difference is a sticky Brownian motion and consequently

$$2\langle X_i, X_j\rangle(t) = \langle X_i, X_i\rangle(t) + \langle X_j, X_j\rangle(t) - \langle X_i - X_j\rangle(t) = 2\int_0^t \mathbf{1}(X_i(s) \neq X_j(s))ds$$

under $\mathbf{P}_{x_0}$. Thus it suffices to show that

$$f(X(t)) - \int_0^t \mathcal{A}_N^\theta f(X(s))ds \tag{25}$$

is a $\mathbf{P}_{x_0}$-martingale for each $f \in L^N$. By the addition of a suitable linear function we may assume that $f \in L_0^N$. In fact we claim that it is enough that for every convex $f \in L_0^N$ the expression at (25) defines a submartingale. We verify this claim as follows. For a general f we may consider $g(x) = c\sum_{i<j}|x_i - x_j| + f(x)$ which for sufficiently large c is convex. We would then have that the corresponding process $g(X(t)) - \int_0^t \mathcal{A}_N^\theta g(X(s))ds$ is a submartingale. But we also know that the difference of each pair of components of X is a sticky Brownian motion with parameter $\theta = 2\theta(1:1)$, and thus,

$$|X_i(t) - X_j(t)| - 4\theta(1:1)\int_0^t \mathbf{1}(X_i(s) = X_j(s))ds$$

is a martingale. Now we also observe that

$$\mathcal{A}_N^\theta g(x) = 4c\theta(1:1)\sum_{i<j}\mathbf{1}(x_i = x_j) + \mathcal{A}_N^\theta f(x).$$

And so we deduce that (25) must be a submartingale. But we can consider $g(x) = c\sum|x_i - x_j| - f(x)$ in the same manner, and hence deduce that (25) is a supermartingale.

We now proceed with the proof of the theorem. The result holds for dimension $N = 2$, and we argue by induction on N. So assume the result holds for dimension $N - 1$, and consider a convex $f \in L_0^N$. By the Meyer decomposition theorem, associated with the $\mathbf{P}_{x_0}$ submartingale $f(X(t))$ is some continuous increasing process $A(t)$. Let $U_\pi = \{x \in \mathbf{R}^N : x_i > x_j \text{ for all } i \in \pi_1, j \in \pi_2\}$ for some ordered partition $\pi = (\pi_1, \pi_2)$ of $\{1, 2, \dots, N\}$ into two parts. According to Lemma 3, on U_π, $f(x)$ can be written as a sum of $f_1(x_j; j \in \pi_1)$ and $f_2(x_j; j \in \pi_2)$. Applying the inductive hypothesis the processes

$$f_i(X_j(t); j \in \pi_i) - \int_0^t \mathcal{A}_{\pi_i}^\theta f_i(X_j(s); j \in \pi_i)ds$$

for $i = 1, 2$ are both martingales. Consequently, the compensator A of $f(X(t))$ must satisfy

$$dA(t) = \big(\mathcal{A}_{\pi_1}^\theta f_1(X_j(t); j \in \pi_1) + \mathcal{A}_{\pi_2}^\theta f_i(X_j(t); j \in \pi_2)\big)dt$$

on the set $\{t : X(t) \in U_\pi\}$. Noting that

$$\left(\mathcal{A}^\theta_{\pi_1} f_1(x_j; j \in \pi_1) + \mathcal{A}^\theta_{\pi_2} f_i(x_j; j \in \pi_2)\right) = \mathcal{A}^\theta_N f(x) \text{ for } x \in U_\pi,$$

and letting π vary we conclude that in fact

$$dA(t) = \mathcal{A}^\theta_N f(X(t))dt \text{ on } \{t : \hat{X}(t) \neq 0\}.$$

Finally applying Lemma 1 we deduce that dA must dominate $\mathcal{A}^\theta_N f(X(t))dt$ on $\{t : \hat{X}(t) = 0\}$ and that (25) must be a submartingale. By our previous discussion since this holds for every convex f$\in L^N_0$ in fact (25) is a martingale and the inductive step is complete. □

4 Some Lemmas

Lemma 2 *Given t and* $\epsilon > 0$ *there exist c, c′ and* n_0 *such that*

$$\mathbf{E}^{N,n}_x \left[\int_0^t \mathbf{1}\left(|X_i(s) - X_j(s)| \in (c/n^2, c')\right) ds \right] \leq \epsilon$$

for all $n \geq n_0$ *and* $x \in \mathbf{R}^N$.

Proof Under $\mathbf{P}^{N,n}_x$, the process $Z = X_i - X_j$ is a diffusion in natural scale with speed measure m_n given by (5). It can thus be represented as a time changed Brownian motion:

$$Z(t) = B(\tau^n_t),$$

where τ^n is the inverse of the increasing functional

$$\frac{1}{2} \int_0^u \frac{ds}{1 + b^2 n^{-2} - \psi(nB(s))}$$

and B a standard Brownian motion starting from $x_i - x_j$. Consequently

$$\int_0^{\tau^n_t} \mathbf{1}_{(c/n^2, c')}\left(|B(s)|\right) \frac{ds}{1 + b^2 n^{-2} - \psi(nB(s))}$$

is a random variable with the same distribution as

$$\int_0^t \mathbf{1}\left(|X_i(s) - X_j(s)| \in (c/n^2), c')\right) ds$$

has under $\mathbf{P}_x^{N,n}$. Note that for all sufficiently large n,

$$\frac{1}{2} \leq \frac{1}{1 + b^2 n^{-2} - \psi(nz))} \text{ for all } z \in \mathbf{R}$$

whence $\tau_t^n \leq 4t$ and

$$\int_0^{\tau_t^n} \mathbf{1}_{(c/n^2, c')}\big(|B(s)|\big) \frac{ds}{1 + b^2 n^{-2} - \psi(nB(s))} \leq \int_0^{4t} f_n(B(s))ds$$

where $f_n(z) = \mathbf{1}_{(c/n^2,c')}(|z|)\big(1 + b^2 n^{-2} - \psi(nz)\big)^{-1}$. Now rewriting this integral using the occupation time formula, and taking expectations we see that it is enough to verify that

$$\int_{\mathbf{R}} f_n(z)dz$$

can be made arbitrarily small for all sufficiently large n c sufficiently large and c' sufficiently small. This is easily checked using the assumptions on ψ and in particular using that there is a $\delta > 0$ and a constant $M < \infty$ so that for all sufficiently large n,

$$\big(1 + b^2 n^{-2} - \psi(nz)\big)^{-1} \leq \frac{2n^2}{2b^2 + a^2 n^4 z^2}, \text{ for } z \in (-\delta/n, \delta/n)$$

whilst

$$\big(1 + b^2 n^{-2} - \psi(nz)\big)^{-1} \leq M \text{ for } z \in \mathbf{R} \setminus (-\delta/n, \delta/n).$$

□

Lemma 3 *Let $\pi = (\pi_1, \pi_2)$ be an ordered partition of $\{1, 2, \ldots, N\}$ into two non-empty parts, and define*

$$U_\pi = \{x \in \mathbf{R}^N : x_i > x_j \textit{ for all } i \in \pi_1, j \in \pi_2\}.$$

Then $f \in L^N$ can be expressed as

$$f(x) = f_1(x_i; i \in \pi_1) + f_2(x_j; j \in \pi_2) \textit{ for all } x \in U_\pi$$

for some $f_1 \in L^{|\pi_1|}$, $f_2 \in L^{|\pi_2|}$.

Proof By subtracting a linear function we can assume $f \in L_0^N$. Now suppose that a given $x \in U_\pi$ satisfies $x_i > 0 > x_j$ for all $i \in \pi_1, j \in \pi_2$. Let $y \in \mathbf{R}^N$ have components $y_i = x_i$ for $i \in \pi_1$ and $y_i = 0$ otherwise. Likewise let $z \in \mathbf{R}^N$ have components $z_i = x_i$ for $i \in \pi_2$ and $z_i = 0$. Then both y and z lie in the closure of the cell that contains x, and by the linearity of f restricted to the closure of that cell,

$$f(x) = f(y) + f(z).$$

Consequently we define $f_1(x_i; i \in \pi_1) = f(y)$ and $f_2(x_j; j \in \pi_2) = f_2(z)$, extending each linearly within cells so as to functions $f_1 \in L^{|\pi_1|}$ and $f_2 \in L^{|\pi_2|}$. □

Lemma 4 *Suppose that $B_1(t) \geq B_2(t) \geq \cdots \geq B_N(t)$ are a system of coalescing Brownian motions on* **R**. *Let $T_R = \inf\{t \geq 0 : B_1(t) - B_N(t) = R\}$, and let r denote $B_1(0) - B_N(0)$. Then there exists a constant C such that for all r and R with $0 \leq r \leq R/2$,*

$$\mathbf{P}\big(T_R < \infty \textit{ and there exists some } i \textit{ with } B_1(T_R) > B_i(T_R) > B_N(T_R)\big) \leq C(r/R)^3.$$

Proof For $i = 2, 3, \ldots, N-1$, let A_i be the event

$$T_R < \infty \text{ and } B_1(T_R) > B_i(T_R) > B_N(T_R)$$

Since the event in question is the union of these events, it is enough to prove the desired estimate holds for each A_i. Projecting the three dimensional process $\big(B_1(t), B_i(t), B_N(t)\big)$ onto the plane $\{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}$ we see A_i can be identified with the event that a two dimensional Brownian motion started at a point satisfying $y_1 = r$ exits the domain

$$\{y \in \mathbf{R}^2 : 0 \leq y_1 \leq R, |y_2| \leq y_1/\sqrt{3}\}$$

via the boundary $y_1 = R$. By comparing with a wedge with a circular outer boundary and interior angle $\pi/3$ and solving the appropriate Dirichlet problem this exit probability is easily seen to by bounded by $C(r/R)^3$. □

5 Stochastic Flows of Kernels

Returning to the motivation coming from Gawędzki and Horvai it is natural to interpret the results from this paper in terms of the stochastic flows. As remarked in the introduction the consistent family of N point motions with generators $\mathcal{G}^{N,n}$ do not correspond to any stochastic flow of maps. However according to the theory developed by Le Jan and Raimomd [9] they are associated with the more general notion of a flow of kernels.

Let $W = \big(W(t,x), t \geq 0, x \in \mathbf{R}\big)$ denote the centred Gaussian process with covariance function $\psi(n(x_1 - x_2)) \min(t_1, t_2)$. Suppose $B_1, B_2, \ldots, B_N$ are real valued Brownian motions, independent of each other and W. Then a diffusion with generator $\mathcal{G}^{N,n}$ can be obtained, at least in a formal sense, by solving the stochastic differential equations

$$X_i(t) = x_i + \int_0^t dW(s, X_i(s)) + \frac{\sigma}{n} B_i(t). \tag{26}$$

The stochastic flow of kernels $\big(K_{s,t}, s \le t\big)$ associated with family $\mathcal{G}^{N,n}$ describes a cloud of infinitesimal particles moving in this manner. It can be obtained by filtering on W,

$$K_{0,t}(x_1, A) = \mathbf{P}\big(X_1(t) \in A | W\big). \tag{27}$$

These kernels have smooth densities which satisfy a stochastic partial differential equation of advection-diffusion type. If $v(t,y)$ denotes the density of $\int v(0,x) K_{0,t}(x,\cdot)dx$ at y, then

$$\begin{aligned} v(t,y) - v(0,y) = & \int_0^t \frac{\partial v}{\partial y}(s,y) dW(s,y) + \int_0^t v(s,y) dW_y(s,y) \\ & + \frac{1}{2}(b^2+1) \int_0^t \frac{\partial^2 v}{\partial y^2}(s,y) ds, \end{aligned} \tag{28}$$

where $W_y(t,y) = \partial W(t,y)/\partial y$. Simulations showing a realization of the density of $K_{0,1}(0,\cdot)$ for two different sets of parameter values are shown in Fig. 1.

As n tends to infinity, the convergence of the N point motions suggests that these flows of kernels should converge to the flow of kernels associated with a consistent family of sticky Brownian motions. See, for example, Theorem 8 of [7] for appropriate notions of convergence for flows of kernels. Flows of kernels

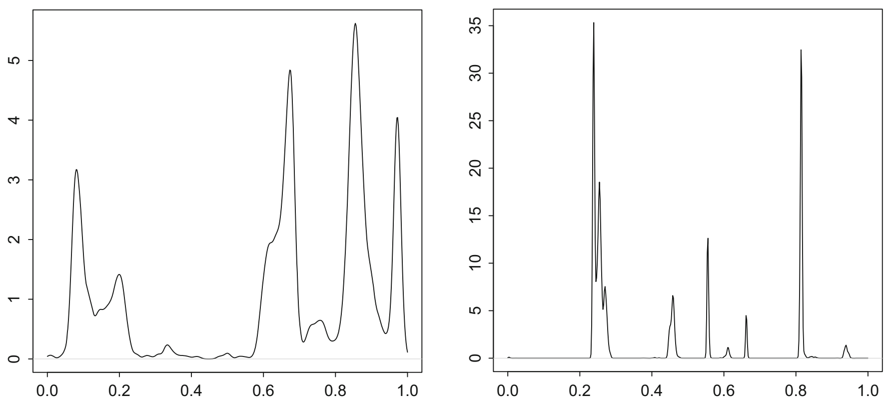

Fig. 1 Simulated realizations of the density of the kernel $K_{0,1}(0,\cdot)$ associated with generators $\mathcal{G}^{N,1}$. The parameters are $a = 20, b = 0.375$ in (**a**), and $a = 60, b = 0.125$ in (**b**)

associated with sticky Brownian motions were first considered by Le Jan and Raimond [10]. For a general splitting rule, they were defined by Howitt and Warren [5], and have subsequently been studied extensively in [12]. In general the parameters of a consistent family of sticky Brownian motions can represented in terms of a splitting measure ν as

$$\theta(k:l) = \int_0^1 q^{k-1}(1-q)^{l-1}\nu(dq) \tag{29}$$

For the parameters $\theta(k:l)$ given by Theorem 1, the measure ν is given by

$$\nu(dq) = \frac{q(1-q)}{\phi(\Phi^{-1}(q))}dq \tag{30}$$

where ϕ denotes the standard Gaussian density, and Φ the corresponding distribution function. The right and left speeds of the flow are defined by

$$\beta_+ = 2\int_0^1 q^{-1}\nu(dq) \text{ and } \beta_- = -2\int_0^1 (1-q)^{-1}\nu(dq) \tag{31}$$

and with ν given by (30) are both infinite. Thus according to the Theorem 2.7 of [12], the support of the corresponding kernels is almost surely equal to $\mathbf{R}$. However, by Theorem 2.8 of [12], for any $s \le t$ and x the measure $K_{s,t}(x,\cdot)$ is purely atomic. This seems consistent with the simulations which show the mass becoming more concentrated as the parameters a and b increase and decrease respectively. It is less evident from these simulations that, in the limit, the set of points carrying the mass is dense.

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Infinitesimal Invariance for the Coupled KPZ Equations

Tadahisa Funaki

Abstract This paper studies the infinitesimal invariance for $\mathbb{R}^d$-valued extension of the Kardar-Parisi-Zhang (KPZ) equation at approximating level.

1 Introduction and Main Result

Once Marc Yor told me that he was interested in the Euclidean quantum field theory at the very beginning of his academic carrier. My guess is that this gave him a motivation to study stochastic processes and martingale problems in infinite-dimensional spaces in [9], though it is not clearly stated. Hairer [8] has recently developed the theory of regularity structures and succeeded to construct solutions of ill-posed stochastic partial differential equations (SPDEs) including the dynamic Φ_d^4 model with $d \leq 3$ and the KPZ equation. The present paper is related to the KPZ equation and deals with infinite-dimensional diffusion operators.

In [6], we studied the KPZ equation especially from a viewpoint of finding its invariant measures. Since the KPZ equation is an ill-posed stochastic partial differential equation, we need to introduce a regularization of the noise and, at the same time, a renormalization for the nonlinear term in an appropriate manner to find invariant measures. The equation studied there was scalar-valued, while we treat $\mathbb{R}^d$-valued coupled equation in this paper. In [6], we first used the lattice approximation and then passed to the continuum limit. In particular, the infinitesimal invariance of the corresponding infinite-dimensional diffusion operator was obtained. In this paper, we show this infinitesimal invariance without relying on the lattice approximation, but by directly approaching to the continuum system. This is done only for the regularized equation. In [6], due to the Cole-Hopf transform and relying on a similar method for establishing Boltzmann-Gibbs principle, the regularization of the noise was eventually removed. For our coupled equation, the Cole-Hopf transform is not available in general so that the same method does not

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work. Our hope is that the methods of [8] or [7] would work in our situation too, but we will not discuss this here.

We consider the following $\mathbb{R}^d$-valued extension of the KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{R}$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \dot{W}^\alpha(t,x), \quad x \in \mathbb{R}, \tag{1}$$

for $1 \le \alpha \le d$, where $\dot{W}(t,x) = (\dot{W}^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise. In particular, it has the correlation function

$$E[\dot{W}^\alpha(t,x)\dot{W}^\beta(s,y)] = \delta^{\alpha\beta}\delta(x-y)\delta(t-s),$$

and $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ are given constants; see [3]. We use Einstein's convention and $\delta^{\alpha\beta}$ denotes Kronecker's δ. From the form of the Eq. (1), the constants $\Gamma^\alpha_{\beta\gamma}$ ought to satisfy $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$.

In this paper, we consider (1) only at approximating level; see (3) below. To show the infinitesimal invariance for the KPZ approximating equation (3), we need the additional condition on $\Gamma^\alpha_{\beta\gamma}$:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\beta_{\gamma\alpha}, \tag{2}$$

for all α, β, γ.

Remark 1.1 To discuss a random evolution of loops on a manifold, [5] considered the SPDE (1) with $x \in \mathbb{S}\,(= [0,1]$ with periodic boundary condition), $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}(h)$, which express the Christoffel symbols on a manifold, and $\dot{W}(t,x)$ replaced by a smooth noise.

We now introduce KPZ approximating equation. Let $\eta \in C_0^\infty(\mathbb{R})$ be a function satisfying $\eta(x) \ge 0, \eta(x) = \eta(-x)$ and $\int_\mathbb{R} \eta(x)dx = 1$. We set $\eta^\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon > 0$, $\eta_2(x) = \eta * \eta(x)$, and $\eta_2^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$. Note that $\eta_2^\varepsilon(x) = \eta^\varepsilon * \eta^\varepsilon(x)$. Define the smeared noise:

$$W^\varepsilon(t,x) \equiv (W^{\varepsilon,\alpha}(t,x))_{\alpha=1}^d = \langle W(t), \eta^\varepsilon(x-\cdot)\rangle,$$

and consider the following $\mathbb{R}^d$-valued KPZ approximating equation for $h = h^\varepsilon(t,x) \equiv (h^{\varepsilon,\alpha}(t,x))_{\alpha=1}^d$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - \xi^\varepsilon\delta^{\beta\gamma}) * \eta_2^\varepsilon + \dot{W}^{\varepsilon,\alpha}(t,x), \quad x \in \mathbb{R}, \tag{3}$$

for $1 \le \alpha \le d$, where

$$\xi^\varepsilon = \int_\mathbb{R} \eta^\varepsilon(y)^2 dy \; (= \eta_2^\varepsilon(0)).$$

Noting that the solution h of the SPDE (3) is smooth in x, we are concerned with the associated tilt process $\partial_x h$.

Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$ on $\mathscr{C} = C(\mathbb{R};\mathbb{R}^d)$, where B is the $\mathbb{R}^d$-valued two-sided Brownian motion satisfying $B(0) = 0$. Note that ν^ε is a probability measure which is independent of the choice of the value of $B(0)$. Then, the main result of this paper is formulated in the following theorem; see Theorem 3.1 below for more detailed statements.

Theorem 1.1 *The probability measure ν^ε on $\mathscr{C}$ is infinitesimally invariant for the tilt process $\partial_x h$ of the SPDE* (3).

2 Generator and Associated Gaussian Random Measure

To state Theorem 1.1 more precisely, we introduce the (formal) generator of the process $h(t)$ determined by (3). Let $\mathscr{D}_H$ be the class of all tame functions Φ on $\mathscr{C}$, that is, those of the form:

$$\Phi(h) = f(\langle h,\varphi_1\rangle,\ldots,\langle h,\varphi_n\rangle), \quad h\in\mathscr{C}, \tag{4}$$

with $n = 1,2,\ldots, f = f(z_1,\ldots,z_n) \in C_b^2(\mathbb{R}^n), \varphi_1 \ldots,\varphi_n \in C_0^\infty(\mathbb{R};\mathbb{R}^d)$, where $\langle h,\varphi\rangle = \sum_\alpha \int_{\mathbb{R}} h^\alpha(x)\varphi^\alpha(x)dx$ for $\varphi = (\varphi^\alpha(x))_{\alpha=1}^d$. We define its functional derivatives by

$$D_\alpha\Phi(x;h) := \sum_{i=1}^n \partial_{z_i} f(\langle h,\varphi_1\rangle,\ldots,\langle h,\varphi_n\rangle)\varphi_i^\alpha(x), \tag{5}$$

$$D_{\alpha\beta}^2\Phi(x_1,x_2;h) := \sum_{i,j=1}^n \partial_{z_i}\partial_{z_j} f(\langle h,\varphi_1\rangle,\ldots,\langle h,\varphi_n\rangle)\varphi_i^\alpha(x_1)\varphi_j^\beta(x_2), \tag{6}$$

for $1\le\alpha,\beta\le d$. Let $\mathscr{D}_{H,\nabla}$ be the class of all $\Phi\in\mathscr{D}_H$ with φ_i satisfying $\int_{\mathbb{R}}\varphi_i dx = 0, 1\le i\le n$. This is a natural class of functions for tilt variables, since, under the equivalence relation $h\sim h+c$ with some $c\in\mathbb{R}^d$, it holds $\Phi(h) = \Phi(h+c)$ if $\Phi\in\mathscr{D}_{H,\nabla}$ so that Φ is a function on the quotient space $\tilde{\mathscr{C}} = \mathscr{C}/\sim$. For the function $\Phi\in\mathscr{D}_{H,\nabla}$, though we write its variable by h, the height h itself has no meaning. In particular, if h is differentiable, $\Phi\in\mathscr{D}_{H,\nabla}$ can be considered as a function of its tilt $h'\equiv\partial_x h$: if $\Phi(h') = f(\langle h',\varphi_1\rangle,\ldots,\langle h',\varphi_n\rangle)$, then $\langle h',\varphi\rangle = -\langle h,\varphi'\rangle$ and φ' satisfies the condition $\int_{\mathbb{R}}\varphi' dx = 0$, which is the additional condition imposed to $\Phi\in\mathscr{D}_{H,\nabla}$ with φ replaced by φ'.

For $\Phi \in \mathscr{D}_H$, define two operators $\mathscr{L}_0^\varepsilon$ and $\mathscr{A}^\varepsilon$ by

$$\mathscr{L}_0^\varepsilon \Phi(h) = \frac{1}{2}\sum_\alpha \int_{\mathbb{R}^2} D^2_{\alpha\alpha}\Phi(x_1,x_2;h)\eta_2^\varepsilon(x_1-x_2)dx_1dx_2 + \frac{1}{2}\int_{\mathbb{R}} \partial_x^2 h^\alpha(x) D_\alpha \Phi(x;h)dx,$$

$$\mathscr{A}^\varepsilon \Phi(h) = \frac{1}{2}\int_{\mathbb{R}} \Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - \xi^\varepsilon \delta^{\beta\gamma}) * \eta_2^\varepsilon(x) D_\alpha \Phi(x;h)dx.$$

Recall that we use Einstein's convention. Then, $\mathscr{L}^\varepsilon := \mathscr{L}_0^\varepsilon + \mathscr{A}^\varepsilon$ is the (formal) generator corresponding to the SPDE (3). In fact, by applying Itô's formula and denoting $h(t)$ by h_t, we have that

$$\partial_t \Phi(h_t) = \langle D_\alpha \Phi(x;h_t), \partial_t h_t^\alpha(x)\rangle_{\mathbb{R}} + \frac{1}{2}\langle D^2_{\alpha\beta}\Phi(x_1,x_2;h_t), \dot{W}^{\varepsilon,\alpha}(t,x_1)\dot{W}^{\varepsilon,\beta}(t,x_2)\rangle_{\mathbb{R}^2}$$

and note that

$$dW_t^{\varepsilon,\alpha}(x_1)dW_t^{\varepsilon,\beta}(x_2) = \delta^{\alpha\beta}\eta_2^\varepsilon(x_1-x_2)dt.$$

The limit as $\varepsilon \downarrow 0$ of ν^ε for tilt variables (and therefore defined on $\tilde{\mathscr{C}}$) can be identified with the $\mathbb{R}^d$-valued Gaussian random measure ν on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ determined from dB. More precisely saying, under ν, $\mathbb{R}^d$-valued random variables $\{M(A) = (M^\alpha(A))_{\alpha=1}^d; A \in \mathscr{B}(\mathbb{R})\}$ are given and

1. $\{M^\alpha(\cdot)\}_{\alpha=1}^d$ are independent system.
2. $M^\alpha(A) \overset{\text{law}}{=} N(0,|A|)$ for each α.
3. If $\{A_i \in \mathscr{B}(\mathbb{R})\}_{i=1}^n$ are disjoint, then $\{M^\alpha(A_i)\}_{i=1}^n$ are independent and $M^\alpha(\cup_{i=1}^n A_i) = \sum_{i=1}^n M^\alpha(A_i)$ holds a.s. for each α.

Such $M(A)$ can be constructed from $M((a,b]) := B(b) - B(a)$ in terms of the $\mathbb{R}^d$-valued two sided Brownian motion $\{B(x); x \in \mathbb{R}\}$ satisfying, for instance, $B(0) = 0$.

Recall that the multiple Wiener integral of order $\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}_+^d, \mathbb{Z}_+ = \{0,1,2,\dots\}$, with a kernel $\varphi_\mathbf{n} \in \hat{L}^2(\mathbb{R}^\mathbf{n})$, $\mathbb{R}^\mathbf{n} := \mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_d}$, that is $\varphi_\mathbf{n} = \varphi_\mathbf{n}(x_1^1,\dots,x_{n_1}^1;\dots;x_1^d,\dots,x_{n_d}^d) \in L^2(\mathbb{R}^\mathbf{n})$ being symmetric in n_α-variables for each $1 \le \alpha \le d$, is defined by

$$I(\varphi_\mathbf{n}) = \frac{1}{\mathbf{n}!}\int_{\mathbb{R}^\mathbf{n}} \varphi_\mathbf{n}(x_1^1,\dots,x_{n_1}^1;\dots;x_1^d,\dots,x_{n_d}^d) \times dB^1(x_1^1)\cdots dB^1(x_{n_1}^1)\cdots dB^d(x_1^d)\cdots dB^d(x_{n_d}^d),$$

where $\mathbf{n}! = n_1! \cdots n_d!$. Set $\mathscr{H}_{\mathbf{n}} = \{I(\varphi_{\mathbf{n}}) \in L^2(\tilde{\mathscr{C}}, \nu); \varphi_{\mathbf{n}} \in \hat{L}^2(\mathbb{R}^{\mathbf{n}})\}$ for $|\mathbf{n}| \geq 1$ and $\mathscr{H}_0 = \{\text{const}\}$. Then, the well-known Wiener-Itô (Wiener chaos) expansion of $\Phi \in \mathscr{H} := L^2(\tilde{\mathscr{C}}, \nu)$ is given by

$$\Phi = \sum_{\mathbf{n}} I(\varphi_{\mathbf{n}}) \in \bigoplus_{\mathbf{n}} \mathscr{H}_{\mathbf{n}}, \tag{7}$$

with some $\varphi_0 \in \mathbb{R}$ and $\varphi_{\mathbf{n}} \in \hat{L}^2(\mathbb{R}^{\mathbf{n}})$, where $I(\varphi_0) = \varphi_0$, and

$$\|\Phi\|^2_{L^2(\nu)} = \sum_{\mathbf{n}} \|I(\varphi_{\mathbf{n}})\|^2_{L^2(\nu)} = \sum_{\mathbf{n}} \frac{1}{\mathbf{n}!} \|\varphi_{\mathbf{n}}\|^2_{L^2(\mathbb{R}^{\mathbf{n}})} \tag{8}$$

holds because of the orthogonality and then by Itô isometry. The expansion (7) identifies $\Phi \in L^2(\tilde{\mathscr{C}}, \nu)$ with the element $\varphi = \{\varphi_{\mathbf{n}}\}_{\mathbf{n}} \in \bigoplus_{\mathbf{n}} \hat{L}^2(\mathbb{R}^{\mathbf{n}})$ of the symmetric Fock space. The reason to do this is that it gives an explicit representation of the functional derivative D_α: For each $x \in \mathbb{R}$, $D_\alpha \Phi(x)$ has representation $\{D_\alpha \varphi_{\mathbf{n}}(x)\}_{\mathbf{n}}$ where $D_\alpha \varphi_{\mathbf{n}}(x; \cdot) \in \hat{L}^2(\mathbb{R}^{\mathbf{n}-\delta^\alpha})$ with $\delta^\alpha \in \mathbb{Z}_+^d$ defined by $(\delta^\alpha)_\beta = \delta^{\alpha\beta}$ is given by

$$\begin{aligned} &D_\alpha \varphi_{\mathbf{n}}(x; x_1^1, \dots, x_{n_1}^1; \dots; x_1^\alpha, \dots, x_{n_\alpha-1}^\alpha; \dots; x_1^d, \dots, x_{n_d}^d) \\ &\quad = -\frac{1}{n_\alpha} \sum_{i=1}^{n_\alpha} \partial_{x_i^\alpha} \varphi_{\mathbf{n}}(x_1^1, \dots, x_{n_1}^1; \dots; x_1^\alpha, \dots, x_{i-1}^\alpha, x, x_i^\alpha, \dots, x_{n_\alpha-1}^\alpha; \dots; x_1^d, \dots, x_{n_d}^d) \\ &\quad = -\partial_{x_1^\alpha} \varphi_{\mathbf{n}}(x_1^1, \dots, x_{n_1}^1; \dots; x, x_1^\alpha, \dots, x_{n_\alpha-1}^\alpha; \dots; x_1^d, \dots, x_{n_d}^d). \end{aligned} \tag{9}$$

The factor $\frac{1}{n_\alpha}$ arises when we replace $\frac{1}{n_\alpha!}$ with $\frac{1}{(n_\alpha-1)!}$, and the second equality is due to the symmetry of $\varphi_{\mathbf{n}}$. The minus sign appears due to the integration by parts $\langle \varphi, \psi' \rangle = -\langle \varphi', \psi \rangle$ for $\mathbb{R}$-valued functions φ and ψ.

We will denote $\mathbf{n}$ whose αth components n_α are given by m_α and non-specified components are all 0 by $\mathbf{n}(\alpha : m_\alpha)$. For example, $\mathbf{n}(\alpha : 3) = (0, \dots, 0, 3, 0, \dots, 0)$ with $n_\alpha = 3$ for a single α and $\mathbf{n}(\alpha : 2, \beta : 1) = (0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0)$ with $n_\alpha = 2, n_\beta = 1$ for $\alpha \neq \beta$.

3 Infinitesimal Invariance

We now prove the following theorem, a kind of integration by parts formula, due to the Wiener-Itô expansion.

Theorem 3.1 *Assume the condition* (2) *on* $\Gamma^\alpha_{\beta\gamma}$. *Then, for every* $\varepsilon > 0$ *and* $\Phi \in \mathscr{D}_{H,\nabla}$, *which has kernels* $\varphi_{\mathbf{n}(\alpha:3)} \in \hat{L}^2(\mathbb{R}^3) \cap C_0^1(\mathbb{R}^3)$, $\varphi_{\mathbf{n}(\alpha:2,\beta:1)} \in \hat{L}^2(\mathbb{R}^{(2,1)}) \cap C_0^1(\mathbb{R}^3)$ *and* $\varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)} \in L^2(\mathbb{R}^3) \cap C_0^1(\mathbb{R}^3)$ *of the third order Wiener chaos determined by*

(7) *for every different* $1 \le \alpha, \beta, \gamma \le d$*, we have that*

$$\int \mathscr{A}^\varepsilon \Phi(h) \nu^\varepsilon(dh) = 0. \tag{10}$$

Moreover,

$$\int \mathscr{L}^\varepsilon \Phi(h) \nu^\varepsilon(dh) = 0. \tag{11}$$

Proof Recalling that $\partial_x h = \partial_x (B * \eta^\varepsilon)$ under ν^ε, by Itô's formula, we have that

$$\begin{aligned} \partial_x h^\beta \partial_x h^\gamma &= \int_{\mathbb{R}} \eta^\varepsilon(x-y) dB^\beta(y) \int_{\mathbb{R}} \eta^\varepsilon(x-y) dB^\gamma(y) \\ &= \Psi^{\beta\gamma}(x) + \xi^\varepsilon \delta^{\beta\gamma}, \end{aligned} \tag{12}$$

where $\Psi^{\beta\gamma}(x) \equiv \Psi^{\varepsilon,\beta\gamma}(x)$ is a Wiener functional of second order given by

$$\Psi^{\beta\gamma}(x) = \int_{\mathbb{R}^2} \eta^\varepsilon(x - x_1^\beta) \eta^\varepsilon(x - x_1^\gamma) dB^\beta(x_1^\beta) dB^\gamma(x_1^\gamma), \tag{13}$$

if $\beta \neq \gamma$ and

$$\Psi^{\beta\beta}(x) = \int_{\mathbb{R}^2} \eta^\varepsilon(x - x_1^\beta) \eta^\varepsilon(x - x_2^\beta) dB^\beta(x_1^\beta) dB^\beta(x_2^\beta). \tag{14}$$

Therefore, we have that

$$\begin{aligned} 2 \int \mathscr{A}^\varepsilon \Phi(h) \nu^\varepsilon(dh) &= \int_{\tilde{\mathscr{C}}} \nu(dB) \int_{\mathbb{R}} \Gamma^\alpha_{\beta\gamma} \Psi^{\beta\gamma} * \eta_2^\varepsilon(x) D_\alpha \Phi(x; B * \eta^\varepsilon) dx \\ &= \int_{\tilde{\mathscr{C}}} \nu(dB) \int_{\mathbb{R}} \Gamma^\alpha_{\beta\gamma} \Psi^{\beta\gamma}(x) \boldsymbol{\Phi}_\alpha(x) dx, \end{aligned} \tag{15}$$

where

$$\boldsymbol{\Phi}_\alpha(x) \equiv \boldsymbol{\Phi}_\alpha^\varepsilon(x) = \int_{\mathbb{R}} \eta_2^\varepsilon(x-y) D_\alpha \Phi(y; B * \eta^\varepsilon) dy.$$

Note that $\Psi^{\beta\gamma} \in \mathscr{H}_{\mathbf{n}(\beta:1,\gamma:1)}$ if $\beta \neq \gamma$ and $\Psi^{\beta\beta} \in \mathscr{H}_{\mathbf{n}(\beta:2)}$. Because of the orthogonality in the Wiener-Itô expansion, to compute the integral in (15), we may just pick up the $\mathscr{H}_{\mathbf{n}(\beta:1,\gamma:1)}$-component (for $\beta \neq \gamma$) and $\mathscr{H}_{\mathbf{n}(\beta:2)}$-component in the Wiener-Itô expansion of $\Phi_\alpha(x)$. Or, we may find the corresponding kernels of these components, which are obtained in the following in five different cases separately.

Case 1 ($\alpha = \beta = \gamma$): The kernel of $\mathscr{H}_{\mathbf{n}(\beta:2)}$-component of $D_\alpha \Phi(y; B * \eta^\varepsilon)$, with variables denoted by (x_1^α, x_2^α), is given by

$$\int_{\mathbb{R}^2} D_\alpha \varphi_{\mathbf{n}(\alpha:3)}(y; z_1^\alpha, z_2^\alpha) \eta^\varepsilon(x_1^\alpha - z_1^\alpha) \eta^\varepsilon(x_2^\alpha - z_2^\alpha) dz_1^\alpha dz_2^\alpha,$$

where, from (9),

$$\begin{aligned} &D_\alpha \varphi_{\mathbf{n}(\alpha:3)}(x; x_1^\alpha, x_2^\alpha) \\ &\quad = -\tfrac{1}{3}\{\partial_{x_1^\alpha} \varphi_{\mathbf{n}(\alpha:3)}(x, x_1^\alpha, x_2^\alpha) + \partial_{x_2^\alpha} \varphi_{\mathbf{n}(\alpha:3)}(x_1^\alpha, x, x_2^\alpha) + \partial_{x_3^\alpha} \varphi_{\mathbf{n}(\alpha:3)}(x_1^\alpha, x_2^\alpha, x)\}. \end{aligned} \tag{16}$$

Case 2 ($\alpha \neq \beta = \gamma$): The kernel of the same as above, with variables denoted by (x_1^β, x_2^β), is given by

$$\int_{\mathbb{R}^2} D_\alpha \varphi_{\mathbf{n}(\alpha:1,\beta:2)}(y; z_1^\beta, z_2^\beta) \eta^\varepsilon(x_1^\beta - z_1^\beta) \eta^\varepsilon(x_2^\beta - z_2^\beta) dz_1^\beta dz_2^\beta,$$

where

$$D_\alpha \varphi_{\mathbf{n}(\alpha:1,\beta:2)}(x; x_1^\beta, x_2^\beta) = -\partial_{x_1^\alpha} \varphi_{\mathbf{n}(\alpha:1,\beta:2)}(x; x_1^\beta, x_2^\beta).$$

Case 3 ($\alpha = \beta \neq \gamma$): The kernel of $\mathscr{H}_{\mathbf{n}(\beta:1,\gamma:1)}$-component of $D_\alpha \Phi(y; B * \eta^\varepsilon)$, with variables denoted by (x_1^β, x_1^γ), is given by

$$\int_{\mathbb{R}^2} D_\alpha \varphi_{\mathbf{n}(\alpha:2,\gamma:1)}(y; z_1^\beta, z_1^\gamma) \eta^\varepsilon(x_1^\beta - z_1^\beta) \eta^\varepsilon(x_1^\gamma - z_1^\gamma) dz_1^\beta dz_1^\gamma,$$

where

$$\begin{aligned} &D_\alpha \varphi_{\mathbf{n}(\alpha:2,\gamma:1)}(x; x_1^\beta, x_1^\gamma) \\ &\quad = -\tfrac{1}{2}\{\partial_{x_1^\beta} \varphi_{\mathbf{n}(\alpha:2,\gamma:1)}(x, x_1^\beta; x_1^\gamma) + \partial_{x_2^\beta} \varphi_{\mathbf{n}(\alpha:2,\gamma:1)}(x_1^\beta, x; x_1^\gamma)\}. \end{aligned}$$

Case 4 ($\alpha = \gamma \neq \beta$): The kernel of the same as above is given by

$$\int_{\mathbb{R}^2} D_\alpha \varphi_{\mathbf{n}(\alpha:2,\beta:1)}(y; z_1^\beta, z_1^\gamma) \eta^\varepsilon(x_1^\beta - z_1^\beta) \eta^\varepsilon(x_1^\gamma - z_1^\gamma) dz_1^\beta dz_1^\gamma,$$

where

$$\begin{aligned} &D_\alpha \varphi_{\mathbf{n}(\alpha:2,\beta:1)}(x; x_1^\beta, x_1^\gamma) \\ &\quad = -\tfrac{1}{2}\{\partial_{x_1^\gamma} \varphi_{\mathbf{n}(\alpha:2,\beta:1)}(x, x_1^\gamma; x_1^\beta) + \partial_{x_2^\gamma} \varphi_{\mathbf{n}(\alpha:2,\beta:1)}(x_1^\gamma, x; x_1^\beta)\}. \end{aligned}$$

Case 5 (α, γ, β are all different): The kernel of the same as above is given by

$$\int_{\mathbb{R}^2} D_\alpha \varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(y; z_1^\beta, z_1^\gamma)\eta^\varepsilon(x_1^\beta - z_1^\beta)\eta^\varepsilon(x_1^\gamma - z_1^\gamma)dz_1^\beta dz_1^\gamma,$$

where

$$D_\alpha \varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x; x_1^\beta, x_1^\gamma) = -\partial_{x_1^\alpha}\varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x; x_1^\beta; x_1^\gamma).$$

The goal is to show that the sum in the right hand side of (15) vanishes. Recalling (14), the contribution from *Case 1* to this sum is given by

$$\begin{aligned}
&\sum_\alpha \Gamma_{\alpha\alpha}^\alpha \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} D_\alpha \varphi_{\mathbf{n}(\alpha:3)}(y; z_1^\alpha, z_2^\alpha)\eta_2^\varepsilon(x-y)\eta_2^\varepsilon(x-z_1^\alpha)\eta_2^\varepsilon(x-z_2^\alpha)dydz_1^\alpha dz_2^\alpha \\
&\quad = \sum_\alpha \Gamma_{\alpha\alpha}^\alpha \int_{\mathbb{R}} dx E[D_\alpha \varphi_{\mathbf{n}(\alpha:3)}(x+R_1; x+R_2, x+R_3)] \\
&\quad = -\frac{1}{3}\sum_\alpha \Gamma_{\alpha\alpha}^\alpha \int_{\mathbb{R}} dx E\left[\frac{d}{dx}\{\varphi_{\mathbf{n}(\alpha:3)}(x+R_1, x+R_2, x+R_3)\}\right] \\
&\quad = 0,
\end{aligned}$$

since $\varphi_{\mathbf{n}(\alpha:3)} \in C_0^1(\mathbb{R}^3)$, where R_1, R_2, R_3 are independent random variables with the same distributions $\eta_2^\varepsilon(x)dx$. For the third line, we have used (16) and the symmetry of the distribution of (R_1, R_2, R_3) under permutations.

From (14), the contribution from *Case 2* is given by

$$\begin{aligned}
&\sum_{\alpha\neq\beta} \Gamma_{\beta\beta}^\alpha \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} D_\alpha \varphi_{\mathbf{n}(\alpha:1,\beta:2)}(y; z_1^\beta, z_2^\beta)\eta_2^\varepsilon(x-y)\eta_2^\varepsilon(x-z_1^\beta)\eta_2^\varepsilon(x-z_2^\beta)dydz_1^\beta dz_2^\beta \\
&\quad = -\sum_{\alpha\neq\beta} \Gamma_{\beta\beta}^\alpha \int_{\mathbb{R}} dx E[\partial_{x_1^\alpha}\varphi_{\mathbf{n}(\alpha:1,\beta:2)}(x+R_1; x+R_2, x+R_3)] \\
&\quad = -\sum_{\beta\neq\gamma} \Gamma_{\beta\beta}^\gamma \int_{\mathbb{R}} dx E[\partial_{x_1^\gamma}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1, x+R_2; x+R_3)].
\end{aligned}$$

The last equality follows by rewriting α into γ and noting the symmetry of the distribution of (R_1, R_2, R_3). From (13), the contribution from *Case 3* is given by

$$\begin{aligned}
&\sum_{\beta\neq\gamma} \Gamma_{\beta\gamma}^\beta \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} D_\beta \varphi_{\mathbf{n}(\beta:2,\gamma:1)}(y; z_1^\beta, z_1^\gamma)\eta_2^\varepsilon(x-y)\eta_2^\varepsilon(x-z_1^\beta)\eta_2^\varepsilon(x-z_1^\gamma)dydz_1^\beta dz_1^\gamma \\
&\quad = -\frac{1}{2}\sum_{\beta\neq\gamma} \Gamma_{\beta\gamma}^\beta \int_{\mathbb{R}} dx E[\partial_{x_1^\beta}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1, x+R_2; x+R_3) \\
&\qquad\qquad + \partial_{x_2^\beta}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1, x+R_2; x+R_3)]
\end{aligned}$$

From (13), the contribution from *Case 4* is given by

$$\sum_{\beta\neq\gamma}\Gamma^{\gamma}_{\beta\gamma}\int_{\mathbb{R}}dx\int_{\mathbb{R}^3}D_\gamma\varphi_{\mathbf{n}(\gamma:2,\beta:1)}(y;z_1^\gamma,z_1^\beta)\eta_2^\varepsilon(x-y)\eta_2^\varepsilon(x-z_1^\beta)\eta_2^\varepsilon(x-z_1^\gamma)dydz_1^\beta dz_1^\gamma$$

$$=-\frac{1}{2}\sum_{\beta\neq\gamma}\Gamma^{\gamma}_{\beta\gamma}\int_{\mathbb{R}}dxE[\partial_{x_1^\gamma}\varphi_{\mathbf{n}(\gamma:2,\beta:1)}(x+R_1,x+R_2;x+R_3)$$
$$+\partial_{x_2^\gamma}\varphi_{\mathbf{n}(\gamma:2,\beta:1)}(x+R_1,x+R_2;x+R_3)]$$
$$=-\frac{1}{2}\sum_{\beta\neq\gamma}\Gamma^{\beta}_{\gamma\beta}\int_{\mathbb{R}}dxE[\partial_{x_1^\beta}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1,x+R_2;x+R_3)$$
$$+\partial_{x_2^\beta}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1,x+R_2;x+R_3)].$$

The last equality follows by interchanging the roles of β and γ. Thus, since $\Gamma^{\beta}_{\gamma\beta}=\Gamma^{\beta}_{\beta\gamma}$, the sum of contributions from *Case 3* and *Case 4* becomes

$$-\sum_{\beta\neq\gamma}\Gamma^{\beta}_{\beta\gamma}\int_{\mathbb{R}}dxE[\partial_{x_1^\beta}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1,x+R_2;x+R_3)$$
$$+\partial_{x_2^\beta}\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1,x+R_2;x+R_3)].$$

Therefore, from the condition (2), we see that the sum of contributions from *Case 2*, *Case 3* and *Case 4* becomes

$$-\sum_{\beta\neq\gamma}\Gamma^{\gamma}_{\beta\beta}\int_{\mathbb{R}}dxE[(\partial_{x_1^\beta}+\partial_{x_2^\beta}+\partial_{x_1^\gamma})\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1,x+R_2;x+R_3)]$$
$$=-\sum_{\beta\neq\gamma}\Gamma^{\gamma}_{\beta\beta}\int_{\mathbb{R}}dxE\left[\frac{d}{dx}\{\varphi_{\mathbf{n}(\beta:2,\gamma:1)}(x+R_1,x+R_2;x+R_3)\}\right]$$
$$=0,$$

since $\varphi_{\mathbf{n}(\beta:2,\gamma:1)}\in C_0^1(\mathbb{R}^3)$.

Finally, from (13), the contribution from *Case 5* is given by

$$\sum\nolimits^{*}\Gamma^{\alpha}_{\beta\gamma}\int_{\mathbb{R}}dx\int_{\mathbb{R}^3}D_\alpha\varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(y;z_1^\beta,z_1^\gamma)\eta_2^\varepsilon(x-y)\eta_2^\varepsilon(x-z_1^\beta)\eta_2^\varepsilon(x-z_1^\gamma)dydz_1^\beta dz_1^\gamma$$
$$=-\sum\nolimits^{*}\Gamma^{\alpha}_{\beta\gamma}\int_{\mathbb{R}}dxE[\partial_{x_1^\alpha}\varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x+R_1;x+R_2;x+R_3)]$$
$$=-\frac{1}{3}\sum\nolimits^{*}\left\{\Gamma^{\alpha}_{\beta\gamma}\int_{\mathbb{R}}dxE[\partial_{x_1^\alpha}\varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x+R_1;x+R_2;x+R_3)]\right.$$

$$+ \Gamma^{\beta}_{\alpha\gamma} \int_{\mathbb{R}} dx E[\partial_{x_1^{\beta}} \varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x+R_1; x+R_2; x+R_3)]$$

$$+ \Gamma^{\gamma}_{\beta\alpha} \int_{\mathbb{R}} dx E[\partial_{x_1^{\gamma}} \varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x+R_1; x+R_2; x+R_3)] \Big\}$$

by interchanging the roles of α, β and γ, where $\sum^*$ means the sum over all different α, β, γ. Therefore, since $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\beta}_{\alpha\gamma} = \Gamma^{\gamma}_{\beta\alpha}$ hold from (2), the above sum becomes

$$= -\frac{1}{3} \sum\nolimits^* \Gamma^{\alpha}_{\beta\gamma} \int_{\mathbb{R}} dx E\left[\frac{d}{dx} \{ \varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)}(x+R_1; x+R_2; x+R_3) \} \right]$$
$$= 0,$$

since $\varphi_{\mathbf{n}(\alpha:1,\beta:1,\gamma:1)} \in C_0^1(\mathbb{R}^3)$.

These prove (10). The other identity (11) follows from (10), since ν^{ε} is reversible for the tilt process $\partial_x h = \partial_x h^{\varepsilon}$ of the $\mathbb{R}^d$-valued SPDE:

$$\partial_t h = \frac{1}{2} \partial_x^2 h + \dot{W}^{\varepsilon}(t,x), \quad x \in \mathbb{R}, \tag{17}$$

which determines an Ornstein-Uhlenbeck process, and this shows that

$$\int \mathscr{L}_0^{\varepsilon} \Phi(h) \nu^{\varepsilon}(dh) = 0,$$

for all $\Phi \in \mathscr{D}_H$. The reversibility of ν^{ε} is shown as follows: First, recall the well-known fact that μ^c, which is a centered Gaussian measure on $\mathscr{C}$ with covariance operator $\big(-\partial_x^2 + c\big)^{-1}$, is reversible for the SPDE with a mass $c > 0$:

$$\partial_t h^c = \frac{1}{2} \partial_x^2 h^c - \frac{c}{2} h^c + \dot{W}(t,x), \quad x \in \mathbb{R},$$

see e.g. [4], Proposition 6.1. Therefore, the distribution $\mu^{c,\varepsilon}$ of $B^c * \eta^{\varepsilon}$ is reversible for $h^{c,\varepsilon} = h^c * \eta^{\varepsilon}$, where B^c is distributed under μ^c. Thus, the distribution $\nu^{c,\varepsilon}$ of $\partial_x(B^c * \eta^{\varepsilon})$ is reversible for the tilt process $\partial_x h^{c,\varepsilon}$. On the other hand, we easily see that $h^{c,\varepsilon}$ satisfies the SPDE:

$$\partial_t h^{c,\varepsilon} = \frac{1}{2} \partial_x^2 h^{c,\varepsilon} - \frac{c}{2} h^{c,\varepsilon} + \dot{W}^{\varepsilon}(t,x), \quad x \in \mathbb{R}.$$

Letting $c \downarrow 0$, since $\nu^{c,\varepsilon}$ and $\partial_x h^{c,\varepsilon}$ converge to ν^{ε} and $\partial_x h^{\varepsilon}$, respectively, we see the reversibility of ν^{ε} for the tilt process of the SPDE (17).

The proof of the theorem is completed.

Remark 3.1

(1) The operators $\mathscr{L}_0^\varepsilon$ and $\mathscr{A}^\varepsilon$ are symmetric and asymmetric with respect to ν^ε, respectively, that is,

$$\int \Psi \mathscr{L}_0^\varepsilon \Phi d\nu^\varepsilon = \int \Phi \mathscr{L}_0^\varepsilon \Psi d\nu^\varepsilon,$$

and

$$\int \Psi \mathscr{A}^\varepsilon \Phi d\nu^\varepsilon = -\int \Phi \mathscr{A}^\varepsilon \Psi d\nu^\varepsilon.$$

Indeed, the asymmetry of $\mathscr{A}^\varepsilon$ follows from (10) by noting that $\mathscr{A}^\varepsilon(\Phi\Psi) = \Psi\mathscr{A}^\varepsilon\Phi + \Phi\mathscr{A}^\varepsilon\Psi$.

(2) If Echeverria's result [2] can be extended in our setting and if the well-posedness of the $\mathscr{L}^\varepsilon$-martingale problem is shown (this is true at least on the torus $\mathbb{S}$ instead of $\mathbb{R}$), Theorem 3.1 proves that ν^ε is invariant under the time evolution determined by the $\mathbb{R}^d$-valued KPZ approximating Eq. (3). The result of [1] extends that of [2] to an infinite-dimensional setting, however the condition assumed in [1] is rather strong and we cannot apply it in our setting.

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Patterns in Random Walks and Brownian Motion

Jim Pitman and Wenpin Tang

Abstract We ask if it is possible to find some particular continuous paths of unit length in linear Brownian motion. Beginning with a discrete version of the problem, we derive the asymptotics of the expected waiting time for several interesting patterns. These suggest corresponding results on the existence/non-existence of continuous paths embedded in Brownian motion. With further effort we are able to prove some of these existence and non-existence results by various stochastic analysis arguments. A list of open problems is presented.

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1 Introduction and Main Results

We are interested in the question of embedding some continuous-time stochastic processes $(Z_u, 0 \leq u \leq 1)$ into a Brownian path $(B_t; t \geq 0)$, without time-change or scaling, just by a random translation of origin in spacetime. More precisely, we ask the following:

Question 1 Given some distribution of a process Z with continuous paths, does there exist a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as $(Z_u, 0 \leq u \leq 1)$?

The question of whether external randomization is allowed to construct such a random time T, is of no importance here. In fact, we can simply ignore Brownian

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motion on $[0, 1]$, and consider only random times $T \geq 1$. Then $(B_t; 0 \leq t \leq 1)$ provides an independent random element which is adequate for any randomization, see e.g. Kallenberg [40, Theorem 6.10].

Note that a continuous-time process whose sample paths have different regularity, e.g. fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$, cannot be embedded into Brownian motion. Given $(B_t; t \geq 0)$ standard Brownian motion, we define $g_1 := \sup\{t < 1; B_t = 0\}$ the time of last exit from 0 before $t = 1$, and $d_1 := \inf\{t > 1; B_t = 0\}$ the first hitting time of 0 after $t = 1$. The following processes, derived from Brownian motion, are of special interest.

- Brownian bridge, which can be defined by

$$\left(b_u^0 := \frac{1}{\sqrt{g_1}} B_{ug_1}; 0 \leq u \leq 1\right),$$

  and its reflected counterpart $(|b_u^0|; 0 \leq u \leq 1)$.
- Normalized Brownian excursion defined by

$$\left(e_u := \frac{1}{\sqrt{d_1 - g_1}} |B_{g_1 + u(d_1 - g_1)}|; 0 \leq u \leq 1\right).$$

- Brownian meander defined as

$$\left(m_u := \frac{1}{\sqrt{1 - g_1}} |B_{g_1 + u(1 - g_1)}|; 0 \leq u \leq 1\right).$$

- Brownian co-meander defined as

$$\left(\tilde{m}_u := \frac{1}{\sqrt{d_1 - 1}} |B_{d_1 - u(d_1 - 1)}|; 0 \leq u \leq 1\right).$$

- The three-dimensional Bessel process

$$\left(R_u := \sqrt{(B_u)^2 + (B'_u)^2 + (B''_u)^2}; 0 \leq u \leq 1\right),$$

  where $(B'_t; t \geq 0)$ and $(B''_u; u \geq 0)$ are two independent copies of $(B_t; t \geq 0)$.
- The first passage bridge through level $\lambda \neq 0$, defined by

$$(F_u^{\lambda,br}; 0 \leq u \leq 1) \stackrel{(d)}{=} (B_u; 0 \leq u \leq 1) \quad \text{conditioned on} \quad \tau_\lambda = 1,$$

  where $\tau_\lambda := \inf\{t \geq 0; B_t = \lambda\}$ is the first time at which Brownian motion hits $\lambda \neq 0$. Note that for $\lambda < 0$, $(F_u^{|\lambda|,br}; 0 \leq u \leq 1) \stackrel{(d)}{=} (-F_u^{\lambda,br}; 0 \leq u \leq 1)$, and $(F_{1-u}^{\lambda,br} + |\lambda|; 0 \leq u \leq 1)$ is distributed as three dimensional Bessel bridge ending at $|\lambda| > 0$, see e.g. Biane and Yor [10].

- The Vervaat transform of Brownian motion, defined as

$$\left(V_u := \begin{cases} B_{\tau+u} - B_\tau & \text{for } 0 \le u \le 1-\tau \\ B_{\tau-1+u} + B_1 - B_\tau & \text{for } 1-\tau \le u \le 1 \end{cases} ; 0 \le u \le 1\right),$$

where $\tau := argmin_{0 \le t \le 1} B_t$, and the Vervaat transform of Brownian bridge with endpoint $\lambda \in \mathbb{R}$

$$\left(V_u^\lambda := \begin{cases} b^\lambda_{\tau+u} - b^\lambda_\tau & \text{for } 0 \le u \le 1-\tau \\ b^\lambda_{\tau-1+u} + \lambda - b^\lambda_\tau & \text{for } 1-\tau \le u \le 1 \end{cases} ; 0 \le u \le 1\right),$$

where $(b_u^\lambda; 0 \le u \le 1)$ is Brownian bridge ending at $\lambda \in \mathbb{R}$ and $\tau := argmin_{0 \le t \le 1} b_t^\lambda$. It was proved by Vervaat [85] that $(V_u^0; 0 \le u \le 1) \stackrel{(d)}{=} (e_u; 0 \le u \le 1)$. For $\lambda < 0$, $(V_u^{|\lambda|}; 0 \le u \le 1)$ has the same distribution as $(V_{1-u}^\lambda + |\lambda|; 0 \le u \le 1)$.

The Brownian bridge, meander, excursion and the three-dimensional Bessel process are well-known. The definition of the co-meander is found in Yen and Yor [91, Chap. 7]. The first passage bridge is studied by Bertoin et al. [8]. The Vervaat transform of Brownian bridges and of Brownian motion are extensively discussed in Lupu et al. [56]. According to the above definitions, the distributions of the Brownian bridge, excursion and (co-)meander can all be achieved in Brownian motion provided some Brownian scaling operation is allowed. Note that the distributions of all these processes are singular with respect to Wiener measure. So it is a non-trivial question whether copies of them can be found in Brownian motion just by a shift of origin in spacetime. Otherwise, for a process $(Z_t, 0 \le t \le 1)$ whose distribution is absolutely continuous with respect to that of $(B_t, 0 \le t \le 1)$, for instance the Brownian motion with drift $Z_t := \vartheta t + B_t$ for a fixed ϑ, the distribution of Z can be easily obtained as that of $(B_{T+t} - B_T, 0 \le t \le 1)$ for a suitable stopping time $T + 1$ by *Rost's filling scheme*. We refer readers to Sect. 3.5 for further development.

The question raised here has some affinity to the question of embedding a given one-dimensional distribution as the distribution of B_T for a random time T. This *Skorokhod embedding problem* traces back to Skorokhod [80] and Dubins [24]– who found integrable stopping times T such that the distribution of B_T coincides with any prescribed one with zero mean and finite second moment. Monroe [64, 65] considered embedding of a continuous-time process into Brownian motion, and showed that every semi-martingale is a time-changed Brownian motion. Rost [76] studied the problem of embedding a one-dimensional distribution in a Markov process with randomized stopping times. We refer readers to the excellent survey of Obloj [69] and references therein. Let $X_t := (B_{t+u} - B_t; 0 \le u \le 1)$ for $t \ge 0$ be the moving-window process associated to Brownian motion. In Question 1, we are concerned with the possibility of embedding a given distribution on $\mathcal{C}[0, 1]$ as that of X_T for some random time T.

Let us present the main results of the paper. We start with a list of continuous-time processes that cannot be embedded into Brownian motion by a shift of origin in spacetime.

Theorem 1 (Impossibility of Embedding of Normalized Excursion, Reflected Bridge, Vervaat Transform of Brownian Motion, First Passage Bridge and Vervaat Bridge) *For each of the following five processes* $Z := (Z_u; 0 \le u \le 1)$*, there is no random time* T *such that* $(B_{T+u} - B_T; 0 \le u \le 1)$ *has the same distribution as* Z*:*

1. *the normalized Brownian excursion* $Z = (e_u; 0 \le u \le 1)$*;*
2. *the reflected Brownian bridge* $Z = (|b_u^0|; 0 \le u \le 1)$*;*
3. *the Vervaat transform of Brownian motion* $Z = (V_u; 0 \le u \le 1)$*;*
4. *the first passage bridge through level* $\lambda \neq 0$*,* $Z = (F_u^{\lambda,br}; 0 \le u \le 1)$*;*
5. *the Vervaat transform of Brownian bridge with endpoint* $\lambda \in \mathbb{R}$*,* Z*=*$(V_u^{\lambda}; 0 \le u \le 1)$.

Note that in Theorem 1(4), (5), it suffices to consider the case of $\lambda < 0$ by time-reversal. As we will see later in Theorem 4, Theorem 1 is an immediate consequence of the fact that typical paths of these processes cannot be found in Brownian motion. The next theorem shows the possibility of embedding into Brownian motion some continuous-time processes whose distributions are singular with respect to Wiener measure.

Theorem 2 (Embeddings of Meander, Co-meander and 3-d Bessel Process) *For each of the following three processes* $Z := (Z_u, 0 \le u \le 1)$ *there is some random time* T *such that* $(B_{T+u} - B_T; 0 \le u \le 1)$ *has the same distribution as* Z*:*

1. *the meander* $Z = (m_u; 0 \le u \le 1)$.
2. *the co-meander* $Z = (\tilde{m}_u; 0 \le u \le 1)$.
3. *the three-dimensional Bessel process* $Z = (R_u; 0 \le u \le 1)$.

The problem of embedding Brownian bridge b^0 into Brownian motion is treated in a subsequent work of Pitman and Tang [73]. Since the proof relies heavily on Palm theory of stationary random measures, we prefer not to include it in the current work.

Theorem 3 ([73]) *There exists a random time* $T \ge 0$ *such that* $(B_{T+u} - B_T; 0 \le u \le 1)$ *has the same distribution as* $(b_u^0; 0 \le u \le 1)$.

In Question 1, we seek to embed a particular continuous-time process Z of unit length into a Brownian path. The distribution of X resides in the infinite-dimensional space $\mathcal{C}_0[0, 1]$ of continuous paths $(w(t); 0 \le t \le 1)$ starting from $w(0) = 0$. So a closely related problem is whether a given subset of $\mathcal{C}_0[0, 1]$ is hit by the path-valued moving-window process $X_t := (B_{t+u} - B_t; 0 \le u \le 1)$ indexed by $t \ge 0$. We formulate this problem as follows.

Question 2 Given a Borel measurable subset $S \subset \mathcal{C}_0[0, 1]$, can we find a random time T such that $X_T := (B_{T+u} - B_T; 0 \le u \le 1) \in S$ with probability one?

Question 2 involves scanning for patterns in a continuous-time process. By the general theory of stochastic processes, assuming that the underlying Brownian motion B is defined on a complete probability space, $\{\exists T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S\}$ is measurable. See e.g. Dellacherie [20, T32, Chap. I], Meyer and Dellacherie [21, Sect. 44, Chap. III], and Bass [2, 3]. Assume that

$$\mathbb{P}(\exists T \geq 0 \text{ such that } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) > 0.$$

Then there exists some fixed $M > 0$ and $p > 0$ such that

$$\mathbb{P}(\exists T : 0 \leq T \leq M \text{ and } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) = p > 0.$$

We start the process afresh at $M + 1$, and then also

$$\mathbb{P}(\exists T : M + 1 \leq T \leq 2M + 1 \text{ and } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S) = p > 0.$$

By repeating the above procedure, we obtain a sequence of i.i.d. Bernoulli(p) random variables. Therefore, the probability that a given measurable set $S \subset \mathcal{C}_0[0, 1]$ is hit by the path-valued process generated by Brownian motion is either 0 or 1:

$$\mathbb{P}[\exists T \geq 0 \text{ such that } (B_{T+u} - B_T; 0 \leq u \leq 1) \in S] = 0 \text{ or } 1. \tag{1}$$

Using various stochastic analysis tools, we are able to show that

Theorem 4 (Impossibility of Embedding of Excursion, Reflected Bridge, Vervaat Transform of Brownian Motion, First Passage Bridge and Vervaat Bridge Paths) *For each of the following five sets of paths S, almost surely, there is no random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$:*

1. *the set of excursion paths, which first return to* 0 *at time* 1*,*

$$S = \mathcal{E} := \{w \in \mathcal{C}_0[0, 1]; w(t) > w(1) = 0 \textit{ for } 0 < t < 1\};$$

2. *the set of reflected bridge paths,*

$$S = \mathcal{RBR}^0 := \{w \in \mathcal{C}_0[0, 1]; w(t) \geq w(1) = 0 \textit{ for } 0 \leq t \leq 1\};$$

3. *the set of paths of Vervaat transform of Brownian motion with a floating negative endpoint,*

$$S = \mathcal{VB}^- := \{w \in \mathcal{C}_0[0, 1]; w(t) > w(1) \textit{ for } 0 \leq t < 1 \textit{ and } \inf\{t > 0; w(t) < 0\} > 0\};$$

4. *the set of first passage bridge paths at fixed level* $\lambda < 0$*,*

$$S = \mathcal{FP}^\lambda := \{w \in \mathcal{C}_0[0, 1]; w(t) > w(1) = \lambda \textit{ for } 0 \leq t < 1\};$$

5. *the set of Vervaat bridge paths ending at fixed level* $\lambda < 0$,

$$S = \mathcal{VB}^{\lambda} := \{w \in \mathcal{FP}^{\lambda}; \inf\{t > 0; w(t) < 0\} > 0\} = \{w \in \mathcal{VB}^{-}; w(1) = \lambda\}.$$

Observe that for each $\lambda < 0$, $\mathcal{VB}^{\lambda}$ is a subset of $\mathcal{VB}^{-}$ and $\mathcal{FP}^{\lambda}$. Then Theorem 4(5) follows immediately from Theorem 4(3) or (4). As we will see in Sect. 3.1, Theorem 4(5) is also reminiscent of Theorem 4(1) in the proof.

It is obvious that for the following two sets of paths S, almost surely, there is a random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ almost surely:

- the set of positive paths,

$$S = \mathcal{M} := \{w \in \mathcal{C}_0[0,1]; w(t) > 0 \text{ for } 0 < t \leq 1\};$$

- the set of bridge paths, which ends at $\lambda \in \mathbb{R}$,

$$S = \mathcal{BR}^{\lambda} := \{w \in \mathcal{C}_0[0,1]; w(1) = \lambda\}.$$

The case of positive paths is easily treated by excursion theory, as discussed in Sect. 3.5. Bridge paths are obtained by simply taking $T := \inf\{t > 0; B_{t+1} = B_t + \lambda\}$, see Pitman and Tang [73] for related discussion. In both cases, $T+1$ is a stopping time relative to the Brownian filtration. For a general measurable $S \subset \mathcal{C}_0[0,1]$, it is easily shown that if there is a random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1) \in S$ almost surely, then for each $\epsilon > 0$ this can be achieved by a random time T such that $T + 1 + \epsilon$ is a stopping time relative to the Brownian filtration.

In the current work, we restrict ourselves to continuous paths in linear Brownian motion. However, the problem is also worth considering in the multi-dimensional case, as discussed briefly in Sect. 4.

At first glance, neither Question 1 nor Question 2 seems to be tractable. To gain some intuition, we start by studying the analogous problem in the random walk setting. We deal with simple symmetric random walks $SW(n)$ of length n with increments ± 1 starting at 0. A typical question is how long it would take, in a random walk, to observe a pattern in a collection of patterns of length n satisfying some common properties. More precisely,

Question 3 Given for each $n \in \mathbb{N}$ a collection $\mathcal{A}^n$ of patterns of length $L(\mathcal{A}^n)$, what is the asymptotics of the expected waiting time $\mathbb{E}T(\mathcal{A}^n)$ until some element of $\mathcal{A}^n$ is observed in a random walk?

We are not aware of any previous study on pattern problems in which some natural definition of the collection of patterns is made for each $n \in \mathbb{N}$. Nevertheless, this question fits into the general theory of runs and patterns in a sequence of discrete trials. This theory dates back to work in 1940s by Wald and Wolfowitz [87] and Mood [66]. Since then, the subject has become important in various areas of science, including industrial engineering, biology, economics and statistics. In the 1960s, Feller [29] treated the problem probabilistically by identifying the occurrence of a single pattern as a renewal event. By the generating function method, the law of the occurrence times of a single pattern is entirely characterized. More advanced study, of the occurrence of patterns in a collection, developed in 1980s by two different methods. Guibas and Odlyzko [37], and Breen et al. [12] followed the steps of Feller [29] by studying the generating functions in pattern-overlapping regimes. An alternative approach was adopted by Li [55], and Gerber and Li [34] using martingale arguments. We also refer readers to the book of Fu and Lou [32] for the Markov chain embedding approach for multi-state trials.

Techniques from the theory of patterns in i.i.d. sequences provide general strategies to Question 3. Here we focus on some special cases where the asymptotics of the expected waiting time is computable. As we will see later, these asymptotics help us predict the existence or non-existence of some particular paths in Brownian motion. The following result answers Question 3 in some particular cases.

Theorem 5 *Let $T(\mathcal{A}^n)$ be the waiting time until some pattern in $\mathcal{A}^n$ appears in the simple random walk. Then*

1. *for the set of discrete positive excursions of length $2n$, whose first return to 0 occurs at time $2n$,*

$$\mathcal{E}^{2n} := \{w \in SW(2n); w(i) > 0 \text{ for } 1 \leq i \leq 2n-1 \text{ and } w(2n) = 0\},$$

we have

$$\mathbb{E}T(\mathcal{E}^{2n}) \sim 4\sqrt{\pi} n^{\frac{3}{2}}; \tag{2}$$

2. *for the set of positive walks of length $2n+1$,*

$$\mathcal{M}^{2n+1} := \{w \in SW(2n+1); w(i) > 0 \text{ for } 1 \leq i \leq 2n+1\},$$

we have

$$\mathbb{E}T(\mathcal{M}^{2n+1}) \sim 4n; \tag{3}$$

3. *for the set of discrete bridges of length n, which end at λ_n for some $\lambda \in \mathbb{R}$, where $\lambda_n := [\lambda\sqrt{n}]$ if $[\lambda\sqrt{n}]$ and n have the same parity, and $\lambda_n := [\lambda\sqrt{n}] + 1$ otherwise,*

$$\mathcal{BR}^{\lambda,n} := \{w \in SW(n); w(n) = \lambda_n\},$$

we have

$$c^{\lambda}_{\mathcal{BR}} n \leq \mathbb{E}T(\mathcal{BR}^{\lambda,n}) \leq C^{\lambda}_{\mathcal{BR}} n \quad \text{for some } c^{\lambda}_{\mathcal{BR}} \text{ and } C^{\lambda}_{\mathcal{BR}} > 0; \tag{4}$$

4. *for the set of negative first passage walks of length n, ending at* λ_n *with* $\lambda < 0$,

$$\mathcal{FP}^{\lambda,n} := \{w \in SW(n); w(i) > w(n) = \lambda_n \text{ for } 0 \le i \le n-1\},$$

we have

$$\sqrt{\frac{\pi}{2\lambda^2}} \exp\left(\frac{\lambda^2}{2}\right) n \le \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \le \sqrt{\frac{4}{\lambda}} \exp\left(\frac{\lambda^2}{2}\right) n^{\frac{5}{4}}. \tag{5}$$

Now we explain how the asymptotics in Theorem 5 suggest answers to Question 1 and Question 2 in some cases. Formula (2) tells that it would take on average $n^{\frac{3}{2}} \gg n$ steps to observe an excursion in a simple random walk. In view of the usual scaling of random walks to converge to Brownian motion, the time scale appears to be too large. Thus we should not expect to find the excursion paths $\mathcal{E}$ in Brownian motion. However, in (3) and (4), the typical waiting time to observe a positive walk or a bridge has the same order n involved in the time scaling for convergence in distribution to Brownian motion. So we can anticipate to observe the positive paths $\mathcal{M}$ and the bridge paths $\mathcal{BR}^{\lambda}$ in Brownian motion. Finally in (5), there is an exponent gap in evaluating the expected waiting time for first passage walks ending at $\lambda_n \sim [\lambda\sqrt{n}]$ with $\lambda < 0$. In this case, we do not know whether it would take asymptotically n steps or much longer to first observe such patterns. This prevents us from predicting the existence of the first passage bridge paths $\mathcal{FP}^{\lambda}$ in Brownian motion.

The scaling arguments used in the last paragraph are quite intuitive but not rigorous since we are not aware of any theory which would justify the existence or non-existence of continuous paths in Brownian motion by taking limits from the discrete setting.

Organization of the Paper The rest of the paper is organized as follows.

- Section 2 treats the asymptotic behavior of the expected waiting time for discrete patterns. There Theorem 5 is proved.
- Section 3 is devoted to the analysis of continuous paths/processes in Brownian motion. Proofs of Theorems 2 and 4 are provided.
- Section 4 discusses the potential theory of continuous paths in Brownian motion.

A selection of open problems is presented in Sects. 2.5 and 4.

2 Expected Waiting Time for Discrete Patterns

Consider the expected waiting time for some collection of patterns

$$\mathcal{A}^n \in \{\mathcal{E}^{2n}, \mathcal{M}^{2n+1}, \mathcal{BR}^{\lambda,n}, \mathcal{FP}^{\lambda,n}\}$$

as defined in the introduction, except that we now encode a simple walk with m steps by its sequence of increments, with each increment a ± 1. We call such an increment sequence a *pattern* of length m. For each of these collections $\mathcal{A}^n$, all patterns in the collection have a common length, say $L(\mathcal{A}^n)$. We are interested in the asymptotic behavior of $\mathbb{E}T(\mathcal{A}^n)$ as $L(\mathcal{A}^n) \to \infty$.

We start by recalling the general strategy to compute the expected waiting time for discrete patterns in a simple random walk. From now on, let $\mathcal{A}^n := \{A_1^n, \cdots, A_{\#\mathcal{A}^n}^n\}$, where A_i^n is a sequence of signs ± 1 for $1 \le i \le \#\mathcal{A}^n$. That is,

$$A_i^n := A_{i1}^n \cdots A_{iL(\mathcal{A}^n)}^n, \quad \text{where } A_{ik}^n = \pm 1 \text{ for } 1 \le k \le L(\mathcal{A}^n).$$

Let $T(A_i^n)$ be the waiting time until the end of the first occurrence of A_i^n, and let $T(\mathcal{A}^n)$ be the waiting time until the first of the patterns in $\mathcal{A}^n$ is observed. So $T(\mathcal{A}^n)$ is the minimum of the $T(A_i^n)$ over $1 \le i \le \#\mathcal{A}^n$.

Define the matching matrix $M(\mathcal{A}^n)$, which accounts for the overlapping phenomenon among patterns within the collection $\mathcal{A}^n$. The coefficients are given by

$$M(\mathcal{A}^n)_{ij} := \sum_{l=0}^{L(\mathcal{A}^n)-1} \frac{\epsilon_l(A_i^n, A_j^n)}{2^l} \quad \text{for } 1 \le i,j \le \#\mathcal{A}^n, \tag{6}$$

where $\epsilon_l(A_i^n, A_j^n)$ is defined for $A_i^n = A_{i1}^n \cdots A_{iL(\mathcal{A}^n)}^n$ and $A_j^n = A_{j1}^n \cdots A_{jL(\mathcal{A}^n)}^n$ as

$$\epsilon_l(A_i^n, A_j^n) := \begin{cases} 1 & \text{if } A_{i1}^n = A_{j1+l}^n, \cdots, A_{iL(\mathcal{A}^n)-l}^n = A_{jL(\mathcal{A}^n)}^n \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

for $0 \le l \le L(\mathcal{A}^n) - 1$. Note that in general for $i \neq j$, $M(\mathcal{A}^n)_{ij} \neq M(\mathcal{A}^n)_{ji}$ and hence the matching matrix $M(\mathcal{A}^n)$ is not necessarily symmetric. The following result, which can be read from Breen et al. [12] is the main tool to study the expected waiting time for the collection of patterns.

Theorem 6 ([12])

1. *The matching matrix $M(\mathcal{A}^n)$ is invertible and the expected waiting times for patterns in $\mathcal{A}^n := \{A_1^n, \cdots, A_{\#\mathcal{A}^n}^n\}$ are given by*

$$\left(\frac{1}{\mathbb{E}T(A_1^n)}, \cdots, \frac{1}{\mathbb{E}T(A_{\#\mathcal{A}^n}^n)} \right)^T = \frac{1}{2^n} M(\mathcal{A}^n)^{-1} (1, \cdots, 1)^T; \tag{8}$$

2. *The expected waiting time till one of the patterns in $\mathcal{A}^n$ is observed is given by*

$$\frac{1}{\mathbb{E}T(\mathcal{A}^n)} = \sum_{l=1}^{\#\mathcal{A}^n} \frac{1}{\mathbb{E}T(A_l^n)} = \frac{1}{2^n} (1, \cdots, 1) M(\mathcal{A}^n)^{-1} (1, \cdots, 1)^T. \tag{9}$$

In Sect. 2.1, we apply the previous theorem to obtain the expected waiting time for discrete excursions $\mathcal{E}^{2n}$, i.e. (1) of Theorem 5. The same problem for positive walks $\mathcal{M}^{2n+1}$, bridge paths $\mathcal{BR}^{0,2n}$ and first passage walks $\mathcal{FP}^{\lambda,n}$ through $\lambda_n \sim \lambda\sqrt{n}$, i.e. (2)–(4) of Theorem 5, is studied in Sects. 2.2–2.4. Finally, we discuss the problem of the exponent gap for some discrete patterns in Sect. 2.5.

2.1 Expected Waiting Time for Discrete Excursions

For $n \in \mathbb{N}$, the number of discrete excursions of length $2n$ is equal to the $n-1$th Catalan number, see e.g. Stanley [82, Exercise 6.19 (i)]. That is,

$$\#\mathcal{E}^{2n} = \frac{1}{n}\binom{2n-2}{n-1} \sim \frac{1}{4\sqrt{\pi}} 2^{2n} n^{-\frac{3}{2}}. \tag{10}$$

Note that discrete excursions never overlap since the starting point and the endpoint are the only two minima. We have then $\epsilon(E_i^n, E_j^n) = \delta_{ij}$ for $1 \leq i, j \leq \#\mathcal{E}^{2n}$ by (7). Thus, the matching matrix defined as in (6) for discrete excursions $\mathcal{E}^{2n}$ has the simple form

$$M(\mathcal{E}^{2n}) = I_{\#\mathcal{E}^{2n}} \quad (\#\mathcal{E}^{2n} \times \#\mathcal{E}^{2n} \text{ identity matrix}).$$

According to Theorem 6,

$$\forall 1 \leq i \leq \#\mathcal{E}^{2n},\ \mathbb{E}T(E_i^n) = 2^{2n} \quad \text{and} \quad \mathbb{E}T(\mathcal{E}^n) = \frac{2^{2n}}{\#\mathcal{E}^{2n}} \sim 4\sqrt{\pi}n^{\frac{3}{2}}, \tag{11}$$

where $\#\mathcal{E}^{2n}$ is given as in (10). This is (2). □

2.2 Expected Waiting Time for Positive Walks

Let $n \in \mathbb{N}$. It is well-known that the number of non-negative walks of length $2n+1$ is $\binom{2n}{n}$, see e.g. Larbarbe and Marckert [52] and Leeuwen [84] for modern proofs. Thus the number of positive walks of length $2n+1$ is given by

$$\#\mathcal{M}^{2n+1} = \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-\frac{1}{2}}. \tag{12}$$

Note that a positive walk of length $2n+1$ is uniquely determined by

- its first $2n$ steps, which is a positive walk of length $2n$;
- its last step, which can be either $+1$ or -1.

As a consequence,

$$\#\mathcal{M}^{2n} = \frac{1}{2}\#\mathcal{M}^{2n+1} \sim \frac{1}{\sqrt{\pi}}2^{2n-1}n^{-\frac{1}{2}}. \tag{13}$$

Now consider the matching matrix $M(\mathcal{M}^{2n+1})$ defined as in (6) for positive walks $\mathcal{M}^{2n+1}$. $M(\mathcal{M}^{2n+1})$ is no longer diagonal since there are overlaps among positive walks. The following lemma presents the particular structure of this matrix.

Lemma 1 *$M(\mathcal{M}^{2n+1})$ is a multiple of some right stochastic matrix (whose row sums are equal to* 1*). The multiplicity is*

$$1 + \sum_{l=1}^{2n} \frac{k(\mathcal{M}^l)}{2^l} \sim \frac{2}{\sqrt{\pi}}\sqrt{n}. \tag{14}$$

Proof Let $1 \le i \le \#\mathcal{M}^{2n+1}$ and consider the sum of the ith row

$$\begin{aligned}\sum_{j=1}^{\#\mathcal{M}^{2n+1}} M(\mathcal{M}^{2n+1})_{ij} &:= \sum_{j=1}^{\#\mathcal{M}^{2n+1}} \sum_{l=0}^{2n} \frac{\epsilon_l(M_i^{2n+1}, M_j^{2n+1})}{2^l} \\ &= \sum_{l=0}^{2n} \frac{1}{2^l} \sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}),\end{aligned} \tag{15}$$

where for $0 \le l \le 2n$ and $M_i^{2n+1}, M_j^{2n+1} \in \mathcal{M}^{2n+1}$, $\epsilon_l(M_i^{2n+1}, M_j^{2n+1})$ is defined as in (7). Note that $\epsilon_0(M_i^{2n+1}, M_j^{2n+1}) = 1$ if and only if $i = j$. Thus,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_0(M_i^{2n+1}, M_j^{2n+1}) = 1. \tag{16}$$

In addition, for $1 \le l \le 2n$,

$$\begin{aligned}&\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}) \\ &\quad = \#\{M_j^{2n+1} \in \mathcal{M}^{2n+1}; M_{i1}^{2n+1} = M_{j1+l}^{2n+1}, \cdots, M_{i2n+1-l}^{2n+1} = M_{j2n+1}^{2n+1}\}.\end{aligned}$$

Note that given $M_{i1}^{2n+1} = M_{j1+l}^{2n+1}, \cdots, M_{i2n+1-l}^{2n+1} = M_{j2n+1}^{2n+1}$, which implies that $M_{j1+l}^{2n+1} \cdots M_{j2n+1}^{2n+1}$ is a positive walk of length $2n - l + 1$, we have

$$M_j^{2n+1} \in \mathcal{M}^{2n+1} \Longleftrightarrow M_{j1}^{2n+1} \cdots M_{jl}^{2n+1} \text{ is a positive walk of length } l.$$

Therefore, for $1 \leq l \leq 2n$,

$$\sum_{j=1}^{\#\mathcal{M}^{2n+1}} \epsilon_l(M_i^{2n+1}, M_j^{2n+1}) = k(\mathcal{M}^l). \tag{17}$$

In view of (15), (16) and (17), we obtain for all $1 \leq i \leq \#\mathcal{M}^{2n+1}$, the sum of ith row of $M(\mathcal{M}^{2n+1})$ is given by (14). Furthermore, by (12) and (13), we know that $k(\mathcal{M}^l) \sim \frac{1}{\sqrt{2\pi}} 2^l l^{-\frac{1}{2}}$ as $l \to \infty$, which yields the asymptotics $\frac{2}{\sqrt{\pi}}\sqrt{n}$. □

Now by Theorem 6(1), $M(\mathcal{M}^{2n+1})$ is invertible and the inverse $M(\mathcal{M}^{2n+1})^{-1}$ is as well the multiple of some right stochastic matrix. The multiplicity is

$$\left(1 + \sum_{l=1}^{n-1} \frac{k(\mathcal{M}^l)}{2^l}\right)^{-1} \sim \frac{\sqrt{\pi}}{2\sqrt{n}}.$$

Then using (9), we obtain

$$\mathbb{E}T(\mathcal{M}^{2n+1}) = \frac{2^{2n+1}}{\left(1 + \sum_{l=1}^{n-1} \frac{k(\mathcal{M}^l)}{2^l}\right)^{-1} \#\mathcal{M}^{2n+1}} \sim 4n. \tag{18}$$

This is (3). □

2.3 Expected Waiting Time for Bridge Paths

In this part, we deal with the expected waiting time for the set of discrete bridges. In order to simplify the notations, we focus on the set of bridges of length $2n$ which end at $\lambda = 0$, that is $\mathcal{BR}^{0,2n}$. Note that the result in the general case for $\mathcal{BR}^{\lambda,n}$, where $\lambda \in \mathbb{R}$, can be derived in a similar way. Using Theorem 6, we prove a weaker version of (4): there exist $\tilde{c}^0_{\mathcal{BR}}$ and $C^0_{\mathcal{BR}} > 0$ such that

$$\tilde{c}^0_{\mathcal{BR}} n^{\frac{1}{2}} \leq \mathbb{E}T(\mathcal{BR}^{0,n}) \leq C^0_{\mathcal{BR}} n. \tag{19}$$

Compared to (4), there is an exponent gap in (19) and the lower bound is not optimal. Nevertheless, the lower bound of (4) follows a soft argument by scaling limit, Proposition 1. We defer the discussion to Sect. 2.5. It is standard that the number of discrete bridges of length $2n$ is

$$\#\mathcal{BR}^{0,2n} = \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-\frac{1}{2}}. \tag{20}$$

Denote $\mathcal{BR}^{0,2n} := \{BR_1^{2n}, \cdots, BR^{2n}_{\#\mathcal{BR}^{0,2n}}\}$ and $M(\mathcal{BR}^{0,2n})$ the matching matrix of $\mathcal{BR}^{0,2n}$. We first establish the LHS estimate of (19). According to (8), we have

$$(1, \cdots, 1)M(\mathcal{BR}^{0,2n})\left(\frac{1}{\mathbb{E}T(BR_1^{2n})}, \cdots, \frac{1}{\mathbb{E}T(BR^{2n}_{\#\mathcal{BR}^{0,2n}})}\right)^T = \frac{\#\mathcal{BR}^{0,2n}}{2^{2n}}. \tag{21}$$

Note that the matching matrix $\mathcal{M}(\mathcal{BR}^{0,2n})$ is non-negative with diagonal elements

$$M(\mathcal{BR}^{0,2n})_{ii} \geq \epsilon_0(BR_i^{2n}, BR_i^{2n}) = 1,$$

for $1 \leq i \leq \#\mathcal{BR}^{0,2n}$. As a direct consequence, the column sums of $M(\mathcal{BR}^{0,2n})$ is larger or equal to 1. Then by (9) and (21),

$$\mathbb{E}T(\mathcal{BR}^{0,2n}) \geq \frac{2^{2n}}{\#\mathcal{BR}^{0,2n}} \sim \sqrt{\pi n},$$

where $\#\mathcal{BR}^{0,2n}$ is defined as in (20). Take then $\tilde{c}^0_{\mathcal{BR}} = \sqrt{\pi}$.

Now we establish the RHS estimate of (19). In view of (21), it suffices to work out an upper bound for the column sums of $M(\mathcal{BR}^{0,2n})$. Similarly as in (15), for $1 \leq j \leq \#\mathcal{BR}^{0,2n}$,

$$\sum_{i=1}^{\#\mathcal{BR}^{0,2n}} M(\mathcal{BR}^{0,2n})_{ij} = 1 + \sum_{l=1}^{2n-1} \frac{1}{2^l} \sum_{i=1}^{\#\mathcal{BR}^{0,2n}} \epsilon_l(BR_i^{2n}, BR_j^{2n}), \tag{22}$$

and

$$\begin{aligned}
&\sum_{i=1}^{\#\mathcal{BR}^{0,2n}} \epsilon_l(BR_i^{2n}, BR_j^{2n}) \\
&\quad = \#\{BR_i^{2n} \in \mathcal{BR}^{0,2n}; BR_{i1}^{2n} = BR_{j1+l}^{2n}, \cdots, BR_{in-l}^{2n} = BR_{jn}^{2n}\}, \\
&\quad = \#\{\text{discrete bridges of length } l \text{ which end at } \sum_{k=1}^{n-l} BR_{jk}^{2n}\} \\
&\quad = \binom{l}{\frac{l+\sum_{k=1}^{n-l} BR_{jk}^{2n}}{2}} \leq \binom{l}{[\frac{l}{2}]},
\end{aligned} \tag{23}$$

where the last inequality is due to the fact that $\binom{l}{k} \leq \binom{l}{[l]/2}$ for $0 \leq k \leq l$. By (22) and (23), the column sums of $M(\mathcal{BR}^{0,2n})$ are bounded from above by

$$1 + \sum_{l=0}^{2n-1} \frac{1}{2^l} \binom{l}{[\frac{l}{2}]} \sim \frac{4}{\sqrt{\pi}} n^{\frac{1}{2}}.$$

Again by (9) and (21),

$$\mathbb{E}T(\mathcal{BR}^{0,2n}) \leq 2^{2n}\frac{4n^{\frac{1}{2}}/\sqrt{\pi}}{\#\mathcal{BR}^{0,2n}} \sim 4n.$$

Hence we take $C^0_{\mathcal{BR}} = 4$. □

2.4 Expected Waiting Time for First Passage Walks

We consider the expected waiting time for first passage walks through $\lambda_n \sim \lambda\sqrt{n}$ for $\lambda < 0$. Following Feller [29, Theorem 2, Chap. III.7], the number of patterns in $\mathcal{FP}^{\lambda,n}$ is

$$\#\mathcal{FP}^{\lambda,n} = \frac{\lambda_n}{n}\binom{n}{\frac{n+\lambda_n}{2}} \sim \lambda \exp\left(-\frac{\lambda^2}{2}\right)\sqrt{\frac{2}{\pi}}2^n n^{-1}. \tag{24}$$

For $\mathcal{FP}^{\lambda,n} := \{FP^n_1, \cdots, FP^n_{\#\mathcal{FP}^{\lambda,n}}\}$ and $M(\mathcal{FP}^{\lambda,n})$ the matching matrix for $\mathcal{FP}^{\lambda,n}$, we have, by (8), that

$$(1,\cdots,1)M(\mathcal{FP}^{\lambda,n})\left(\frac{1}{\mathbb{E}T(FP^n_1)},\cdots,\frac{1}{\mathbb{E}T(FP^n_{\#\mathcal{FP}^{\lambda,n}})}\right)^T = \frac{\#\mathcal{FP}^{\lambda,n}}{2^n}. \tag{25}$$

The LHS bound of (5) can be derived in a similar way as in Sect. 2.3.

$$\mathbb{E}T(\mathcal{FP}^{\lambda,n}) \geq \frac{2^n}{\#\mathcal{FP}^{\lambda,n}} \sim \sqrt{\frac{\pi}{2\lambda^2}}\exp\left(\frac{\lambda^2}{2}\right)n,$$

where $\#\mathcal{FP}^{\lambda,n}$ is defined as in (24). We get the lower bound of (5).

For the upper bound of (5), we aim to obtain an upper bound for the column sums of $M(\mathcal{FP}^{\lambda,n})$. Note that for $1 \leq j \leq k^\lambda_{\mathcal{FP}^n}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij} = 1 + \sum_{l=1}^{n-1}\frac{1}{2^l}\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP_i, FP_j) \tag{26}$$

and

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP^n_i, FP^n_j)$$
$$= \#\{FP^n_i \in \mathcal{FP}^{\lambda,n}; FP^n_{i1} = FP^n_{j1+l}, \cdots, FP^n_{in-l} = FP^n_{jn}\}.$$

Observe that $\{FP_i^n \in \mathcal{FP}^{\lambda,n}; FP_{i1}^n = FP_{j1+l}^n, \cdots, FP_{in-l}^n = FP_{jn}^n\} \neq \emptyset$ if and only if $\sum_{k=1}^{l} FP_{jk}^n < 0$ (otherwise $\sum_{k=1}^{n-l} FP_{ik}^n = \sum_{k=1+l}^{n} FP_{jk}^n = \lambda_n - \sum_{k=1}^{l} FP_{jk}^n < \lambda_n$, which implies $FP_i^n \notin \mathcal{FP}^{\lambda,n}$). Then given $FP_{i1}^n = FP_{j1+l}^n, \cdots, FP_{in-l}^n = FP_{jn}^n$ and $\sum_{k=1}^{l} FP_{jk}^n < 0$,

$$FP_i^n \in \mathcal{FP}^{\lambda,n}$$

$$\Longleftrightarrow FP_{in-l+1}^n \cdots FP_{in}^n \text{ is a first passage walk of length } l \text{ through } \sum_{k=1}^{l} FP_{jk}^n < 0.$$

Therefore, for $1 \leq l \leq n-1$ and $1 \leq j \leq k_{\mathcal{FP}^n}^{\lambda}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} \epsilon_l(FP_i^n, FP_j^n) = 1_{\sum_{k=1}^{l} FP_{jk}^n < 0} \frac{|\sum_{k=1}^{l} FP_{jk}^n|}{l} \binom{l}{\frac{l+\sum_{k=1}^{l} FP_{jk}^n}{2}}. \tag{27}$$

From the above discussion, it is easy to see for $1 \leq j \leq \#\mathcal{FP}^{\lambda,n}$,

$$\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij} \leq \sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij^*},$$

where $FP_{j^*}^n$ is defined as follows: $FP_{j^*k}^n = -1$ if $1 \leq k \leq \lambda_n - 1$; $\lambda_n - 1 < k \leq n-1$ and $k - \lambda_n$ is odd; $k = n$. Otherwise $FP_{j^*k}^n = 1$.

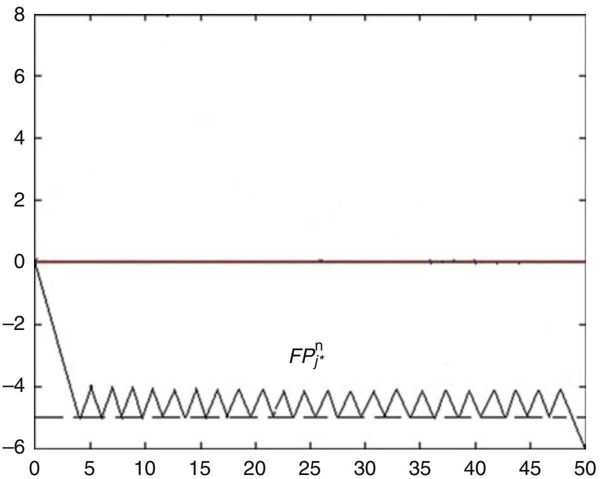

Fig. 1 Extreme patterns $FP_{j^*}^n$

The rest of this part is devoted to estimating $\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij*}$. By (26) and (27),

$$\begin{aligned}
&\sum_{i=1}^{\#\mathcal{FP}^{\lambda,n}} M(\mathcal{FP}^{\lambda,n})_{ij*} \\
&= \sum_{l=0}^{|\lambda_n|-1} \frac{1}{2^l} + \sum_{\substack{l=\lambda_n \\ l-|\lambda_n| \text{ odd}}}^{n-1} \frac{|\lambda_n|-1}{l \cdot 2^l} \binom{l}{\frac{l-|\lambda_n|+1}{2}} + \sum_{\substack{l=\lambda_n \\ l-|\lambda_n| \text{ even}}}^{n-1} \frac{|\lambda_n|-2}{l \cdot 2^l} \binom{l}{\frac{l-|\lambda_n|+2}{2}} \\
&\leq 2 + |\lambda_n| \sum_{l=|\lambda_n|}^{n-1} \frac{1}{2^l l} \binom{l}{[\frac{l}{2}]} \sim \sqrt{\frac{8\lambda}{\pi}} n^{\frac{1}{4}}.
\end{aligned}$$

Thus, the column sums of $M(\mathcal{FP}^{\lambda,n})$ are bounded from above by $\sqrt{\frac{8\lambda}{\pi}} n^{\frac{1}{4}}$. By (9) and (25),

$$\mathbb{E}T(\mathcal{FP}^{\lambda,n}) \leq \frac{2^n \sqrt{8\lambda/\pi}\, n^{\frac{1}{4}}}{\#\mathcal{FP}^{\lambda,n}} \sim \sqrt{\frac{4}{\lambda}} \exp\left(\frac{\lambda^2}{2}\right) n^{\frac{5}{4}}.$$

This is the upper bound of (5). □

2.5 Exponent Gaps for $\mathcal{BR}^{\lambda,n}$ and $\mathcal{FP}^{\lambda,n}$

It can be inferred from (19) (resp. (5)) that the expected waiting time for $\mathcal{BR}^{\lambda,n}$ where $\lambda \in \mathbb{R}$ (resp. $\mathcal{FP}^{\lambda,n}$ where $\lambda < 0$) is bounded from below by order $n^{\frac{1}{2}}$ (resp. n) and from above by order n (resp. $n^{\frac{5}{4}}$). The exponent gap in the estimates of first passage walks $\mathcal{FP}^{\lambda,n}$ is frustrating, since we do not know whether the waiting time is exactly of order n, or is of order $\gg n$. This prevents the prediction of the existence of first passage bridge patterns $\mathcal{FP}^{\lambda}$ in Brownian motion.

From (9), we see that the most precise way to compute $\mathbb{E}T(\mathcal{BR}^{\lambda,n})$ and $\mathbb{E}T(\mathcal{FP}^{\lambda,n})$ consists in evaluating the sum of all entries in the inverse matching matrices $M(\mathcal{BR}^{\lambda,n})^{-1}$ and $M(\mathcal{FP}^{\lambda,n})^{-1}$. But the task is difficult since the structures of $M(\mathcal{BR}^{\lambda,n})$ and $M(\mathcal{FP}^{\lambda,n})$ are more complex than the structures of $M(\mathcal{E}^{2n})$ and $M(\mathcal{M}^{2n+1})$. We do not understand well the exact form of the inverse matrices $M(\mathcal{BR}^{\lambda,n})^{-1}$ and $M(\mathcal{FP}^{\lambda,n})^{-1}$.

The technique used in Sects. 2.3 and 2.4 is to bound the column sums of the matching matrix $M(\mathcal{BR}^{\lambda,n})$ (resp. $M(\mathcal{FP}^{\lambda,n})$). More precisely, we have proved that

$$\mathcal{O}(1) \leq \text{ column sums of } M(\mathcal{BR}^{\lambda,n}) \leq \mathcal{O}(n^{\frac{1}{2}}) \quad \text{for each fixed } \lambda \in \mathbb{R}; \tag{28}$$

$$\mathcal{O}(1) \leq \text{ column sums of } M(\mathcal{FP}^{\lambda,n}) \leq \mathcal{O}(n^{\frac{1}{4}}) \quad \text{for each fixed } \lambda < 0. \tag{29}$$

For the bridge pattern $\mathcal{BR}^{0,2n}$, the LHS bound of (28) is obtained by any excursion path of length $2n$, while the RHS bound of (28) is achieved by the sawtooth path with consecutive ± 1 increments. In the first passage pattern $\mathcal{FP}^{\lambda,n}$ where $\lambda < 0$, the LHS bound of (29) is achieved by some excursion-like path, which starts with an excursion and goes linearly to $\lambda\sqrt{n} < 0$ at the end. The RHS bound of (29) is given by the extreme pattern defined in Sect. 2.4, see Fig. 1. However, the above estimations are not accurate, since there are only few columns in $\mathcal{BR}^{\lambda,n}$ which sum up either to $\mathcal{O}(1)$ or to $\mathcal{O}(n^{\frac{1}{2}})$, and few columns of $\mathcal{FP}^{\lambda,n}$ which sum up either to $\mathcal{O}(1)$ or to $\mathcal{O}(n^{\frac{1}{4}})$.

Open Problem 1

1. Determine the exact asymptotics for $\mathbb{E}T(\mathcal{BR}^{\lambda,n})$ where $\lambda \in \mathbb{R}$, as $n \to \infty$.
2. Determine the exact asymptotics for $\mathbb{E}T(\mathcal{FP}^{\lambda,n})$ where $\lambda < 0$, as $n \to \infty$.

As we prove below, for $\lambda \in \mathbb{R}$, $\mathbb{E}T(\mathcal{BR}^{\lambda,n}) \asymp n$ by a scaling limit argument. Nevertheless, to obtain this result only by discrete analysis would be of independent interest. Table 1 provides the simulations of the expected waiting time $\mathbb{E}T(\mathcal{FP}^{-1,n})$ for some large n.

Table 1 Estimation of ζ by $\log \frac{\mathbb{E}T(\mathcal{FP}^{n_2}_{-1})}{\mathbb{E}T(\mathcal{FP}^{n_1}_{-1})} / \log(\frac{n_2}{n_1})$, where n_2 is the next to n_1 in the table

n	100	200	500	1000	2000	5000	10,000
$\mathbb{E}T(\mathcal{FP}^{n}_{-1})$	179.805	358.249	893.041	1800.002	3682.022	8549.390	12231.412
Estimated ζ		0.9945	0.9968	1.0112	1.0375	1.0205	1.0335

The result suggests that $\mathbb{E}T(\mathcal{FP}^{-1,n})$ be linear, but possibly with some log-correction. Yuval Peres made the following conjecture:

Conjecture 1 (Peres (personal communications)) For $\lambda < 0$, there exist $c^{\lambda}_{\mathcal{FP}}$ and $C^{\lambda}_{\mathcal{FP}} > 0$ such that

$$c^{\lambda}_{\mathcal{FP}} n \ln n \le \mathbb{E}T(\mathcal{FP}^{\lambda,n}) \le C^{\lambda}_{\mathcal{FP}} n \ln n. \tag{30}$$

This is consistent with Theorem 4(4), that we cannot find a first passage bridge with fixed negative endpoint in Brownian motion.

Now let us focus on the lower bound (4) of expected waiting time for bridge pattern $\mathcal{BR}^0$. For $n \in 2\mathbb{N}$, we run a simple random walk $(RW_k)_{k\in\mathbb{N}}$ until the first level bridge of length n appears. That is, we consider

$$(RW_{F_n+k} - RW_{F_n})_{0\le k\le n}, \quad \text{where } F_n := \inf\{k \ge 0; RW_{k+n} = RW_k\}. \tag{31}$$

For simplicity, let RW_k for non-integer k be defined by the usual linear interpolation of a simple random walk. For background on the weak convergence in $\mathcal{C}[0, 1]$, we refer readers to Billingsley [11, Chap. 2].

Proposition 1 *The process*

$$\left(\frac{RW_{F_n+nu} - RW_{F_n}}{\sqrt{n}}; 0 \le u \le 1\right)$$

converges weakly in $\mathcal{C}[0, 1]$ *to the bridge-like process*

$$(B_{F+u} - B_F; 0 \le u \le 1), \quad \textit{where } F := \inf\{t > 0; B_{t+1} - B_t = 0\}. \tag{32}$$

The process $(S_t := B_{t+1} - B_t; t \ge 0)$ is a stationary Gaussian process, first studied by Slepian [81] and Shepp [79]. The following result, which can be found in Pitman and Tang [73, Lemma 2.3], is needed for the proof of Proposition 1.

Lemma 2 ([71, 78]) *For each fixed* $t \ge 0$*, the distribution of* $(S_u; t \le u \le t+1)$ *is mutually absolutely continuous with respect to the distribution of*

$$(\tilde{B}_u := \sqrt{2}(\xi + B_u); t \le u \le t+1), \tag{33}$$

where $\xi \sim \mathcal{N}(0, 1)$*. In particular, the distribution of the Slepian zero set restricted to* $[t, t+1]$*, i.e.* $\{u \in [t, t+1]; S_u = 0\}$ *is mutually absolutely continuous with respect to that of* $\{u \in [t, t+1]; \xi + B_u = 0\}$*, the zero set of Brownian motion starting at* $\xi \sim \mathcal{N}(0, 1)$.

Proof of Proposition 1 Let $\mathbb{P}^{\mathbf{W}}$ be Wiener measure on $\mathcal{C}[0, \infty)$. Let $\mathbb{P}^{\mathbf{S}}$ (resp. $\mathbb{P}^{\tilde{\mathbf{W}}}$) be the distribution of the Slepian process S (resp. the distribution of $\tilde{B}$ defined as in (33)). We claim that

$$F := \inf\{t \ge 0; w_{t+1} = w_t\},$$

is a functional of the coordinate process $w := \{w_t; t \ge 0\} \in \mathcal{C}[0, \infty)$ that is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. Note that the distribution of $(x_t := w_{t+1} - w_t; t \ge 0)$ under $\mathbb{P}^{\mathbf{W}}$ is the same as that of $(w_t; t \ge 0)$ under $\mathbb{P}^{\mathbf{S}}$. In addition, $x \in \mathcal{C}[0, \infty)$ is a functional of $w \in \mathcal{C}[0, \infty)$ that is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. By composition, it is equivalent to show that

$$F' := \inf\{t \ge 0; w_t = 0\},$$

is a functional of $w \in \mathcal{C}[0, \infty)$ that is continuous $\mathbb{P}^{\mathbf{S}}$ a.s. Consider the set

$$\mathcal{Z} := \{w \in \mathcal{C}[0, \infty); F' \text{ is not continuous at } w\} = \cup_{p \in \mathbb{Q}} \mathcal{Z}_p,$$

where $\mathcal{Z}_p := \{w \in \mathcal{C}[0,\infty); F' \in [p, p+1]$ and F' is not continuous at $w\}$. It is obvious that $\mathbb{P}^{\tilde{\mathbf{W}}}(\mathcal{Z}) = 0$ and thus $\mathbb{P}^{\tilde{\mathbf{W}}}(\mathcal{Z}_p) = 0$ for all $p \geq 0$. By Lemma 2, $\mathbb{P}^{\mathbf{S}}$ is locally absolutely continuous relative to $\mathbb{P}^{\tilde{\mathbf{W}}}$, which implies that $\mathbb{P}^{\mathbf{S}}(\mathcal{Z}_p) = 0$ for all $p \geq 0$. As a countable union of null events, $\mathbb{P}^{\mathbf{S}}(\mathcal{Z}) = 0$, and the claim is proved. Thus, the mapping

$$\Xi_F : \mathcal{C}[0,\infty) \ni (w_t; t \geq 0) \longrightarrow (w_{F+u} - w_F; 0 \leq u \leq 1) \in \mathcal{C}[0,1]$$

is continuous $\mathbb{P}^{\mathbf{W}}$ a.s. According to Donsker's theorem [23], see e.g. Billingsley [11, Sect. 10] or Kallenberg [40, Chap. 16], the linearly interpolated simple random walks

$$\left(\frac{RW_{[nt]}}{\sqrt{n}}; t \geq 0\right) \text{ converges weakly in } \mathcal{C}[0,1] \text{ to } (B_t; t \geq 0).$$

So by the continuous mapping theorem, see e.g. Billingsley [11, Theorem 5.1],

$$\Xi_F \circ \left(\frac{RW_{[nt]}}{\sqrt{n}}; t \geq 0\right) \text{ converges weakly to } \Xi_F \circ (B_t; t \geq 0). \quad \square$$

Note that $T(\mathcal{BR}^{0,n}) = F_n + n$. Following the above analysis, we know that $T(\mathcal{BR}^{0,n})/n$ converges weakly to $F + 1$, where $T(\mathcal{BR}^{0,n})$ is the waiting time until an element of $\mathcal{BR}^{0,n}$ occurs in a simple random walk and F is the random time defined as in (32). As a consequence,

$$\liminf_{n\to\infty} \mathbb{E}\frac{T(\mathcal{BR}^{0,n})}{n} \geq \mathbb{E}F + 1, \quad \text{since } \mathbb{E}F < \infty.$$

In particular, $\mathbb{E}F \leq C^0_{\mathcal{BR}} - 1 = 3$ as in Sect. 2.3. We refer readers to Pitman and Tang [73] for further discussion on first level bridges and the structure of the Slepian zero set.

3 Continuous Paths in Brownian Motion

This section is devoted to the proof of Theorems 2 and 4 regarding continuous paths and the distribution of continuous-time processes embedded in Brownian motion. In Sect. 3.1, we show that there is no normalized excursion in a Brownian path, i.e. Theorem 4(1). A slight modification of the proof allows us to exclude the existence of the Vervaat bridges with negative endpoint, i.e. Theorem 4(5). Furthermore, we prove in Sect. 3.2 that there is even no reflected bridge in Brownian motion, i.e. Theorem 4(2). In Sects. 3.3 and 3.4, we show that neither the Vervaat transform of Brownian motion nor first passage bridges with negative endpoint can be found in Brownian motion, i.e. Theorem 4(3), (4). We make use of the potential theory of *additive Lévy processes*, which is recalled in Sect. 3.3. Finally in Sect. 3.5, we provide a proof for the existence of Brownian meander, co-meander and three-dimensional Bessel process in Brownian motion, i.e. Theorem 2, using the filling scheme.

3.1 No Normalized Excursion in a Brownian Path

In this part, we provide two proofs for Theorem 4(1), though similar, from different viewpoints. The first proof is based on a fluctuation version of *Williams' path decomposition* of Brownian motion, originally due to Williams [88], and later extended in various ways by Millar [61, 62], and Greenwood and Pitman [36]. We also refer readers to Pitman and Winkel [72] for a combinatorial explanation and various applications.

Theorem 7 ([36, 88]) *Let $(B_t; t \geq 0)$ be standard Brownian motion and ξ be exponentially distributed with rate $\frac{1}{2}\vartheta^2$, independent of $(B_t; t \geq 0)$. Define $M := \operatorname{argmin}_{[0,\xi]} B_t$, $H := -B_M$ and $R := B_\xi + H$. Then H and R are independent exponential variables, each with the same rate ϑ. Furthermore, conditionally given H and R, the path $(B_t; 0 \leq t \leq \xi)$ is decomposed into two independent pieces:*

- *$(B_t; 0 \leq t \leq M)$ is Brownian motion with drift $-\vartheta < 0$ running until it first hits the level $-H < 0$;*
- *$(B_{\xi-t} - B_\xi; 0 \leq t \leq \xi - M)$ is Brownian motion with drift $-\vartheta < 0$ running until it first hits the level $-R < 0$.*

Now we introduce the notion of first passage process, which will be used in the proof of Theorem 4(1). Given a càdlàg process $(Z_t; t \geq 0)$ starting at 0, we define the first passage process $(\tau_{-x}; x \geq 0)$ associated to X to be the first time that the level $-x < 0$ is hit:

$$\tau_{-x} := \inf\{t \geq 0; Z_t < -x\} \quad \text{for } x > 0.$$

When Z is Brownian motion, the distribution of the first passage process is well-known:

Lemma 3

1. *Let $\mathbf{W}$ be Wiener measure on $\mathcal{C}[0, \infty)$. Then the first passage process $(\tau_{-x}; x \geq 0)$ under $\mathbf{W}$ is a stable$(\frac{1}{2})$ subordinator, with*

$$\mathbb{E}^{\mathbf{W}}[\exp(-\alpha\tau_{-x})] = \exp(-x\sqrt{2\alpha}) \quad \textit{for } \alpha > 0.$$

2. *For $\vartheta \in \mathbb{R}$, let $\mathbf{W}^\vartheta$ be the distribution on $\mathcal{C}[0, \infty)$ of Brownian motion with drift ϑ.*

Then for each fixed $L > 0$, on the event $\tau_{-L} < \infty$, the distribution of the first passage process $(\tau_{-x}; 0 \leq x \leq L)$ under $\mathbf{W}^{\vartheta}$ is absolutely continuous with respect to that under $\mathbf{W}$, with density $D_L^{\vartheta} := \exp(-\vartheta L - \frac{\vartheta^2}{2}\tau_{-L})$.

Proof The part (1) of the lemma is a well known result of Lévy, see e.g. Bertoin et al. [8, Lemma 4]. The part (2) is a direct consequence of Girsanov's theorem, see e.g. Revuz and Yor [74, Chap. VIII] for background. □

Proof of Theorem 4(1) Suppose by contradiction that $\mathbb{P}(T < \infty) > 0$, where T is a random time at which some excursion appears. Take ξ exponentially distributed with rate $\frac{1}{2}$, independent of $(B_t; t \geq 0)$. We have then

$$\mathbb{P}(T < \xi < T + 1) > 0. \tag{34}$$

Now $(T, T+1)$ is inside the excursion of Brownian motion above its past-minimum process, which straddles ξ. See Fig. 2. Define

- $(\tau_{-x}; x \geq 0)$ to be the first passage process of $(B_{\xi+t} - B_{\xi}; t \geq 0)$.

By the strong Markov property of Brownian motion, $(B_{\xi+t} - B_{\xi}; t \geq 0)$ is still Brownian motion. Thus, $(\tau_{-x}; x \geq 0)$ is a stable$(\frac{1}{2})$ subordinator by part (1) of Lemma 3. Also, define

- $(\sigma_{-x}; x \geq 0)$ to be the first passage process derived from the process $(B_{\xi-t} - B_{\xi}; 0 \leq t \leq \xi - M)$ followed by an independent Brownian motion with drift -1 running forever.

According to Theorem 7, $(B_{\xi-t} - B_{\xi}; 0 \leq t \leq \xi - M)$ is Brownian motion with drift -1 running until it first hits the level $-R < 0$. Then $(\sigma_{-x}; x \geq 0)$ is the first

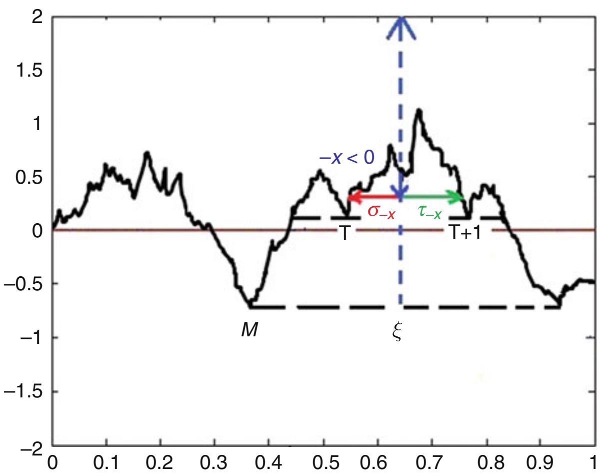

Fig. 2 No excursion of length 1 in a Brownian path

passage process of Brownian motion with drift -1, whose distribution is absolutely continuous on any compact interval $[0, L]$, with respect to that of $(\tau_{-x}; 0 \leq x \leq L)$ by part (2) of Lemma 3.

Thus, the distribution of $(\sigma_{-x} + \tau_{-x}; 0 \leq x \leq L)$ is absolutely continuous relative to that of $(\tau_{-2x}; 0 \leq x \leq L)$. It is well known that a real stable$(\frac{1}{2})$ process does not hit points, see e.g. Bertoin [5, Theorem 16, Chap. II.5]. As a consequence,

$$\mathbb{P}(\sigma_{-x} + \tau_{-x} = 1 \text{ for some } x \geq 0) = 0,$$

which contradicts (34). □

Proof of Theorem 4(5) Impossibility of embedding the Vervaat bridge paths $\mathcal{VB}^{\lambda}$ with endpoint $\lambda < 0$. We borrow the notations from the preceding proof. Observe that, for fixed $\lambda < 0$,

$$\mathbb{P}(\sigma_{-x} + \tau_{-x+\lambda} = 1 \text{ for some } x \geq 0) = 0.$$

The rest of the proof is just a duplication of the preceding one. □

We give yet another proof of Theorem 4(1), which relies on Itô's excursion theory, combined with Bertoin's self-similar fragmentation theory. For general background on fragmentation processes, we refer to the monograph of Bertoin [7]. The next result, regarding a normalized Brownian excursion, follows Bertoin [6, Corollary 2].

Theorem 8 ([6]) *Let $e := (e_u; 0 \leq u \leq 1)$ be normalized Brownian excursion and $F^e := (F^e_t; t \geq 0)$ be the associated interval fragmentation defined as $F^e_t := \{u \in (0,1); e_u > t\}$. Introduce*

- $\lambda := (\lambda_t; t \geq 0)$ *the length of the interval component of F^e that contains U, independent of the excursion and uniformly distributed;*
- $\xi := \{\xi_t; t \geq 0\}$ *a subordinator, the Laplace exponent of which is given by*

$$\Phi^{ex}(q) := q\sqrt{\frac{8}{\pi}} \int_0^1 t^{q-\frac{1}{2}}(1-t)^{-\frac{1}{2}} = q\sqrt{\frac{8}{\pi}} B(q+\frac{1}{2}, \frac{1}{2}); \tag{35}$$

Then $(\lambda_t; t \geq 0)$ has the same law as $(\exp(-\xi_{\rho_t}); t \geq 0)$, where

$$\rho_t := \inf\left\{u \geq 0; \int_0^u \exp\left(-\frac{1}{2}\xi_r\right) dr > t\right\}. \tag{36}$$

Alternative Proof of Theorem 4(1) Consider the reflected process $(B_t - \underline{B}_t; t \geq 0)$, where $\underline{B}_t := \inf_{0 \leq u \leq t} B_u$ is the past-minimum process of the Brownian motion. For $\mathbf{e}$ the first excursion of $B - \underline{B}$ that contains some excursion pattern $\mathcal{E}$ of length 1, let $\Lambda_{\mathbf{e}}$ be the length of such excursion, and $\mathbf{e}^*$ be the normalized Brownian excursion. Following Itô's excursion theory, see e.g. Revuz and Yor [74, Chap. XII], $\Lambda_{\mathbf{e}}$ is independent of the distribution of the normalized excursion $\mathbf{e}^*$.

As a consequence, the fragmentation associated to $\mathbf{e}^*$ produces an interval of length $\frac{1}{\Lambda_{\mathbf{e}}}$. Now choose U uniformly distributed on [0, 1] and independent of the Brownian motion. According to Theorem 8, there exists a subordinator ξ characterized as in (35) and a time-change ρ defined as in (36) such that $(\lambda_t; t \geq 0)$, the process of the length of the interval fragmentation which contains U, has the same distribution as $(\exp(-\xi_{\rho_t}); t \geq 0)$. Note that $(\lambda_t; t \geq 0)$ depends only on the normalized excursion $\mathbf{e}^*$ and U, so $(\lambda_t; t \geq 0)$ is independent of $\Lambda_{\mathbf{e}}$. It is a well known result of Kesten [41] that a subordinator without drift does not hit points. Therefore,

$$\mathbb{P}\left(\lambda_t = \frac{1}{\Lambda_{\mathbf{e}}} \text{ for some } t \geq 0\right) = 0,$$

which yields the desired result. □

3.2 No Reflected Bridge in a Brownian Path

This part is devoted to proving Theorem 4(2). The main difference between Theorem 4(1) and (2) is that the strict inequality $B_{T+u} > B_T$ for all $u \in (0, 1)$ is relaxed by the permission of equalities $B_{T+u} = B_T$ for some $u \in (0, 1)$. Thus, there are paths in $\mathcal{C}[0, 1]$ which may contain reflected bridge paths but not excursion paths. Nevertheless, such paths form a null set for Wiener measure. Below is a slightly stronger version of this result.

Lemma 4 *Almost surely, there are no random times $S < T$ such that $B_T = B_S$, $B_u \geq B_S$ for $u \in (S, T)$ and $B_v = B_w = B_S$ for some $S < v < w < T$.*

Proof Consider the following two sets

$$\mathcal{T} := \{\text{there exist } S \text{ and } T \text{ which satisfy the conditions in the lemma}\}$$

and

$$\mathcal{U} := \bigcup_{s,t \in \mathbb{Q}} \{B \text{ attains its minimum for more than once on } [s, t]\}.$$

It is straightforward that $\mathcal{T} \subset \mathcal{U}$. In addition, it is well-known that almost surely Brownian motion has a unique minimum on any fixed interval $[s, t]$ for all $s, t \in \mathbb{R}$. As a countable union of null events, $\mathbb{P}(\mathcal{U}) = 0$ and thus $\mathbb{P}(\mathcal{T}) = 0$. □

Remark 1 The previous lemma has an interesting geometric interpretation in terms of Brownian trees, see e.g. Pitman [71, Sect. 7.4] for background. Along the lines of the second proof of Theorem 4(1) in Sect. 3.1, we only need to show that the situation in Lemma 4 cannot happen in a Brownian excursion either of an independent and diffuse length or of normalized unit length. But this is just another way to state that Brownian trees have only binary branch points, which follows readily from Aldous' stick-breaking construction of the continuum random trees, see e.g. Aldous [1, Sect. 4.3] and Le Gall [53].

According to Theorem 4(1) and Lemma 4, we see that almost surely, there are neither excursion paths of length 1 nor reflected bridge paths of any length with at least two intermediate returns in Brownian motion. To prove the desired result, it suffices to exclude the possibility of reflected bridge paths with exactly one reflection. This is done by the following lemma.

Lemma 5 *Assume that* $0 \le S < T < U$ *are random times such that* $B_S = B_T = B_U$ *and* $B_u > B_S$ *for* $u \in (S,T) \cup (T,U)$. *Then the distribution of* $U - S$ *is absolutely continuous with respect to the Lebesgue measure.*

Proof Suppose by contradiction that the distribution of $U - S$ is not absolutely continuous with respect to the Lebesgue measure. Then there exists $p, q \in \mathbb{Q}$ such that $U - S$ fails to have a density on the event $\{S < p < T < q < U\}$. In fact, if $U - S$ has a density on $\{S < p < T < q < U\}$ for all $p, q \in \mathbb{Q}$, Radon-Nikodym theorem guarantees that $U - S$ has a density on $\{S < T < U\} = \cup_{p,q \in \mathbb{Q}} \{S < p < T < q < U\}$. Note that on the event $\{S < p < T < q < U\}$, U is the first time after q such that the Brownian motion B attains $\inf_{u \in [p,q]} B_u$ and obviously has a density. Again by Radon-Nikodym theorem, the distribution of $U - S$ has a density on $\{S < p < T < q < U\}$, which leads to a contradiction. □

Remark 2 The previous lemma can also be inferred from a fine study on local minima of Brownian motion. Neveu and Pitman [68] studied the renewal structure of local extrema in a Brownian path, in terms of Palm measure, see e.g. Kallenberg [40, Chap. 11].

More precisely, denote

- $\mathcal{C}$ to be the space of continuous paths on $\mathbb{R}$, equipped with Wiener measure $\mathbf{W}$;
- E to be the space of excursions with lifetime ζ, equipped with Itô measure $\mathbf{n}$.

Then the Palm measure of all local minima is the image of $\frac{1}{2}(\mathbf{n} \times \mathbf{n} \times \mathbf{W})$ by the mapping $E \times E \times \mathcal{C} \ni (e, e', w) \to \tilde{w} \in \mathcal{C}$ given by

$$\tilde{w}_t = \begin{cases} w_{t+\zeta(e')} & \text{if } t \le -\zeta(e'), \\ e'_{-t} & \text{if } -\zeta(e') \le t \le 0, \\ e_t & \text{if } 0 \le t \le \zeta(e), \\ w_{t-\zeta(e)} & \text{if } t \ge \zeta(e). \end{cases}$$

See Fig. 3. Using the notations of Lemma 5, an in-between reflected position T corresponds to a Brownian local minimum. Then the above discussion implies that $U - S$ is the sum of two independent random variables with densities and hence is diffuse. See also Tsirelson [83] for the i.i.d. uniform sampling construction of Brownian local minima, which reveals the diffuse nature of $U - S$.

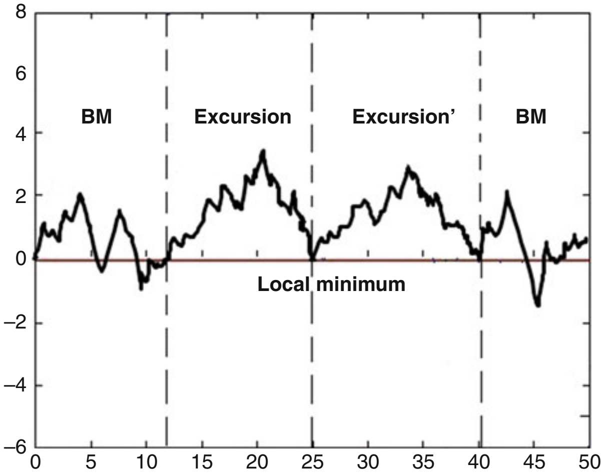

Fig. 3 Structure of local minima in Brownian motion

3.3 No Vervaat Transform of Brownian Motion in a Brownian Path

In the current section, we aim to prove Theorem 4(3). That is, there is no random time T such that

$$(B_{T+u} - B_T; 0 \le u \le 1) \in \mathcal{VB}^-.$$

A similar argument shows that there is no random time T such that

$$(B_{T+u} - B_T; 0 \le u \le 1) \in \mathcal{VB}^+,$$

where $\mathcal{VB}^+ := \{w \in \mathcal{C}[0,1]; w(t) > 0 \text{ for } 0 < t \le 1 \text{ and } \sup\{t < 1; w(t) < w(1)\} < 1\}$. Observe that $(V_u; 0 \le u \le 1)$ is supported on $\mathcal{VB}^+ \cup \mathcal{VB}^-$. Thus, the Vervaat transform of Brownian motion cannot be embedded into Brownian motion.

In Sect. 3.1, we showed that for each fixed $\lambda < 0$, there is no random time T such that $(B_{T+u} - B_T; 0 \le u \le 1) \in \mathcal{VB}^\lambda$. However, there is no obvious way to pass from the non-existence of the Vervaat bridges to that of the Vervaat transform of Brownian motion, due to an uncountable number of possible final levels.

To get around the problem, we make use of an additional tool–potential theory of additive Lévy processes, developed by Khoshnevisan et al. [43, 44, 47–49]. We now recall some results of this theory that we need in the proof of Theorem 4(3). For a more extensive overview of the theory, we refer readers to the survey of Khoshnevisan and Xiao [45].

Definition 1 An N-parameter, $\mathbb{R}^d$-valued additive Lévy process $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ with Lévy exponent $(\Psi^1, \ldots, \Psi^N)$ is defined as

$$Z_{\mathbf{t}} := \sum_{i=1}^{N} Z_{t_i}^i \quad \text{for } \mathbf{t} = (t_1, \ldots, t_N) \in \mathbb{R}_+^N, \tag{37}$$

where $(Z_{t_1}^1; t_1 \geq 0), \ldots, (Z_{t_N}^N; t_N \geq 0)$ are N independent $\mathbb{R}^d$-valued Lévy processes with Lévy exponent $\Psi^1, \ldots, \Psi^N$.

The following result regarding the range of additive Lévy processes is due to Khoshnevisan et al. [49, Theorem 1.5], [47, Theorem 1.1], and Yang [89, 90, Theorem 1.1].

Theorem 9 ([47, 49, 89]) *Let* $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ *be an additive Lévy process defined as in* (37)*. Then*

$$\mathbb{E}[Leb(Z(\mathbb{R}_+^N))] > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{i=1}^{N} Re\left(\frac{1}{1+\Psi^i(\zeta)}\right) d\zeta < \infty,$$

where $Leb(\cdot)$ *is the Lebesgue measure on* $\mathbb{R}^d$*, and* $Re(\cdot)$ *is the real part of a complex number.*

The next result, which is read from Khoshnevisan and Xiao [46, Lemma 4.1], makes a connection between the range of an additive Lévy process and the polarity of single points. See also Khoshnevisan and Xiao [45, Lemma 3.1].

Theorem 10 ([43, 46]) *Let* $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ *be an additive Lévy process defined as in* (37)*. Assume that for each* $\boldsymbol{t} \in \mathbb{R}_+^N$*, the distribution of* $Z_{\boldsymbol{t}}$ *is mutually absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^d$*. Let* $z \in \mathbb{R}^d \setminus \{0\}$*, then*

$$\mathbb{P}(Z_{\boldsymbol{t}} = z \text{ for some } \boldsymbol{t} \in \mathbb{R}_+^N) > 0 \Longleftrightarrow \mathbb{P}(Leb(Z(\mathbb{R}_+^N) > 0) > 0.$$

Note that $\mathbb{P}(Leb(Z(\mathbb{R}_+^N) > 0) > 0$ is equivalent to $\mathbb{E}[Leb(Z(\mathbb{R}_+^N))] > 0$. Combining Theorems 9 and 10, we have:

Corollary 1 *Let* $(Z_{\mathbf{t}}; \mathbf{t} \in \mathbb{R}_+^N)$ *be an additive Lévy process defined as in* (37)*. Assume that for each* $\boldsymbol{t} \in \mathbb{R}_+^N$*, the distribution of* $Z_{\boldsymbol{t}}$ *is mutually absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}^d$*. Let* $z \in \mathbb{R}^d \setminus \{0\}$*, then*

$$\mathbb{P}(Z_{\boldsymbol{t}} = z \text{ for some } \boldsymbol{t} \in \mathbb{R}_+^N) > 0 \Longleftrightarrow \int_{\mathbb{R}^d} \prod_{i=1}^{N} Re\left(\frac{1}{1+\Psi^i(\zeta)}\right) d\zeta < \infty.$$

Proof of Theorem 4(3) We borrow the notations from the proof of Theorem 4(1) in Sect. 3.1. It suffices to show that

$$\mathbb{P}(\sigma_{-t_1} + \tau_{-t_2} = 1 \text{ for some } t_1, t_2 \geq 0) = 0, \tag{38}$$

where $(\sigma_{-t_1}; t_1 \geq 0)$ is the first passage process of Brownian motion with drift -1, and $(\tau_{-t_2}; t_2 \geq 0)$ is a stable($\frac{1}{2}$) subordinator independent of $(\sigma_{-t_1}; t_1 \geq 0)$. Let $Z_{\mathbf{t}} = Z^1_{t_1} + Z^2_{t_2} := \sigma_{-t_1} + \tau_{-t_2}$ for $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$. By Definition 1, Z is a 2-parameter, real-valued additive Lévy process with Lévy exponent (Ψ^1, Ψ^2) given by

$$\Psi^1(\zeta) = \sqrt[4]{1 + 4\zeta^2} \exp\left[-i\frac{\arctan(2\zeta)}{2}\right] - 1 \quad \text{and} \quad \Psi^2(\zeta) = \sqrt{|\zeta|}(1 - i\, sgn\zeta)$$

for $\zeta \in \mathbb{R}$, which is derived from the formula in Cinlar [16, Chap. 7, Page 330] and Lemma 3(2). Hence,

$$Re\left(\frac{1}{1 + \Psi^1(\zeta)}\right) = \frac{1}{\sqrt[4]{1 + 4\zeta^2}} \sqrt{\frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + 4\zeta^2}}\right)}$$

and

$$Re\left(\frac{1}{1 + \Psi^2(\zeta)}\right) = \frac{1 + \sqrt{|\zeta|}}{1 + 2\sqrt{|\zeta|} + 2|\zeta|}.$$

Clearly, $\Xi : \zeta \to Re\left(\frac{1}{1+\Psi^1(\zeta)}\right) Re\left(\frac{1}{1+\Psi^2(\zeta)}\right)$ is not integrable on $\mathbb{R}$ since $\Xi(\zeta) \sim \frac{1}{4|\zeta|}$ as $|\zeta| \to \infty$. In addition, for each $\mathbf{t} \in \mathbb{R}^2_+$, Z_t is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Applying Corollary 1, we obtain (38). □

3.4 No First Passage Bridge in a Brownian Path

We prove Theorem 4(4), i.e. there is no first passage bridge in Brownian motion by a spacetime shift. The main difference between Vervaat bridges with fixed endpoint $\lambda < 0$ and first passage bridges ending at $\lambda < 0$ is that the former start with an excursion piece, while the latter return to the origin infinitely often on any small interval $[0, \epsilon]$, $\epsilon > 0$. Thus, the argument used in Sect. 3.1 to prove the non-existence of Vervaat bridges is not immediately applied in case of first passage bridges. Nevertheless, the potential theory of additive Lévy processes helps to circumvent the difficulty.

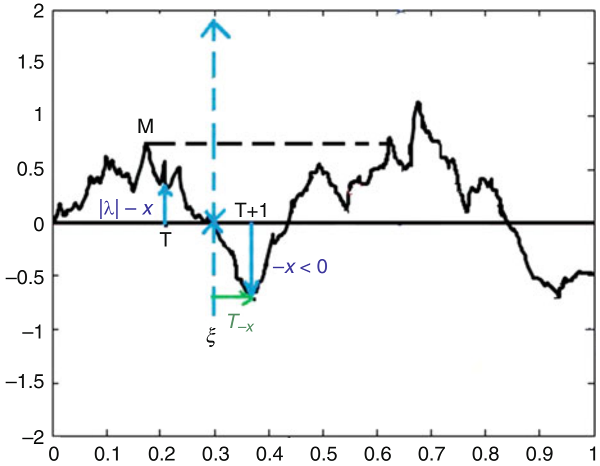

Fig. 4 No first passage bridge of length 1 in a Brownian path

Proof of Theorem 4(4) Suppose by contradiction that $\mathbb{P}(T < \infty) > 0$, where T is a random time that some first passage bridge through a fixed level appears. Take ξ exponentially distributed with rate $\frac{1}{2}$, independent of $(B_t; t \geq 0)$. We have then

$$\mathbb{P}(T < \xi < T + 1) > 0. \tag{39}$$

Now $(T, T+1)$ is inside the excursion of Brownian motion below its past-maximum process, which straddles ξ. See Fig. 4. Define

- $(\tau_{-x}; x \geq 0)$ to be the first passage process of $(B_{\xi+t} - B_\xi; t \geq 0)$.

By strong Markov property of Brownian motion, $(B_{\xi+t} - B_\xi; t \geq 0)$ is still Brownian motion. Thus, $(\tau_{-x}; x \geq 0)$ is a stable($\frac{1}{2}$) subordinator. Let $M := argmax_{[0,\xi]} B_t$. By a variant of Theorem 7, $(B_{\xi-t} - B_\xi; 0 \leq t \leq \xi - M)$ is Brownian motion with drift 1 running until it first hits the level $B_M - B_\xi > 0$, independent of $(\tau_{-x}; x \geq 0)$.

As a consequence, (39) implies that

$$\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x, l) \in \mathbb{R}_+ \times [0, 1]) > 0, \tag{40}$$

where $(B^{\uparrow}_t; t \geq 0)$ is Brownian motion with drift 1, independent of $\frac{1}{2}$-stable subordinator $(\tau_{-x}; x \geq 0)$. By setting $t_1 := x$ and $t_2 := 1 - l$, we have:

$$\begin{aligned}
&\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x, l) \in \mathbb{R}_+ \times [0, 1]) \\
&\quad = \mathbb{P}(\tau_{-t_1} + t_2 = 1 \text{ and } B^{\uparrow}_{t_2} + t_1 = |\lambda| \text{ for some } (t_1, t_2) \in \mathbb{R}_+ \times [0, 1]) \\
&\quad \leq \mathbb{P}[(\tau_{-t_1}, t_1) + (t_2, B^{\uparrow}_{t_2}) = (1, |\lambda|) \text{ for some } (t_1, t_2) \in \mathbb{R}^2_+]
\end{aligned} \tag{41}$$

Let $Z_{\mathbf{t}} = Z^1_{t_1} + Z^2_{t_2} := (\tau_{-t_1}, t_1) + (t_2, B^{\uparrow}_{t_2})$ for $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2_+$. By Definition 1, Z is a 2-parameter, $\mathbb{R}^2$-valued additive Lévy process with Lévy exponent (Ψ^1, Ψ^2) given by

$$\Psi^1(\zeta_1, \zeta_2) := \sqrt{|\zeta_1|} - i(\sqrt{|\zeta_1|}sgn\zeta_1 + \zeta_2) \quad \text{and} \quad \Psi^2(\zeta_1, \zeta_2) := \frac{\zeta_2^2}{2} - i(\zeta_1 + \zeta_2),$$

for $(\zeta_1, \zeta_2) \in \mathbb{R}^2$. Hence,

$$Re\left(\frac{1}{1+\Psi^1(\zeta_1,\zeta_2)}\right) Re\left(\frac{1}{1+\Psi^2(\zeta_1,\zeta_2)}\right)$$
$$= \frac{(1+\sqrt{|\zeta_1|})\left(1+\frac{\zeta_2^2}{2}\right)}{\left[(1+\sqrt{|\zeta_1|})^2 + (\sqrt{|\zeta_1|}sgn\zeta_1 + \zeta_2)^2\right]\left[\left(1+\frac{\zeta_2^2}{2}\right)^2 + (\zeta_1+\zeta_2)^2\right]} := \Xi(\zeta_1, \zeta_2).$$

Observe that $\zeta \to \Xi(\zeta_1, \zeta_2)$ is not integrable on $\mathbb{R}^2$, which is clear by passage to polar coordinates $(\zeta_1, \zeta_2) = (\rho\cos\theta, \sqrt{\rho}\sin\theta)$ for $\rho \geq 0$, $\theta \in [0, 2\pi)$. In addition, for each $\mathbf{t} \in \mathbb{R}^2_+$, $Z_{\mathbf{t}}$ is mutually absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$. Applying Corollary 1, we know that

$$\mathbb{P}(Z_{\mathbf{t}} = (1, |\lambda|) \text{ for some } \mathbf{t} \in \mathbb{R}^2_+) = 0.$$

Combining with (41), we obtain:

$$\mathbb{P}(\tau_{-x} = l \text{ and } B^{\uparrow}_{1-l} = |\lambda| - x \text{ for some } (x, l) \in \mathbb{R}_+ \times [0,1]) = 0,$$

which contradicts (40). □

It is not hard to see that the above argument, together with those in Sect. 3.2 works for Bessel bridge of any dimension.

Corollary 2 (Impossibility of Embedding of Reflected Bridge Paths/Bessel Bridge) *For each fixed $\lambda > 0$, almost surely, there is no random time T such that*

$$(B_{T+u} - B_T; 0 \leq u \leq 1) \in \mathcal{RBR}^{\lambda}$$
$$:= \{w \in \mathcal{C}[0,1]; w(t) \geq 0 \textit{ for } 0 \leq t \leq 1 \textit{ and } w(1) = \lambda\}.$$

In particular, there is no random time $T \geq 0$ such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has the same distribution as Bessel bridge ending at λ.

3.5 Meander, Co-meander and 3-d Bessel Process in a Brownian Path

We prove Theorem 2 in this section using Itô's excursion theory, combined with *Rost's filling scheme* [13, 75] solution to the Skorokhod embedding problem.

The existence of Brownian meander in a Brownian path is assured by the following well-known result, which can be read from Maisoneuve [58, Sect. 8], with explicit formulas due to Chung [15]. An alternative approach was provided by Greenwood and Pitman [35], and Pitman [70, Sects. 4 and 5]. See also Biane and Yor [9, Theorem 6.1], or Revuz and Yor [74, Exercise 4.18, Chap. XII].

Theorem 11 ([9, 35, 58]) *Let $(e^i)_{i\in\mathbb{N}}$ be the sequence of excursions, whose length exceeds 1, in the reflected process $(B_t - \underline{B}_t; t \geq 0)$, where $\underline{B}_t := \inf_{0\leq u\leq t} B_u$ is the past-minimum process of the Brownian motion. Then $(e^i_u; 0 \leq u \leq 1)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed paths, each distributed as Brownian meander $(m_u; 0 \leq u \leq 1)$.*

Let us recall another basic result due to Imhof [38], which establishes the absolute continuity relation between Brownian meander and the three-dimensional Bessel process. Their relation with Brownian co-meander is studied in Yen and Yor [91, Chap. 7].

Theorem 12 ([38, 91]) *The distributions of Brownian meander $(m_u; 0 \leq u \leq 1)$, Brownian co-meander $(\tilde{m}_u; 0 \leq u \leq 1)$ and the three-dimensional Bessel process $(R_u; 0 \leq u \leq 1)$ are mutually absolutely continuous with respect to each other. For $F : \mathcal{C}[0,1] \to \mathbb{R}^+$ a measurable function,*

1. $\mathbb{E}[F(m_u; 0 \leq u \leq 1)] = \mathbb{E}\left[\sqrt{\frac{\pi}{2}}\frac{1}{R_1}F(R_u; 0 \leq u \leq 1)\right]$;
2. $\mathbb{E}[F(\tilde{m}_u; 0 \leq u \leq 1)] = \mathbb{E}\left[\frac{1}{R_1^2}F(R_u; 0 \leq u \leq 1)\right]$.

According to Theorem 11, there exist $T_1, T_2, \cdots$ such that

$$m^i := (B_{T_i+u} - B_{T_i}; 0 \leq u \leq 1) \tag{42}$$

form a sequence of i.i.d. Brownian meanders. Since Brownian co-meander and the three-dimensional Bessel process are absolutely continuous relative to Brownian meander, it is natural to think of *von Neumann's acceptance-rejection algorithm* [86], see e.g. Rubinstein and Kroese [77, Sect. 2.3.4] for background and various applications. However, von Neumann's selection method requires that the Radon-Nikodym density between the underlying probability measures is essentially bounded, which is not satisfied in the cases suggested by Theorem 12. Nevertheless, we can apply the filling scheme of Chacon and Ornstein [13] and Rost [75].

We observe that sampling Brownian co-meander or the three-dimensional Bessel process from i.i.d. Brownian meanders $(m^i)_{i\in\mathbb{N}}$ fits into the general theory of Rost's filling scheme applied to the Skorokhod embedding problem. In the sequel, we follow the approach of Dellacherie and Meyer [22, Sects. 63–74, Chap. IX], which is based on the seminal work of Rost [75], to construct a stopping time N such that m^N achieves the distribution of $\tilde{m}$ or R. We need some notions from potential theory for the proof.

Definition 2

1. Given a Markov chain $X := (X_n)_{n\in\mathbb{N}}$, a function f is said to be excessive relative to X if

$$(f(X_n))_{n\in\mathbb{N}} \text{ is } \mathcal{F}_n - \text{supermartingale},$$

   where $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is the natural filtration of X.
2. Given two positive measures μ and λ, μ is said to be a balayage/sweeping of λ if

$$\mu(f) \leq \lambda(f) \quad \text{for all bounded excessive functions } f.$$

Proof of Theorem 2 Let μ^m (resp. μ^R) be the distribution of Brownian meander (resp. the three-dimensional Bessel process) on the space $(\mathcal{C}[0,1],\mathcal{F})$. By the filling scheme, the sequence of measures $(\mu_i^m,\mu_i^R)_{i\in\mathbb{N}}$ is defined recursively as

$$\mu_0^m := (\mu^m - \mu^R)^+ \quad \text{and} \quad \mu_0^R := (\mu^m - \mu^R)^-, \tag{43}$$

and for each $i \in \mathbb{N}$,

$$\mu_{i+1}^m := (\mu_i^m(1)\cdot\mu^m - \mu_i^R)^+ \quad \text{and} \quad \mu_{i+1}^R := (\mu_i^m(1)\cdot\mu^m - \mu_i^R)^-, \tag{44}$$

where $\mu_i^m(1)$ is the total mass of the measure μ_i^m. It is not hard to see that the bounded excessive functions of the i.i.d. meander sequence are constant μ^m a.s. Since μ^R is absolutely continuous with respect to μ^m, for each μ^m a.s. constant function c, $\mu^R(c) = \mu^m(c) = c$. Consequently, μ^R is a balayage/sweeping of μ^m by Definition 2. According to Theorem 69 of Dellacherie and Meyer [22],

$$\mu_\infty^R = 0, \quad \text{where } \mu_\infty^R :=\downarrow \lim_{i\to\infty} \mu_i^R.$$

Now let d_0 be the Radon-Nikodym density of μ_0^m relative to μ^m, and for $i > 0$, d_i be the Radon-Nikodym density of μ_i^m relative to $\mu_{i-1}^m(1)\cdot\mu^m$. We have

$$\begin{aligned}\mu^R &= (\mu^R - \mu_0^R) + (\mu_0^R - \mu_1^R) + \cdots \\ &= (\mu^m - \mu_0^m) + (\mu_0^m(1)\cdot\mu^m - \mu_1^m) + \cdots \\ &= (1-d_0)\mu^m + d_0\mu^m(1)\cdot(1-d_1)\mu^m + \cdots .\end{aligned} \tag{45}$$

Consider the stopping time N defined by

$$N := \inf\left\{n \geq 0; -\sum_{i=0}^{n} \log d_i(m^i) > \xi\right\}, \tag{46}$$

where $(d_i)_{i\in\mathbb{N}}$ is the sequence of Radon-Nikodym densities defined as in the preceding paragraph, $(m^i)_{i\in\mathbb{N}}$ is the sequence of i.i.d. Brownian meanders defined as in (42), and ξ is exponentially distributed with rate 1, independent of $(m^i)_{i\in\mathbb{N}}$.

From the computation of (45), for all bounded measurable function f and all $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}[f(m^N); N = k] &= \mathbb{E}[f(m^k); -\sum_{i=0}^{k-1} \log d_i(m^i) \leq \xi < -\sum_{i=0}^{k} \log d_i(m^i)] \\
&= \mathbb{E}[d_0(m^0) \cdots d_{k-1}(m^{k-1}) f(m^k)(1 - d_k(m^k))] \\
&= (\mu^m_{k-1}(1) \cdot \mu^m - \mu^m_k) f \\
&= (\mu^R_{k-1} - \mu^R_k) f,
\end{aligned}$$

where $(\mu^m_i, \mu^R_i)_{i\in\mathbb{N}}$ are the filling measures defined as in (43) and (44). By summing over all k, we have

$$\mathbb{E}[f(m^N); N < \infty] = \mu^R f.$$

That is, m^N has the same distribution as R. As a summary,

$$(B_{T_N+u} - B_{T_N}; 0 \leq u \leq 1) \text{ has the same distribution as } (R_u; 0 \leq u \leq 1),$$

where $(T_i)_{i\in\mathbb{N}}$ are defined by (42) and N is the stopping time as in (46). Thus we achieve the distribution of the three-dimensional Bessel process in Brownian motion. The embedding of Brownian co-meander into Brownian motion is obtained in the same vein. □

Remark 3 Note that the stopping time N defined as in (46) has infinite mean, since

$$\mathbb{E}N = \sum_{i\in\mathbb{N}} \mu^m_i(1) = \infty.$$

The problem whether Brownian co-meander or the three-dimensional Bessel process can be embedded in finite expected time, remains open. More generally, Rost [76] was able to characterize all stopping distributions of a continuous-time Markov process, given its initial distribution. In our setting, let $(P_t)_{t\geq 0}$ be the semi-group of the moving window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$, and μ^W be its initial distribution, corresponding to Wiener measure on $\mathcal{C}[0, 1]$. Following Rost

[76], for any distribution μ on $\mathcal{C}[0,1]$, one can construct the continuous-time filling measures $(\mu_t, \mu_t^W)_{t \geq 0}$ and a suitable stopping time T such that

$$\mu - \mu_t + \mu_t^W = \mu^W P_{t \wedge T}.$$

Thus, the distribution μ is achieved if and only if $\mu_\infty = 0$, where $\mu_\infty :=\downarrow \lim_{t\to\infty} \mu_t$. In particular, Brownian motion with drift $(\vartheta t + B_t; 0 \leq t \leq 1)$ for a fixed ϑ, can be obtained for a suitable stopping time $T + 1$.

4 Potential Theory for Continuous-Time Patterns

In Question 2, we ask for any Borel measurable subset S of $\mathcal{C}_0[0,1]$ whether S is hit by the moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$, at some random time T. Related studies of the moving window process appear in several contexts. Knight [50, 51] introduced the prediction processes, where the whole past of the underlying process is tracked to anticipate its future behavior. The relation between Knight's prediction processes and our problems is discussed briefly at the end of the section. Similar ideas are found in stochastic control theory, where certain path-dependent stochastic differential equations were investigated, see e.g. the monograph of Mohammed [63] and Chang et al. [14]. More recently, Dupire [25] worked out a functional version of Itô's calculus, in which the underlying process is path-valued and notions as time-derivative and space-derivative with respect to a path, are proposed. We refer readers to the thesis of Fournié [31] as well as Cont and Fournié [17–19] for further development.

Indeed, Question 2 is some issue of potential theory. In Benjamini et al. [4] a potential theory was developed for transient Markov chains on any countable state space E. They showed that the probability for a transient chain to ever visit a given subset $S \subset E$, is estimated by $Cap_M(S)$—the *Martin capacity* of the set S. See also Mörters and Peres [67, Sect. 8.3] for a detailed exposition. As pointed out by Steven Evans (personal communications), such a framework still works well for our discrete patterns. For $0 < \alpha < 1$, define the α-potential of the discrete patterns/strings of length n as

$$\begin{aligned} G^\alpha(\epsilon', \epsilon'') &:= \sum_{k=0}^{\infty} \alpha^k P^k(\epsilon', \epsilon'') \\ &= \sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k 1\{\sigma_k(\epsilon') = \tau_k(\epsilon'')\} + \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\right)^k, \end{aligned}$$

where $\epsilon', \epsilon'' \in \{-1, 1\}^n$, and $P(\cdot, \cdot)$ is the transition kernel of discrete patterns/strings of length n in a simple random walk, and σ_k (resp. τ_k): $\{-1,1\}^n \to \{-1,1\}^{n-k}$ the restriction operator to the last $n-k$ strings (resp. to the first $n-k$ strings). The following result is a direct consequence of the first/second moment method, and we leave the detail to readers.

Proposition 2 (Evans (personal communications)) *Let T be an $\mathbb{N}$-valued random variable with $\mathbb{P}(T > n) = \alpha^n$, independent of the simple random walk. For $\mathcal{A}^n$ a collection of discrete patterns of length n, we have*

$$\frac{1}{2}\frac{2^{-n}}{1-\alpha}Cap_\alpha(\mathcal{A}^n) \leq \mathbb{P}(T(\mathcal{A}^n) < T) \leq \frac{2^{-n}}{1-\alpha}Cap_\alpha(\mathcal{A}^n),$$

where for $A \subset \{-1, 1\}^n$,

$$Cap_\alpha(A) := \left[\inf\left\{\sum_{\epsilon',\epsilon''\in\{-1,1\}^n} G^\alpha(\epsilon',\epsilon'')g(\epsilon')g(\epsilon''); g \geq 0, g(A^c) = \{0\} \right.\right.$$
$$\left.\left. \text{and} \sum_{\epsilon\in\{0,1\}^n} g(\epsilon) = 1\right\}\right]^{-1}.$$

Now let us mention some previous work regarding the potential theory for path-valued Markov processes. There has been much interest in developing a potential theory for the Ornstein-Uhlenbeck process in the Wiener space $\mathcal{C}_0[0,\infty)$, defined as

$$Z_t := U(t,\cdot) \quad \text{for } t \geq 0,$$

where $U(t,\cdot) := e^{-t/2}W(e^t,\cdot)$ is the Ornstein-Uhlenbeck Brownian sheet. Note that the continuous-time process $(Z_t; t \geq 0)$ takes values in the Wiener space $\mathcal{C}_0[0,\infty)$ and starts at $Z_0 := W(1,\cdot)$ as standard Brownian motion. Following Williams [60], a Borel measurable set $S \subset \mathcal{C}_0[0,\infty)$ is said to be quasi-sure if $\mathbb{P}(\forall t \geq 0, Z_t \in S) = 1$, which is known to be equivalent to

$$Cap_{OU}(S^c) = 0, \tag{47}$$

where

$$Cap_{OU}(S^c) := \int_0^\infty e^{-t}\mathbb{P}(\exists T \in [0,t] \text{ such that } Z_T \in S^c)dt \tag{48}$$

is the *Fukushima-Malliavin capacity* of S^c, that is the probability that Z hits S^c before an independent exponential random time with parameter 1. Taking advantage of the well-known *Wiener-Itô decomposition* of the Ornstein-Uhlenbeck semigroup, Fukushima [33] provided an alternative construction of (47) via the Dirichlet form. The approach allows the strengthening of many Brownian almost sure properties to quasi-sure properties. See also the survey of Khoshnevisan [42] for recent development.

Note that the definition (48) can be extended to any (path-valued) Markov process. Within this framework, a related problem to Question 2 is

Question 4 Given a Borel measurable set $S_\infty \subset \mathcal{C}_0[0,\infty)$, is

$$Cap_{MW}(S_\infty) := \int_0^\infty e^{-t}\mathbb{P}[\exists T \in [0,t] \text{ such that } \Theta_T \circ B \in S_\infty]dt$$
$$= 0 \text{ or } > 0?$$

where $(\Theta_t)_{t\geq 0}$ is the family of spacetime shift operators defined as

$$\Theta_t \circ B := (B_{t+u} - B_t; u \geq 0) \quad \text{for all } t \geq 0. \tag{49}$$

It is not difficult to see that the set function Cap_{MW} is a Choquet capacity associated to the shifted process $(B_{t+u} - B_t; u \geq 0)$ for $t \geq 0$, or the moving-window process $X_t := (B_{t+u} - B_t; 0 \leq u \leq 1)$ for $t \geq 0$. For a Borel measurable subset S of $\mathcal{C}_0[0,1]$, if $Cap_{MW}(S \otimes_1 \mathcal{C}_0[0,\infty)) = 0$, where

$$S \otimes_1 \mathcal{C}_0[0,\infty) := \{(w_t 1_{t<1} + (w_1 + w'_t)1_{t\geq 1})_{t\geq 0}; w \in S \text{ and } w' \in \mathcal{C}_0[0,\infty)\} \tag{50}$$

is the usual path-concatenation, then

$$\mathbb{P}(\exists T > 0 \text{ such that } X_T \in S] = 0,$$

i.e. almost surely the set S is not hit by the moving-window process X. Otherwise,

$$\mathbb{P}[\exists T \in [0,t] \text{ such that } X_T \in S] > 0 \quad \text{for some } t \geq 0,$$

and an elementary argument leads to $\mathbb{P}[\exists T \geq 0 \text{ such that } X_T \in S] = 1$.

As context for this question, we note that path-valued Markov processes have also been extensively investigated in the superprocess literature. In particular, Le Gall [54] characterized the polar sets for the Brownian snake, which relies on earlier work on the potential theory of symmetric Markov processes by Fitzsimmons and Getoor [30] among others.

There has been much progress in the development of potential theory for symmetric path-valued Markov processes. However, the shifted process, or the moving-window process, is not time-reversible and the transition kernel is more complicated than that of the Ornstein-Uhlenbeck process in Wiener space. So working with a non-symmetric Dirichlet form, see e.g. the monograph of Ma and Röckner [57], seems to be far from obvious.

Open Problem 2

1. Is there any relation between the two capacities Cap_X and Cap_{MW} on Wiener space?
2. Propose a non-symmetric Dirichlet form for the shifted process $(\Theta_t \circ B)_{t\geq 0}$, which permits to compute the capacities of the sets of paths $\mathcal{E}$, $\mathcal{M}$, $\mathcal{BR}^{\lambda}$ …etc.

This problem seems substantial already for one-dimensional Brownian motion. But it could of course be posed also for higher dimensional Brownian motion, or a still more general Markov process. Following are some well-known examples of non-existing patterns in d-dimensional Brownian motion for $d \geq 2$.

- $d = 2$ (Evans [28]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ has a two-sided cone point with angle $\alpha < \pi$;
- $d = 3$ (Dvoretzky et al. [27]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ contains a triple point;
- $d \geq 4$ (Kakutani [39], Dvoretzky et al. [26]): There is no random time T such that $(B_{T+u} - B_T; 0 \leq u \leq 1)$ contains a double point.

We refer readers to the book of Mörters and Peres [67, Chaps. 9 and 10] for historical notes and further discussions on sample path properties of Brownian motion in all dimensions.

Finally, we make some connections between Knight's prediction processes and our problems. For background, readers are invited to Knight [50, 51] as well as the commentary of Meyer [59] on Knight's work. To avoid heavy measure theoretic discussion, we restrict ourselves to the classical Wiener space $(\mathcal{C}_0[0,\infty), \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^{\mathbf{W}})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the augmented Brownian filtrations satisfying the usual hypothesis of right-continuity.

The prediction process is defined as, for all $t \geq 0$ and S_∞ a Borel measurable set of $\mathcal{C}_0[0,\infty)$,

$$Z_t^{\mathbf{W}}(S_\infty) := \mathbb{P}^{\mathbf{W}}[\Theta_t \circ B \in S_\infty | \mathcal{F}_t],$$

where $\Theta_t \circ B$ is the shifted path defined as in (49). Note that $(Z_t^{\mathbf{W}})_{t\geq 0}$ is a strong Markov process, which takes values in the space of probability measure on the Wiener space $(\mathcal{C}_0[0,\infty), \mathcal{F})$. In terms of the prediction process, Question 2 can be reformulated as

Question 5 Given a Borel measurable set $S \subset \mathcal{C}_0[0,1]$, can we find a random time T such that

$$\mathbb{E}Z_T^{\mathbf{W}}(S \otimes_1 \mathcal{C}_0[0,\infty)) = 1?$$

where $S \otimes_1 \mathcal{C}_0[0,\infty)$ is defined as in (50).

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Bessel Processes, the Brownian Snake and Super-Brownian Motion

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Abstract We prove that, both for the Brownian snake and for super-Brownian motion in dimension one, the historical path corresponding to the minimal spatial position is a Bessel process of dimension −5. We also discuss a spine decomposition for the Brownian snake conditioned on the minimizing path.

1 Introduction

Marc Yor used to say that "Bessel processes are everywhere". Partly in collaboration with Jim Pitman [13, 14], he wrote several important papers, which considerably improved our knowledge of Bessel processes and of their numerous applications. A whole chapter of Marc Yor's celebrated book with Daniel Revuz [15] is devoted to Bessel processes and their applications to Ray-Knight theorems. As a matter of fact, Bessel processes play a major role in the study of properties of Brownian motion, and, in particular, the three-dimensional Bessel process is a key ingredient of the famous Williams decomposition of the Brownian excursion at its maximum. In the present work, we show that Bessel processes also arise in similar properties of super-Brownian motion and the Brownian snake. Informally, we obtain that, both for the Brownian snake and for super-Brownian motion, the (historical) path reaching the minimal spatial position is a Bessel process of negative dimension.

Let us describe our results in a more precise way. We write $(W_s)_{s\geq 0}$ for the Brownian snake whose spatial motion is one-dimensional Brownian motion. Recall that $(W_s)_{s\geq 0}$ is a Markov process taking values in the space of all finite paths in $\mathbb{R}$, and for every $s \geq 0$, write ζ_s for the lifetime of W_s. We let $\mathbb{N}_0$ stand for the σ-finite excursion measure of $(W_s)_{s\geq 0}$ away from the trivial path with initial point 0 and zero lifetime (see Sect. 2 for the precise normalization of $\mathbb{N}_0$). We let W_* be the minimal

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spatial position visited by the paths W_s, $s \geq 0$. Then the "law" of W_* under $\mathbb{N}_0$ is given by

$$\mathbb{N}_0(W_* \leq -a) = \frac{3}{2a^2}, \tag{1}$$

for every $a > 0$ (see [9, Sect. VI.1] or [12, Lemma 2.1]). Furthermore, it is known that, $\mathbb{N}_0$ a.e., there is a unique instant $s_\mathbf{m}$ such that $W_* = W_{s_\mathbf{m}}(\zeta_{s_\mathbf{m}})$. Our first main result (Theorem 1) shows that, conditionally on $W_* = -a$, the random path $a + W_{s_\mathbf{m}}$ is a Bessel process of dimension $d = -5$ started from a and stopped upon hitting 0. Because of the relations between the Browian snake and super-Brownian motion, this easily implies a similar result for the unique historical path of one-dimensional super-Brownian motion that attains the minimal spatial value (Corollary 1). Our second result (Theorem 2) provides a "spine decomposition" of the Brownian snake under $\mathbb{N}_0$ given the minimizing path $W_{s_\mathbf{m}}$. Roughly speaking, this decomposition involves Poisson processes of Brownian snake excursions branching off the minimizing path, which are conditioned not to attain the minimal value W_*. See Theorem 2 for a more precise statement.

Our proofs depend on various properties of the Brownian snake, including its strong Markov property and the "subtree decomposition" of the Brownian snake ([9, Lemma V.5], see Lemma 3 below) starting from an arbitrary finite path w. We also use the explicit distribution of the Brownian snake under $\mathbb{N}_0$ at its first hitting time of a negative level: If $b > 0$ and S_b is the first hitting time of $-b$ by the Brownian snake, the path $b + W_{S_b}$ is distributed under $\mathbb{N}_0(\cdot \mid S_b < \infty)$ as a Bessel process of dimension $d = -3$ started from b and stopped upon hitting 0 (see Lemma 4 below). Another key ingredient (Lemma 1) is a variant of the absolute continuity relations between Bessel processes that were discovered by Yor [17] and studied in a more systematic way in the paper [13] by Pitman and Yor.

Let us briefly discuss connections between our results and earlier work. As a special case of a famous time-reversal theorem due to Williams [16, Theorem 2.5] (see also Pitman and Yor [14, Sect. 3], and in particular the examples treated in subsection (3.5) of [14]), the time-reversal of a Bessel process of dimension $d = -5$ started from a and stopped upon hitting 0 is a Bessel process of dimension $d = 9$ started from 0 and stopped at its last passage time at a – This property can also be found in [15, Exercise XI.1.23]. Our results are therefore related to the appearance of nine-dimensional Bessel processes in limit theorems derived in [12] and [11]. Note however that in contrast with [12] and [11], Theorem 1 gives an exact identity in distribution and not an asymptotic result. As a general remark, Theorem 2 is related to a number of "spine decompositions" for branching processes that have appeared in the literature in various contexts. We finally note that a strong motivation for the present work came from the forthcoming paper [2], which uses Theorems 1 and 2 to provide a new construction of the random metric space called the Brownian plane [1] and to give a number of explicit calculations of distributions related to this object.

The paper is organized as follows. Section 2 presents a few preliminary results about Bessel processes and the Brownian snake. Section 3 contains the statement and proof of our main results Theorems 1 and 2. Finally Sect. 4 gives our applications to super-Brownian motion, which are more or less straightforward consequences of the results of Sect. 3.

2 Preliminaries

2.1 Bessel Processes

We will be interested in Bessel processes of negative index. We refer to [13] for the theory of Bessel processes, and we content ourselves with a brief presentation limited to the cases of interest in this work. We let $B = (B_t)_{t\geq 0}$ be a linear Brownian motion and for every $\alpha > 0$, we will consider the nonnegative process $R^{(\alpha)} = (R_t^{(\alpha)})_{t\geq 0}$ that solves the stochastic differential equation

$$\mathrm{d}R_t^{(\alpha)} = \mathrm{d}B_t - \frac{\alpha}{R_t^{(\alpha)}}\,\mathrm{d}t, \tag{2}$$

with a given (nonnegative) initial condition. To be specific, we require that Eq. (2) holds up to the first hitting time of 0 by $R^{(\alpha)}$,

$$T^{(\alpha)} := \inf\{t \geq 0 : R_t^{(\alpha)} = 0\},$$

and that $R_t^{(\alpha)} = 0$ for $t \geq T^{(\alpha)}$. Note that uniqueness in law and pathwise uniqueness hold for (2).

In the standard terminology (see e.g. [13, Sect. 2]), the process $R^{(\alpha)}$ is a Bessel process of index $\nu = -\alpha - \frac{1}{2}$, or dimension $d = 1 - 2\alpha$. We will be interested especially in the cases $\alpha = 2$ ($d = -3$) and $\alpha = 3$ ($d = -5$).

For notational convenience, we will assume that, for every $r \geq 0$, there is a probability measure P_r such that both the Brownian motion B and the Bessel processes $R^{(\alpha)}$ start from r under P_r.

Let us fix $r > 0$ and argue under the probability measure P_r. Fix $\delta \in (0, r)$ and set

$$T_\delta^{(\alpha)} := \inf\{t \geq 0 : R_t^{(\alpha)} = \delta\},$$

and

$$T_\delta := \inf\{t \geq 0 : B_t = \delta\}.$$

The following absolute continuity lemma is very closely related to results of [17] (Lemma 4.5) and [13] (Proposition 2.1), but we provide a short proof for the sake of completeness. If E is a metric space, $C(\mathbb{R}_+, E)$ stands for the space of all continuous functions from $\mathbb{R}_+$ into E, which is equipped with the topology of uniform convergence on every compact interval.

Lemma 1 *For every nonnegative measurable function F on $C(\mathbb{R}_+, \mathbb{R}_+)$,*

$$E_r\Big[F\Big(\Big(R^{(\alpha)}_{t\wedge T^{(\alpha)}_\delta}\Big)_{t\geq 0}\Big)\Big] = \Big(\frac{r}{\delta}\Big)^{\alpha} E_r\Big[F((B_{t\wedge T_\delta})_{t\geq 0}) \exp\Big(-\frac{\alpha(1+\alpha)}{2}\int_0^{T_\delta}\frac{\mathrm{d}s}{B_s^2}\Big)\Big].$$

Proof Write $(\mathscr{F}_t)_{t\geq 0}$ for the (usual augmentation of the) filtration generated by B. For every $t \geq 0$, set

$$M_t := \Big(\frac{r}{B_{t\wedge T_\delta}}\Big)^{\alpha} \exp\Big(-\frac{\alpha(1+\alpha)}{2}\int_0^{t\wedge T_\delta}\frac{\mathrm{d}s}{B_s^2}\Big).$$

An application of Itô's formula shows that $(M_t)_{t\geq 0}$ is an $(\mathscr{F}_t)$-local martingale. Clearly, $(M_t)_{t\geq 0}$ is bounded by $(r/\delta)^\alpha$ and is thus a uniformly integrable martingale, which converges as $t \to \infty$ to

$$M_\infty = \Big(\frac{r}{\delta}\Big)^{\alpha} \exp\Big(-\frac{\alpha(1+\alpha)}{2}\int_0^{T_\delta}\frac{\mathrm{d}s}{B_s^2}\Big).$$

We define a probability measure Q absolutely continuous with respect to P_r by setting $Q = M_\infty \cdot P_r$. An application of Girsanov's theorem shows that the process

$$B_t + \alpha\int_0^{t\wedge T_\delta}\frac{\mathrm{d}s}{B_s}$$

is an $(\mathscr{F}_t)$-Brownian motion under Q. It follows that the law of $(B_{t\wedge T_\delta})_{t\geq 0}$ under Q coincides with the law of $\Big(R^{(\alpha)}_{t\wedge T^{(\alpha)}_\delta}\Big)_{t\geq 0}$ under P_r. This gives the desired result. □

The formula of the next lemma is probably known, but we could not find a reference.

Lemma 2 *For every $r > 0$ and $a > 0$,*

$$E_r\Big[\exp\Big(-3\int_0^{T^{(2)}}\mathrm{d}t\,(a + R^{(2)}_t)^{-2}\Big)\Big] = 1 - \Big(\frac{r}{r+a}\Big)^2.$$

Proof An application of Itô's formula shows that

$$M_t := \left(1 - \left(\frac{R^{(2)}_t}{R^{(2)}_t + a}\right)^2\right) \exp\Big(-3\int_0^{t\wedge T^{(2)}}\mathrm{d}s\,(a + R^{(2)}_s)^{-2}\Big)$$

is a local martingale. Clearly, M_t is bounded by 1 and is thus a uniformly integrable martingale. Writing $E_r[M_{T^{(2)}}] = E_r[M_0]$ yields the desired result. □

Remark An alternative proof of the formula of Lemma 2 will follow from forthcoming calculations: just use formula (4) below with $G = 1$, noting that the left-hand side of this formula is then equal to $\mathbb{N}_0(-b-\varepsilon < W_* \leq -b)$, which is computed using (1). So strictly speaking we do not need the preceding proof. Still it seems a bit odd to use the Brownian snake to prove the identity of Lemma 2, which has to do with Bessel processes only.

2.2 The Brownian Snake

We refer to [9] for the general theory of the Brownian snake, and only give a short presentation here. We write $\mathscr{W}$ for the set of all finite paths in $\mathbb{R}$. An element of $\mathscr{W}$ is a continuous mapping $\mathrm{w} : [0,\zeta] \longrightarrow \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})} \geq 0$ depends on w and is called the lifetime of w. We write $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$ for the endpoint of w. For $x \in \mathbb{R}$, we set $\mathscr{W}_x := \{\mathrm{w} \in \mathscr{W} : \mathrm{w}(0) = x\}$. The trivial path w such that $\mathrm{w}(0) = x$ and $\zeta_{(\mathrm{w})} = x$ is identified with the point x of $\mathbb{R}$, so that we can view $\mathbb{R}$ as a subset of $\mathscr{W}$. The space $\mathscr{W}$ is equipped with the distance

$$d(\mathrm{w},\mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geq 0} |\mathrm{w}(t\wedge\zeta_{(\mathrm{w})}) - \mathrm{w}'(t\wedge\zeta_{(\mathrm{w}')})|.$$

The Brownian snake $(W_s)_{s\geq 0}$ is a continuous Markov process with values in $\mathscr{W}$. We will write $\zeta_s = \zeta_{(W_s)}$ for the lifetime process of W_s. The process $(\zeta_s)_{s\geq 0}$ evolves like a reflecting Brownian motion in $\mathbb{R}_+$. Conditionally on $(\zeta_s)_{s\geq 0}$, the evolution of $(W_s)_{s\geq 0}$ can be described informally as follows: When ζ_s decreases, the path W_s is shortened from its tip, and, when ζ_s increases, the path W_s is extended by adding "little pieces of linear Brownian motion" at its tip. See [9, Chap. IV] for a more rigorous presentation.

It is convenient to assume that the Brownian snake is defined on the canonical space $C(\mathbb{R}_+,\mathscr{W})$, in such a way that, for $\omega = (\omega_s)_{s\geq 0} \in C(\mathbb{R}_+,\mathscr{W})$, we have $W_s(\omega) = \omega_s$. The notation $\mathbb{P}_{\mathrm{w}}$ then stands for the law of the Brownian snake started from w.

For every $x \in \mathbb{R}$, the trivial path x is a regular recurrent point for the Brownian snake, and so we can make sense of the excursion measure $\mathbb{N}_x$ away from x, which is a σ-finite measure on $C(\mathbb{R}_+,\mathscr{W})$. Under $\mathbb{N}_x$, the process $(\zeta_s)_{s\geq 0}$ is distributed according to the Itô measure of positive excursions of linear Brownian motion, which is normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x\Big(\sup_{s\geq 0}\zeta_s > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

We write $\sigma := \sup\{s \geq 0 : \zeta_s > 0\}$ for the duration of the excursion under $\mathbb{N}_x$. In a way analogous to the classical property of the Itô excursion measure [15, Corollary XII.4.3], $\mathbb{N}_x$ is invariant under time-reversal, meaning that $(W_{(\sigma-s)\vee 0})_{s\geq 0}$ has the same distribution as $(W_s)_{s\geq 0}$ under $\mathbb{N}_x$.

Recall the notation

$$W_* := \inf_{0\leq s\leq \sigma} \hat{W}_s = \inf_{0\leq s\leq \sigma} \inf_{0\leq t\leq \zeta_s} W_s(t),$$

and formula (1) determining the law of W_* under $\mathbb{N}_0$. It is known (see e.g. [12, Proposition 2.5]) that $\mathbb{N}_x$ a.e. there is a unique instant $s_{\mathbf{m}} \in [0, \sigma]$ such that $\hat{W}_{s_{\mathbf{m}}} = W_*$. One of our main objectives is to determine the law of $W_{s_{\mathbf{m}}}$. We start with two important lemmas.

Our first lemma is concerned with the Brownian snake started from $\mathbb{P}_{\mathrm{w}}$, for some fixed $\mathrm{w} \in \mathscr{W}$, and considered up to the first hitting time of 0 by the lifetime process, that is

$$\eta_0 := \inf\{s \geq 0 : \zeta_s = 0\}.$$

Then the values of the Brownian snake between times 0 and η_0 can be classified according to "subtrees" branching off the initial path w. To make this precise, let (α_i, β_i), $i \in I$ be the excursion intervals away from 0 of the process

$$\zeta_s - \min_{0\leq r\leq s} \zeta_r$$

before time η_0. In other words, the intervals (α_i, β_i) are the connected components of the open set $\{s \in [0, \eta_0] : \zeta_s > \min_{0\leq r\leq s} \zeta_r\}$. Using the properties of the Brownian snake, it is easy to verify that $\mathbb{P}_{\mathrm{w}}$ a.s. for every $i \in I$, $W_{\alpha_i} = W_{\beta_i}$ is just the restriction of w to $[0, \zeta_{\alpha_i}]$, and the paths W_s, $s \in [\alpha_i, \beta_i]$ all coincide over the time interval $[0, \zeta_{\alpha_i}]$. In order to describe the behavior of these paths beyond time ζ_{α_i} we introduce, for every $i \in I$, the element $W^i = (W^i_s)_{s\geq 0}$ of $C(\mathbb{R}_+, \mathscr{W})$ obtained by setting, for every $s \geq 0$,

$$W^i_s(t) := W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i} + t) \ , \quad 0 \leq t \leq \zeta^i_s := \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i}.$$

Lemma 3 *Under $\mathbb{P}_{\mathrm{w}}$, the point measure*

$$\sum_{i\in I} \delta_{(\zeta_{\alpha_i}, W^i)}(\mathrm{d}t, \mathrm{d}\omega)$$

is a Poisson point measure on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathscr{W})$ with intensity

$$2\,\mathbf{1}_{[0,\zeta_{(\mathrm{w})}]}(t)\,\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega).$$

We refer to [9, Lemma V.5] for a proof of this lemma. Our second lemma deals with the distribution of the Brownian snake under $\mathbb{N}_0$ at the first hitting time of a negative level. For every $b > 0$, we set

$$S_b := \inf\{s \geq 0 : \hat{W}_s = -b\}$$

with the usual convention $\inf \varnothing = \infty$.

Lemma 4 *The law of the random path* W_{S_b} *under the probability measure* $\mathbb{N}_0(\cdot \mid S_b < \infty)$ *is the law of the process* $(R_t^{(2)} - b)_{0 \leq t \leq T^{(2)}}$ *under* P_b.

This lemma can be obtained as a very special case of Theorem 4.6.2 in [6]. Alternatively, the lemma is also a special case of Proposition 1.4 in [5], which relied on explicit calculations of capacitary distributions for the Brownian snake found in [8]. Let us briefly explain how the result follows from [6]. For every $x > -b$, set

$$u_b(x) := \mathbb{N}_x(S_b < \infty) = \frac{3}{2(x+b)^2}$$

where the second equality is just (1). Following the comments at the end of Sect. 4.6 in [6], we get that the law of W_{S_b} under the probability measure $\mathbb{N}_0(\cdot \mid S_b < \infty)$ is the distribution of the process X solving the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{u_b'}{u_b}(X_t)\,\mathrm{d}t\ , \quad X_0 = 0,$$

and stopped at its first hitting time of $-b$. Since $\frac{u_b'}{u_b}(x) = -\frac{2}{x+b}$ we obtain the desired result.

3 The Main Results

Our first theorem identifies the law of the minimizing path $W_{s_\mathbf{m}}$.

Theorem 1 *Let* $a > 0$. *Under* $\mathbb{N}_0$, *the conditional distribution of* $W_{s_\mathbf{m}}$ *knowing that* $W_* = -a$ *is the distribution of the process* $(R_t^{(3)} - a)_{0 \leq t \leq T^{(3)}}$, *where* $R^{(3)}$ *is a Bessel process of dimension* -5 *started from* a, *and* $T^{(3)} = \inf\{t \geq 0 : R_t^{(3)} = 0\}$.

In an integral form, the statement of the theorem means that, for any nonnegative measurable function F on $\mathscr{W}_0$,

$$\mathbb{N}_0\big(F(W_{s_\mathbf{m}})\big) = 3\int_0^\infty \frac{\mathrm{d}a}{a^3}\, E_a\Big[F\Big((R_t^{(3)} - a)_{0 \leq t \leq T^{(3)}}\Big)\Big]$$

where we recall that the process $R^{(3)}$ starts from a under P_a.

Proof We fix three positive real numbers δ, K, K' such that $\delta < K < K'$, and we let G be a bounded nonnegative continuous function on $\mathscr{W}_0$. For every $\mathrm{w} \in \mathscr{W}_0$, we then set

$$\tau_\delta(\mathrm{w}) := \inf\{t \geq 0 : \mathrm{w}(t) = -\delta\}$$

and $F(\mathrm{w}) := G((\mathrm{w}(t))_{0\leq t\leq\tau_\delta(\mathrm{w})})$ if $\tau_\delta(\mathrm{w}) < \infty$, $F(\mathrm{w}) := 0$ otherwise.

For every real x and every integer $n \geq 1$, write $[x]_n$ for the largest real number of the form $k2^{-n}$, $k \in \mathbb{Z}$, smaller than or equal to x. Using the special form of F and the fact that $S_{[-W_*]_n} \uparrow s_{\mathbf{m}}$ as $n \uparrow \infty$, $\mathbb{N}_0$ a.e., we easily get from the properties of the Brownian snake that $F(W_{S_{[-W_*]_n}}) = F(W_{s_{\mathbf{m}}})$, for all n large enough, $\mathbb{N}_0$ a.e. on the event $\{W_* < -\delta\}$. By dominated convergence, we have then

$$\begin{aligned}
&\mathbb{N}_0(F(W_{s_{\mathbf{m}}})\mathbf{1}\{-K' \leq W_* \leq -K\}) \\
&\quad = \lim_{n\to\infty} \mathbb{N}_0(F(W_{S_{[-W_*]_n}})\mathbf{1}\{K \leq [-W_*]_n \leq K'\}) \\
&\quad = \lim_{n\to\infty} \sum_{K2^n\leq k\leq K'2^n} \mathbb{N}_0\Big(F(W_{S_{k2^{-n}}})\,\mathbf{1}\{S_{k2^{-n}} < \infty\}\,\mathbf{1}\Big\{\min_{S_{k2^{-n}}\leq s\leq\sigma} \hat{W}_s > -(k+1)2^{-n}\Big\}\Big).
\end{aligned} \tag{3}$$

Let $b > \delta$ and $\varepsilon > 0$. We use the strong Markov property of the Brownian snake at time S_b, together with Lemma 3, to get

$$\begin{aligned}
&\mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b < \infty\}\,\mathbf{1}\Big\{\min_{S_b\leq s\leq\sigma} \hat{W}_s > -b-\varepsilon\Big\}\Big) \\
&\quad = \mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b < \infty\}\,\exp\Big(-2\int_0^{\zeta_{S_b}} \mathrm{d}t\,\mathbb{N}_{W_{S_b}(t)}(W_* > -b-\varepsilon)\Big)\Big) \\
&\quad = \mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b < \infty\}\,\exp\Big(-3\int_0^{\zeta_{S_b}} \mathrm{d}t\,(b+\varepsilon+W_{S_b}(t))^{-2}\Big)\Big) \\
&\quad = \frac{3}{2b^2}\,E_b\Big[F((R^{(2)}_t - b)_{0\leq t\leq T^{(2)}})\,\exp\Big(-3\int_0^{T^{(2)}} \mathrm{d}t\,(\varepsilon + R^{(2)}_t)^{-2}\Big)\Big]
\end{aligned} \tag{4}$$

using (1) in the second equality, and Lemma 4 and (1) again in the third one. Recall the definition of the stopping times $T^{(\alpha)}_\delta$ before Lemma 1. From the special form of the function F, and then the strong Markov property of the process $R^{(2)}$ at time $T^{(2)}_{b-\delta}$, we obtain that

$$\begin{aligned}
&E_b\Big[F((R^{(2)}_t - b)_{0\leq t\leq T^{(2)}})\,\exp\Big(-3\int_0^{T^{(2)}} \mathrm{d}t\,(\varepsilon + R^{(2)}_t)^{-2}\Big)\Big] \\
&\quad = E_b\Big[G((R^{(2)}_t - b)_{0\leq t\leq T^{(2)}_{b-\delta}})\,\exp\Big(-3\int_0^{T^{(2)}} \mathrm{d}t\,(\varepsilon + R^{(2)}_t)^{-2}\Big)\Big]
\end{aligned}$$

$$= E_b\Big[G((R^{(2)}_t - b)_{0\le t\le T^{(2)}_{b-\delta}})\ \exp\Big(-3\int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t\,(\varepsilon + R^{(2)}_t)^{-2}\Big)$$

$$\times E_{b-\delta}\Big[\exp\Big(-3\int_0^{T^{(2)}} \mathrm{d}t\,(\varepsilon + R^{(2)}_t)^{-2}\Big)\Big]\Big]. \tag{5}$$

Using the formula of Lemma 2 and combining (4) and (5), we arrive at

$$\mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b<\infty\}\,\mathbf{1}\Big\{\min_{S_b\le s\le\sigma}\hat{W}_s > -b-\varepsilon\Big\}\Big)$$

$$= \frac{3}{2b^2}\Big(1-\Big(\frac{b-\delta}{b-\delta+\varepsilon}\Big)^2\Big)$$

$$\times E_b\Big[G((R^{(2)}_t - b)_{0\le t\le T^{(2)}_{b-\delta}})\ \exp\Big(-3\int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t\,(\varepsilon + R^{(2)}_t)^{-2}\Big)\Big].$$

Hence,

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b<\infty\}\,\mathbf{1}\Big\{\min_{S_b\le s\le\sigma}\hat{W}_s > -b-\varepsilon\Big\}\Big)$$

$$= \Big(\frac{3}{b^2(b-\delta)}\Big)\,E_b\Big[G((R^{(2)}_t - b)_{0\le t\le T^{(2)}_{b-\delta}})\ \exp\Big(-3\int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t\,(R^{(2)}_t)^{-2}\Big)\Big].$$

At this stage we use Lemma 1 twice to see that

$$E_b\Big[G((R^{(2)}_t - b)_{0\le t\le T^{(2)}_{b-\delta}})\ \exp\Big(-3\int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t\,(R^{(2)}_t)^{-2}\Big)\Big]$$

$$= \Big(\frac{b}{b-\delta}\Big)^2 E_b\Big[G((B_t - b)_{0\le t\le T_{b-\delta}})\,\exp\Big(-6\int_0^{T_{b-\delta}}\frac{\mathrm{d}s}{B_s^2}\Big)\Big]$$

$$= \Big(\frac{b}{b-\delta}\Big)^{-1} E_b\Big[G((R^{(3)}_t - b)_{0\le t\le T^{(3)}_{b-\delta}})\Big]$$

Summarizing, we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b<\infty\}\,\mathbf{1}\Big\{\min_{S_b\le s\le\sigma}\hat{W}_s > -b-\varepsilon\Big\}\Big)$$

$$= \frac{3}{b^3}\,E_b\Big[G((R^{(3)}_t - b)_{0\le t\le T^{(3)}_{b-\delta}})\Big].$$

Note that the right-hand side of the last display is a continuous function of $b \in (\delta,\infty)$. Furthermore, a close look at the preceding arguments shows that the convergence is uniform when b varies over an interval of the form $[\delta',\infty)$, where

$\delta' > \delta$. We can therefore return to (3) and obtain that

$$\begin{aligned}
&\mathbb{N}_0(F(W_{s_{\mathbf{m}}})\mathbf{1}\{-K' \le W_* \le -K\}) \\
&= \lim_{n\to\infty} \int_K^{K'} \mathrm{d}b\, 2^n \mathbb{N}_0\Big(F(W_{S_{[b]_n}})\,\mathbf{1}\{S_{[b]_n} < \infty\}\,\mathbf{1}\Big\{\min_{S_{[b]_n}\le s\le\sigma} \hat{W}_s > -[b]_n - 2^{-n}\Big\}\Big) \\
&= 3\int_K^{K'} \frac{\mathrm{d}b}{b^3}\, E_b\Big[G((R^{(3)}_t - b)_{0\le t\le T^{(3)}_{b-\delta}})\Big].
\end{aligned}$$

The result of the theorem now follows easily. □

We turn to a statement describing the structure of subtrees branching off the minimizing path $W_{s_{\mathbf{m}}}$. In a sense, this is similar to Lemma 3 above (except that we will need to consider separately subtrees branching *before* and *after* time $s_{\mathbf{m}}$, in the time scale of the Brownian snake). Since $s_{\mathbf{m}}$ is not a stopping time of the Brownian snake, it is of course impossible to use the strong Markov property in order to apply Lemma 3. Still this lemma will play an important role.

We argue under the excursion measure $\mathbb{N}_0$ and, for every $s \ge 0$, we set

$$\hat{\zeta}_s := \zeta_{(s_{\mathbf{m}}+s)\wedge\sigma}\ , \quad \check{\zeta}_s := \zeta_{(s_{\mathbf{m}}-s)\vee 0}.$$

We let $(\hat{a}_i, \hat{b}_i)$, $i \in I$ be the excursion intervals of $\hat{\zeta}_s$ above its past minimum. Equivalently, the intervals $(\hat{a}_i, \hat{b}_i)$, $i \in I$ are the connected components of the set

$$\Big\{s \ge 0 : \hat{\zeta}_s > \min_{0\le r\le s} \hat{\zeta}_r\Big\}.$$

Similarly, we let $(\check{a}_j, \check{b}_j)$, $j \in J$ be the excursion intervals of $\check{\zeta}_s$ above its past minimum. We may assume that the indexing sets I and J are disjoint. In terms of the tree $\mathscr{T}_\zeta$ coded by the excursion $(\zeta_s)_{0\le s\le\sigma}$ under $\mathbb{N}_0$ (see e.g. [10, Sect. 2]), each interval $(\hat{a}_i, \hat{b}_i)$ or $(\check{a}_j, \check{b}_j)$ corresponds to a subtree of $\mathscr{T}_\zeta$ branching off the ancestral line of the vertex associated with $s_{\mathbf{m}}$. We next consider the spatial displacements corresponding to these subtrees. For every $i \in I$, we let $W^{(i)} = (W^{(i)}_s)_{s\ge 0} \in C(\mathbb{R}_+, \mathscr{W})$ be defined by

$$W^{(i)}_s(t) = W_{s_{\mathbf{m}}+(\hat{a}_i+s)\wedge\hat{b}_i}(\zeta_{s_{\mathbf{m}}+\hat{a}_i} + t)\ , \quad 0 \le t \le \zeta_{s_{\mathbf{m}}+(\hat{a}_i+s)\wedge\hat{b}_i} - \zeta_{s_{\mathbf{m}}+\hat{a}_i}.$$

Similarly, for every $j \in J$,

$$W^{(j)}_s(t) = W_{s_{\mathbf{m}}-(\check{a}_j+s)\wedge\check{b}_j}(\zeta_{s_{\mathbf{m}}-\check{a}_j} + t)\ , \quad 0 \le t \le \zeta_{s_{\mathbf{m}}-(\check{a}_j+s)\wedge\check{b}_j} - \zeta_{s_{\mathbf{m}}-\check{a}_j}.$$

We finally introduce the point measures on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathscr{W})$ defined by

$$\hat{\mathscr{N}} = \sum_{i\in I} \delta_{(\zeta_{s_{\mathbf{m}}+\hat{a}_i}, W^{(i)})}\ , \quad \check{\mathscr{N}} = \sum_{j\in J} \delta_{(\zeta_{s_{\mathbf{m}}-\check{a}_j}, W^{(j)})}.$$

If $\omega = (\omega_s)_{s\geq 0}$ belongs to $C(\mathbb{R}_+, \mathscr{W})$, we set $\omega_* := \inf\{\omega_s(t) : s \geq 0, 0 \leq t \leq \zeta_{(\omega_s)}\}$.

Theorem 2 *Under $\mathbb{N}_0$, conditionally on the minimizing path $W_{s_\mathbf{m}}$, the point measures $\hat{\mathscr{N}}(\mathrm{d}t, \mathrm{d}\omega)$ and $\check{\mathscr{N}}(\mathrm{d}t, \mathrm{d}\omega)$ are independent and their common conditional distribution is that of a Poisson point measure with intensity*

$$2\,\mathbf{1}_{[0,\zeta_{s_\mathbf{m}}]}(t)\,\mathbf{1}_{\{\omega_* > \hat{W}_{s_\mathbf{m}}\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_\mathbf{m}}(t)}(\mathrm{d}\omega).$$

Clearly, the constraint $\omega_* > \hat{W}_{s_\mathbf{m}}$ corresponds to the fact that none of the spatial positions in the subtrees branching off the ancestral line of $p_\zeta(s_\mathbf{m})$ can be smaller than $W_* = \hat{W}_{s_\mathbf{m}}$, by the very definition of W_*.

Proof We will first argue that the conditional distribution of $\hat{\mathscr{N}}$ given $W_{s_\mathbf{m}}$ is as described in the theorem. To this end, we fix again δ, K, K' such that $0 < \delta < K < K'$, and we use the notation $\tau_\delta(\mathrm{w})$ introduced in the proof of Theorem 1. On the event where $W_* < -\delta$, we also set

$$\hat{\mathscr{N}_\delta} = \sum_{\substack{i\in I \\ \zeta_{s_\mathbf{m}}+\hat{a}_i \leq \tau_\delta(W_{s_\mathbf{m}})}} \delta_{(\zeta_{s_\mathbf{m}}+\hat{a}_i, W^{(i)})}.$$

Informally, considering only the subtrees that occur after $s_\mathbf{m}$ in the time scale of the Brownian snake, $\hat{\mathscr{N}_\delta}$ corresponds to those subtrees that branch off the minimizing path $W_{s_\mathbf{m}}$ before this path hits the level $-\delta$.

Next, let Φ be a bounded nonnegative measurable function on the space of all point measures on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathscr{W})$ – we should restrict this space to point measures satisfying appropriate σ-finiteness conditions, but we omit the details – and let Ψ be a bounded continuous function on $C(\mathbb{R}_+, \mathscr{W})$. To simplify notation, we write $W_{\leq s_\mathbf{m}}$ for the process $(W_{s\wedge s_\mathbf{m}})_{s\geq 0}$ viewed as a random element of $C(\mathbb{R}_+, \mathscr{W})$, and we use the similar notation $W_{\leq S_b}$. For every $b > 0$, let the point measure $\hat{\mathscr{N}}_\delta^{(b)}$ be defined (only on the event where $S_b < \infty$) in a way analogous to $\hat{\mathscr{N}_\delta}$ but replacing the path $W_{s_\mathbf{m}}$ with the path W_{S_b}: To be specific, $\hat{\mathscr{N}}_\delta^{(b)}$ accounts for those subtrees (occurring after S_b in the time scale of the Brownian snake) that branch off W_{S_b} before this path hits $-\delta$.

As in (3), we have then

$$\begin{aligned}
&\mathbb{N}_0\Big(\Psi(W_{\leq s_\mathbf{m}})\mathbf{1}\{-K' \leq W_* \leq -K\}\,\Phi(\hat{\mathscr{N}_\delta})\Big) \\
&= \lim_{n\to\infty} \sum_{K2^n \leq k \leq K'2^n} \mathbb{N}_0\Big(\Psi(W_{\leq S_{k2^{-n}}})\,\mathbf{1}\{S_{k2^{-n}} < \infty\} \\
&\qquad\qquad \mathbf{1}\Big\{\min_{S_{k2^{-n}} \leq s \leq \sigma} \hat{W}_s > -(k+1)2^{-n}\Big\}\,\Phi(\hat{\mathscr{N}}_\delta^{(k2^{-n})})\Big).
\end{aligned} \tag{6}$$

The point in (6) is the fact that, $\mathbb{N}_0$ a.e., if n is sufficiently large, and if $k \geq K2^{-n}$ is the largest integer such that $S_{k2^{-n}} < \infty$, the paths $W_{s_{\mathbf{m}}}$ and $W_{S_{k2^{-n}}}$ are the same up to a time which is greater than $\tau_\delta(W_{s_{\mathbf{m}}})$, and the point measures $\hat{\mathscr{N}}_\delta$ and $\hat{\mathscr{N}}_\delta^{(k2^{-n})}$ coincide.

Next fix $b > \delta$ and, for $\varepsilon > 0$, consider the quantity

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\,\mathbf{1}\{S_b < \infty\}\,\mathbf{1}\Big\{\min_{S_b \leq s \leq \sigma} \hat{W}_s > -b-\varepsilon\Big\}\,\Phi(\hat{\mathscr{N}}_\delta^{(b)})\Big). \tag{7}$$

To evaluate this quantity, we again apply the strong Markov property of the Brownian snake at time S_b. For notational convenience, we suppose that, on a certain probability space, we have a random point measure $\mathscr{M}$ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathscr{W})$ and, for every $\mathrm{w} \in \mathscr{W}_0$, a probability measure Π_{w} under which $\mathscr{M}(\mathrm{d}t, \mathrm{d}\omega)$ is Poisson with intensity

$$2\,\mathbf{1}_{[0,\zeta_{\mathrm{w}}]}(t)\,\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega).$$

By the strong Markov property at S_b and Lemma 3, the quantity (7) is equal to

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\mathbf{1}\{S_b < \infty\}\Pi_{W_{S_b}}\Big(\mathbf{1}\{\mathscr{M}(\{(t,\omega)\colon \omega_* \leq -b-\varepsilon\}) = 0\}\Phi(\mathscr{M}_{\leq \tau_\delta(W_{S_b})})\Big)\Big),$$

where $\mathscr{M}_{\leq \tau_\delta(W_{S_b})}$ denotes the restriction of the point measure $\mathscr{M}$ to $[0, \tau_\delta(W_{S_b})] \times C(\mathbb{R}_+, \mathscr{W})$. Write $W_{S_b}^{(\delta)}$ for the restriction of the path W_{S_b} to $[0, \tau_\delta(W_{S_b})]$. We have then

$$\begin{aligned}
&\Pi_{W_{S_b}}\Big(\mathbf{1}\{\mathscr{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0\}\,\Phi(\mathscr{M}_{\leq \tau_\delta(W_{S_b})})\Big)\\
&\quad = \Pi_{W_{S_b}}(\mathscr{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0)\\
&\qquad \times \Pi_{W_{S_b}}\Big(\Phi(\mathscr{M}_{\leq \tau_\delta(W_{S_b})})\,\Big|\,\mathscr{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0\Big)\\
&\quad = \Pi_{W_{S_b}}(\mathscr{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0)\\
&\qquad \times \Pi_{W_{S_b}^{(\delta)}}\Big(\Phi(\mathscr{M})\,\Big|\,\mathscr{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0\Big),
\end{aligned}$$

using standard properties of Poisson measures in the last equality. Summarizing, we see that the quantity (7) coincides with

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\,H(W_{S_b}, b+\varepsilon)\,\mathbf{1}\{S_b < \infty\}\,\Pi_{W_{S_b}}(\mathscr{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0)\Big), \tag{8}$$

where, for every $\mathrm{w} \in \mathscr{W}_0$ such that $\tau_\delta(\mathrm{w}) < \infty$, for every $a > \delta$, $H(\mathrm{w}, a) := \tilde{H}((\mathrm{w}(t))_{0 \leq t \leq \tau_\delta(\mathrm{w})}, a)$, and the function $\tilde{H}$ is given by

$$\tilde{H}(\mathrm{w}, a) := \Pi_{\mathrm{w}}\Big(\Phi(\mathscr{M})\,\Big|\,\mathscr{M}(\{(t,\omega) : \omega_* \leq -a\}) = 0\Big),$$

this definition making sense if $\mathrm{w} \in \mathscr{W}_0$ does not hit $-a$. By the strong Markov property at S_b and again Lemma 3, the quantity (8) is also equal to

$$\mathbb{N}_0\Big(\Psi(W_{\le S_b})\,H(W_{S_b}, b+\varepsilon)\,\mathbf{1}\{S_b<\infty\}\,\mathbf{1}\Big\{\min_{S_b\le s\le\sigma}\hat{W}_s > -b-\varepsilon\Big\}\Big).$$

We may now come back to (6), and get from the previous observations that

$$\begin{aligned}
&\mathbb{N}_0\Big(\Psi(W_{\le s_{\mathbf{m}}})\mathbf{1}\{-K'\le W_*\le -K\}\,\Phi(\hat{\mathscr{N}_\delta})\Big)\\
&= \lim_{n\to\infty}\sum_{K2^n\le k\le K'2^n}\mathbb{N}_0\Big(\Psi(W_{\le S_{k2^{-n}}})\,H(W_{S_{k2^{-n}}},(k+1)2^{-n})\\
&\qquad\qquad\mathbf{1}\{S_{k2^{-n}}<\infty\}\,\mathbf{1}\Big\{\min_{S_{k2^{-n}}\le s\le\sigma}\hat{W}_s > -(k+1)2^{-n}\Big\}\Big)\\
&= \lim_{n\to\infty}\mathbb{N}_0\Big(\Psi(W_{\le S_{[-W_*]_n}})\,H(W_{S_{[-W_*]_n}},[-W_*]_n-2^{-n})\,\mathbf{1}\{K\le[-W_*]_n\le K'\}\Big)\\
&= \mathbb{N}_0\Big(\Psi(W_{\le s_{\mathbf{m}}})\,H(W_{s_{\mathbf{m}}},-W_*)\,\mathbf{1}\{-K'\le W_*\le -K\}\Big).
\end{aligned}$$

To verify the last equality, recall that the paths $W_{S_{[-W_*]_n}}$ and $W_{s_{\mathbf{m}}}$ coincide up to their first hitting time of $-\delta$, for all n large enough, $\mathbb{N}_0$ a.e., and use also the fact that the function $H(\mathrm{w},a)$ is Lipschitz in the variable a on every compact subset of (δ,∞), uniformly in the variable w.

From the definition of H, we have then

$$\begin{aligned}
&\mathbb{N}_0\Big(\Psi(W_{\le s_{\mathbf{m}}})\mathbf{1}\{-K'\le W_*\le -K\}\,\Phi(\hat{\mathscr{N}_\delta})\Big)\\
&\quad= \mathbb{N}_0\Big(\Psi(W_{\le s_{\mathbf{m}}})\mathbf{1}\{-K'\le W_*\le -K\}\,\Pi_{W^{(\delta)}_{s_{\mathbf{m}}}}\Big(\Phi(\mathscr{M})\Big|\mathscr{M}(\{(t,\omega):\omega_*\le W_*\})=0\Big)\Big),
\end{aligned}$$

where $W^{(\delta)}_{s_{\mathbf{m}}}$ denotes the restriction of $W_{s_{\mathbf{m}}}$ to $[0,\tau_\delta(W_{s_{\mathbf{m}}})]$. From this, and since $W_* = \hat{W}_{s_{\mathbf{m}}}$, we obtain that the conditional distribution of $\hat{\mathscr{N}_\delta}$ given $W_{\le s_{\mathbf{m}}}$ is (on the event where $W_* < -\delta$) the law of a Poisson point measure with intensity

$$2\,\mathbf{1}_{[0,\tau_\delta(W_{s_{\mathbf{m}}})]}(t)\,\mathbf{1}_{\{\omega_*>\hat{W}_{s_{\mathbf{m}}}\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_{\mathbf{m}}}(t)}(\mathrm{d}\omega).$$

Since δ is arbitrary, it easily follows that the conditional distribution of $\hat{\mathscr{N}}$ given $W_{\le s_{\mathbf{m}}}$ is that of a Poisson measure with intensity

$$2\,\mathbf{1}_{[0,\zeta_{W_{s_{\mathbf{m}}}}]}(t)\,\mathbf{1}_{\{\omega_*>\hat{W}_{s_{\mathbf{m}}}\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_{\mathbf{m}}}(t)}(\mathrm{d}\omega).$$

Note that this conditional distribution only depends on $W_{s_{\mathbf{m}}}$, meaning that $\hat{\mathscr{N}}$ is conditionally independent of $W_{\le s_{\mathbf{m}}}$ given $W_{s_{\mathbf{m}}}$.

Since the measure $\mathbb{N}_0$ is invariant under time-reversal, we also get that the conditional distribution of $\check{\mathscr{N}}$ given $W_{s_\mathbf{m}}$ is the same as the conditional distribution of $\hat{\mathscr{N}}$ given $W_{s_\mathbf{m}}$. Finally, $\check{\mathscr{N}}$ is a measurable function of $W_{\leq s_\mathbf{m}}$ and since $\hat{\mathscr{N}}$ is conditionally independent of $W_{\leq s_\mathbf{m}}$ given $W_{s_\mathbf{m}}$, we get that $\check{\mathscr{N}}$ and $\hat{\mathscr{N}}$ are conditionally independent given $W_{s_\mathbf{m}}$. □

4 Applications to Super-Brownian Motion

We will now discuss applications of the preceding results to super-Brownian motion. Let μ be a (nonzero) finite measure on $\mathbb{R}$. We denote the topological support of μ by $\mathrm{supp}(\mu)$ and always assume that

$$m := \inf \mathrm{supp}(\mu) > -\infty.$$

We then consider a super-Brownian motion $X = (X_t)_{t\geq 0}$ with quadratic branching mechanism $\psi(u) = 2u^2$ started from μ. The particular choice of the normalization of ψ is motivated by the connection with the Brownian snake. Let us recall this connection following Sect. IV.4 of [9]. We consider a Poisson point measure $\mathscr{P}(\mathrm{d}x, \mathrm{d}\omega)$ on $\mathbb{R} \times C(\mathbb{R}_+, \mathscr{W})$ with intensity

$$\mu(\mathrm{d}x)\, \mathbb{N}_x(\mathrm{d}\omega).$$

Write

$$\mathscr{P}(\mathrm{d}x, \mathrm{d}\omega) = \sum_{i\in I} \delta_{(x^i,\omega^i)}(\mathrm{d}x, \mathrm{d}\omega)$$

and for every $i \in I$, let $\zeta^i_s = \zeta_{(\omega^i_s)}$, $s \geq 0$, stand for the lifetime process associated with ω^i. Also, for every $r \geq 0$ and $s \geq 0$, let $\ell^r_s(\zeta^i)$ be the local time at level r and at time s of the process ζ^i. We may and will construct the super-Brownian motion X by setting $X_0 = \mu$ and for every $r > 0$, for every nonnegative measurable function φ on $\mathbb{R}$,

$$\langle X_r, \varphi\rangle = \sum_{i\in I} \int_0^\infty \mathrm{d}_s\ell^r_s(\zeta^i)\, \varphi(\hat{\omega}^i_s), \tag{9}$$

where the notation $\mathrm{d}_s\ell^r_s(\zeta^i)$ refers to integration with respect to the increasing function $s \to \ell^r_s(\zeta^i)$.

A major advantage of the Brownian snake construction is the fact that it also yields an immediate definition of the historical super-Brownian motion $Y = (Y_r)_{r\geq 0}$ associated with X (we refer to [4] or [7] for the general theory of historical superprocesses). For every $r \geq 0$, Y_r is a finite measure on the subset of $\mathscr{W}$

consisting of all stopped paths with lifetime r. We have $Y_0 = \mu$ and for every $r > 0$,

$$\langle Y_r, \Phi \rangle = \sum_{i \in I} \int_0^\infty \mathrm{d}_s \ell_s^r(\zeta^i)\, \Phi(\omega_s^i), \tag{10}$$

for every nonnegative measurable function Φ on $\mathscr{W}$. Note the relation $\langle X_r, \varphi \rangle = \int Y_r(\mathrm{dw})\, \varphi(\hat{\mathrm{w}})$.

The range $\mathscr{R}^X$ is the closure in $\mathbb{R}$ of the set

$$\bigcup_{r \geq 0} \mathrm{supp}(X_r),$$

and, similarly, we define $\mathscr{R}^Y$ as the closure in $\mathscr{W}$ of

$$\bigcup_{r \geq 0} \mathrm{supp}(Y_r).$$

We note that

$$\mathscr{R}^X = \mathrm{supp}(\mu) \cup \left(\bigcup_{i \in I} \{\hat{\omega}_s^i : s \geq 0\} \right)$$

and

$$\mathscr{R}^Y = \mathrm{supp}(\mu) \cup \left(\bigcup_{i \in I} \{\omega_s^i : s \geq 0\} \right).$$

We set

$$m_X := \inf \mathscr{R}^X.$$

From the preceding formulas and the uniqueness of the minimizing path in the case of the Brownian snake, it immediately follows that there is a unique stopped path $\mathrm{w}_{\min} \in \mathscr{R}^Y$ such that $\hat{\mathrm{w}}_{\min} = m_X$. Our goal is to describe the distribution of $\mathrm{w}_{\min}$. We first observe that the distribution of m_X is easy to obtain from (1) and the Brownian snake representation: We have obviously $m_X \leq m$ and, for every $x < m$,

$$P(m_X \geq x) = \exp\Big(-\frac{3}{2} \int \frac{\mu(\mathrm{d}u)}{(u-x)^2} \Big). \tag{11}$$

Note that this formula is originally due to [3, Theorem 1.3]. It follows that

$$P(m_X = m) = \exp\Big(-\frac{3}{2} \int \frac{\mu(\mathrm{d}u)}{(u-m)^2} \Big).$$

Therefore, if $\int (u-m)^{-2} \mu(\mathrm{d}u) < \infty$, the event $\{m_X = m\}$ occurs with positive probability. If this event occurs, $\mathrm{w}_{\min}$ is just the trivial path m with zero lifetime.

Proposition 1 *The joint distribution of the pair* $(\mathrm{w}_{\min}(0), m_X)$ *is given by the formulas*

$$P(\mathrm{w}_{\min}(0) \le a,\, m_X \le x) = 3\int_{-\infty}^{x} \mathrm{d}y \Big(\int_{[m,a]} \frac{\mu(\mathrm{d}u)}{(u-y)^3}\Big) \exp\Big(-\frac{3}{2}\int \frac{\mu(\mathrm{d}u)}{(u-y)^2}\Big),$$

for every $a \in [m,\infty)$ *and* $x \in (-\infty, m)$*, and*

$$P(m_X = m) = P(m_X = m, \mathrm{w}_{\min}(0) = m) = \exp\Big(-\frac{3}{2}\int \frac{\mu(\mathrm{d}u)}{(u-m)^2}\Big).$$

Proof Fix $a \in [m,\infty)$, and let μ', respectively μ'' denote the restriction of μ to $[m,a]$, resp. to (a,∞). Define X', respectively X'', by setting $X'_0 = \mu'$, resp. $X''_0 = \mu''$, and restricting the sum in the right-hand side of (9) to indices $i \in I$ such that $x^i \in [m,a]$, resp. $x^i \in (a,\infty)$. Define Y' and Y'' similarly using (10) instead of (9). Then X', respectively X'' is a super-Brownian motion started from μ', resp. from μ'', and Y', resp. Y'' is the associated historical super-Brownian motion. Furthermore, (X',Y') and (X'',Y'') are independent.

By (11), the law of $m_{X'}$ has a density on $(-\infty,m)$ given by

$$f_{m_{X'}}(y) = 3\Big(\int_{[m,a]} \frac{\mu(\mathrm{d}u)}{(u-y)^3}\Big) \exp\Big(-\frac{3}{2}\int_{[m,a]} \frac{\mu(\mathrm{d}u)}{(u-y)^2}\Big), \quad y \in (-\infty,m).$$

On the other hand, if $x \in (-\infty,m)$,

$$\begin{aligned} P(\mathrm{w}_{\min}(0) \le a,\, m_X \le x) &= P(m_{X'} \le x,\, m_{X''} > m_{X'}) \\ &= \int_{-\infty}^{x} \mathrm{d}y\, f_{m_{X'}}(y)\, P(m_{X''} > y), \end{aligned}$$

and we get the first formula of the proposition using (11) again. The second formula is obvious from the remarks preceding the proposition. □

Together with Proposition 1, the next corollary completely characterizes the law of $\mathrm{w}_{\min}$. Recall that the case where $m_X = m$ is trivial, so that we do not consider this case in the following statement.

Corollary 1 *Let* $x \in (-\infty,m)$ *and* $a \in [m,\infty)$. *Then conditionally on* $m_X = x$ *and* $\mathrm{w}_{\min}(0) = a$*, the path* $\mathrm{w}_{\min}$ *is distributed as the process* $(x + R^{(3)}_t)_{0\le t\le T^{(3)}}$ *under* P_{a-x}.

Proof On the event $\{m_X < m\}$, there is a unique index $i_{\min} \in I$ such that

$$m_X = \min\{\hat\omega^{i_{\min}}_s : s \ge 0\}.$$

Furthermore, if $s_{\min}$ is the unique instant such that $m_X = \hat\omega^{i_{\min}}_{s_{\min}}$, we have $\mathrm{w}_{\min} = \omega^{i_{\min}}_{s_{\min}}$, and in particular $x_{i_{\min}} = \mathrm{w}_{\min}(0)$.

Standard properties of Poisson measures now imply that, conditionally on $m_X = x$ and $\mathrm{w}_{\min}(0) = a$, $\omega^{i_{\min}}$ is distributed according to $\mathbb{N}_a(\cdot \mid W_* = x)$. The assertions of the corollary then follow from Theorem 1. □

We could also have obtained an analog of Theorem 2 in the superprocess setting. The conditional distribution of the process X (or of Y) given the minimizing path $\mathrm{w}_{\min}$ is obtained by the sum of two contributions. The first one (present only if $\hat{\mathrm{w}}_{\min} < m$) corresponds to the minimizing "excursion" $\omega^{i_{\min}}$ introduced in the previous proof, whose conditional distribution given $\mathrm{w}_{\min}$ is described by Theorem 2. The second one is just an independent super-Brownian motion $\tilde{X}$ started from μ and conditioned on the event $m_{\tilde{X}} \geq \hat{\mathrm{w}}_{\min}$. We leave the details of the statement to the reader.

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On Inversions and Doob h-Transforms of Linear Diffusions

Larbi Alili, Piotr Graczyk, and Tomasz Żak

Abstract Let X be a regular linear diffusion whose state space is an open interval $E \subseteq \mathbb{R}$. We consider the dual diffusion X^* whose probability law is obtained as a Doob h-transform of the law of X, where h is a positive harmonic function for the infinitesimal generator of X on E. We provide a construction of X^* as a deterministic inversion $I(X)$ of X, time changed with some random clock. Such inversions generalize the Euclidean inversions that intervene when X is a Brownian motion. The important case where X^* is X conditioned to stay above some fixed level is included. The families of deterministic inversions are given explicitly for the Brownian motion with drift, Bessel processes and the three-dimensional hyperbolic Bessel process.

1 Motivations and Main Results

One of the main incentives for carrying out this work was the paper [27] where Marc Yor studied involutions which are different from the classical inversion with respect to the unit sphere on R^d , $d \geq 1$. We had with him interesting discussions and received from him many encouragements to finalize this work, particularly during the conference *Analyse harmonique et Probabilités, Angers 2012*. We would like to thank him for that and for the dear time and invaluable advices he gave to the first and second authors.

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Since then he passed away. May his soul rest in peace!

The main objective of this paper is to study analytical aspects of the stochastic Doob duality. We elucidate a striking equivalence between stochastic Doob duality of one-dimensional diffusions X and X^* and a simple analytical transformation $I(X)$ of trajectories of a diffusion via a deterministic inversion I.

The construction was known for the case where X is a three-dimensional Bessel process started at a positive X_0, the dual process X^* is a Brownian motion killed when it hits 0 and the inversed process is $1/X$, which is a Brownian motion conditioned via a Doob h-transform to stay positive. The processes X^* and $1/X$ coincide up to a time change, see [18].

It was also known [18] that the three-dimensional hyperbolic Bessel process can be realized via a Doob transform as a Brownian motion with negative unit drift conditioned to stay positive. This work was inspired by the search and discovery of an inversion

$$I(X_t) = \frac{1}{2} \ln \coth X_t$$

of the three-dimensional hyperbolic Bessel process (X_t). When $I(X_t)$ is appropriately time changed, we obtain a Brownian motion with negative unit drift, see Sect. 5.2.

The main result of this paper says that an analytical inversion I can be constructed for any pair of dual linear diffusions X and X^*, see Theorem 1.

A direct application of this result is a better understanding of the conditioned diffusions X^*: they are obtained, up to a time change, as an analytical transformation $I(X)$ of the original diffusion X. Both the families of inversions and the random clocks involved in the construction have interesting features and deserve their own right of mathematical interest.

Our original motivations for the search of deterministic inversions of stochastic processes come from potential theory, where a crucial role is played by the Kelvin transformation, related to the inversion with respect to the unit sphere $I(x) = x/\|x\|^2$, see e.g. [3, 4]. One of the other reasons why we worked on this topic is a strong need of such analytical tools to develop the potential theory of various important processes, e.g. hyperbolic Brownian motions and hyperbolic Bessel processes.

Taking into account the results of [3] for stable processes, it is natural to ask whether such analytical constructions of conditioned processes should also be available for one-dimensional self-similar processes. In a work in progress, this question and some related topics are studied in collaboration with L. Chaumont.

The paper is organized as follows. In Sect. 2 we introduce basic notions and notations on diffusions X with a state space E and we explain precisely the objectives of the paper. We start Sect. 3 with the construction of a family of inversions associated with a diffusion X. The construction involves a reference scale function s and not the speed measure of X. We note that the inversion of the state space E in the direction of s is uniquely characterized by the fixed point x_0. Also, among the set of inversions in the direction of s, the s-inversion with fixed point x_0 is uniquely characterized by the associated positive harmonic function h. Thus, the family of inversions we obtain is a one parameter family of involutions indexed by the fixed point x_0. In Sect. 4, we state and prove our main result. That gives the path construction of X^* in terms of the inverse of X in the direction of s with respect to a point $x_0 \in E$. Section 5 is devoted to applications. We point out in Corollaries 3 and 4 some new results that we obtain for Bessel processes and the hyperbolic Bessel process of dimension 3.

2 Preliminaries on Dual Processes and Inversions

Let $X := (X_t, t \leq \zeta)$ be a regular diffusion with life time ζ and state space $E = (l, r) \subseteq \mathbb{R}$ which is defined on complete probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Unless otherwise specified, we assume that X is killed, i.e. sent to a cemetery point Δ, as soon as it hits one of the boundaries; that is $\zeta = \inf\{s,\ X_s = l \text{ or } X_s = r\}$ with the usual convention $\inf\{\emptyset\} = +\infty$.

Our objectives in this paper are summarized as follows. Given a positive function h which is harmonic for the infinitesimal generator L of X, i.e. $Lh = 0$, we give an explicit construction of the dual X^* of X with respect to $h(x)m(dx)$ where $m(dx)$ is the speed measure of X. The distribution of X^* is obtained by a Doob h-transform change of measure of the distribution of X. We shall see that X^* is either the process itself, i.e. the process is self-dual, or the original diffusion conditioned to have opposite behaviors at the boundaries when started from a specific point x_0 in the state space; this is explained in details in Proposition 4 below. We refer to the original paper [7] by Doob for h-transforms and to [5] where this topic is surveyed. The procedure consists in first constructing the inverse of the diffusion with respect to a point $x_0 \in E$ which is a deterministic involution of the original diffusion. Time changing then with an appropriate clock gives a realization of the dual process. In order to say more, let us fix the mathematical setting. Suppose that X satisfies the s.d.e.

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds, \quad t < \zeta, \tag{1}$$

where $X_0 \in E$, $(W_t, t \leq \zeta)$ is a standard Brownian motion and σ, $b : E \to \mathbb{R}$ are measurable real valued functions. Assume that σ and b satisfy the Engelbert-Schmidt conditions

$$\sigma \neq 0 \text{ and } 1/\sigma^2,\ b/\sigma^2 \in L^1_{loc}(E), \tag{2}$$

where $L^1_{loc}(E)$ is the space of locally integrable functions on E. Condition (2) implies that (1) has a unique solution in law, see Proposition 5.15 in [12]. We write $(\mathscr{F}^X_t, t \geq 0)$ for the natural filtration generated by X and denote by $\mathscr{D}\mathscr{F}(E)$ the set of diffusions satisfying the aforementioned conditions. For background on diffusion processes, we refer to [1, 5, 10, 12, 17–19].

Let $X \in \mathscr{DF}(E)$. For $y \in E$, let $H_y = \inf\{t > 0;\ X_t = y\}$ be the first hitting time of y by X. Recall that the scale function of X is any continuous strictly increasing function on E satisfying

$$\mathbb{P}_x(H_\alpha < H_\beta) = (s(x) - s(\beta))/(s(\alpha) - s(\beta)) \tag{3}$$

for all $l < \alpha < x < \beta < r$. This is a reference function which is strictly increasing and given modulo an affine transformation by $s(x) = \int_c^x \exp\left(-2\int_c^z b(r)/\sigma^2(r)dr\right)dz$ for some $c \in E$. For convenience, we distinguish, as in Proposition 5.22 in [12], the following four different subclasses of diffusions which exhibit different forms of inversions, i.e. mappings $I : E \to E$ such that $I \circ I(x) = x$, for all $x \in E$, and $I(E) = E$ (for a more precise definition of an s-inversion, see Definition 1). We say that $X \in \mathscr{DF}(E)$ is of

- Type 1 if $-\infty < s(l)$ and $s(r) < +\infty$;
- Type 2 if $-\infty < s(l)$ and $s(r) = +\infty$;
- Type 3 if $s(l) = -\infty$ and $s(r) < +\infty$;
- Type 4 if $s(l) = -\infty$ and $s(r) = +\infty$.

Type 4 corresponds to recurrent diffusions while Types 1–3 correspond to transient ones. Recall that the infinitesimal generator of X is given by $Lf = (\sigma^2/2)f'' + bf'$ where f is in the domain $\mathscr{D}(X)$ which is appropriately defined for example in [10]. For $x_0 \in E$ let h be the unique positive harmonic function for L satisfying $h(x_0) = 1$ and either

$$h(l) = \begin{cases} 1/h(r) & \text{if } X \text{ is of type 1 with } 2s(x_0) \neq s(l) + s(r); \\ 1 & \text{if } X \text{ is of type 1 and } 2s(x_0) = s(l) + s(r) \text{ or type 4}; \\ 0 & \text{if } X \text{ is of type 2}; \end{cases}$$

or

$$h(r) = 0 \text{ if } X \text{ is of type 3.}$$

If X is of type 4 or of type 1 with $2s(x_0) = s(l) + s(r)$ then $h \equiv 1$ otherwise h is specifically given by (9) which is displayed in Sect. 3 below.

Let X^* be the dual of X, with respect to $h(x)m(dx)$, in the following classical sense. For all $t > 0$ and all Borel functions f and g, we have

$$\int_E f(x)P_t g(x)h(x)m(dx) = \int_E g(x)P_t^* f(x)h(x)m(dx)$$

where P_t and P_t^* are the semigroup operators of X and X^*, respectively. The probability law of X^* is related to that of X by a Doob h-transform, see for example [8]. To be more precise, assuming that $X_0^* = x \in E$ then the distribution $\mathbb{P}_x^*$ of X^* is obtained from the distribution $\mathbb{P}_x$ of X by the change of measure

$$d\mathbb{P}_x^*|_{\mathscr{F}_t} = \frac{h(X_t)}{h(x)} d\mathbb{P}_x|_{\mathscr{F}_t}, \quad t < \zeta. \tag{4}$$

We shall denote by $\mathbb{E}_x^*$ the expectation under the probability measure $\mathbb{P}_x^*$; X^* has the infinitesimal generator $L^*f = L(hf)/h$ for $f \in \mathscr{D}(X^*) = \{g : E \to E,\ hg \in \mathscr{D}(X)\}$. The two processes $(h(X_t), t \le \zeta)$ and $(1/h(X_t^*), t \le \zeta^*)$ are continuous local martingales. We shall show that the former process can be realized as the latter when time changed with an appropriate random clock. Thus, the expression of either the process X^* or X, which are both E-valued, in terms of the other, involves the function $I(x) = h^{-1}(1/h(x))$ which is clearly an involution where it is well defined. We will show in Proposition 2 that $I : E \to E$.

For general properties of real valued involutions, we refer to [24, 25]. Now, indeed, an intuitive formulation of our main result is that, when h is not constant, the processes $(1/h(X_t^*), t \le \zeta^*)$ and $(h(X_{\tau_t}), t \le A_\zeta)$, if both are started at x_0, have the same law, where, for $t > 0$,

$$A_t := \int_0^t I'^2(X_s)\sigma^2(X_s)/\sigma^2 \circ I(X_s)\, ds, \tag{5}$$

τ_t is the inverse of A_t, and I' stands for the derivative of I. Two interesting features of the involved clocks are described as follows. First, $\tau_t = A_t^*$ where A_t^* is defined as above with X replaced by X^*. Second, ζ^* (resp. ζ) and A_ζ (resp. $A^*_{\zeta^*}$) have the same distribution; these new identities in distribution for killed diffusions resemble the Ciesielski-Taylor and Biane identities, see [2, 6]. We call the process $(I(X_t), t < \zeta)$ the inverse of X with respect to x_0. We know that Doob h-transforming X amounts to conditioning it to behave in a particular way at the boundaries. Our construction sheds light on the exact behaviour of the Doob h-transformed process at the boundaries.

In the transient case, the general construction of X^* from X discovered by Nagasawa, in [16], applies to linear diffusions; see also [14, 18]. We mention that this powerful method is used by Sharpe in [23] and by Williams in [26] for the study of path transformations of some diffusions. While the latter path transform involves time reversal from cooptional times, such as last passage time, the construction we present here involves only deterministic inversions and time changes with random clocks of the form (5). Although we only consider one dimensional diffusions in this paper, inversions of stable processes and Brownian motion in higher dimensions are studied in [3] and [27], respectively.

3 Conditioned Diffusions and Inversions

Let $X \in \mathscr{DF}(E)$. To start with, assume that X is of type 1 and let $h : E \to \mathbb{R}_+$ be a positive harmonic function for the infinitesimal generator L of X satisfying $0 < h(l) < h(r) < \infty$. Let X^* be the dual of X with respect to $h(x)m(dx)$.

We are ready to state the following result which motivates the construction of inversions; to our best knowledge, the described role of the h-geometric mean x_0 and the h-arithmetic mean x_1, which are defined below, for X and X^*, has not been known.

Proposition 1 *Suppose that X is of type 1. The following assertions hold true.*

1) *There exists a unique $x_0 \in E$ such that $h^2(x_0) = h(l)h(r)$. We call x_0 the h-geometric mean of $\{l, r\}$. Furthermore, for $x \in E$, we have $\mathbb{P}_x(H_l < H_r) = \mathbb{P}^*_x(H_r < H_l)$ if and only if $x = x_0$.*
2) *There exists a unique $x_1 \in E$ such that $2h(x_1) = h(l) + h(r)$. We call x_1 the h-arithmetic mean of $\{l, r\}$. Furthermore, for $x \in E$, we have $\mathbb{P}_x(H_l < H_r) = \mathbb{P}_x(H_r < H_l) = 1/2$ if and only if $x = x_1$.*

Proof

1) Since h is continuous and monotone, because it is an affine function of s, if h is increasing (resp. decreasing) then the inequality $h(l) < \sqrt{h(l)h(r)} < h(r)$ (resp. $h(r) < \sqrt{h(l)h(r)} < h(l)$) implies the existence and uniqueness of x_0. Because $-1/h$ is a scale function for X^*, see for example [5], applying (3) yields

$$\mathbb{P}_x(H_l < H_r) = (h(x) - h(r)) \,/\, (h(l) - h(r))$$

and

$$\mathbb{P}^*_x(H_l > H_r) = (1/h(x) - 1/h(l)) \,/\, (1/h(r) - 1/h(l))\,.$$

These are equal if and only if $x = x_0$.
2) The proof is omitted since it is very similar. ■

As h is monotone, we find $r = h^{-1}\left(h^2(x_0)/h(l)\right)$. This expression of r in terms of x_0 and l allows us to introduce the mappings we are interested in. The last formula exhibits the function $I : x \to h^{-1}\left(h^2(x_0)/h(x)\right)$ which is well defined on E by monotonicity of h. Clearly, I is a decreasing involution of E with fixed point x_0. Next, observe that $h \circ I \circ h^{-1} : x \to h^2(x_0)/x$ is the Euclidian inversion with fixed point $h(x_0)$.

We return now to the general case and assume that X is of one of the types 1–4. Our aim is to determine the set of all involutions associated to X which lead to the set of Möbius real involutions

$$\mathscr{MI} := \left\{\omega : \mathbb{R}\backslash\{a/c\} \to \mathbb{R}\backslash\{a/c\}; \omega(x) = \frac{ax+b}{cx-a}, a^2 + bc > 0,\ a, b, c \in \mathbb{R}\right\}.$$

Note that the condition $a^2 + bc > 0$ for $\omega \in \mathscr{MI}$ ensures that ω, when restricted to either of the intervals $(-\infty, a/c)$ and $(a/c, +\infty)$, is a decreasing involution. Let us settle the following definition.

Definition 1 Let s and s^{-1} be, respectively, a reference scale function for X and its inverse function, and $x_0 \in E$. A mapping $I : E \to E$ is called the inversion in the direction of s, or s-inversion, with fixed point x_0 if the following hold:

1) $I \circ I(x) = x$ for $x \in E$;
2) $s \circ I \circ s^{-1} \in \mathscr{MI}$;
3) $I(E) = E$;
4) $I(x_0) = x_0$.

If $s \circ I \circ s^{-1}$ is the Euclidian reflection in x_0 then I is called the s-reflection in x_0.

Since I is defined on the whole of the open interval E, it is necessarily continuous. This, in turn, implies that it is a decreasing involution such that $I(l) = r$. The objective of our next result is to show the existence of the inversion of E in the direction of s in case when s is bounded on E i.e. for diffusions of type 1.

Proposition 2 *Let $x_0 \in E$ and assume that s is bounded on E. Then, the following assertions hold.*

1) The inversion of E in the direction of s with fixed point x_0 is given by

$$I(x) = \begin{cases} s^{-1}\left(s^2(x_0)/s(x)\right) & \text{if } s^2(x_0) = s(l)s(r); \\ s^{-1}\left(2s(x_0) - s(x)\right) & \text{if } 2s(x_0) = s(l) + s(r); \\ s^{-1}\left((s(x) + a)/(bs(x) - 1)\right) & \text{otherwise,} \end{cases}$$

where

$$a = (2s(l)s(r) - s(x_0)\,(s(l) + s(r)))\left(s^2(x_0) - s(l)s(r)\right)^{-1} s(x_0)$$

and

$$b = (2s(x_0) - (s(l) + s(r)))\left(s^2(x_0) - s(l)s(r)\right)^{-1}.$$

2) If $2s(x_0) \neq s(l) + s(r)$ then $I = h^{-1}(1/h)$ where

$$h(x) = \begin{cases} (bs(x) - 1)/(bs(x_0) - 1) & \text{if } s^2(x_0) \neq s(l)s(r); \\ s(x)/s(x_0) & \text{otherwise.} \end{cases}$$

Furthermore, h is continuous, strictly monotonic and satisfies $h \neq 0$ on E.

Proof

1) We shall first assume that $s^2(x_0) \neq s(l)s(r)$. We look for I such that $s \circ I \circ s^{-1}(x) = (x + a)/(bx - 1)$ where a and b are reals satisfying $ab + 1 \neq 0$. Since the images

of l and x_0 by I are respectively r and x_0, we get the following linear system of equations

$$\begin{cases} bs^2(x_0) - a &= 2s(x_0); \\ bs(l)s(r) - a &= s(l) + s(r). \end{cases} \tag{6}$$

Solving it yields a and b. We need to show that $ab + 1 \neq 0$. A manipulation of the first equation of (6) shows that $1 + ab = (bs(x_0) - 1)^2$. In fact, we even have the stronger fact that $s(x) \neq 1/b$ on E which is seen from $1/b > s(r)$ if $2s(x_0) > s(l) + s(r)$ and $1/b < s(l)$ if $2s(x_0) < s(l) + s(r)$. Finally, if $s^2(x_0) = s(l)s(r)$ then clearly $I(x) = s^{-1}(s^2(x_0)/s(x))$.

2) Assume that $2s(x_0) \neq s(l) + s(r)$. Let us first consider the case $s^2(x_0) \neq s(l)s(r)$. Setting $h(x) = (s(x) - 1/b)/\delta$ we then obtain

$$h^{-1}(1/h(x)) = s^{-1}\left((s(x) + (b\delta^2 - 1/b))/(bs(x) - 1)\right).$$

Thus, the equality $I(x) = h^{-1}(1/h)$ holds if and only if $\delta = \pm\sqrt{1 + ab}/b$ which, in turn, implies that $h(x) = \pm(bs(x) - 1)/(bs(x_0) - 1)$. Since h is positive, we take the solution with plus sign. Since h is an affine transformation of s, it is strictly monotone and continuous on E. Finally, because $s \neq 1/b$, as seen in the proof of (1), we conclude that h does not vanish on E. The case $s^2(x_0) = s(l)s(r)$ is completed by observing that this corresponds to letting $b \to \infty$ and $\delta = s(x_0)$ above which gives the desired expression for h. ■

Now, we are ready to fully describe the set of inversions associated to the four types of diffusions described in the introduction. The proof of the following result is omitted, keeping in mind that when s is unbounded on E, by approximating E by a family of intervals $(\alpha, \beta) \subset E$ where s is bounded, using continuity and letting $\alpha \to l$ and $\beta \to r$ we obtain an expression for I.

Proposition 3 *All kinds of inversions of E in the direction of s with fixed point $x_0 \in E$ are described as follows.*

1) X is of type 1 with $2s(x_0) \neq s(l) + s(r)$ then the inversion is given in Proposition 2.

2) X is of type 2 then we have

$$I(x) = s^{-1}\left(s(l) + (s(x_0) - s(l))^2/(s(x) - s(l))\right). \tag{7}$$

3) X is of type 3 then we have

$$I(x) = s^{-1}\left(s(r) - (s(r) - s(x_0))^2/(s(r) - s(x))\right). \tag{8}$$

4) X is of type 4 or type 1 with $2s(x_0) = s(l) + s(r)$ then I is the s-reflection in x_0.

Remark 1 Observe that the s-inversions described in Proposition 3 solve $G(x, y) = 0$ in y, where G is the symmetric function $G(x, y) = As(x)s(y) - B(s(x) + s(y)) - C$ for some reals A, B and C. This is in agreement with the fact that I is an involution, see [25].

Remark 2 The inversion in the direction of s with fixed point x_0 does not depend on the particular choice we make of s. Tedious calculations show that the inversion of E in the direction of s is invariant under a Möbius transformation of s.

Going back to Proposition 1 we can express the inversions of Proposition 3 in terms of the harmonic function h instead of the reference scale function s. For that we need to compute the positive harmonic function h described in the introduction for each of the types 1–4 of diffusions. We easily get

$$h(x) = \begin{cases} \frac{bs(x)-1}{bs(x_0)-1} & \text{if } X \text{ is of type 1 and } 2s(x_0) \neq s(l) + s(r); \\ 1 & \text{if } X \text{ is of type 1 and } 2s(x_0) = s(l) + s(r) \text{ or type 4;} \\ \frac{s(x)-s(l)}{s(x_0)-s(l)} & \text{if } X \text{ is of type 2;} \\ \frac{s(r)-s(x)}{s(r)-s(x_0)} & \text{if } X \text{ is of type 3.} \end{cases} \tag{9}$$

Note that the case where X is of type 1 and x_0 is the s-geometric mean is covered in the first case by letting $b \to \infty$ to obtain $h(x) = s(x)/s(x_0)$. In the following result, which generalizes Proposition 1, we note that the first assertion could serve as the probabilistic definition for s-inversions.

Proposition 4 *The following assertions hold true.*

1) A function $I : E \to E$ is the s-inversion with fixed point $x_0 \in E$ if and only if $I(E) = E$ and for all $x \in E$

$$\mathbb{P}_{x_0}(H_x < H_{I(x)}) = \mathbb{P}^*_{x_0}(H_{I(x)} < H_x) \tag{10}$$

where $\mathbb{P}^*$ is the distribution of the Doob transform of X by some positive harmonic function k. Furthermore, the s-inversion and the k-inversion of E with fixed point x_0 are equal and $k = h$.

2) Let $\mathbb{Q}_{x_0}$ be the probability law of $(I(X_t), t \le \zeta)$ when $X_0 = x_0$. Then formula (10) holds true when $\mathbb{P}_{x_0}$ is replaced by $\mathbb{Q}_{x_0}$. We call the process $(I(X_t), t \le \zeta)$ the inverse with respect to x_0 of $(X_t, t \le \zeta)$. The fixed point x_0 of the involution I is seen to be the unique level at which the paths of the latter processes intersect.

Proof

1) If x_0 is the s-arithmetic mean of $\{l, r\}$ or X is of type 4 then we are looking for $I : E \to E$ such that $\mathbb{P}_{x_0}(H_x < H_{I(x)}) = 1/2$. Using (3) we get $(s(x_0) - s \circ I(x))/(s(x) - s \circ I(x)) = 1/2$ which gives that I is the s-reflection. For the other cases, using (10) and the fact that $-1/h$ is a scale function for X^*, we find that $I(x) = h^{-1}(h(x_0)^2/h(x))$ so that I is an h-inversion with fixed point x_0. The "only if" part is straightforward following a similar reasoning to that of the proof of

Proposition 1 giving I to be either the h-reflection or the h-inversion with fixed point x_0.

2) The first part is easily seen by using the first assertion. The interpretation for the fixed point x_0 follows from the fact that x_0 is the unique fixed point of I. ■

For completeness, we explain now how to define rigorously a diffusion Y_t obtained by conditioning a transient diffusion X_t to hit one boundary of an interval before another with a prescribed probability a. By a natural definition, it holds if for any bounded $\mathscr{F}_t^X$-measurable functional G and $t > 0$, we have

$$\mathbb{E}_{x_0}[G(Y_s, s \le t), t < \zeta] = a\mathbb{E}_{x_0}[G(X_s, s \le t), t < \zeta | H_l < H_r] \\ +(1-a)\mathbb{E}_{x_0}[G(X_s, s \le t), t < \zeta | H_r < H_l].$$

For conditioning a transient diffusion to avoid one of the boundaries we refer, for example, to [9, 13, 22].

We show in the following Proposition that the dual process X^* can be realized as X conditioned in the sense of Doob to exits the segment $[l, r]$ at the endpoints l and r with some specified probabilities.

Proposition 5 *Assume that X is transient and let h be given by (9). Let p be the probability that X, when started at x_0, exits $[l, r]$ at l. X conditioned to exit $[l, r]$ at l with probability $q = 1 - p$ is a realization of the dual X^* of X with respect to $h(x)m(dx)$.*

Proof By construction, we have $h(x_0) = 1$. Assume at first that X is of type 1. Let us decompose h, in terms of h_l and h_r which are defined below, as follows

$$h(x) = q^* \frac{h(x) - h(l)}{h(x_0) - h(l)} + p^* \frac{h(r) - h(x)}{h(r) - h(x_0)} := q^* h_r(x) + p^* h_l(x)$$

where

$$q^* = \frac{h(x_0) - h(l)}{h(r) - h(l)} h(r) = \frac{h^*(x_0) - h^*(l)}{h^*(r) - h^*(l)} = \mathbb{P}^*_{x_0}(H_r < H_l)$$

and

$$p^* = \frac{h(r) - h(x_0)}{h(r) - h(l)} h(l) = \frac{h^*(r) - h^*(x_0)}{h^*(r) - h^*(l)} = \mathbb{P}^*_{x_0}(H_l < H_r).$$

But, we have that $q^* = p$ and $p^* = q$ when X and X^* are started at x_0. Thus, for any bounded $\mathscr{F}_t^X$-measurable functional G and $t > 0$, we can write

$$\begin{aligned} \mathbb{E}^*_{x_0}[G(X_s, s \le t), t < \zeta] &= \mathbb{E}_{x_0}[h(X_t)G(X_s, s \le t), t < \zeta] \\ &= q\mathbb{E}_{x_0}[h_l(X_t)G(X_s, s \le t), t < \zeta] \\ &\quad +p\mathbb{E}_{x_0}[h_r(X_t)G(X_s, s \le t), t < \zeta]. \end{aligned}$$

Next, since our assumptions imply that $p = \mathbb{P}_{x_0}(H_l < H_r) \in (0, 1)$, we have

$$\begin{aligned}\mathbb{E}_{x_0}[h_l(X_t)G(X_s, s < t), t < \zeta] &= \mathbb{E}_{x_0}[G(X_s, s \leq t)\frac{\mathbb{P}_{X_t}[H_l < H_r]}{\mathbb{P}_{x_0}(H_l < H_r)}, t < \zeta] \\ &= \mathbb{E}_{x_0}[G(X_s, s \leq t), t < \zeta | H_l < H_r]\end{aligned}$$

where we used the strong Markov property for the last equality. Similarly, for the other term, since $q \in (0, 1)$ we get

$$\mathbb{E}_{x_0}[(h_r(X_t)G(X_s, s \leq t), t < \zeta] = \mathbb{E}_{x_0}[G(X_s, s \leq t), t < \zeta | H_r < H_l].$$

The last two equations imply our assertion. Assume now that $h(r) = \infty$. Then $h(l) = 0$ and $\mathbb{P}_{x_0}$-a.s. all trajectories of the process X tend to l and $p = \mathbb{P}_{x_0}(H_r < H_l) = 0$. We follow [22] to define X conditioned to avoid l (i.e. never to hit l in a positive time) as follows. For any bounded $\mathscr{F}_t^X$-measurable functional G and $t > 0$, we set

$$\begin{aligned}\mathbb{E}^*_{x_0}[G(X_s, s \leq t), t < \zeta] &= \lim_{a \to r} \mathbb{E}_{x_0}[G(X_s, s \leq t), t < \zeta | H_a < H_l] \\ &= \lim_{a \to r} \mathbb{E}_{x_0}[G(X_s, s \leq t), t < H_a < H_l]/\mathbb{P}_{x_0}(H_a < H_l) \\ &= \lim_{a \to r} \mathbb{E}_{x_0}[h_r(X_t)G(X_s, s < t), t < H_a \wedge H_l] \\ &= \mathbb{E}_{x_0}[h(X_t)G(X_s, s \leq t), t < \zeta]\end{aligned}$$

where we used the strong Markov property for the third equality and the monotone convergence theorem for the last one. In this case $\mathbb{P}^*_{x_0}$-a.s all trajectories of the process X^* tend to r and $p^* = \mathbb{P}^*_{x_0}(H_l < H_r) = 0$ which completes the proof of the statement. The case $h(l) = -\infty$ and $h(r) = 0$ can be treated similarly. ■

Remark 3 From the point of view of Martin boundaries, the functions h_l and h_r which appear in the proof of Proposition 5 are the minimal excessive functions attached to the boundary points l and r, see [5, 20, 21]. That is the Doob h-transformed processes obtained by using h_l and h_r tend a.s., respectively, to l and r. Harmonic functions having a representing measure with support not included in the boundary set of E are not considered in this paper since we do not allow killings inside E.

4 Inversion of Diffusions

Let $X \in \mathscr{D}\mathscr{F}(E)$ and s be a scale function for X. For $x_0 \in E$, let $I : E \to E$ be the inversion of E in the direction of s with fixed point x_0. Let h be the positive harmonic function specified by (9). Let X^* be the dual of X with respect to $h(x)m(dx)$. As aforementioned, the distribution of X^* is obtained as a Doob h-transform of the distribution of X by using the harmonic function h, as given in (4). Clearly, if X is of type 1 (resp. of type 2 and drifts thus to l, of type 3 and drifts thus to r or of type 4) then X^* is of type 1 (resp. of type 3 and drifts thus to r, of type 2 and drifts thus to l or of type 4). It is easy to see that the inversions of E in the direction of s given in Proposition 3 are differentiable on E. Recall that for a fixed $t < \zeta$, τ_t is the inverse of the strictly increasing and continuous additive functional $A_t := \int_0^t I'^2(X_s)\sigma^2(X_s)/\sigma^2 \circ I(X_s)\, ds$; τ_t^* and A_t^* are the analogue objects associated to the dual X^*. Recall that the speed measure $m(dy) = 2dy/(\sigma^2 s')$ of X is uniquely determined by

$$\mathbb{E}_x[H_\alpha \wedge H_\beta] = \int_J G_J(x, y)\, m(dy)$$

where

$$G_J(x, y) = c(s(x \wedge y) - s(\alpha))(s(\beta) - s(x \vee y))/(s(\beta) - s(\alpha))$$

for any $J = (\alpha, \beta) \subsetneq E$ and all $x, y \in J$, where c is a normalization constant and $G_J(.,.)$ is the potential kernel density relative to $m(dy)$ of X killed when it exits J; see for instance [18, 19]. Recall that $-1/h$ is a scale function and $m^*(dx) := h^2(x)m(dx)$ is the speed measure of X^*. We are ready to state the main result in this paper. The proof we give is based on the resolvent method for the identification of the speed measure, see [18, 19, 23]. Other possible methods of proof are commented in Remarks 6 and 7.

Theorem 1 *With the previous setting, let I be the s-inversion of E with fixed point $x_0 \in E$. Assuming that X_0, $X_0^* \in E$ are such that $I(X_0) = X_0^*$ then the following assertions hold true.*

1) For all $t < \zeta$, τ_t and A_t^* have the same distribution.
2) The processes $(X_t^*, t \le \zeta^*)$ and $(I(X_{\tau_t}), t \le A_\zeta)$ have the same law.
3) The processes $(X_t, t \le \zeta)$ and $(I(X^*_{\tau_t^*}), t \le A^*_{\zeta^*})$ have the same law.

Remark 4 Note that if the starting point is the fixed point of I, i.e. $X_0 = x_0$, then both processes X and X^* start from x_0. However, our result holds true and is proven in the general case $X_0 \in E$ provided that $X_0^* = I(X_0)$.

Proof (of Theorem 1)

1) Let $t > 0$ be fixed and set $\eta_t = I(X_{\tau_t})$. Because τ_t is the inverse of A_t, we can write $A_{\tau_t} = t$. Differentiating and extracting the derivative of τ_t yields

$$\frac{d}{dt}\tau_t = \frac{\sigma^2 \circ I(X_{\tau_t})}{I'^{\,2}(X_{\tau_t})\sigma^2(X_{\tau_t})}.$$

Integrating yields

$$\tau_t = \int_0^t \left(I'\sigma/(\sigma \circ I)\right)^{-2} (X_{\tau_s}) ds = A_t^{\eta}.$$

The proof of the first assertion is complete once we have shown that η and X^* have the same distribution which will be done in the next assertion.

2) First, assume that h is not constant. In this case, $x \to -1/h(x)$ is a scale function for η since $-1/h \circ I(X_{\tau_t}) = -h(X_{\tau_t})$ is a continuous local martingale. Next, let $J = (\alpha, \beta)$ be an arbitrary subinterval of E. We proceed by identifying the speed measure of η on J. By using the fact that the hitting time H_y^{η} of y by η equals $A_{H_{I(y)}}$ for $y \in E$, we can write

$$\begin{aligned}
\mathbb{E}_{I(x)} \left[H_\alpha^\eta \wedge H_\beta^\eta\right] &= \mathbb{E}_{I(x)} \left[\int_0^{H_{I(\alpha)} \wedge H_{I(\beta)}} dA_r\right] \\
&= \int_{I(\beta)}^{I(\alpha)} G_{I(J)}(I(x), y) I'^2(y) \sigma^2(y) \{\sigma^2 \circ I(y)\}^{-1} m(dy) \\
&= 2 \int_{I(\beta)}^{I(\alpha)} G_{I(J)}(I(x), y) I'^2(y) \{\sigma^2 \circ I(y) s'(y)\}^{-1} dy \\
&= 2 \int_\alpha^\beta G_{I(J)}(I(x), I(y)) \{\sigma^2(y)(-h \circ I)'(y)\}^{-1} dy.
\end{aligned}$$

On the one hand, we readily check that $\{\sigma^2(y)(s \circ I)'(y)\}^{-1} dy = h^2(y) m(dy) = m^*(dy)$ for $y \in J$. On the other hand, we have

$$\begin{aligned}
G_{I(J)}(I(x), I(y)) &= (h(I(x) \wedge I(y)) - h(I(\beta))) \frac{h(I(\alpha)) - h(I(x) \vee I(y))}{h(I(\alpha)) - h(I(\beta))} \\
&= \left(-h^*(x \vee y) + h^*(\beta)\right) \frac{-h^*(\alpha) + h^*(x \wedge y)}{-h^*(\alpha) + h^*(\beta)} \\
&= G_J^*(x, y),
\end{aligned}$$

where $h^* = -1/h$ and G_J^* is the potential kernel density of X^* relative to $m^*(dy)$. The case when h is constant can be dealt with similarly but by working with s instead of h. This shows that the speed measure of η is the same as that of X^* in all cases. Now, since $\tau_t^X = A_t^\eta$ we get $\tau_t^X = A_t^*$ which, in turn, implies that $A_t^X = \tau_t^*$. Finally, using the fact that I is an involution gives

$$H_y^* = \inf\{s > 0, I(X_{A_s^*}) = y\} = \tau_{H_{I(y)}}^* = A_{H_{I(y)}}$$

for $y \in E$. The assertion is completed by letting y tend to either of the boundaries to find $\zeta^* = A_\zeta$ and $\zeta = A_{\zeta^*}^*$ as desired.

3) The proof is easy using (1) and (2), the fact that I is an involution and time changes. ■

Remark 5 The resolvent method used in the proof of Theorem 1 suggests that it could be not necessary to suppose that X solves a diffusion s.d.e. We conjecture that variants of Theorem 1 are true for "nice" Markov processes. For example, analogue inversions are known for symmetric stable processes, see [3]. As mentioned in the introduction, inversions of a more general class of self-similar Markov processes are studied in a joint work with L. Chaumont.

Remark 6 Since X satisfies the s.d.e. (1), by Girsanov's theorem, we see that X^* satisfies $Y_t = X_0^* + \int_0^t \sigma(Y_s)dB_s + \int_0^t (b + \sigma^2 h'/h)(Y_s)ds$ for $t < \zeta^*$ where B is a Brownian motion which is measurable with respect to the filtration generated by X^* and $\zeta^* = \inf\{s,\ Y_s = l \text{ or } Y_s = r\}$, see for example [8]. Long calculations show that η also satisfies the above s.d.e. which, by Engelbert-Schmidt condition (2), has a unique solution in law. This gives a second proof of Theorem 1. Note that the use of Itô's formula for η is licit since $I \in \mathscr{C}^2(E)$.

Remark 7 Another way to view the main statements of Theorem 1 is the equality of generators

$$\frac{1}{I'^2(x)}\frac{\sigma^2(x)}{\sigma^2(I(x))}L(f\circ I)(I(x)) = \frac{1}{h(x)}L(hf)(x) = L^*f(x)$$

and

$$\frac{1}{I'^2(x)}\frac{\sigma^2(x)}{\sigma^2(I(x))}L^*(g\circ I)\ (I(x)) = Lg(x)$$

for all $x \in E$, $g \in \mathscr{D}(X)$ and $f \in \mathscr{D}(X^*)$. However, the main difficulty of this method of proof is the precise description of domains of generators.

The focus now is on the cases where X is transient and drifts a.s. either to l or to r as $t \to \zeta$, i.e. X is of type 2 or 3. Clearly, if X is of type 2 (resp. type 3) then X^* is of type 3 (resp. 2). Hence, for our purpose, it is enough to consider the case where X is of type 2, in which case formula (3) gives that l is hit a.s. before r.

Corollary 1 *If $X \in \mathscr{DF}(E)$ is of type 2 then $\zeta = H_l$ and $A^*_{\zeta^*} = A^*_{H_r^*}$ have the same distribution. Furthermore, $A^*_{H_r^*} < \infty$ with probability 1 if and only if*

$$\int_l^c (s(x) - s(l))m(dx) < \infty, \quad l < c < r. \tag{11}$$

Proof The first claim is a straightforward consequence of Theorem 1. Next, $\zeta^* = H_r^*$ because $X_t^* \to r$ a.s. as $t \to \zeta$. Now by Feller's classification of boundaries, $\zeta < \infty$ a.s., and hence $A^*_{H_r^*} < \infty$ a.s., if and only if (11) holds, see e.g. p. 745 in [26]. ■

So far we made the assumption that attainable boundaries are killing and the process cannot be started from such points. We stress out however that, in the following result, we assume that l is an entrance not an exit (and not an absorbing or killing) point for the dual diffusion X^*. The following result gives a path construction of the diffusion which we obtain when we apply the time reversal property to X^* (or X).

Corollary 2 *Assume that $X \in \mathcal{DF}(E)$ is of type 2 and satisfies (11). Introduce the last passage time of X at the fixed point $x_0 \in E$ of the s-inversion I, i.e.*

$$L_{x_0} = \sup\{t : X_t = x_0\}.$$

Then the time inverted process $(X^*_{L^*_{x_0}-t}, t \le L^*_{x_0} | X^*_0 = l)$ and $(I(X^*_{\tau^*_t}), t < A^*_{H^*_r} | X_0 = x_0)$ are identical in law.

Proof Let h be given by (9). Then $s^*(x) = -1/h(x)$ is a scale function for X^* and $s^*(l) = -\infty$. Thus, condition (11) implies that l is an entrance not exit point for X^*, see [11]. Furthermore, the process X^*, when started at l, is X conditioned never to return to l in a positive time. Now, on the one hand, we have that $(X_t, t \le H_l | X_0 = x_0)$ and $(X^*_{L^*_{x_0}-t}, t \le L^*_{x_0} | X^*_0 = l)$ have the same distribution, see for example Theorem 2.5 in [26]. On the other hand, we know by Theorem 1 that $(X_t, t < \zeta | X_0 = x_0)$ and $(I(X^*_{\tau^*_t}), t < A^*_{\zeta^*} | X_0 = x_0)$ have the same law. ■

Remark 8 Theorem 2.11 of [15] states that if $X \in \mathcal{DF}(E)$ with $s(r) < \infty$ and $f : E \to \mathbb{R}$ is a non-negative Borel function, then it holds that $\int_0^\zeta f(X_s)ds < \infty$ a.s., on the event $\{\lim_{t\to\zeta} X_t = r\}$, if and only if

$$\int_c^r (s(r) - s(x)) f(x) m(dx) < \infty, \quad l < c < r.$$

Keeping the setting of Corollary 1 and applying the above to X^*, with

$$m^*(x) = (s(x) - s(l))^2 m(dx) \quad \text{and} \quad f(x) = I'^2(x) \frac{\sigma^2(x)}{\sigma^2 \circ I(x)},$$

we obtain the necessary and sufficient condition

$$\int_c^r (s^*(r) - s^*(x)) f(x) m^*(dx)$$
$$= \frac{1}{s(x_0) - s(l)} \int_l^{I(c)} (s(x) - s(l)) m(dx) < \infty, \quad l < c < r,$$

for the finiteness of both $A^*_{\zeta^*}$ and ζ. This is in agreement with the aforementioned corollary.

5 Applications

5.1 Inversions of Brownian Motions Killed upon Exiting Intervals

Assume that X is a Brownian motion killed upon exiting the interval $E = (l, r)$. Let $X_0 = x_0 \in E$. If E is bounded then we obtain the inversions

$$I(x) = \begin{cases} x_0^2/x, & \text{if } x_0^2 = lr; \\ 2x_0 - x, & \text{if } 2x_0 = l + r; \\ (x+a)/(bx-1), & \text{otherwise,} \end{cases}$$

where

$$a = \frac{2lr - x_0(l+r)}{x_0^2 - lr} x_0 \quad \text{and} \quad b = \frac{2x_0 - (l+r)}{x_0^2 - lr}.$$

We distinguish three cases appearing in the form of the inversion I. In the first case, equation $x_0^2 = lr$ implies that l and r are of the same sign. If $l > 0$ then X^* is the three-dimensional Bessel process killed upon exiting E. If $l < 0$ then $-X^*$ is the three-dimensional Bessel process.

In the second case, X^* is a Brownian motion killed when it exits E. In the third case, X^* satisfies the s.d.e. $X_t^* = B_t + x_0 + \int_0^t (X_s^* - 1/b)^{-1} ds$ for $t < \zeta^*$. If x_0 is below the arithmetic mean i.e. $x_0 < (l+r)/2$ then $x_0 - 1/b = (x_0 - r)(x_0 - l)/(2x_0 - (l+r))$. By uniqueness of the solution to the s.d.e. $R_t = R_0 + B_t + \int_0^t (1/R_s) ds$ driving the three-dimensional Bessel process R, we get that $X_t^* = 1/b + R_t$ with $R_0 = x_0 - 1/b$ killed when R exits the interval $(l - 1/b, r - 1/b)$. If x_0 is above the arithmetic mean then we find $X_t^* = 1/b - R_t$, where R is a three-dimensional Bessel process started at $1/b - x_0$, where R is killed as soon as it exits the interval $(1/b - r, 1/b - l)$.

If l is finite and $r = \infty$ or $l = -\infty$ and r is finite then by Proposition 3 we respectively obtain

$$I(x) = l + \frac{(x_0 - l)^2}{x - l} \text{ and } I(x) = r - \frac{(r - x_0)^2}{r - x}.$$

If $r = +\infty$ then a similar reasoning as above gives that $X_t^* = l + R_t$ where R is a three-dimensional Bessel process started at $x_0 - l$. If $l = -\infty$ we obtain $X_t^* = r - R_t$ where R is a three-dimensional Bessel process started at $r - x_0$. If $2x_0 = l + r$ or $l = -\infty$ and $r = +\infty$ then we obtain the Euclidian reflection in x_0, i.e. $x \to 2x_0 - x$, and X^* is a Brownian motion killed when it exits E. Note that for $E = (0, \infty)$ the conclusion from our Theorem 1 is found in Lemma 3.12 on p. 257 of [18]. That is $(X_t, t \le H_0)$ is distributed as $(1/X^*_{\tau_t^*}, t \le A^*_\infty)$, where τ_t^* is the inverse of $A_t^* = \int_0^t (X_s^*)^{-4} ds$. In this case X^* is the three-dimensional Bessel process; see our last example given in Sect. 5.3 for Bessel processes of other dimensions. If $E = \mathbb{R}$

then I is the Euclidian reflection in x_0. Finally, observe that the set of inversions of E we obtain for the Brownian motion killed when it exits E is precisely $\mathscr{M}\mathscr{I}(E) := \{I \in \mathscr{M}\mathscr{I}\,;\, I(E) = E\}$.

5.2 *Inversions of Drifted Brownian Motion and Hyperbolic Bessel Process of Dimension 3*

Set $B_t^{(\mu)} = B_t + \mu t$, $t \geq 0$, where B is a standard Brownian motion and $\mu \in \mathbb{R}$, $\mu \neq 0$. Thus, $B^{(\mu)}$ is a transient diffusion which drifts to $+\infty$ (resp. to $-\infty$) if $\mu > 0$ (resp. $\mu < 0$). Let us take the reference scale function $s(x) = -e^{-2\mu x}/(2\mu)$. Observe that s is increasing for all $\mu \neq 0$. Moreover $\lim_{x\to\infty} s(x) = 0$ if $\mu > 0$ and $\lim_{x\to-\infty} s(x) = 0$ if $\mu < 0$. We take X to be $B^{(\mu)}$ killed when it exits $(l, r) \subseteq \mathbb{R}$. Let us fix $x_0 \in E$.

If we take $E = \mathbb{R}$ then by Proposition 3, even though X is of type 2 if $\mu < 0$ and of type 3 if $\mu > 0$, the inversion of E in the direction of s is the Euclidian reflection in x_0. X^* is the Brownian motion with drift $\mu^* = -\mu$ in this case. If $s(l)$ and $s(r)$ are finite then using Proposition 2 we obtain appropriate, but in most cases complicated, formulas for I. For Brownian motion with drift the case of the half-line is the most interesting. Take for instance $E = (0, \infty)$ and process $X_t = x_0 + B_t + \mu t$ starting from some point $x_0 > 0$ and killed at zero. We consider two cases: if $\mu < 0$ then $s(0) = -1/(2\mu) > 0$, $s(\infty) = \infty$ and X is of type 2; if $\mu > 0$ then $s(0) = -1/(2\mu) < 0$, $s(\infty) = 0$ and X is of type 1.

First, let $\mu < 0$. If X is of type 2 then, by Proposition 3, we have only one possible inversion: $I(x) = s^{-1}\left(s(l) + (s(x_0) - s(l))^2/(s(x) - s(l))\right)$, which gives

$$I(x) = (2|\mu|)^{-1} \ln\left((e^{-2\mu x} - 1 + (1 - e^{-2\mu x_0})^2)/(e^{-2\mu x} - 1)\right).$$

If we choose $x_0 = (2|\mu|)^{-1} \ln(1 + \sqrt{2})$ then the above formula simplifies to

$$I(x) = (2|\mu|)^{-1} \ln\left((e^{-2\mu x} + 1)/(e^{-2\mu x} - 1)\right) = (2|\mu|)^{-1} \ln \coth(|\mu| x).$$

Now, if $\mu > 0$ then X is of type 1. Because $e^{-2\mu x} \neq 0$ implies $s^2(x_0) \neq s(0)s(\infty) = 0$ only two cases are possible. Either $2s(x_0) = s(0) + s(\infty) = s(0)$, which gives $x_0 = \frac{1}{2\mu}\ln 2$, and then we have s-reflection $I(x) = -\frac{1}{2\mu}\ln(1 - e^{-2\mu x})$ or $2s(x_0) \neq s(0) + s(\infty)$ and then the formula from Proposition 3 gives the inversion

$$I(x) = \frac{1}{2\mu} \ln\left(\frac{1 + e^{-2\mu x}(e^{4\mu x_0} - 2e^{2\mu x_0})}{1 - e^{-2\mu x}}\right).$$

This in turn simplifies if we choose $e^{4\mu x_0} - 2e^{2\mu x_0} = 1$, that is, if $x_0 = \frac{1}{2\mu}\ln(1 + \sqrt{2})$.

Then

$$I(x) = \frac{1}{2\mu} \ln \left(\frac{1 + e^{-2\mu x}}{1 - e^{-2\mu x}} \right) = \frac{1}{2\mu} \ln \coth(\mu x).$$

It is easy to check that if $\mu < 0$, then $h(x) = \frac{e^{-2\mu x}-1}{\sqrt{2}}$ and then X^*, being an h-process, has generator $L^* f(x) = \frac{1}{2} f''(x) + |\mu| \coth(|\mu| x) f'(x)$. In particular, if $\mu = -1$ then $I(x) = \frac{1}{2} \ln \coth x$ and X^* has generator $L^* f(x) = \frac{1}{2} f''(x) + \coth(x) f'(x)$. This recovers the well-known fact that $B_t - t$ conditioned to avoid zero is a three-dimensional hyperbolic Bessel process. The novelty here is that we get X^* as a time changed inversion of $B_t - t$. If $\mu > 0$, then $h(x) = \frac{e^{-2\mu x}+1}{\sqrt{2}}$ and X^* has the generator $L^* f(x) = \frac{1}{2} f''(x) + \mu \tanh(\mu x) f'(x)$. Note that we have shown the following particular result.

Corollary 3 *Let X be a three-dimensional hyperbolic Bessel process on $(0, \infty)$ and $I(x) = \frac{1}{2} \ln \coth x$. Then, $I(X)$ is a time-changed drifted Brownian motion $B_t - t$ conditioned to avoid 0. In particular, the functional*

$$A_\infty = \frac{1}{4} \int_0^\infty \frac{ds}{(\cosh(X_s) \sinh(X_s))^2}$$

has the same distribution as the first hitting time of 0 *by the Brownian motion with minus unity drift. In other words, we have*

$$\mathbb{P}\left(A_\infty \in dt\right) = e^{x_0 - \frac{t}{2}} \frac{x_0}{\sqrt{2\pi t^3}} e^{-\frac{x_0^2}{2t}} dt, \qquad t > 0,$$

where $x_0 = \frac{1}{2} \ln(1 + \sqrt{2})$.

5.3 Inversions of Bessel Processes

It is known that two Bessel processes of dimensions δ and $4-\delta$, respectively, are dual one to another, see e.g. [5, 8, 18]. Our focus here is on the construction of the latter dual. To say more, let X be a Bessel process of dimension $\delta \geq 2$. Thus, $E = (0, \infty)$, 0 is polar and X has the infinitesimal generator $Lf(x) = \frac{1}{2} f''(x) + (\delta - 1)/(2x) f'(x)$ for $x \in E$, see [8, 18]. Let $\nu = (\delta/2) - 1$ be the index of X. The scale function of X may then be chosen to be

$$s(x) = \begin{cases} -x^{-2\nu} & \text{if } \nu > 0; \\ 2 \log x & \text{if } \nu = 0; \\ x^{-2\nu} & \text{otherwise.} \end{cases}$$

X is recurrent if and only if $\delta = 2$, see for example Proposition 5.22 on p. 355 of [12]. First, when $\delta = 2$ the inversion of E in the direction of s is the s-reflection $x \to x_0^2/x$. For the other cases, the inversion of E in the direction of s with fixed point 1 is found to be $x \to 1/x$. Furthermore, X^* is a Bessel process of dimension $4 - \delta < 2$. Observe that in the two considered cases the involved clock is $A_t = \int_0^t (x_0/X_s)^4 \, ds$. Hence, we have shown the following result which is a particular case of Proposition 1.11 on p. 447 of [18].

Corollary 4 *Let X be a Bessel process of dimension $\delta \in \mathbb{R}$, killed at 0 if $\delta < 2$, starting from $X_0 > 0$ and $I(x) = 1/x$. Then $I(X)$ is a time-changed Bessel process of dimension $4-\delta$. In particular, the functional $\int_0^\infty ds/X_s^4$, when X is a Bessel process of dimension $\delta > 2$, has the same distribution as the first hitting time of zero by the Bessel process of dimension $4 - \delta$.*

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On h-Transforms of One-Dimensional Diffusions Stopped upon Hitting Zero

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In memoriam, Marc Yor

Abstract For a one-dimensional diffusion on an interval for which 0 is the regular-reflecting left boundary, three kinds of conditionings to avoid zero are studied. The limit processes are h-transforms of the process stopped upon hitting zero, where h's are the ground state, the scale function, and the renormalized zero-resolvent. Several properties of the h-transforms are investigated.

1 Introduction

For the reflecting Brownian motion $\{(X_t), (\mathbb{P}_x)_{x\in[0,\infty)}\}$ and its excursion measure $\boldsymbol{n}$ away from 0, it is well-known that $\mathbb{P}^0_x[X_t] = x$ for all $x \geq 0$ and all $t > 0$, where $\{(X_t), (\mathbb{P}^0_x)_{x\in[0,\infty)}\}$ denotes the process stopped upon hitting 0, and $t \mapsto \boldsymbol{n}[X_t]$ is constant in $t > 0$. Here and throughout this paper we adopt the notation $\mu[F] = \int F \mathrm{d}\mu$ for a measure μ and a function F. The process conditioned to avoid zero may be regarded as the h-transform with respect to $h(x) = x$ of the Brownian motion stopped upon hitting zero. The obtained process coincides with the three-dimensional Bessel process and appears in various aspects of $\boldsymbol{n}$ (see, e.g., [11, 21]).

We study three analogues of conditioning to avoid zero for one-dimensional diffusion processes. Adopting the natural scale $s(x) = x$, we let $M = \{(X_t)_{t\geq 0}, (\mathbb{P})_{x\in I}\}$ be a $D_m D_s$-diffusion on I where $I' = [0, l')$ or $[0, l']$ and $I = I'$ or $I' \cup \{l\}$; the choices of I' and I depend on m (see Sect. 2). We suppose that 0 for M is regular-reflecting. Let $M^0 = \{(X_t)_{t\geq 0}, (\mathbb{P}^0)_{x\in I}\}$ denote the process M stopped upon hitting zero. We focus on three functions which are involved in conditionings to avoid zero. The first one is the natural scale $s(x) = x$. The second one is given as follows. When l' is natural, we set $\gamma_* = 0$ and $h_* = s$. When l' is not natural, it was shown in

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[17, Theorem 3.1] that the q-resolvent operator G_q^0 on $L^2(\mathrm{d}m)$ for M^0 is compact and is represented by the eigenfunction expansion $G_q^0 = \sum_n (q-\gamma_n)^{-1} f_n \otimes f_n$ with $0 \geq \gamma_1 > \gamma_2 > \cdots \downarrow -\infty$; in this case we write $\gamma_* = \gamma_1$ and $h_* = f_1$. The obtained function h_* is the second one. The third one is

$$h_0(x) = \lim_{q \downarrow 0} \{r_q(0,0) - r_q(x,0)\}, \tag{1}$$

where $r_q(x,y)$ denotes the resolvent density with respect to the speed measure. We will prove h_0 always exists and we call h_0 the *renormalized zero-resolvent*.

We now state three theorems concerning conditionings of M to avoid zero. Their proofs will be given in Sect. 5. We write $(\mathscr{F}_t)_{t \geq 0}$ for the natural filtration. Let T_a denote the first hitting time of a. The first conditioning is a slight generalization of a formula found in [24, Sect. 2.2].

Theorem 1.1 *Let $x \in I' \setminus \{0\}$. Let T be a stopping time and F_T be a bounded $\mathscr{F}_T$-measurable functional. Then*

$$\lim_{a \uparrow \sup I} \frac{\mathbb{P}_x[F_T; T < T_a < T_0]}{\mathbb{P}_x(T_a < T_0)} = \mathbb{P}_x^0\left[F_T \frac{X_T}{x}; T < T_*\right], \tag{2}$$

where $T_* = \sup_{a \in I} T_a$. (If l is an isolated point in I, we understand that the symbol $\lim_{a \uparrow \sup I}$ means the evaluation at $a = l$.)

The second conditioning is essentially due to McKean [17, 18].

Theorem 1.2 *Let $x \in I' \setminus \{0\}$. Let T be a stopping time and F_T be a bounded $\mathscr{F}_T$-measurable functional. Then*

$$\lim_{t \to \infty} \frac{\mathbb{P}_x[F_T; T < t < T_0]}{\mathbb{P}_x(t < T_0)} = \mathbb{P}_x^0\left[F_T \frac{\mathrm{e}^{-\gamma_* T} h_*(X_T)}{h_*(x)}; T < \infty\right]. \tag{3}$$

The third conditioning is an analogue of Doney [6, Sect. 8] (see also Chaumont–Doney [3]) for Lévy processes. For $q > 0$, we write $\boldsymbol{e}_q$ for the exponential variable independent of M.

Theorem 1.3 *Let $x \in I' \setminus \{0\}$. Let T be a stopping time and F_T be a bounded $\mathscr{F}_T$-measurable functional. Then*

$$\lim_{q \downarrow 0} \frac{\mathbb{P}_x[F_T; T < \boldsymbol{e}_q < T_0]}{\mathbb{P}_x(\boldsymbol{e}_q < T_0)} = \mathbb{P}_x^0\left[F_T \frac{h_0(X_T)}{h_0(x)}; T < \infty\right]. \tag{4}$$

The aim of this paper is to investigate several properties of the three functions h_*, h_0 and s and of the corresponding h-transforms.

We summarize some properties of the h-transforms of M^0 as follows (see Sect. 2 for the definition of the boundary classification and see the end of Sect. 4 for the classification of recurrence of 0; here we note that $m(\infty) < \infty$ if and only if 0 is positive recurrent):

(i) If $m(\infty) = \infty$, we have that s, h_* and h_0 all coincide. If l' for M is natural with $m(\infty) < \infty$, we have that s and h_* coincide.
(ii) For the h-transform of M^0 for $h = s$, h_* or h_0, the boundary 0 is entrance.
(iii) For the h-transform of M^0 for $h = s$,

 (a) the process explodes to ∞ in finite time when l' for M is entrance;
 (b) the process has no killing inside the interior of I and is elastic at l' when l' for M is regular-reflecting;
 (c) the process is conservative otherwise.

(iv) For the h-transform of M^0 for $h = h_*$, the process is conservative.
(v) For the h-transform of M^0 for $h = h_0$ when $m(\infty) < \infty$, the process has killing inside.

Let us give an example where the three functions are distinct from each other. Let M be a reflecting Brownian motion on $[0, l']$ where both boundaries 0 and l' are regular-reflecting. Then we have

$$h_*(x) = \frac{2l'}{\pi} \sin \frac{\pi x}{2l'}, \quad h_0(x) = x - \frac{x^2}{2l'}, \quad x \in [0, l']. \tag{5}$$

We shall come back to this example in Example 4.2.

We give several remarks about earlier studies related to the h-transforms for the three functions.

1°). The h-transform of M^0 for $h = s$ is sometimes used to obtain a integral representation of the excursion measure: see Salminen [23], Yano [29] and Salminen–Vallois–Yor [24].
2°). The penalization problems for one-dimensional diffusions which generalize Theorem 1.2 were studied in Profeta [19, 20].
3°). The counterpart of h_0 for one-dimensional symmetric Lévy processes where every point is regular for itself has been introduced by Salminen–Yor [25] who proved an analogue of the Tanaka formula. Yano–Yano–Yor [33] and Yano [30, 31] investigated the h-transform of M^0 and studied the penalisation problems and related problems. For an approach to asymmetric cases, see Yano [32].

This paper is organized as follows. We prepare notation and several basic properties for one-dimensional generalized diffusions in Sect. 2 and for excursion measures in Sect. 3. In Sect. 4, we prove existence of h_0. Section 5 is devoted to the proofs of Theorems 1.1, 1.2 and 1.3. In Sect. 6, we study invariance and excessiveness of h_0 and s. In Sect. 7, we study several properties of the h-transforms.

2 Notation and Basic Properties for Generalized Diffusions

Let $\tilde{m}$ and $\tilde{s}$ be strictly-increasing functions $(0, l') \to \mathbb{R}$ such that $\tilde{m}$ is right-continuous and $\tilde{s}$ is continuous. We fix a constant $0 < c < l'$ (the choice of c does not affect the subsequent argument at all). We set

$$F_1 = \iint_{l'>y>x>c} \mathrm{d}\tilde{m}(x)\mathrm{d}\tilde{s}(y), \quad F_2 = \iint_{l'>y>x>c} \mathrm{d}\tilde{s}(x)\mathrm{d}\tilde{m}(y). \tag{6}$$

We adopt Feller's classification of the right boundary l' with a slight refinement as follows:

(i) If $F_1 < \infty$ and $F_2 < \infty$, then l' is called *regular*. In this case we have $\tilde{s}(l'-) < \infty$.
(ii) If $F_1 < \infty$ and $F_2 = \infty$, then l' is called *exit*. In this case we have $\tilde{s}(l'-) < \infty$.
(iii) If $F_1 = \infty$ and $F_2 < \infty$, then l' is called *entrance*. In this case we have $\tilde{m}(l'-) < \infty$.
(iv) If $F_1 = \infty$ and $F_2 = \infty$, then l' is called *natural*. In this case we have either $\tilde{s}(l'-) = \infty$ or $\tilde{m}(l'-) = \infty$. There are three subcases as follows:

   (a) If $\tilde{s}(l'-) = \infty$ and $\tilde{m}(l'-) = \infty$, then l' is called *type-1-natural*.
   (b) If $\tilde{s}(l'-) = \infty$ and $\tilde{m}(l'-) < \infty$, then l' is called *type-2-natural*.
   (c) If $\tilde{s}(l'-) < \infty$ and $\tilde{m}(l'-) = \infty$, then l' is called *type-3-natural* or *natural-approachable*.

The classification of the left boundary 0 is defined in a similar way.

Let m be a function $[0, \infty) \to [0, \infty]$ which is non-decreasing, right-continuous and $m(0) = 0$. We assume that there exist l' and l with $0 < l' \le l \le \infty$ such that

$$m \text{ is } \begin{cases} \text{strictly-increasing on } [0, l'), \\ \text{flat and finite on } [l', l), \\ \text{infinite on } [l, \infty). \end{cases} \tag{7}$$

We take $\tilde{m} = m|_{(0,l')}$ and the natural scale $\tilde{s}(x) = s(x) = x$ on $(0, l')$ to adopt the classification of the boundaries 0 and l'. We choose the intervals I' and I as follows:

(i) If l' is regular, there are three subcases related to the boundary condition as follows:

   (a) If $l' < l = \infty$, then l' is called *regular-reflecting* and $I' = I = [0, l']$.
   (b) If $l' < l < \infty$, then l' is called *regular-elastic*, $I' = [0, l']$ and $I = [0, l'] \cup \{l\}$.
   (c) If $l' = l < \infty$, then l' is called *regular-absorbing*, $I' = [0, l)$ and $I = [0, l]$.

(ii) If l' is exit, then $l' = l < \infty$, $I' = [0, l)$ and $I = [0, l]$.

(iii) If l' is entrance, then $l' = l = \infty$ and $I' = I = [0, \infty)$.
(iv) If l' is natural, then $l' = l \leq \infty$ and $I' = I = [0, l)$.

We always write $(X_t)_{t \geq 0}$ for the coordinate process on the space of paths $\omega : [0, \infty) \to \mathbb{R} \cup \{\partial\}$ with $\zeta(\omega) \in [0, \infty)$ such that $\omega : [0, \zeta(\omega)) \to \mathbb{R}$ is continuous and $\omega(t) = \partial$ for all $t \geq \zeta(\omega)$. We always adopt the canonical representation for each process and the right-continuous filtration $(\mathscr{F}_t)_{t \geq 0}$ defined by $\mathscr{F}_t = \bigcap_{s > t} \sigma(X_u : u \leq s)$.

We study a $D_m D_s$-generalized diffusion on I where 0 is the regular-reflecting boundary (see Watanabe [28, Sect. 3]). Such a process can be constructed from the Brownian motion via the time-change method. Let $\{(X_t)_{t \geq 0}, (\mathbb{P}^B_x)_{x \in \mathbb{R}}\}$ denote the Brownian motion on $\mathbb{R}$ and let $\ell(t, x)$ denote its jointly-continuous local time. Set $A(t) = \int_I \ell(t, x) \mathrm{d}m(x)$ and write A^{-1} for the right-continuous inverse of A. Then the process $\{(X_{A^{-1}(t)})_{t \geq 0}, (\mathbb{P}^B_x)_{x \in I}\}$ is a realization of the desired generalized diffusion.

Let $M = \{(X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in I}\}$ denote the $D_m D_s$-generalized diffusion. We denote the resolvent operator of M by

$$R_q f(x) = \mathbb{P}_x \left[\int_0^\infty \mathrm{e}^{-qt} f(X_t) \mathrm{d}t \right], \quad q > 0. \tag{8}$$

For $x \in I$, we write

$$T_x = \inf\{t > 0 : X_t = x\}. \tag{9}$$

Then, for $a, x, b \in I$ with $a < x < b$, we have

$$\mathbb{P}_x(T_a > T_b) = \frac{x - a}{b - a}. \tag{10}$$

Note that, whenever $l \in I$, we have $\mathbb{P}_x(T_l < \infty) = 1$ for all $x \in I$ and l is a trap for M.

For a function $f : [0, l) \to \mathbb{R}$, we define

$$Jf(x) = \int_0^x \mathrm{d}y \int_{(0,y]} f(z) \mathrm{d}m(z). \tag{11}$$

We sometimes write $s(x) = x$ to emphasize the natural scale. For $q \in \mathbb{C}$, we write ϕ_q and ψ_q for the unique solutions of the integral equations

$$\phi_q = 1 + qJ\phi_q \quad \text{and} \quad \psi_q = s + qJ\psi_q \quad \text{on } [0, l), \tag{12}$$

respectively. They can be represented as

$$\phi_q = \sum_{n=0}^\infty q^n J^n 1 \quad \text{and} \quad \psi_q = \sum_{n=0}^\infty q^n J^n s. \tag{13}$$

Let $q > 0$. Note that ϕ_q and ψ_q are non-negative increasing functions. Set

$$H(q) = \lim_{x \uparrow l} \frac{\psi_q(x)}{\phi_q(x)} = \int_0^l \frac{1}{\phi_q(x)^2} \mathrm{d}x. \tag{14}$$

Then there exist σ-finite measures σ and σ^* on $[0, \infty)$ such that

$$H(q) = \int_{[0,\infty)} \frac{1}{q+\xi} \sigma(\mathrm{d}\xi) \quad \text{and} \quad \frac{1}{qH(q)} = \int_{[0,\infty)} \frac{1}{q+\xi} \sigma^*(\mathrm{d}\xi). \tag{15}$$

Note that

$$l = \lim_{q \downarrow 0} H(q) = \int_{[0,\infty)} \frac{\sigma(\mathrm{d}\xi)}{\xi} = \frac{1}{\sigma^*(\{0\})} \in (0, \infty]. \tag{16}$$

If we write $m(\infty) = \lim_{x \to \infty} m(x)$, we have

$$\pi_0 := \lim_{q \downarrow 0} qH(q) = \sigma(\{0\}) = \frac{1}{\int_{[0,\infty)} \frac{\sigma^*(\mathrm{d}\xi)}{\xi}} = \frac{1}{m(\infty)} \in [0, \infty). \tag{17}$$

Note that $\pi_0 = 0$ whenever $l < \infty$. We define

$$\rho_q(x) = \phi_q(x) - \frac{1}{H(q)} \psi_q(x). \tag{18}$$

Then the function ρ_q is a non-negative decreasing function on $[0, l)$ which satisfies

$$\rho_q = 1 - \frac{s}{H(q)} + qJ\rho_q. \tag{19}$$

We define

$$r_q(x, y) = r_q(y, x) = H(q)\phi_q(x)\rho_q(y) \quad 0 \le x \le y, \ x, y \in I'. \tag{20}$$

In particular, we have $r_q(0, x) = r_q(x, 0) = H(q)\rho_q(x)$ and $r_q(0, 0) = H(q)$. It is well-known (see, e.g., [13]) that

$$\mathbb{P}_x[\mathrm{e}^{-qT_y}] = \frac{r_q(x, y)}{r_q(y, y)}, \quad x, y \in I', \ q > 0. \tag{21}$$

In particular, we have

$$\rho_q(x) = \frac{r_q(x, 0)}{r_q(0, 0)} = \mathbb{P}_x[\mathrm{e}^{-qT_0}], \quad x \in I', \ q > 0. \tag{22}$$

We write $M' = \{(X_t)_{t\geq 0}, (\mathbb{P}'_x)_{x\in I'}\}$ for the process M killed upon hitting l. We write R'_q for the resolvent operator of M'. It is well-known (see, e.g., [13]) that $r_q(x, y)$ is the resolvent density of M' with respect to dm, or in other words,

$$R'_q f(x) = \int_{I'} f(y) r_q(x, y) \mathrm{d}m(y). \tag{23}$$

We have the resolvent equation

$$\int_{I'} r_q(x, y) r_p(y, z) \mathrm{d}m(y) = \frac{r_p(x, z) - r_q(x, z)}{q - p}, \quad x, z \in I',\ q, p > 0. \tag{24}$$

If $l \in I$, we define

$$r_q(l, y) = 0 \quad \text{for } y \in I', \tag{25}$$

$$r_q(x, l) = \frac{1}{q} - R'_q 1(x) \quad \text{for } x \in I', \tag{26}$$

$$r_q(l, l) = \frac{1}{q}, \tag{27}$$

and define a measure $\tilde{m}$ on I by

$$\tilde{m}(\mathrm{d}y) = 1_{I'}(y) \mathrm{d}m(y) + \delta_l(\mathrm{d}y). \tag{28}$$

We emphasize that $r_q(x, y)$ is no longer symmetric when either x or y equals l.

Proposition 2.1 *The formulae* (21) *and* (23) *extend to*

$$\mathbb{P}_x[\mathrm{e}^{-qT_y}] = \frac{r_q(x, y)}{r_q(y, y)}, \quad x, y \in I,\ q > 0, \tag{29}$$

$$R_q f(x) = \int_I f(y) r_q(x, y) \tilde{m}(\mathrm{d}y), \quad x \in I,\ q > 0. \tag{30}$$

Proof Suppose $l \in I$.

First, we let $x = l$. Then we have $\mathbb{P}_l[\mathrm{e}^{-qT_y}] = 0 = \frac{r_q(l,y)}{r_q(y,y)}$ for $y \in I'$ and $\mathbb{P}_l[\mathrm{e}^{-qT_l}] = 1 = \frac{r_q(l,l)}{r_q(l,l)}$. We also have $R_q f(l) = \mathbb{P}_l\big[\int_0^\infty \mathrm{e}^{-qt} f(X_t) \mathrm{d}t\big] = f(l)/q = f(l) r_q(l, l) \tilde{m}(\{l\})$. Hence we obtain (29) and (30) in this case.

Second, we assume $x \in I'$. On one hand, we have

$$\int_0^\infty \mathrm{e}^{-qt} \mathbb{P}_x(t \geq T_l) \mathrm{d}t = \frac{1}{q} \mathbb{P}_x[\mathrm{e}^{-qT_l}] = r_q(l, l) \mathbb{P}_x[\mathrm{e}^{-qT_l}]. \tag{31}$$

On the other hand, we have

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(t \geq T_l)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt}\left\{1 - \mathbb{P}_x(X_t \in I')\right\}\mathrm{d}t = \frac{1}{q} - R_q'1(x) = r_q(x,l). \tag{32}$$

Hence we obtain (29) for $y = l$. Using (23), we obtain

$$R_q f(x) = R_q' f(x) + f(l)\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(t \geq T_l)\mathrm{d}t \tag{33}$$

$$= \int_{I'} f(y) r_q(x,y)\mathrm{d}m(y) + f(l) r_q(x,l)\tilde{m}(\{l\}), \tag{34}$$

which implies (30). □

3 The Excursion Measure Away from 0

For $y \in I$, we write $(L_t(y))_{t\geq 0}$ for the local time at y normalized as follows (see [9]):

$$\mathbb{P}_x\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t(y)\right] = r_q(x,y), \quad x \in I,\ q > 0. \tag{35}$$

We write L_t for $L_t(0)$. Let $\boldsymbol{n}$ denote the excursion measure away from 0 corresponding to $(L_t)_{t\geq 0}$ (see [1]), where we adopt the convention that

$$X_t = 0 \text{ for all } t \geq T_0, \quad \boldsymbol{n}\text{-a.e.} \tag{36}$$

We define the functional N_q by

$$N_q f = \boldsymbol{n}\left[\int_0^\infty \mathrm{e}^{-qt} f(X_t)\mathrm{d}t\right], \quad q > 0. \tag{37}$$

Then it is well-known (see [22]) that $\boldsymbol{n}$ can be characterized by the following identity:

$$N_q f = \frac{R_q f(0)}{r_q(0,0)} \quad \text{whenever } f(0) = 0. \tag{38}$$

In particular, taking $f = 1_{I\setminus\{0\}}$, we have

$$\boldsymbol{n}\left[1 - \mathrm{e}^{-qT_0}\right] = \frac{1}{r_q(0,0)} = \frac{1}{H(q)} \tag{39}$$

and, by (14), we have

$$\boldsymbol{n}(T_0 = \infty) = \lim_{q \downarrow 0} \frac{1}{H(q)} = \frac{1}{l}. \tag{40}$$

We write $M^0 = \{(X_t)_{t \ge 0}, (\mathbb{P}_x^0)_{x \in I}\}$ for the process M stopped upon hitting 0 and write R_q^0 for the resolvent operator of M^0. By the strong Markov property of M, we have

$$R_q f(x) = R_q^0 f(x) + \mathbb{P}_x[\mathrm{e}^{-qT_0}] R_q f(0). \tag{41}$$

The resolvent density with respect to $\tilde{m}(\mathrm{d}y)$ is given as

$$r_q^0(x,y) = r_q(x,y) - \frac{r_q(x,0) r_q(0,y)}{r_q(0,0)} \quad \text{for } x,y \in I. \tag{42}$$

Note that $r_q^0(x,y) = \psi_q(x)\rho_q(y)$ for $x \le y$ and that

$$\mathbb{P}_x^0[\mathrm{e}^{-qT_y}] = \frac{r_q^0(x,y)}{r_q^0(y,y)} = \frac{\psi_q(x)}{\psi_q(y)} \quad \text{for } x,y \in I,\, x \le y. \tag{43}$$

Note also that $(L_t(y))_{t \ge 0}$ is the local time at y such that

$$\mathbb{P}_x^0\left[\int_0^\infty \mathrm{e}^{-qt} \mathrm{d}L_t(y)\right] = r_q^0(x,y), \quad x,y \in I \setminus \{0\},\ q > 0. \tag{44}$$

The strong Markov property of $\boldsymbol{n}$ may be stated as

$$\boldsymbol{n}[F_T\, G \circ \theta_T] = \boldsymbol{n}\big[F_T \mathbb{P}_{X_T}^0[G]\big], \tag{45}$$

where T is a stopping time, F_T is a non-negative $\mathscr{F}_T$-measurable functional, G is a non-negative measurable functional such that $0 < \boldsymbol{n}[F_T] < \infty$ or $0 < \boldsymbol{n}[G \circ \theta_T] < \infty$.

Let $x, y \in I$ be such that $0 < x < y$. Because of the properties of excursion paths of a generalized diffusion, we see that X under $\boldsymbol{n}$ hits y if and only if X hits x and in addition $X \circ \theta_{T_x}$ hits y. Hence, by the strong Markov property of $\boldsymbol{n}$, we have,

$$\begin{aligned} \boldsymbol{n}(T_y < \infty) &= \boldsymbol{n}\big(\{T_y < \infty\} \circ \theta_{T_x} \cap \{T_x < \infty\}\big) && (46) \\ &= \mathbb{P}_x^0(T_y < \infty)\boldsymbol{n}(T_x < \infty) && (47) \\ &= \mathbb{P}_x(T_y < T_0)\boldsymbol{n}(T_x < \infty) && (48) \\ &= \frac{x}{y}\boldsymbol{n}(T_x < \infty). && (49) \end{aligned}$$

This shows that $x\boldsymbol{n}(T_x < \infty)$ equals a constant C in $x \in I \setminus \{0\}$, so that we have

$$\boldsymbol{n}(T_x < \infty) = \frac{C}{x}, \quad x \in I \setminus \{0\}. \tag{50}$$

If $l \in I$, then we have

$$C = l\boldsymbol{n}(T_l < \infty) = l\boldsymbol{n}(T_0 = \infty) = 1. \tag{51}$$

The following theorem generalizes this fact and a result of [4].

Theorem 3.1 (See also [4]) *In any case,* $C = 1$.

Theorem 3.1 will be proved at the end of Sect. 6.

The following lemma is the first step of the proof of Theorem 3.1.

Lemma 3.2 (See also [4]) *The constant* C *may be represented as*

$$C = \lim_{t \downarrow 0} \boldsymbol{n}[X_t]. \tag{52}$$

Proof By definition of C, we have

$$C = \sup_{x \in I \setminus \{0\}} x\boldsymbol{n}(T_x < \infty). \tag{53}$$

Since $\boldsymbol{n}(t < T_x < \infty) \uparrow \boldsymbol{n}(T_x < \infty)$ as $t \downarrow 0$, we have

$$C = \sup_{x \in I \setminus \{0\}} x \sup_{t>0} \boldsymbol{n}(t < T_x < \infty) \tag{54}$$

$$= \sup_{t>0} \sup_{x \in I \setminus \{0\}} x\boldsymbol{n}(t < T_x < \infty) \tag{55}$$

$$= \lim_{t \downarrow 0} \sup_{x \in I \setminus \{0\}} x\boldsymbol{n}(t < T_x < \infty). \tag{56}$$

Because of the properties of excursion paths of a generalized diffusion, we see that X under $\boldsymbol{n}$ hits T_x after t if and only if X does not hit x nor 0 until t and $X \circ \theta_t$ hits x. Hence, by the strong Markov property of $\boldsymbol{n}$, we have,

$$\sup_{x \in I \setminus \{0\}} x\boldsymbol{n}(t < T_x < \infty) = \sup_{x \in I \setminus \{0\}} x\boldsymbol{n}(\{T_x < \infty\} \circ \theta_t \cap \{t < T_x \wedge T_0\}) \tag{57}$$

$$= \sup_{x \in I \setminus \{0\}} \boldsymbol{n}[x\mathbb{P}_{X_t}(T_x < T_0); t < T_x \wedge T_0] \tag{58}$$

$$= \sup_{x \in I \setminus \{0\}} \boldsymbol{n}[X_t; t < T_x \wedge T_0]\,. \tag{59}$$

We divide the remainder of the proof into three cases.

(i) The case $l < \infty$. Since $T_x \le T_l$ for $x \in I \setminus \{0\}$, we have

$$(59) = \boldsymbol{n}[X_t; t < T_l \wedge T_0] \tag{60}$$

$$= \boldsymbol{n}[X_t; t < T_0] - \boldsymbol{n}[X_t; T_l \le t < T_0]\,. \tag{61}$$

Since $\boldsymbol{n}(T_l < \infty) < \infty$, we may apply the dominated convergence theorem to obtain

$$\boldsymbol{n}[X_t; T_l \le t < T_0] \le \boldsymbol{n}[X_t; T_l < \infty] \underset{t \downarrow 0}{\longrightarrow} 0, \tag{62}$$

which implies Equality (52), since $\boldsymbol{n}[X_t; t < T_0] = \boldsymbol{n}[X_t]$.

(ii) The case $l' < l = \infty$. The proof of Case (i) works if we replace l by l'.

(iii) The case $l' = l = \infty$. Since $T_x \uparrow \infty$ as $x \to \infty$, we have

$$(59) = \lim_{x \to \infty} \boldsymbol{n}[X_t; t < T_x \wedge T_0] = \boldsymbol{n}[X_t; t < T_0] \tag{63}$$

by the monotone convergence theorem. This implies Equality (52). □

4 The Renormalized Zero Resolvent

For $q > 0$ and $x \in I$, we set

$$h_q(x) = r_q(0,0) - r_q(x,0). \tag{64}$$

Note that $h_q(x)$ is always non-negative, since we have, by (29),

$$\frac{h_q(x)}{H(q)} = \mathbb{P}_x[1 - \mathrm{e}^{-qT_0}]. \tag{65}$$

The following theorem asserts that the limit $h_0 := \lim_{q \downarrow 0} h_q$ exists, which will be called the *renormalized zero resolvent*.

Theorem 4.1 *For $x \in I$, the limit $h_0(x) := \lim_{q \downarrow 0} h_q(x)$ exists and is represented as*

$$h_0(x) = s(x) - g(x) = x - g(x), \tag{66}$$

where

$$g(x) = \pi_0 J1(x) = \pi_0 \int_0^x m(y)\mathrm{d}y. \tag{67}$$

The function $h_0(x)$ is continuous increasing in $x \in I$, positive in $x \in I \setminus \{0\}$ and zero at $x = 0$. In particular, if $\pi_0 = 0$, then h_0 coincides with the scale function, i.e., $h_0(x) = s(x) = x$.

Proof For $x \in I'$, we have

$$h_q(x) = H(q)\{1 - \rho_q(x)\} = x - qH(q)J\rho_q(x) \underset{q\downarrow 0}{\longrightarrow} x - \pi_0 J1(x), \tag{68}$$

where we used the facts that $0 \leq \rho_q(x) \leq 1$ and $\rho_q(x) \to 1 - \frac{x}{l}(= 1 \text{ if } \pi_0 > 0)$ as $q \downarrow 0$ and used the dominated convergence theorem. If $l \in I$, we have

$$h_q(l) = r_q(0,0) = H(q) \underset{q\downarrow 0}{\longrightarrow} l, \tag{69}$$

and hence we obtain $h_0(l) = l$, which shows (66) for $x = l$, since $\pi_0 = 0$ in this case.

It is obvious that h_0 is continuous. If $\pi_0 = 0$, then $h_0(x) = x$ is increasing in $x \in I$ and positive in $x \in I \setminus \{0\}$. If $\pi_0 > 0$, then we have $\pi_0 m(y) \leq 1$ for all $y \in I$ and $\pi_0 m(y) < 1$ for all $y < l'$, so that $h_0(x)$ is increasing in $x \in I$ and positive in $x \in I \setminus \{0\}$. The proof is now complete. □

Example 4.2 Let $0 < l' < l = \infty$ and let $m(x) = \min\{x, l'\}$. In this case, M is a Brownian motion on $[0, l']$ where both boundaries 0 and l' are regular-reflecting. Then we have

$$h_*(x) = \frac{2l'}{\pi} \sin \frac{\pi x}{2l'}, \quad h_0(x) = x - \frac{x^2}{2l'}, \quad x \in [0, l']. \tag{70}$$

Note that we have $\pi_0 = 1/m(\infty) = 1/l'$ and

$$\phi_q(x) = \begin{cases} \cosh \sqrt{q}x & \text{for } x \in [0, l'], \\ \phi_q(l') + \phi_q'(l')(x - l') & \text{for } x \in (l', \infty), \end{cases} \tag{71}$$

$$\psi_q(x) = \begin{cases} \sinh \sqrt{q}x / \sqrt{q} & \text{for } x \in [0, l'], \\ \psi_q(l') + \psi_q'(l')(x - l') & \text{for } x \in (l', \infty), \end{cases} \tag{72}$$

$$H(q) = \frac{1}{\sqrt{q} \tanh \sqrt{q}l'}. \tag{73}$$

We study recurrence and transience of 0.

Theorem 4.3 *For M, the following assertions hold:*

(i) 0 is transient if and only if $l < \infty$. In this case, it holds that

$$\mathbb{P}_x(T_0 = \infty) = \frac{x}{l} \quad \text{for } x \in I. \tag{74}$$

(ii) 0 is positive recurrent if and only if $\pi_0 > 0$. *In this case, it holds that*

$$\mathbb{P}_x[T_0] = \frac{h_0(x)}{\pi_0} \quad \text{for } x \in I. \tag{75}$$

(iii) 0 is null recurrent if and only if $l = \infty$ *and* $\pi_0 = 0$.

Although this theorem seems well-known, we give the proof for completeness of the paper.

Proof

(i) By the formula (21), we have, for $x \in I'$,

$$\mathbb{P}_x(T_0 = \infty) = \lim_{q\downarrow 0} \mathbb{P}_x[1 - \mathrm{e}^{-qT_0}] = \lim_{q\downarrow 0} \left\{ \frac{\psi_q(x)}{H(q)} - \{\phi_q(x) - 1\} \right\} = \frac{x}{l}. \tag{76}$$

Hence 0 is transient if and only if $l < \infty$. If $x = l \in I$, it is obvious that $\mathbb{P}_l(T_0 = \infty) = 1$. This proves the claim.

(ii) Since $(1 - \mathrm{e}^{-x})/x \uparrow 1$ as $x \downarrow 0$, we may apply the monotone convergence theorem to see that

$$\mathbb{P}_x[T_0] = \lim_{q\downarrow 0} \frac{1}{q} \mathbb{P}_x[1 - \mathrm{e}^{-qT_0}] = \lim_{q\downarrow 0} \frac{h_q(x)}{q r_q(0,0)} = \frac{h_0(x)}{\pi_0}, \tag{77}$$

for $x \in I$. This shows that $\mathbb{P}_x[T_0] < \infty$ if and only if $\pi_0 > 0$, which proves the claim.

(iii) This is obvious by (i) and (ii). □

We illustrate the classification of recurrence of 0 of Theorem 4.3 as follows:

	$l = \infty$	$l < \infty$
$\pi_0 = 0$	(1) Null recurrent	(3) Transient
$\pi_0 > 0$	(2) Positive recurrent	Impossible

(1) l' is type-1-natural
(2) l' is type-2-natural, entrance or regular-reflecting
(3) l' is type-3-natural, exit, regular-elastic or regular-absorbing

5 Various Conditionings to Avoid Zero

We prove the three theorems concerning conditionings to avoid zero. We need the following lemma in later use.

Lemma 5.1 *For any stopping time T and for any $x \in I$, it holds that*

$$\mathbb{P}_x^0[X_T; T < \infty] \le x. \tag{78}$$

Proof By [2, Proposition II.2.8], it suffices to prove that $\mathbb{P}_x^0[X_t] \le x$ for all $t > 0$.

Note that $x \le \liminf_{t \downarrow 0} \mathbb{P}_x^0[X_t]$ for all $x \in I$ by Fatou's lemma. By the help of [2, Corollary II.5.3], it suffices to prove that

$$\mathbb{P}_x^0[X_{T_K}; T_K < \infty] \le x \quad \text{for } x \in I \setminus K \tag{79}$$

for all compact subset K of I.

Let K be a compact subset of I and let $x \in I \setminus K$. Let $a = \sup(K \cap (0, x)) \cup \{0\}$ and $b = \inf(K \cap (x, l)) \cup \{l\}$. Since 0 and l are traps for $\mathbb{P}_x^0$, we have $T_K = T_a \wedge T_b$ on $\{T_K < \infty\}$, $\mathbb{P}_x^0$-a.e. and thus we obtain

$$\mathbb{P}_x^0[X_{T_K}; T_K < \infty] \le \mathbb{P}_x^0[X_{T_a \wedge T_b}] = a\mathbb{P}_x(T_a < T_b) + b\mathbb{P}_x(T_a > T_b) = x, \tag{80}$$

which proves (79) for $x \notin K$. Hence we obtain the desired result. □

First, we prove Theorem 1.1.

Proof of Theorem 1.1

(i) Suppose that $l'(= l)$ is entrance or natural. By the strong Markov property, we have

$$a\mathbb{P}_x[F_T; T < T_a < T_0] = a\mathbb{P}_x[F_T \mathbb{P}_{X_T}(T_a < T_0); T < T_a \wedge T_0] \tag{81}$$

$$= \mathbb{P}_x[F_T X_T; T < T_a \wedge T_0] \tag{82}$$

$$= \mathbb{P}_x^0[F_T X_T; T < T_a] \tag{83}$$

since $X_T = 0$ on $\{T \ge T_0\}$, $\mathbb{P}_x^0$-a.s. By the fact that $1_{\{T<T_a\}} \to 1_{\{T<\infty\}}$, $\mathbb{P}_x^0$-a.s. and by Lemma 5.1, we may thus apply the dominated convergence theorem to see that (83) converges as $a \uparrow l$ to $\mathbb{P}_x^0[F_T X_T; T < \infty]$. Since $a\mathbb{P}_x(T_a < T_0) = x$, we obtain (2).

(ii) Suppose that l' is regular-elastic, regular-absorbing or exit. By the strong Markov property, we have

$$l\mathbb{P}_x[F_T; T < T_l < T_0] = l\mathbb{P}_x[F_T \mathbb{P}_{X_T}(T_l < T_0); T < T_l \wedge T_0] \tag{84}$$

$$= \mathbb{P}_x^0[F_T X_T; T < T_l]. \tag{85}$$

Since $\mathbb{P}_x(T_l < T_0) = x/l$, we obtain (2).

(iii) In the case where l' is regular-reflecting, the proof is the same as (ii) if we replace l by l', and so we omit it. □

Second, we prove Theorem 1.2.

Proof of Theorem 1.2 By McKean [17] (see also [29]), we have the following facts. For $\gamma \in \mathbb{R}$, let ψ_γ be the solution of the integral equation $\psi_\gamma = s + \gamma J\psi_\gamma$. Then we have the eigendifferential expansion

$$r_q(x, y) = \int_{(-\infty,0)} (q-\gamma)^{-1}\psi_\gamma(x)\psi_\gamma(y)\theta(\mathrm{d}\gamma) \tag{86}$$

for the spectral measure θ. We now have

$$\frac{\mathbb{P}_x(T_0 \in \mathrm{d}t)}{\mathrm{d}t} = \int_{(-\infty,0)} \mathrm{e}^{\gamma t}\psi_\gamma(x)\theta(\mathrm{d}\gamma), \quad \frac{\boldsymbol{n}(T_0 \in \mathrm{d}t)}{\mathrm{d}t} = \int_{(-\infty,0)} \mathrm{e}^{\gamma t}\theta(\mathrm{d}\gamma), \tag{87}$$

and, for $r > 0$,

$$\lim_{t\to\infty} \frac{\mathbb{P}_x(T_0 > t)}{\boldsymbol{n}(T_0 > t)} = h_*(x), \quad \lim_{t\to\infty} \frac{\boldsymbol{n}(T_0 > t - r)}{\boldsymbol{n}(T_0 > t)} = \mathrm{e}^{-\gamma_* r}. \tag{88}$$

We note that γ_* equals the supremum of the support of θ and that $h_* = \psi_{\gamma_*}$. If l' is natural, exit, regular-absorbing or regular-elastic, we see that $\gamma_* = 0$ and $h_* = s$.

By the strong Markov property, we have

$$\mathbb{P}_x[F_T; T < t < T_0] = \mathbb{P}_x^0\left[F_T\ \mathbb{P}_{X_T}(T_0 > t - r)|_{r=T}; T < t\right]. \tag{89}$$

Since we have

$$\boldsymbol{n}(T_0 > t) \geq \boldsymbol{n}\big(T_y < T_0,\ T_0 \circ \theta_{T_y} > t\big) = \frac{1}{y}\mathbb{P}_y(T_0 > t), \tag{90}$$

we have $\mathbb{P}_y(T_0 > t - r) \leq y\boldsymbol{n}(T_0 > t - r)$. Hence, by Lemma 5.1 and by the dominated convergence theorem, we obtain

$$\lim_{t\to\infty} \frac{1}{\boldsymbol{n}(T_0 > t)}\mathbb{P}_x[F_T; T < t < T_0] = \mathbb{P}_x^0\left[F_T\mathrm{e}^{-\gamma_* T}h_*(X_T); T < \infty\right]. \tag{91}$$

Dividing both sides of (91) by those of the first equality of (88), we obtain (3). □

Third, we prove Theorem 1.3.

Proof of Theorem 1.3 By (65), we have

$$H(q)\mathbb{P}_x(\boldsymbol{e}_q < T_0) = h_q(x) \underset{q\downarrow 0}{\longrightarrow} h_0(x). \tag{92}$$

Note that

$$\mathbb{P}_x[F_T; T < \boldsymbol{e}_q < T_0] = \mathbb{P}_x\left[F_T \int_T^\infty 1_{\{t<T_0\}} q\mathrm{e}^{-qt}\mathrm{d}t\right] \tag{93}$$

$$= \mathbb{P}_x\left[F_T \mathrm{e}^{-qT} \int_0^\infty 1_{\{t+T<T_0\}} q\mathrm{e}^{-qt}\mathrm{d}t\right] \tag{94}$$

$$= \mathbb{P}_x\big[F_T \mathrm{e}^{-qT}; \boldsymbol{e}_q + T < T_0\big] \tag{95}$$

$$= \mathbb{P}_x\big[F_T \mathrm{e}^{-qT} 1_{\{\boldsymbol{e}_q<T_0\}} \circ \theta_T; T < T_0\big]. \tag{96}$$

By the strong Markov property, we have

$$H(q)\mathbb{P}_x[F_T; T < \boldsymbol{e}_q < T_0] = H(q)\mathbb{P}_x\big[F_T \mathrm{e}^{-qT} \mathbb{P}_{X_T}(\boldsymbol{e}_q < T_0); T < T_0\big] \tag{97}$$

$$= \mathbb{P}_x^0\big[F_T \mathrm{e}^{-qT} h_q(X_T); T < \infty\big], \tag{98}$$

since $h_q(X_T) = 0$ on $\{T \geq T_0\}$, $\mathbb{P}_x^0$-a.s. Once the interchange of the limit and the integration is justified, we see that (98) converges as $q \downarrow 0$ to $\mathbb{P}_x^0[F_T h_0(X_T); T < \infty]$, and hence we obtain (4).

Let us prove $h_q(x) \leq x$ for $q > 0$ and $x \in I$. If $x \in I'$, we use (19) and we have

$$h_q(x) = H(q)\{1 - \rho_q(x)\} = x - qH(q)J\rho_q(x) \leq x. \tag{99}$$

If $l \in I$, we have $h_q(l) = H(q) \leq l$. We thus see that the integrand of (98) is dominated by X_T. By Lemma 5.1, we thus see that we may apply the dominated convergence theorem, and therefore the proof is complete. □

6 Invariance and Excessiveness

Let us introduce notation of invariance and excessiveness. Let h be a non-negative measurable function on E.

(i) We say h is *α-invariant* for M^0 (resp. for $\boldsymbol{n}$) ($\alpha \in \mathbb{R}$) if $\mathrm{e}^{-\alpha t}\mathbb{P}_x^0[h(X_t)] = h(x)$ for all $x \in E$ and all $t > 0$ (resp. there exists a positive constant C such that $\mathrm{e}^{-\alpha t}\boldsymbol{n}[h(X_t)] = C$ for all $t > 0$).

(ii) We say h is *α-excessive* for M^0 (resp. for $\boldsymbol{n}$) ($\alpha \geq 0$) if $\mathrm{e}^{-\alpha t}\mathbb{P}_x^0[h(X_t)] \leq h(x)$ for all $x \in E$ and all $t > 0$ and $\mathrm{e}^{-\alpha t}\mathbb{P}_x^0[h(X_t)] \to h(x)$ as $t \downarrow 0$ (resp. there exists a positive constant C such that $\mathrm{e}^{-\alpha t}\boldsymbol{n}[h(X_t)] \leq C$ for all $t > 0$ and $\boldsymbol{n}[h(X_t)] \to C$ as $t \downarrow 0$).

(iii) We say h is *invariant* (resp. *excessive*) when h is 0-invariant (resp. 0-excessive).

We give the following remarks.

(i) As a corollary of Theorem 1.2, the function h_* is γ_*-invariant for M^0.
(ii) As a corollary of (i), the function s is invariant for M^0 when l' for M is natural, exit, regular-absorbing or regular-elastic.
(iii) As a corollary of Lemma 5.1, the function s is excessive for M^0 when l' for M is entrance or regular-reflecting.
(iv) As a corollary of Theorem 1.3, the function h_0 is excessive for M^0.

In this section, we prove several properties to complement these statements. Following [8, Sect. 2], we introduce the operators

$$D_m f(x) = \lim_{\varepsilon,\varepsilon'\downarrow 0} \frac{f(x+\varepsilon) - f(x-\varepsilon')}{m(x+\varepsilon) - m(x-\varepsilon')} \tag{100}$$

whenever the limit exist. Note that $f(x) = \psi_q(x)$ (resp. $f(x) = \rho_q(x)$) is an increasing (resp. decreasing) solution of the differential equation $D_m D_s f = qf$ satisfying $f(0) = 0$ and $D_s f(0) = 1$ (resp. $f(0) = 1$ and $D_s f(0) = -1/H(q)$).

Theorem 6.1 *The function h_* is γ_*-invariant for $\boldsymbol{n}$ when l' for M is entrance or regular-reflecting.*

Proof By [7, Sect. 12]), we see that if $D_m D_s f = F$ and $D_m D_s g = G$ then

$$D_m\{g D_s f - f D_s g\} = gF - fG. \tag{101}$$

Hence we have

$$(q-\gamma_*)\psi_{\gamma_*}\rho_q = D_m\left\{\psi_{\gamma_*} D_s \rho_q - \rho_q D_s \psi_{\gamma_*}\right\}. \tag{102}$$

Integrate both sides on I' with respect to dm, we obtain

$$(q-\gamma_*)\int_{I'} \psi_{\gamma_*}(x)\rho_q(x)\mathrm{d}m(x) = 1. \tag{103}$$

where we used the facts that $\rho_q(0) = \psi'_{\gamma_*}(0) = 1$, $\psi_{\gamma_*}(0) = \psi'_{\gamma_*}(l') = 0$, $\psi_{\gamma_*}(l') < \infty$ and $\rho'_q(l') = 0$. This shows that

$$N_q h_* = \frac{R_q h_*(0)}{H(q)} = \int_{I'} \rho_q(x)\psi_{\gamma_*}(x)\mathrm{d}m(x) = \frac{1}{q-\gamma_*}. \tag{104}$$

Hence we obtain $\mathrm{e}^{-\gamma_* t}\boldsymbol{n}[h_*(X_t)] = 1$ for a.e. $t > 0$. For $0 < s < t$, we see, by the γ_*-invariance of h_* for M^0, that

$$\mathrm{e}^{-\gamma_* t}\boldsymbol{n}[h_*(X_t)] = \mathrm{e}^{-\gamma_* t}\boldsymbol{n}[\mathbb{P}^0_{X_s}[h_*(X_{t-s})]] = \mathrm{e}^{-\gamma_* s}\boldsymbol{n}[h_*(X_s)], \tag{105}$$

which shows that $t \mapsto \mathrm{e}^{-\gamma_* t}\boldsymbol{n}[h_*(X_t)]$ is constant in $t > 0$. Thus we obtain the desired result. □

For later use, we need the following lemma.

Lemma 6.2 *For $0 < p < q$, it holds that*

$$\int_{(0,l')} \rho_q(y)\psi_p(y)\mathrm{d}m(y) \leq \frac{H(p)}{H(q)(q-p)}. \tag{106}$$

Consequently, it holds that $R'_q\psi_p(x) < \infty$.

Proof Let $x < l'$. Using the fact that $\rho_p \geq 0$ and the resolvent equation, we have

$$\int_{(0,x]} \rho_q(y)\psi_p(y)\mathrm{d}m(y) \leq \int_{(0,x]} \rho_q(y)H(p)\phi_p(y)\mathrm{d}m(y) \tag{107}$$

$$\leq \frac{1}{H(q)\rho_p(x)} \int_{I'} r_q(0,y)r_p(y,x)\mathrm{d}m(y) \tag{108}$$

$$= \frac{1}{H(q)\rho_p(x)} \cdot \frac{r_p(0,x) - r_q(0,x)}{q-p} \tag{109}$$

$$\leq \frac{r_p(0,x)}{H(q)\rho_p(x)(q-p)} = \frac{H(p)}{H(q)(q-p)}. \tag{110}$$

Letting $x \uparrow l'$, we obtain (106). □

We compute the image of the resolvent operators of h_0.

Proposition 6.3 *For $q > 0$ and $x \in I$, it holds that*

$$R_q h_0(x) = \frac{h_0(x)}{q} + \frac{r_q(x,0)}{q} - \frac{\pi_0}{q^2}, \tag{111}$$

$$R_q^0 h_0(x) = \frac{h_0(x)}{q} - \frac{\pi_0}{q^2}\mathbb{P}_x[1 - \mathrm{e}^{-qT_0}], \tag{112}$$

$$N_q h_0 = \frac{1}{q} - \frac{\pi_0}{q^2 H(q)}. \tag{113}$$

Proof Suppose $x \in I'$. Let $0 < p < q/2$. On one hand, by the resolvent equation, we have

$$R_q h_p(x) = r_p(0,0)\int_I r_q(x,y)\tilde{m}(\mathrm{d}y) - \int_I r_q(x,y)r_p(y,0)\tilde{m}(\mathrm{d}y) \tag{114}$$

$$= \frac{r_p(0,0)}{q} - \frac{r_p(x,0) - r_q(x,0)}{q-p} \tag{115}$$

$$= \frac{h_p(x)}{q-p} + \frac{r_q(x,0)}{q-p} - \frac{pH(p)}{q(q-p)} \tag{116}$$

$$\underset{p\downarrow 0}{\longrightarrow} \frac{h_0(x)}{q} + \frac{r_q(x,0)}{q} - \frac{\pi_0}{q^2}. \tag{117}$$

On the other hand, for $y \in I'$, we have

$$h_p(y) = H(p)\{1-\rho_p(y)\} = \psi_p(y) - H(p)\{\phi_p(y)-1\} \leq \psi_{q/2}(y). \tag{118}$$

By Lemma 6.2, we see by the dominated convergence theorem that $R_q h_p(x) \to R_q h_0(x)$ as $p \downarrow 0$. Hence we obtain (111) for $x \in I'$.

Suppose $l \in I$ and $x = l$. Then we have

$$qR_q h_0(l) = qh_0(l) r_q(l,l)\tilde{m}(\{l\}) = h_0(l), \tag{119}$$

which shows (111) for $x = l$, since $r_q(l,0) = 0$ and $\pi_0 = 0$ in this case. Thus we obtain (111). Using (41), (38), (111) and (29), we immediately obtain (112) and (113). □

We now obtain the image of the transition operators of h_0.

Theorem 6.4 *For $t > 0$ and $x \in I$, it holds that*

$$\mathbb{P}_x^0[h_0(X_t)] = h_0(x) - \pi_0 \int_0^t \mathbb{P}_x(s < T_0)\mathrm{d}s, \tag{120}$$

$$\boldsymbol{n}[h_0(X_t)] = 1 - \pi_0 \int_0^t \mathrm{d}s \int_{[0,\infty)} \mathrm{e}^{-s\xi}\sigma^*(\mathrm{d}\xi). \tag{121}$$

Consequently, for M^0 and $\boldsymbol{n}$, it holds that h_0 is invariant when $\pi_0 = 0$ and that h_0 is excessive but non-invariant when $\pi_0 > 0$.

Proof By (112), we have

$$R_q^0 h_0(x) = \frac{h_0(x)}{q} - \frac{\pi_0}{q}\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x(t < T_0)\mathrm{d}t, \tag{122}$$

which proves (120) for a.e. $t > 0$. By Fatou's lemma, we see that $\mathbb{P}_x^0[h_0(X_t)] \leq h_0(x)$ holds for all $t > 0$ and all $x \in I$. For $0 < s < t$, we have

$$\mathbb{P}_x^0[h_0(X_t)] = \mathbb{P}_x^0\big[\mathbb{P}_{X_s}^0[h_0(X_{t-s})]\big] \leq \mathbb{P}_x^0[h_0(X_s)]. \tag{123}$$

This shows that $t \mapsto \mathbb{P}_x^0[h_0(X_t)]$ is non-increasing. Since the right-hand side of (120) is continuous in $t > 0$, we see that (120) holds for all $t > 0$.

By (113), we have

$$N_q h_0 = \frac{1}{q} - \frac{\pi_0}{q} \int_{[0,\infty)} \frac{1}{q+\xi} \sigma^*(\mathrm{d}\xi) \tag{124}$$

$$= \frac{1}{q} - \frac{\pi_0}{q} \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-qt} \int_{[0,\infty)} \mathrm{e}^{-t\xi} \sigma^*(\mathrm{d}\xi), \tag{125}$$

which proves (121) for a.e. $t > 0$. For $0 < s < t$, we have

$$\boldsymbol{n}[h_0(X_t)] = \boldsymbol{n}\big[\mathbb{P}^0_{X_s}[h_0(X_{t-s})]\big] \le \boldsymbol{n}[h_0(X_s)], \tag{126}$$

from which we can conclude that (121) holds for all $t > 0$. □

We have already proved that s is invariant for M^0 and $\boldsymbol{n}$ when $\pi_0 = 0$. We now study properties of s in the case where $\pi_0 > 0$. In the case $l'(= \infty)$ is entrance, the measure $\mathbb{P}_{l'}$ denotes the extension of M starting from l' constructed by a scale transform (see also Fukushima [10, Sect. 6]).

Theorem 6.5 *Suppose $\pi_0 > 0$. Then the following assertions hold:*

(i) If l' is type-2-natural, then the scale function $s(x) = x$ is invariant for M^0 and $\boldsymbol{n}$.

(ii) If l' is entrance or regular-reflecting, then, for any $q > 0$ and any $t > 0$,

$$R^0_q s(x) = \frac{x}{q} - \frac{\psi_q(x)}{q} \chi_q(l'), \tag{127}$$

$$N_q s = \frac{1}{q} \mathbb{P}_{l'}[1 - \mathrm{e}^{-qT_0}], \tag{128}$$

$$\boldsymbol{n}[X_t] = \mathbb{P}_{l'}(t < T_0), \tag{129}$$

where

$$\chi_q(l') = \begin{cases} \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}] & \text{if } l' \text{ for } M \text{ is entrance,} \\ \frac{1}{q}\left\{ \frac{l'}{\psi_q(l')} - \rho_q(l') \right\} & \text{if } l' \text{ for } M \text{ is regular-reflecting.} \end{cases} \tag{130}$$

Consequently, $s(x) = x$ is excessive but non-invariant for M^0 and $\boldsymbol{n}$.

Proof By (19), we have, for $x \in I'$,

$$\rho'_q(x) = -\frac{1}{H(q)} + q \int_{(0,x]} \rho_q(y) \mathrm{d}m(y). \tag{131}$$

Since $I' = I$ when $\pi_0 > 0$, we have

$$\int_{I'} \rho_q(y)\mathrm{d}m(y) = \frac{1}{H(q)} R_q 1(0) = \frac{1}{qH(q)}. \tag{132}$$

We write $\rho_q(l') = \lim_{x \uparrow l'} \rho_q(x)$. Recalling g is defined by (67) and using (132), we obtain

$$N_q g = \pi_0 \int_0^{l'} \mathrm{d}x\, m(x) \int_{I' \setminus (0,x]} \rho_q(y)\mathrm{d}m(y) \tag{133}$$

$$= -\frac{\pi_0}{q} \int_0^{l'} \mathrm{d}x\, m(x) \rho_q'(x) \tag{134}$$

$$= -\frac{\pi_0}{q} \int_{I'} \mathrm{d}m(y) \int_y^{l'} \rho_q'(x)\mathrm{d}x \tag{135}$$

$$= \frac{\pi_0}{q} \int_{I'} \mathrm{d}m(y) \{\rho_q(y) - \rho_q(l')\} \tag{136}$$

$$= \frac{\pi_0}{q} \left\{ \frac{1}{qH(q)} - m(\infty)\rho_q(l') \right\}. \tag{137}$$

(i) If l' is type-2-natural, then, by [12, Theorem 5.13.3], we have $\rho_q(l') = 0$. By (113), we obtain $N_q s = 1/q$. Since $t \mapsto \boldsymbol{n}[X_t]$ is non-decreasing, we obtain $\boldsymbol{n}[X_t] = 1$ for all $t > 0$. We thus conclude that s is invariant for $\boldsymbol{n}$. The invariance of s for M^0 has already been remarked in the beginning of this section.

(ii) We postpone the proof of (127) until the end of the proof of Theorem 7.5. Let us prove (128) and (129).

If l' is regular-reflecting, we have $\rho_q(l') = \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}]$. If l' is entrance, then we may take limit as $x \uparrow l'$ and obtain

$$\rho_q(l') := \lim_{x \uparrow l'} \rho_q(x) = \lim_{x \uparrow l'} \mathbb{P}_x[\mathrm{e}^{-qT_0}] = \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}] \tag{138}$$

(see Kent [14, Sect. 6]). Since $\pi_0 m(\infty) = 1$, we obtain

$$N_q s = N_q h_0 + N_q g = \frac{1}{q}\mathbb{P}_{l'}[1 - \mathrm{e}^{-qT_0}] = \int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_{l'}(t < T_0)\mathrm{d}t. \tag{139}$$

This proves (128) and $\boldsymbol{n}[X_t] = \mathbb{P}_{l'}(t < T_0)$ for a.e. $t > 0$. Since $t \mapsto \mathbb{P}_{l'}(t < T_0)$ is continuous (see Kent [14, Sect. 6]) and by Lemma 5.1, we can employ the same argument as the proof of Theorem 6.4, and therefore we obtain (129).

Suppose that s were invariant for M^0. Then we would see that $\boldsymbol{n}[X_t] = \boldsymbol{n}[\mathbb{P}^0_{X_s}[X_{t-s}]] = \boldsymbol{n}[X_s]$ for $0 < s < t$, which would lead to the invariance of s for $\boldsymbol{n}$. This would be a contradiction. □

Remark 6.6 An excessive function h is called *minimal* if, whenever u and v are excessive and $h = u + v$, both u and v are proportional to h. It is known (see Salminen [23]) that s is minimal. We do not know whether h_0 is minimal or not in the positive recurrent case.

We now prove Theorem 3.1.

Proof of Theorem 3.1 In the case where $\pi_0 = 0$, we have $h_0(x) = x$ by Theorem 4.1. Hence, by Theorem 6.4, we see that $\boldsymbol{n}[X_t] \to 1$ as $t \downarrow 0$, which shows $C = 1$ in this case.

In the case where $\pi_0 > 0$, we obtain $C = 1$ by Theorem 6.5 and Lemma 3.2. The proof is therefore complete. □

7 The *h*-Transforms of the Stopped Process

We study h-transforms in this section. For a measure μ and a function f, we write $f\mu$ for the measure defined by $f\mu(A) = \int_A f \mathrm{d}\mu$.

Since h_* is γ_*-invariant, there exists a conservative strong Markov process $M^{h_*} = \{(X_t)_{t \geq 0}, (\mathbb{P}_x^{h_*})_{x \in I}\}$ such that

$$\mathbb{P}_x^{h_*} = \frac{\mathrm{e}^{-\gamma_* t} h_*(X_t)}{h_*(x)} \mathbb{P}_x^0 \quad \text{on } \mathscr{F}_t \text{ for } t > 0 \text{ and } x \in I \setminus \{0\}, \tag{140}$$

$$\mathbb{P}_0^{h_*} = \mathrm{e}^{-\gamma_* t} h(X_t) \boldsymbol{n} \quad \text{on } \mathscr{F}_t \text{ for } t > 0. \tag{141}$$

We set

$$m^{h_*}(x) = \int_{(0,x]} h_*(y)^2 \tilde{m}(\mathrm{d}y), \quad s^{h_*}(x) = \int_c^x \frac{\mathrm{d}y}{h_*(y)^2}, \tag{142}$$

where $0 < c < l'$ is a fixed constant, We define, for $q > 0$,

$$r_q^{h_*}(x,y) = \begin{cases} \dfrac{r_{q+\gamma_*}^0(x,y)}{h(x)h(y)} & \text{for } x, y \in I \setminus \{0\}, \\ \dfrac{r_{q+\gamma_*}(0,y)}{h(y) r_{q+\gamma_*}(0,0)} & \text{for } x = 0 \text{ and } y \in I \setminus \{0\}. \end{cases} \tag{143}$$

Then, we see that $r_q^{h_*}(x,y)$ is a density of the resolvent $R_q^{h_*}$ for M^{h_*}.

Theorem 7.1 *For M^{h_*}, the following assertions hold:*

(i) For $q > 0$, $\phi_q^{h_*} = \frac{\psi_{q+\gamma_*}}{h_*}$ (resp. $\rho_q^{h_*} = \frac{\rho_{q+\gamma_*}}{h_*}$) is an increasing (resp. decreasing) solution of $D_{m^{h_*}} D_{s^{h_*}} f = qf$ satisfying $f(0) = 1$ and $D_{s^{h_*}} f(0) = 0$ (resp. $f(0) = \infty$ and $D_{s^{h_*}} f(0) = -1$).

(ii) M^{h_*} is the $D_{m^{h_*}} D_{s^{h_*}}$-diffusion.

(iii) *0 for M^{h*} is entrance.*
(iv) *l' for M^{h*} is entrance when l' for M is entrance;*
l' for M^{h*} is regular-reflecting when l' for M is regular-reflecting.

Proof (i) For $q \in \mathbb{R}$ and for any function h such that $D_m D_s h$ exists, we see that

$$D_{m^h} D_{s^h} \left(\frac{\psi_{q+\alpha}}{h} \right) = \frac{1}{h^2} D_m \left\{ h^2 D_s \left(\frac{\psi_{q+\alpha}}{h} \right) \right\} \tag{144}$$

$$= \frac{1}{h^2} D_m \{ h D_s \psi_{q+\alpha} - \psi_{q+\alpha} D_s h \} \tag{145}$$

$$= \left(q + \alpha - \frac{D_m D_s h}{h} \right) \frac{\psi_{q+\alpha}}{h}. \tag{146}$$

Taking $h = h_*$ and $\alpha = \gamma_*$. we obtain $D_{m^{h*}} D_{s^{h*}} \phi_q^{h*} = q \phi_q^{h*}$. In the same way we obtain $D_{m^{h*}} D_{s^{h*}} \rho_q^{h*} = q \rho_q^{h*}$. The initial conditions can be obtained easily.

Claims (ii) and (iii) are obvious from (i).

(iv) Suppose that l' for M is entrance or regular-reflecting. Then h_* is bounded, so that l' for M^{h*} is of the same classification as l' for M. Since M^{h*} is conservative, we obtain the desired result. □

We now develop a general theory for the h-transform with respect to an excessive function. Let $\alpha \geq 0$ and let h be a function on I which is α-excessive for M^0 and $\boldsymbol{n}$ which is positive on $I \setminus \{0\}$. Then it is well-known (see, e.g., [5, Theorem 11.9]) that there exists a (possibly non-conservative) strong Markov process $M^h = \{(X_t)_{t \geq 0}, (\mathbb{P}_x^h)_{x \in I}\}$ such that

$$1_{\{t<\zeta\}} \mathbb{P}_x^h = \frac{e^{-\alpha t} h(X_t)}{h(x)} \mathbb{P}_x^0 \quad \text{on } \mathscr{F}_t \text{ for } t > 0 \text{ and } x \in I \setminus \{0\}, \tag{147}$$

$$1_{\{t<\zeta\}} \mathbb{P}_0^h = e^{-\alpha t} h(X_t) \boldsymbol{n} \quad \text{on } \mathscr{F}_t \text{ for } t > 0. \tag{148}$$

We note that M^h becomes a diffusion when killed upon hitting l if $l \in I$. If $\alpha \geq 0$, we see by [5, Theorem 11.9] that the identities (147) and (148) are still valid if we replace the constant time t by a stopping time T and restrict both sides on $\{T < \infty\}$. We set

$$m^h(x) = \int_{(0,x]} h(y)^2 \tilde{m}(dy), \quad s^h(x) = \int_c^x \frac{dy}{h(y)^2}, \tag{149}$$

where $0 < c < l'$ is a fixed constant, We define, for $q > 0$,

$$r_q^h(x,y) = \begin{cases} \dfrac{r_{q+\alpha}^0(x,y)}{h(x)h(y)} & \text{for } x, y \in I \setminus \{0\}, \\ \dfrac{r_{q+\alpha}(0,y)}{h(y) r_{q+\alpha}(0,0)} & \text{for } x = 0 \text{ and } y \in I \setminus \{0\}. \end{cases} \tag{150}$$

Then, we see that $r_q^h(x,y)$ is a density of the resolvent R_q^h for M^h.

Lemma 7.2 *Suppose that* $h(x) \le \psi_{q+\alpha}(x)$ *for all* $q > 0$ *and all* $x \in I$. *Define* $L_t^h(y) = L_t(y)/h(y)^2$ *for* $y \in I \setminus \{0\}$. *Then the process* $(L_t^h(y))_{t\ge 0}$ *is the local time at* y *for* M^h *normalized as*

$$\mathbb{P}_x^h\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right] = r_q^h(x,y), \quad x \in I,\ y \in I \setminus \{0\}. \tag{151}$$

It also holds that

$$\mathbb{P}_x^h\left[\mathrm{e}^{-qT_y}\right] = \frac{r_q^h(x,y)}{r_q^h(y,y)}, \quad x \in I,\ y \in I \setminus \{0\}. \tag{152}$$

Proof Since $\mathbb{P}_x^h$ is locally absolutely continuous with respect to $\mathbb{P}_x^0$, we see that $(L_t^h(y))_{t\ge 0}$ is the local time at y for M^h. Let $x, y \in I \setminus \{0\}$. For $u \ge 0$, we note that $\eta_u(y) = \inf\{t > 0 : L_t(y) > u\}$ is a stopping time and that $X_{\eta_u(y)} = y$ if $\eta_u(y) < \infty$. Let $0 = u_0 < u_1 < \ldots < u_n$. Then, by the strong Markov property, we have

$$\mathbb{P}_x^h\left[\int_{\eta_{u_{j-1}}(y)}^{\eta_{u_j}(y)} f(t)\mathrm{d}L_t^h(y)\right] = \frac{1}{h(x)h(y)}\mathbb{P}_x^0\left[\mathrm{e}^{-\alpha\eta_{u_j}(y)}\int_{\eta_{u_{j-1}}(y)}^{\eta_{u_j}(y)} f(t)\mathrm{d}L_t(y)\right]; \tag{153}$$

in fact, we have (153) with restriction on $\{\eta_{u_n}(y) \le T_{\varepsilon x}\}$ and then we obtain (153) by letting $\varepsilon \downarrow 0$. Hence, by the monotone convergence theorem, we obtain

$$\mathbb{P}_x^h\left[\int_0^\infty f(t)\mathrm{d}L_t^h(y)\right] = \frac{1}{h(x)h(y)}\mathbb{P}_x^0\left[\int_0^\infty \mathrm{e}^{-\alpha t}f(t)\mathrm{d}L_t(y)\right]. \tag{154}$$

Letting $f(t) = \mathrm{e}^{-qt}$, we obtain (151) for $x \in I \setminus \{0\}$.

Let $x = 0$ and $y \in I \setminus \{0\}$. For $p > 0$, we write $\boldsymbol{e}_p$ for an independent exponential time of parameter p. By the strong Markov property, we have

$$\mathbb{P}_0^h\left[\int_{\boldsymbol{e}_p}^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right] = \mathbb{P}_0^h\left[\mathrm{e}^{-q\boldsymbol{e}_p}\, r_q^h(X_{\boldsymbol{e}_p}, y)\right]. \tag{155}$$

On one hand, we have

$$(155) \le \mathbb{P}_0^h\left[r_q^h(X_{\boldsymbol{e}_p}, y)\right] = p\int_I r_q^h(0,x)r_q^h(x,y)m^h(\mathrm{d}x) \tag{156}$$

$$= \frac{p}{p-q}\cdot\left\{r_q^h(0,y) - r_p^h(0,y)\right\} \underset{p\to\infty}{\longrightarrow} r_q^h(0,y). \tag{157}$$

On the other hand, since we have $h(x) \le \psi_{q+\alpha}(x)$, we have

$$(155) \ge \mathbb{P}_0^h\left[\mathrm{e}^{-q\boldsymbol{e}_p}; \boldsymbol{e}_p < T_y\right] \frac{\rho_{q+\alpha}(y)}{h(y)} \underset{p\to\infty}{\longrightarrow} r_q^h(0,y). \tag{158}$$

By the monotone convergence theorem, we obtain (151) for $x = 0$.

Using (151) and using the strong Markov property, we obtain

$$\mathbb{P}_x^h\left[\mathrm{e}^{-qT_y}\right] = \frac{\mathbb{P}_x^h\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right]}{\mathbb{P}_y^h\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t^h(y)\right]} = \frac{r_q^h(x,y)}{r_q^h(y,y)}. \tag{159}$$

This shows (152). □

Theorem 7.3 *For M^s, i.e., the h-transform for $h = s$, the following assertions hold:*

(i) For $q > 0$, $\phi_q^s = \frac{\psi_q}{s}$ (resp. $\rho_q^s = \frac{\rho_q}{s}$) is an increasing (resp. decreasing) solution of $D_{m^s}D_{s^s}f = qf$ satisfying $f(0) = 1$ and $D_{s^s}f(0) = 0$ (resp. $f(0) = \infty$ and $D_{s^s}f(0) = -1$).

(ii) M^s is the $D_{m^s}D_{s^s}$-diffusion.

(iii) 0 for M^s is entrance.

(iv) l' for M^s is of the same classification as l' for M when $l' < \infty$, i.e., l' for M is exit, regular-absorbing, regular-elastic or type-3-natural;
l' for M^s is type-3-natural when l' for M is natural;
l' for M^s is exit when $l'(= \infty)$ for M is entrance with $\int_c^\infty x^2\mathrm{d}m(x) = \infty$;
l' for M^s is regular-absorbing when $l'(= \infty)$ for M is entrance with $\int_c^\infty x^2\mathrm{d}m(x) < \infty$;
l' for M^s is regular-elastic when l' for M is regular-reflecting.

Proof Claim (i) can be obtained in the same way as the proof of (i) of Theorem 7.1. Claims (ii) and (iii) are obvious from (i).

(iv) Suppose l' for M is exit, regular-absorbing or regular-elastic. Then we have $l' < \infty$, and hence it is obvious that l' for M^s is of the same classification as l' for M.

Suppose l' for M is natural. Then we have

$$\iint_{l'>y>x>c} \mathrm{d}m^s(x)\mathrm{d}s^s(y) = \int_{l'>x>c} x\mathrm{d}m(x) \ge \iint_{l'>y>x>c} \mathrm{d}x\mathrm{d}m(y) = \infty \tag{160}$$

and

$$\iint_{l'>y>x>c} \mathrm{d}s^s(x)\mathrm{d}m^s(y) = \int_{l'>y>c} \left(\frac{1}{c} - \frac{1}{y}\right) y^2\mathrm{d}m(y) \ge \int_{l'>y>2c} y\mathrm{d}m(y) = \infty. \tag{161}$$

Thus we see that l' for M^s is natural. Since $s^s(l') = 1/c - 1/l' < \infty$, we see that l' for M^s is type-3-natural.

Suppose $l'(=\infty)$ for M is entrance. Then we have

$$\iint_{\infty>y>x>c} \mathrm{d}m^s(x)\mathrm{d}s^s(y) = \iint_{\infty>y>x>c} \mathrm{d}x\mathrm{d}m(y) + c\{m(\infty) - m(c)\} < \infty. \tag{162}$$

In addition, we have

$$\iint_{\infty>y>x>c} \mathrm{d}s^s(x)\mathrm{d}m^s(y) = \int_{\infty>y>c} \left(\frac{1}{c} - \frac{1}{y}\right) y^2 \mathrm{d}m(y), \tag{163}$$

which is finite if and only if $\int_c^\infty x^2 \mathrm{d}m(x)$ is finite.

Suppose l' for M is regular-reflecting. Then it is obvious that l' for M^s is regular. Since M^s has no killing inside $[0, l')$ and since M^s is not conservative, we see that M^s has killing at l'. Since we have

$$\mathbb{P}^s_{l'}(T_x < \zeta) = \frac{x}{l'}\mathbb{P}^0_{l'}(T_x < T_0) = \frac{x}{l'} < 1 \quad \text{for all } x < l', \tag{164}$$

we see that M^s has killing at l'. Thus we see that l' for M^s is regular-elastic. □

Remark 7.4 When $l' = \infty$ and $\int_{\infty>x>c} x^2 \mathrm{d}m(x) < \infty$, the left boundary ∞ is called of *limit circle type*. See Kotani [15] for the spectral analysis involving Herglotz functions.

Theorem 7.5 *Suppose l' for M is entrance or regular-reflecting. For M^s, it holds that*

$$\mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = \frac{\psi_q(x)}{x}\chi_q(l'), \quad q > 0,\ x \in I' \setminus \{0\}, \tag{165}$$

where $\chi_q(l')$ is given by (130).

Proof Suppose l' is entrance. Then we have

$$\mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = \lim_{y\uparrow l'} \mathbb{P}^s_x[\mathrm{e}^{-qT_y}] = \lim_{y\uparrow l'} \frac{y}{x} \cdot \frac{\psi_q(x)}{\psi_q(y)}. \tag{166}$$

By [12, Theorem 5.13.3], we have

$$\lim_{y\uparrow l'} \frac{y}{\psi_q(y)} = \lim_{y\uparrow l'} \frac{1}{\psi_q'(y)} = \rho_q(l') = \mathbb{P}_{l'}[\mathrm{e}^{-qT_0}]. \tag{167}$$

Suppose $l'(=\infty)$ is regular-reflecting. Then we have

$$\mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = \mathbb{P}^s_x[\mathrm{e}^{-qT_{l'}}]\mathbb{P}^s_{l'}[\mathrm{e}^{-q\zeta}] = \frac{r^s_q(x,l')}{r^s_q(l',l')} \cdot \frac{1}{l'} R^0_q s(l') \tag{168}$$

$$= \frac{\psi_q(x)}{x} \cdot \frac{\rho_q(l')}{\psi_q(l')} \cdot \int_{(0,l']} \psi_q(x) x \mathrm{d}m(x). \tag{169}$$

Since $D_m\{\psi_q'(x)x - \psi_q(x)\} = q\psi_q(x)x$, we obtain

$$\int_{(0,l']} \psi_q(x) x \mathrm{d}m(x) = \frac{1}{q}\{\psi_q'(l')l' - \psi_q(l')\} = \frac{1}{q}\left\{ \frac{l'}{\rho_q(l')} - \psi_q(l') \right\}. \tag{170}$$

Thus we obtain (165). □

We now give the proof of (127).

Proof of (127) Note that

$$1 - \mathbb{P}^s_x[\mathrm{e}^{-q\zeta}] = q \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-qt} \mathbb{P}^s_x(\zeta > t) = \frac{q}{x} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-qt} \mathbb{P}^0_x[X_t] = \frac{1}{x} q R^0_q s(x). \tag{171}$$

Combining this fact with (165), we obtain (127). □

Theorem 7.6 *For M^{h_0}, i.e., the h-transform for $h = h_0$, the following assertions hold:*

(i) For $q > 0$, $\phi^{h_0}_q = \frac{\psi_q}{h_0}$ (resp. $\rho^{h_0}_q = \frac{\rho_q}{h_0}$) is an increasing (resp. decreasing) solution of $D_{m^{h_0}} D_{s^{h_0}} f = qf$ satisfying $f(0) = 1$ and $D_{s^{h_0}} f(0) = 0$ (resp. $f(0) = \infty$ and $D_{s^{h_0}} f(0) = -1$).
(ii) M^{h_0} is the $D_{m^{h_0}} D_{s^{h_0}}$-diffusion with killing measure $\frac{\pi_0}{h_0}\mathrm{d}m^{h_0}$.
(iii) 0 for M^{h_0} is entrance;
(iv) l' for M^{h_0} is natural when l' for M is type-2-natural;
l' for M^{h_0} is entrance when l' for M is entrance;
l' for M^{h_0} is regular when l' for M is regular-reflecting.

(For the boundary classifications for diffusions with killing measure, see, e.g., [13, Chap. 4].)

Proof Claim (i) can be obtained in the same way as the proof of (i) of Theorem 7.1.
(ii) For $f = \frac{\psi_q}{h_0}$ or $f = \frac{\rho_q}{h_0}$, we have

$$\left(D_{m^{h_0}} D_{s^{h_0}} - \frac{\pi_0}{h_0} \right) f = qf, \tag{172}$$

since $D_m D_s h_0 = -\pi_0$. This shows (ii).

Claim (iii) is obvious from (i).

(iv) Suppose l' for M is type-2-natural. Then it is obvious that $\lim_{x\uparrow l'} \rho_q^{h_0}(x) = 0$. Since we have $D_m\{h_0\rho_q' - \rho_q h_0'\} = (qh_0 + \pi_0)\rho_q$, we have

$$D_{s^{h_0}} \rho_q^{h_0}(x) = (h_0\rho_q' - \rho_q h_0')(x) = -1 + \int_{(0,x]} (qh_0(x) + \pi_0)\rho_q(x)\mathrm{d}m(x). \tag{173}$$

Hence, by Proposition 6.3, we obtain

$$\lim_{x\uparrow l'} D_{s^{h_0}} \rho_q^{h_0}(x) = -1 + \frac{1}{H(q)} R_q(qh_0 + \pi_0)(0) = 0. \tag{174}$$

Thus we see that l' for M^{h_0} is natural.

Suppose l' for M is entrance. Note that

$$\frac{h_0(x)}{\pi_0} = x \int_{(x,\infty)} \mathrm{d}m(z) + \int_{(0,x]} z\mathrm{d}m(z). \tag{175}$$

Since we have $\int_{(0,\infty)} z\mathrm{d}m(z) < \infty$, we see that

$$h_0(l') := \lim_{x\uparrow l'} h_0(x) = \pi_0 \int_{(0,\infty)} z\mathrm{d}m(z) < \infty. \tag{176}$$

This shows that l' for M^{h_0} is of the same classification as l' for M.

The last statement is obvious. □

Remark 7.7 General discussions related to Theorems 7.3 and 7.6 can be found in Maeno [16], Tomisaki [27] and Takemura [26].

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h-Transforms and Orthogonal Polynomials

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Abstract We describe some examples of classical and explicit h-transforms as particular cases of a general mechanism, which is related to the existence of symmetric diffusion operators having orthogonal polynomials as spectral decomposition.

MSC classification: 33C52, 31C35, 35K05, 60J60, 60J45

1 Introduction

When the first author of this paper was a student, he was attending the DEA course of Marc Yor, about Brownian motions and the many laws that one would compute explicitly for various transformations on the trajectories. It looked like magic, and was indeed. In particular, the fact that conditioning a real Brownian motion to remain positive would turn it into a Bessel process in dimension 3, that is the norm of a three-dimensional Brownian motion, seemed miraculous. Of course, there are much more striking identities concerning the laws of Brownian motion that one may find in the numerous papers or books of Marc Yor (see [26] for a large collection of such examples). The same kind of conditioning appears in many similar situations, and specially in geometric models. This is related to the fact that we then have explicit h (or Doob)-transforms.

This relation between conditioning and h-transform was first put forward by Doob [11], and is described in full generality in Doob's book [12]. However, this kind of conditioning has been extended in various contexts, and very reader friendly explained by Marc Yor and his co-authors, in particular in [27, 29]. The fact that conditioning a d-dimensional model to remain in some set produces a new model in the same family (whatever the meaning of "family"), moreover with dimension

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$d+2$, appears to be a general feature worth to be further understood. It turns out that the most known models have a common explanation, due to an underlying structure related to orthogonal polynomials. The scope of this short note is to shed light on these connections.

The paper is organized as follows. In Sect. 2, we present the langage of symmetric diffusion operators that we shall use in the core of the text, and explain what h-transforms are. Section 3 gives a few classical and known examples (some of them less well known indeed). They all follow the same scheme, explained in Sect. 4, which provides the general framework, related to the study of orthogonal polynomials which are eigenvectors of diffusion operators. The last Sect. 5 provides further examples, as applications of the main result, inspired from random matrix theory.

2 Symmetric Diffusion Operators, Images and *h*-Transforms

2.1 Symmetric Diffusion Operators

We give here a brief account of the tools and notations that we shall be using throughout this paper, most of them following the general setting described in [4]. A symmetric diffusion process (ξ_t) on a measurable space E may be described by its generator $\mathscr{L}$, acting on a good algebra $\mathscr{A}$ of real valued functions (we shall be more precise about this below). The diffusion property is described through the so-called change of variable formula. Namely, whenever $f=(f_1,\cdots,f_p)\in\mathscr{A}^p$, and if $\Phi:\mathbb{R}^p\mapsto\mathbb{R}$ is a smooth function such that $\Phi(f)\in\mathscr{A}$ together with $\partial_i\Phi(f)$ and $\partial_{ij}\Phi(f)$, $\forall i,j=1\cdots n$, then

$$\mathscr{L}(\Phi(f))=\sum_i\partial_i\Phi(f)\mathscr{L}(f_i)+\sum_{ij}\partial_{ij}\Phi(f)\Gamma(f_i,f_j), \tag{1}$$

where $\Gamma(f,g)$ is the square field operator (or carré du champ), defined on the algebra $\mathscr{A}$ through

$$\Gamma(f,g)=\frac{1}{2}\big(\mathscr{L}(fg)-f\mathscr{L}(g)-g\mathscr{L}(f)\big).$$

This change of variable formula (1) is some "abstract" way of describing a second order differential operator with no 0-order term. It turns out that the operators associated with diffusion processes satisfy $\Gamma(f,f)\geq 0$ for any $f\in\mathscr{A}$, and that the operator Γ is a first order differential operator in each of its argument, that is, with the same conditions as before,

$$\Gamma(\Phi(f),g))=\sum_i\partial_i\Phi(f)\Gamma(f_i,g), \tag{2}$$

In most cases, our set E is an open subset $\Omega \subset \mathbb{R}^n$, and the algebra $\mathscr{A}$ is the set of smooth (that is $\mathscr{C}^\infty$) functions $\Omega \mapsto \mathbb{R}$. Then, using formula (1) for a smooth function $f : \Omega \mapsto \mathbb{R}$ instead of Φ and $(x_1, \cdots, x_n)$ instead of $(f_1, \cdots, f_n)$, we see that $\mathscr{L}$ may be written as

$$\mathscr{L}(f) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f, \tag{3}$$

and similarly

$$\Gamma(f, g) = \sum_{ij} g^{ij}(x)\partial_i f \partial_j g.$$

In this system of coordinates, $g^{ij} = \Gamma(x_i, x_j)$ and $b^i = \mathscr{L}(x_i)$. The positivity of the operator Γ just says that the symmetric matrix $(g^{ij})(x)$ is non negative for any $x \in \Omega$, which is usually translated into the fact that the operator is semi-elliptic. In the same way, the absence of constant term translates into the fact that for the constant function $\mathbf{1}$, that we always assume to belong to the set $\mathscr{A}$, one has $\mathscr{L}(\mathbf{1}) = 0$, which is an easy consequence of (3).

It is not always wise to restrict to diffusion operators defined on some open subsets of $\mathbb{R}^n$. We may have to deal with operators defined on manifolds, in which case one may describe the same objects in a local system of coordinates. However, using such local system of coordinates in not a good idea. In Sect. 5.1 for example, we shall consider the group $SO(d)$ of d-dimensional orthogonal matrices. The natural algebra $\mathscr{A}$ that we want to use is then the algebra of polynomial functions in the entries (m_{ij}) of the matrix, and the natural functions Φ acting on it are the polynomial functions. Since the polynomial structure will play an important rôle in our computations, it is not wise in this context to consider local system of coordinates (the entries of the matrix cannot play this rôle since they are related through algebraic relations).

Coming back to the general situation, the link between the process (ξ_t) and the operator $\mathscr{L}$ is that, for any $f \in \mathscr{A}$,

$$f(\xi_t) - f(\xi_0) - \int_0^t \mathscr{L}(f)(\xi_s)ds$$

is a local martingale, and this is enough to describe the law of the process starting from some initial point $\xi_0 = x \in E$, provided the set of functions $\mathscr{A}$ is large enough, for example when $\mathscr{A}$ contains a core of the so-called domain of the operator $\mathscr{L}$, see [4], Chap. 3, for more details.

The law of a single variable ξ_t, when $\xi_0 = x$, is then described by a Markov operator P_t, as

$$P_t(f)(x) = \mathbb{E}_x(f(\xi_t)),$$

and, at least at a formal level, $P_t = \exp(t\mathscr{L})$ is the semigroup generated by $\mathscr{L}$.

In most of the cases that we are interested in, the operator $\mathscr{L}$ will be symmetric in some $\mathbb{L}^2(\mu)$ space. That is, for some subset $\mathscr{A}_0$ of $\mathscr{A}$, which is rich enough to describe P_t from the knowledge of $\mathscr{L}$ (technically, as mentioned above, a core in the domain $\mathscr{D}(\mathscr{L})$), one has, for f, g in $\mathscr{A}_0$

$$\int f\mathscr{L}(g)\,d\mu = \int g\mathscr{L}(f)\,d\mu.$$

This translates into the integration by parts formula

$$\int f\mathscr{L}(g)\,d\mu = -\int \Gamma(f,g)\,d\mu. \tag{4}$$

For an operator given in an open set $\Omega \subset \mathbb{R}^n$ by the formula (3), and when the coefficients g^{ij} and b^i are smooth, one may identify the density $\rho(x)$ of the measure μ, when $\rho(x) > 0$, by the formula

$$\mathscr{L}(f) = \frac{1}{\rho(x)}\sum_{ij} \partial_i(\rho g^{ij}\partial_j f),$$

which gives

$$b^i = \sum_j (g^{ij}\partial_j \log\rho + \partial_j g^{ij}), \tag{5}$$

an easy way to recover ρ up to a multiplicative constant provided (g^{ij}) is non degenerate, that is when $\mathscr{L}$ is elliptic. We call this measure μ the reversible measure. Indeed, whenever the measure μ is a probability measure, and under this symmetry property, then the associated process (ξ_t) has the property that, whenever the law of ξ_0 is μ, then for any $t > 0$ the law of $(\xi_{t-s}, s \in [0,t])$ is identical to the law of $(\xi_s, s \in [0,t])$. This justifies in this case the name "reversible", which we keep in the infinite mass case, following [4].

Through the integration by parts formula, the operator $\mathscr{L}$ (and therefore the process and the semigroup themselves, provided we know something about a core in the domain), is entirely described by the triple (Ω, Γ, μ), called a Markov triple in [4].

Thanks to the change of variable formula (1), it is enough to describe an operator in a given system of coordinates (x^i) to describe $\mathscr{L}(x^i) = b^i$ and $\Gamma(x^i, x^j) = g^{ij}$. Indeed, this determines $\mathscr{L}(\Phi(x^i))$, for any Φ at least $\mathscr{C}^2$. As outlined earlier, we do not even require that these functions x^i form a coordinate system. They may be redundant (that is more variables than really necessary, as for example in the $SO(d)$ mentioned above), or not sufficient, provided the computed expressions depend only on those variables, as we do for example in Sect. 5.

Moreover, it may be convenient in even dimension to use complex variables, that is, for a pair (x, y) of functions in the domain, to set $z = x + iy$ and describe $\mathscr{L}(z) = \mathscr{L}(x) + i\mathscr{L}(y)$, $\Gamma(z, z) = \Gamma(x, x) - \Gamma(y, y) + 2i\Gamma(x, y)$ and $\Gamma(z, \bar{z}) = \Gamma(x, x) + \Gamma(y, y)$, and similarly for many pairs of real variables, or a pair of a real variable and a complex one. This will be used for example in Sects. 3.4 and 5.2. However, we shall be careful in this case to apply $\mathscr{L}$ only to polynomial functions in the variables (x, y), replacing x by $\frac{1}{2}(z+\bar{z})$ and y by $\frac{1}{2i}(z-\bar{z})$. Then, the various change of variable formulae (on $\mathscr{L}$ and Γ) apply when considering z and $\bar{z}$ as independent variables.

As we already mentioned, it may happen that we can find some functions X_i, $i = 1, \cdots, k$ such that, for any i, $\mathscr{L}(X_i)$ depend only on $(X_1, \cdots, X_k)$ and that the same is true for $\Gamma(X_i, X_j)$ for any pair (i, j). Then, writing $X = (X_1, \cdots, X_k) \in \mathbb{R}^k$, setting $B^i(X) = \mathscr{L}(X^i)$ and $G^{ij}(X) = \Gamma(X_i, X_j)$, one writes for any smooth function $\Phi : \mathbb{R}^k \mapsto \mathbb{R}$, $\mathscr{L}\big(\Phi(X)\big) = \hat{\mathscr{L}}(\Phi)(X)$, where

$$\hat{\mathscr{L}} = \sum_{ij} G^{ij}(X)\partial^2_{ij} + \sum_i B_i(X)\partial_i,$$

which is a direct consequence of formula (1). When such happens, the image of the process (ξ_t) with generator $\mathscr{L}$ under the map X is again a diffusion process $(\hat{\xi}_t)$ with generator $\hat{\mathscr{L}}$. In this situation, we say that $\hat{\mathscr{L}}$ is the image of $\mathscr{L}$ through the map X.

Some caution should be taken in this assertion concerning the domains of the operators, but in the examples below all this will be quite clear (our operators will mostly act on polynomials). When $\mathscr{L}$ is symmetric with respect to some probability measure μ, then $\hat{\mathscr{L}}$ is symmetric with respect to the image measure $\hat{\mu}$ of μ through X. With the help of formula (5), it may be an efficient way to compute $\hat{\mu}$.

2.2 h-Transforms

Given some diffusion operator $\mathscr{L}$ on some open set in $\mathbb{R}^d$, we may sometimes find an explicit function h, defined on some subset Ω_1 of Ω, with values in $(0, \infty)$ such that $\mathscr{L}(h) = \lambda h$, for some real parameter $\lambda > 0$. We then look at the new operator $\mathscr{L}^{(h)}$, acting on functions defined on Ω_1, described as

$$\mathscr{L}^{(h)}(f) = \frac{1}{h}\mathscr{L}(hf) - \lambda f$$

is another diffusion operator with the same square field operator than $\mathscr{L}$. This is the so-called h (or Doob's) transform, see [4, 11, 12]. Indeed, thanks to formula (1), one has

$$\mathscr{L}^{(h)}(f) = \mathscr{L}(f) + 2\Gamma(\log h, f).$$

When the operator $\mathscr{L}$ is symmetric with respect to some measure μ, then $\mathscr{L}^{(h)}$ is symmetric with respect to $d\mu_h = h^2 d\mu$.

Considering functions with support in Ω_1, the application $D : f \mapsto hf$ is an isometry between $\mathbb{L}^2(\mu_h)$ and $\mathbb{L}^2(\mu)$. It is worth to observe that $\mathscr{L}^{(h)} = D^{-1}(\mathscr{L} - \lambda \mathrm{Id})D$: every spectral property (discreteness of the spectrum, cores, etc.) is preserved through this transformation.

For example, if $f \in \mathbb{L}^2(\mu)$ is an eigenvector of $\mathscr{L}$ with eigenvalue $-\lambda_1$, then f/h is an eigenvector of $\mathscr{L}^{(h)}$ with eigenvalue $-(\lambda_1 + \lambda)$.

Also, at least formally, for the semigroup $P_t^{(h)}$ associated with $\mathscr{L}^{(h)}$, one has

$$P_t^{(h)}(f) = e^{-\lambda t} \frac{1}{h} P_t(hf).$$

In general, one looks for positive functions h which vanish at the boundary of Ω_1, and there is a unique such function h satisfying $\mathscr{L}(h) = -\lambda h$, usually called the ground state for $\mathscr{L}$ on Ω_1. This situation appears in general in the following context. When $\mathscr{L}$ is elliptic on $\Omega \subset \mathbb{R}^n$, and whenever Ω_1 is bounded, with $\bar{\Omega}_1 \subset \Omega$, there one may consider the restriction of $\mathscr{L}$ on Ω_1. If we impose Dirichlet boundary conditions, then the spectrum of this operator consists of a discrete sequence $0 > \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_n \cdots$. The eigenvector h associated with λ_0 is strictly positive in Ω_1 and vanishes on the boundary $\partial \Omega_1$. This is the required ground state h of the operator $\mathscr{L}$ on Ω_1.

In probabilistic terms, the operator $\mathscr{L}^{(h)}$ is the generator of the process (ξ_t), conditioned to stay forever in the subset Ω_1. However, this interpretation is not that easy to check in the general diffusion case. We shall not be concerned here with this probabilist aspect of this transformation, which is quite well documented in the literature (see [12] for a complete account on the subject, and also [27, 29] for many examples on conditioning), but rather play around some algebraic aspects of it in concrete examples. However, for the sake of completeness, we shall briefly explain the flavor of this conditioning argument in the simplest example of finite discrete Markov chains, where all the analysis for justification of the arguments involved may be removed.

For this, let us consider a finite Markov chain (X_n) on some finite space E, with probability transition matrix $P(x, y)$, $(x, y) \in E^2$, which would play the rôle of P_1 in the diffusion context. For simplicity, let us assume that $P(x, y) > 0$ for any $(x, y) \in E^2$. Consider now a subset $A \in E$, and look at the restriction P_A of the matrix P to $A \times A$. The Perron-Frobenius theorem asserts that there exists a unique eigenvector V_0 for P_A, associated with a positive eigenvalue μ_0, which is everywhere positive. This eigenvector V_0 corresponds to the ground state h described above in the diffusion context. Then, one may look at the matrix Q on $A \times A$, defined through

$$Q(x, y) = \frac{V_0(y)}{\mu_0 V_0(x)} P(x, y),$$

which is a Markov matrix on $A \times A$. This Markov matrix Q plays on A the rôle of $\exp(\mathscr{L}^{(h)})$ when h is the ground state on Ω_1.

Fix now $n > 0$ and $N > n$. Let A_N be the event $(X_0 \in A, \cdots, X_N \in A)$. For the initial Markov chain (X_n) with transition matrix P and for $X_0 = x \in A$, consider now the law of $(X_0, \cdots, X_n)$ conditioned on A_N. When $F(X_0, \cdots, X_n) = f_0(X_0)\cdots f_n(X_n)$, it is quite easy to check that

$$\frac{E(F(X_0,\cdots X_n)\mathbf{1}_{A_N})}{E(1_{A_N})} = \frac{1}{Q^N(1/V_0)(x)}\tilde{E}\Big(F(X_0,\cdots,X_n)Q^{N-n}(1/V_0)(X_n)\Big),$$

where $\tilde{E}$ denotes the expectation for the law of a Markov chain with matrix transition Q.

Now, using the irreducibility of the Markov matrix Q, one sees that, when N goes to infinity, both $Q^{N-n}(1/V_0)(X_n)$ and $Q^N(1/V_0)(x)$ converge to $\int \frac{1}{V_0} d\nu$, where ν is the (unique) invariant measure for the matrix Q. In the limit, we recover the interpretation of the transition matrix transition Q as a matrix of the conditioning of the Markov chain (X_n) to stay forever in A.

Coming back to the general case, it is worth to observe that, at least formally, the transformation $\mathscr{L} \mapsto \mathscr{L}^{(h)}$ is an involution. Indeed, $\mathscr{L}^{(h)}(\frac{1}{h}) = -\frac{\lambda}{h}$ and $(\mathscr{L}^{(h)})^{(1/h)} = \mathscr{L}$. However, in the usual context of ground states, the interpretation of the associated process as a conditioning is more delicate, since $1/h$ converges to infinity at the boundary of the domain Ω_1.

It is not in general easy to exhibit explicit examples of such ground states h, but there are many very well known examples in the literature. We shall show that in the realm of diffusion processes which are associated to families of orthogonal polynomials, there is a generic argument to provide them, and that this family of examples cover most of the known ones, either directly, either as limiting cases.

Remark 1 Observe that, beyond the case where h is a positive eigenvector for $\mathscr{L}$, one may use the same transformation for any positive function h. One may then look at $\mathscr{L}^{(h)}(f) = \frac{1}{h}\mathscr{L}(hf) = \mathscr{L}(f) + 2\Gamma(\log h, f) + Vh$, where $V = \frac{\mathscr{L}(h)}{h}$. In particular, with operators in $\mathbb{R}^n$ of the form $\mathscr{L}(f) = \Delta(f) + \nabla \log V \cdot \nabla f$, which have reversible measure Vdx, one may use $h = V^{-1/2}$, which transforms in an isospectral way $\mathscr{L}$ into a Shrödinger type operator $\Delta f + Vf$, associated with Feynman-Kac semigroups. This allows to remove a gradient vector field, the price to pay is that one adds a potential term. This technique may be used to analyse spectral properties of such symmetric diffusion operators through the techniques used for Shrödinger operator (see [4], for example).

3 Some Examples

3.1 Bessel Operators

We start form the Brownian motion in $\mathbb{R}$. The operator $\mathscr{L}$ is given by $\mathscr{L}(f) = \frac{1}{2}f''$. Here, $\Gamma(f,f) = \frac{1}{2}f'^2$ and μ is the Lebesgue measure. If we consider $\Omega = (0, \infty)$

and $h = x$, one has $\lambda = 0$ and $\mathscr{L}^{(h)}(f'') = \frac{1}{2}(f'' + \frac{2}{x}f')$. This last operator is a Bessel operator $\mathscr{B}_3$. More generally, a Bessel process $Bes(n)$ with parameter n has a generator in $(0,\infty)$ given by

$$\mathscr{B}_n(f) = \frac{1}{2}(f'' + \frac{n-1}{x}f'),$$

and it is easily seen, when $n \geq 1$ is an integer, to be the generator of $\|B_t\|$, where (B_t) is an n-dimensional Brownian motion (indeed, $\mathscr{B}_n$ is the image of the Laplace operator $\frac{1}{2}\Delta$ under $x \mapsto \|x\|$, in the sense described in Sect. 2). This $\mathscr{B}_3$ operator is also the generator of a real Brownian motion conditioned to remain positive. Observe however that the function h is this case does not vanish at the infinite boundary of the set $(0,\infty)$, and that the probabilistic interpretation would require some further analysis than the one sketched in the previous section.

From formula (5), it is quite clear that a reversible measure for the operator $\mathscr{B}_n$ is $x^{n-1}dx$ on $(0,\infty)$, which for $n \in \mathbb{N}^*$, is also, up to a constant, the image of the Lebesgue measure in $\mathbb{R}^n$ through the map $x \mapsto \|x\|$.

This h-transform may be extended to the general Bessel operator. Indeed, for any $n > 0$, one may consider the function $h_n(x) = x^{2-n}$, for which $\mathscr{B}_n(h_n) = 0$, and then $\mathscr{B}_n^{(h_n)} = \mathscr{B}_{4-n}$.

The change of $\mathscr{B}_n$ into $\mathscr{B}_{4-n}$ is perhaps more clear if we consider the generator through the change of variable $x \mapsto x^2$, that is if we consider the generator of the process (ξ_t^2) instead of the process (ξ_t) with generator $\mathscr{B}_n$. A simple change of variable provides the image operator

$$\hat{\mathscr{B}}_n(f) = 2xf'' + nf', \tag{6}$$

for which the reversible measure has density $\rho(x) = x^{(n-2)/2}$, and the function h is nothing else than $1/\rho$.

Under this form, we shall see that it is a particular case of a phenomenon related to orthogonal polynomials, developed in Sect. 4, although here there are no polynomials involved here, the reversible measure being infinite.

Remark 2 It is not hard to observe that for $0 < n < 2$, the process (ξ_t) with associated generator $\mathscr{B}_n$, and starting from $x > 0$ reaches 0 in finite time. Then, $\mathscr{B}_{4-n}$ is the generator of this process conditioned to never reach 0. However, it is well known that the Bessel operator is essentially self-adjoint on $(0,\infty)$ as soon as $n > 3$ (see [4], p. 98, for example). This means that the set of smooth function compactly supported in $(0,\infty)$ is dense in the $\mathbb{L}^2$ domain of $\mathscr{B}_n$. Since this is a spectral property, it is preserved through h-transform and this also shows that it is also essentially self-adjoint for any $n < 1$. In particular, there is a unique symmetric semigroup for which the generator coincides with $\mathscr{B}_n$ on the set of smooth compactly supported functions. On the other hand, for $1 \leq n < 2$, since the associated operator hits the boundary in finite time, there are at least two such semigroups with $\mathscr{B}_n$ as generator acting on smooth functions, compactly supported in $(0, \infty)$: the one corresponding to the Dirichlet boundary condition, corresponding to the process killed at the boundary $\{x = 0\}$, and the one corresponding to the Neumann boundary condition, corresponding to the process reflected at the boundary. Through h-transforms, one sees then that there are also at least two positivity preserving semi groups in the case $2 < n \leq 3$, which may be a bit surprising since then the associated process does not touch the boundary. However, although the Dirichlet semigroup is Markov ($P_t(\mathbf{1}) < \mathbf{1}$), its h-transform is Markov ($P_t(\mathbf{1}) = \mathbf{1}$), while the h-transform of the Neumann semigroup (which is Markov), satisfies $P_t(\mathbf{1}) \geq \mathbf{1}$.

3.2 Jacobi Operators

This is perhaps the most celebrated case of known explicit h-transform, since it is closely related in some special case to the Fourier transform on an interval. The Jacobi operator on the interval $(-1, 1)$ has generator

$$\mathscr{J}_{\alpha,\beta}(f) = (1 - x^2)f'' - \big((\alpha + \beta)x + \alpha - \beta\big)f'$$

and is symmetric with respect to the Beta distribution on $(-1, 1)$ which is $C_{\alpha,\beta}(1 - x)^{\alpha-1}(1 + x)^{\beta-1}dx$, $C_{\alpha,\beta}$ being the normalizing constant. We always assume that $\alpha, \beta > 0$. There is a duality through h-transforms exchanging $\mathscr{J}_{\alpha,\beta}$ and $\mathscr{J}_{2-\alpha,2-\beta}$, the function h being $(1 - x)^{1-\alpha}(1 - x)^{1-\beta}$, that is, as in the Bessel case in the appropriate coordinate system, the inverse of the density measure.

In a similar way that the Bessel process may be described as a norm of a Brownian motion, one may see the symmetric Jacobi operator ($\alpha = \beta$) as an image of a spherical Brownian motion in dimension 2α. Namely, if one considers the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, and looks at the Brownian motion on it (with generator $\Delta_{\mathbb{S}^n}$ being the Laplace operator on the sphere), and then one looks at its first component, one gets a process on $(-1, 1)$ with generator $\mathscr{L}^{n/2,n/2}$. (We refer to Sect. 3.5 for details about the spherical Laplacian, from which this remark follows easily, see also [4, 31].) One may also provide a similar description in the asymmetric case, when the parameters α and β are half integers. In this case, $\mathscr{L}_{\alpha,\beta}$ is, up to a factor 4, the image of the spherical Laplace operator acting on the unit sphere $\mathbb{S}^{2\alpha+2\beta-1}$ through the function $X : \mathbb{S}^{2\alpha+2\beta-1} \mapsto [-1, 1]$ defined, for $x = (x_1, \cdots, x_{2\alpha+2\beta}) \in \mathbb{R}^{2\alpha+2\beta}$ as

$$X(x) = -1 + 2\sum_{i=1}^{2\alpha} x_i^2.$$

The operator $\mathscr{J}_{\alpha,\beta}$ may be diagonalized in a basis of orthogonal polynomials, namely the Jacobi polynomials. They are deeply related to the analysis on the Euclidean case in the geometric cases described above. For example, when $\alpha = \beta$

is an half-integer, then, for each degree k, and up to a multiplicative constant, there exists a unique function on the sphere which depends only on the first coordinate and which is the restriction to the sphere of an homogeneous degree k harmonic polynomial in the corresponding Euclidean space: this is the corresponding degree k Jacobi polynomial (see [4, 31] for more details). In other words, if $P_k(x)$ is one of these Jacobi polynomials with degree k corresponding to the case $\alpha = \beta = n/2$, then the function $(x_1, \cdots, x_{n+1}) \mapsto \|x\|^k P_k(\frac{x_1}{\|x\|})$ is an homogeneous harmonic polynomial in $\mathbb{R}^{n+1}$. A similar interpretation is valid in the asymmetric case, whenever the parameters α and β are half-integers, if one reminds that the eigenvectors of the Laplace operator on the sphere are restriction to the sphere of harmonic homogeneous polynomials in the ambient Euclidean space (see [31]).

For $\alpha = \beta = 1/2$, $\mathcal{J}_{\alpha,\beta}$ this is just the image of the usual operator f'' on $(0, \pi)$ through the change of variables $\theta \mapsto \cos(\theta) = x$. More generally, in the variable θ, $\mathcal{J}_{\alpha,\beta}$ may be written as

$$\mathcal{J}_{\alpha,\beta} = \frac{d^2}{d\theta^2} + \frac{(\alpha + \beta - 1)\cos(\theta) + \alpha - \beta}{\sin(\theta)} \frac{d}{d\theta}.$$

For $\alpha = \beta = 1/2$, corresponding to the arcsine law, the associated orthogonal polynomials $P_n^{1/2,1/2}$ are the Chebyshev polynomials of the first kind, satisfying

$$P_n^{1/2,1/2}(\cos(\theta)) = \cos(n\theta).$$

For $\alpha = \beta = 3/2$, corresponding to the semicircle law, they correspond to the Chebyshev polynomials of the second kind, satisfying the formula

$$\sin(\theta) P_n^{3/2,3/2}(\cos(\theta)) = \sin(n\theta).$$

These formulae indeed reflect the h-transform between $\mathcal{J}^{1/2,1/2}$ and $\mathcal{J}^{3/2,3/2}$. While $P_n^{1/2,1/2}(\cos(\theta))$ is a basis of $\mathbb{L}^2\big((0,\pi), dx\big)$ with Neumann boundary conditions, $\sin(\theta)P_n^{3/2,3/2}(\cos(\theta))$ is another basis of $\mathbb{L}^2\big((0,\pi), dx\big)$, corresponding to the Dirichlet boundary condition. This is the image of the eigenvector basis for $\mathcal{L}^{3/2,3/2}$ through the inverse h transform, the function h being in this system of coordinates nothing else than $(\sin\theta)^{-1}$.

For $n = 1$, one gets the projection of the Brownian motion on the circle, which is locally a Brownian motion on the real line, up to a change of variables. The first coordinate x_1 on the sphere plays the rôle of a distance to the point $(1, 0, \cdots, 0)$ (more precisely, $\arccos(x_1)$ is the Riemannian distance on the sphere from $(1, 0, \cdots, 0)$ to any point with first coordinate x_1), and we have a complete analogue of the case of the one dimensional Brownian motion. Namely,

Proposition 1 *The Brownian motion on the half interval (identified with the circle) conditioned to never reach the boundaries is, up to a change of variable, the radial part of a Brownian motion on a three dimensional sphere.*

3.3 Laguerre Operators

This is the family of operator on $(0, \infty)$ with generator

$$\mathscr{L}_{(\alpha)}(f) = xf'' + (\alpha - x)f',$$

which is symmetric with respect to the Gamma distribution

$$d\mu^{(\alpha)} = C_\alpha x^{\alpha-1} e^{-x} dx.$$

For $\alpha > 0$, the Laguerre family of operators is another instance of diffusion operators on the real line which may be diagonalized in a basis of orthogonal polynomials: these polynomials are the Laguerre polynomials, and are one of the three families, together with Jacobi polynomials and Hermite polynomials, of orthogonal polynomials in dimension 1 which are at the same time eigenvectors of a diffusion operator, see [2]. The Laguerre operator is closely related to the Ornstein-Uhlenbeck operator defined in (7), and plays for this operator the same rôle that the one played by Bessel operators for the Euclidean Brownian motion.

It is indeed quite close to the Bessel generator under the form (6), and in fact the Bessel operator may be seen as a limit of Laguerre operators under proper rescaling. It is also a limit of asymmetric Jacobi operators, also under proper rescaling (see [4]). The function $h = x^{1-\alpha}$ satisfies $\mathscr{L}_{(\alpha)}(h) = (\alpha - 1)h$, and the h-transform of $\mathscr{L}_{(\alpha)}$ is $\mathscr{L}_{(2-\alpha)}$.

As mentioned above, when α is a half-integer $n/2$, the Laguerre operator may be seen as the radial part of the Ornstein-Uhlenbeck operator in $\mathbb{R}^n$ with generator

$$\mathscr{L}^{OU} = \Delta - x\nabla, \tag{7}$$

which is symmetric with respect to the standard Gaussian measure. More precisely, for $\alpha = n/2$, $\mathscr{L}^{OU} f(\frac{\|x\|^2}{2}) = 2\Big(\mathscr{L}_{(\alpha)} f\Big)(\frac{\|x\|^2}{2})$. It is therefore an image of the n-dimensional Ornstein-Uhlenbeck operator in the sense of Sect. 2. In other words, the Laguerre process with generator $2\mathscr{L}_{(n/2)}$ is nothing else than the squared norm of an Ornstein-Uhlenbeck process in $\mathbb{R}^n$. For $\alpha = 1/2$, this corresponds to the modulus of a one dimensional Ornstein-Uhlenbeck, that is the one dimensional Ornstein-Uhlenbeck operator itself on $(0, \infty)$, and we get, as the particular case for $n = 1/2$,

Proposition 2 *The law of an Ornstein-Uhlenbeck operator in dimension 1, conditioned to remain positive is the same as the law of the norm of a three-dimensional Orntein-Uhlenbeck operator.*

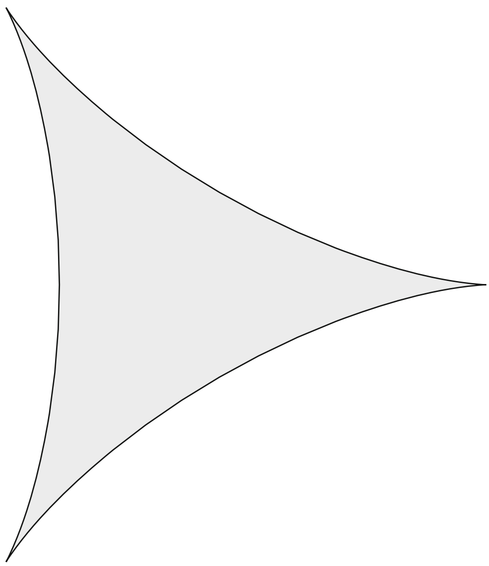
3.4 An Example in $\mathbb{R}^2$

The following example, less well known, had been pointed out by Koornwinder [22], not exactly under this form of h-transform, but in terms of duality between two families of orthogonal polynomials in dimension 2. It shows that the law of a Brownian motion in the plane, conditioned not to reach the boundaries of an equilateral triangle, has the law of the spectrum of an Brownian $SU(3)$ matrix.

This example, closely related to root systems and reflection groups in the plane, consists in observing the image of a planar Brownian motion reflected along the edges of an equilateral triangle. This triangle generates a triangular lattice in the plane, and this image is observed through some function $Z : \mathbb{R}^2 \mapsto \mathbb{R}^2$ which has the property that any function $\mathbb{R}^2 \mapsto \mathbb{R}$ which is invariant under the symmetries among the lines of the lattice is a function of Z. This image of $\mathbb{R}^2$ through the function Z is a bounded domain in $\mathbb{R}^2$, with boundary the Steiner's hypocycloid.

The Steiner hypocycloid (also called deltoid curve) is the curve obtained in the plane by rotating (from inside) a circle with radius 1 on a circle with radius 3. Is is the boundary of a bounded open region in the plane which we call the deltoid domain Ω_D. It is an algebraic curve of degree 4. Its equation may be written in complex coordinates as $\{D(Z, \bar{Z}) = 0\}$, where D is defined in Proposition 5.

Fig. 1 The deltoid domain



Consider the following application $\mathbb{R}^2 \mapsto \mathbb{R}^2$, which is defined as follows. Let $(1, j, \bar{j})$ be the three third roots of units in the complex plane $\mathbb{C}$, and, identifying $\mathbb{R}^2$ with $\mathbb{C}$, let $Z(z) : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the function

$$Z(z) = \frac{1}{3}\Big(\exp(i(1 \cdot z)) + \exp(i(j \cdot z)) + \exp(i(\bar{j} \cdot z))\Big),$$

where $z_1 \cdot z_2$ denotes the scalar product in $\mathbb{R}^2$.

We have

Proposition 3 *Let L be the lattice generated in the plane by the points with coordinates $M_1 = (0, 4\pi/3)$ and $M_2 = (2\pi/3, 2\pi/\sqrt{3})$, and T the (equilateral) triangle with edges $\{(0,0), M_1, M_2\}$.*

1. *The image of $\mathbb{R}^2$ under the function Z is the closure $\bar{\Omega}_D$ of the deltoid domain.*
2. *$Z : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is invariant under the symmetries along all the lines of the lattice L. Moreover, it is injective on the triangle T.*

We shall not give a proof of this, which may be checked directly. We refer to [32] for details. As a consequence, any measurable function $\mathbb{R}^2 \mapsto \mathbb{R}$ which is invariant under the symmetries of L may be written $f(Z)$, for some measurable function $f : \Omega_D \mapsto \mathbb{R}$.

The particular choice of this function Z is due to the fact that the Laplace operator in $\mathbb{R}^2$ has a nice expression through it. Using complex coordinates as described in Sect. 2, one has

Proposition 4 *For the Laplace operator Δ in $\mathbb{R}^2$ and its associated square field operator Γ, one has*

$$\begin{cases} \Gamma(Z,Z) = \bar{Z} - Z^2, \Gamma(\bar{Z},\bar{Z}) = Z - \bar{Z}^2, \\ \Gamma(\bar{Z},Z) = 1/2(1 - Z\bar{Z}), \\ \Delta(Z) = -Z, \Delta(\bar{Z}) = -\bar{Z}. \end{cases} \tag{8}$$

This may be checked directly. One sees that the Laplace operator in $\mathbb{R}^2$ has an image through Z in the sense described in Sect. 2, given in Proposition 4. This describes the generator of the Brownian motion in the plane, reflected along the lines of this lattice, coded through this change of variables. One may express the image measure of the Lebesgue measure on the triangle in this system of coordinates. With the help of formula (5), we get

Proposition 5 *Let $D(Z,\bar{Z}) = \Gamma(Z,\bar{Z})^2 - \Gamma(Z,Z)\Gamma(\bar{Z},\bar{Z})$, where Γ is given by Eq. (8). Then,*

1. *$D(Z,\bar{Z})$ is positive on Ω_D.*
2. *$\{D(Z,\bar{Z}) = 0\}$ is the deltoid curve (that is the boundary of Ω_D).*
3. *The reversible measure for the image operator described by (8) has density $D(Z,\bar{Z})^{-1/2}$ with respect to the Lebesgue measure.*
4. *If we write $z_1 = \exp(i(1\cdot z))$, $z_2 = \exp(i(j\cdot z))$, $z_3 = \exp(i(\bar{j}\cdot z))$, then*

$$D(Z,\bar{Z}) = -(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2/(2^2 3^3).$$

Remark 3 Observe that thanks to the fact that $|z_i| = 1$ and $z_1 z_2 z_3 = 1$, the expression $(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_1)^2$ is always non positive. Moreover, given a complex number Z in the deltoid domain Ω_D, there exist three different complex numbers (z_1, z_2, z_3) with $|z_i| = 1$ and $z_1 z_2 z_3 = 1$ such that $Z = \frac{1}{3}(z_1 + z_2 + z_3)$. They are unique up to permutation, and are the solutions of $X^3 - 3ZX^2 + 3\bar{Z}X - 1 = 0$. Indeed, for such numbers z_1, z_2, z_3,

$$3\bar{Z} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = z_2z_3 + z_1z_3 + z_1z_2.$$

One may now consider the family of operator $\mathscr{L}^{(\lambda)}$ defined through

$$\begin{cases} \Gamma(Z,Z) = \bar{Z} - Z^2, \Gamma(\bar{Z},\bar{Z}) = Z - \bar{Z}^2, \\ \Gamma(\bar{Z},Z) = 1/2(1 - Z\bar{Z}), \\ \mathscr{L}^{(\lambda)}(Z) = -\lambda Z, \mathscr{L}^{(\lambda)}(\bar{Z}) = -\lambda \bar{Z}, \end{cases} \tag{9}$$

which is symmetric with respect to the measure $\mu_\lambda = D(Z,\bar{Z})^{(2\lambda-5)/6}dZ$, with support the set $\{D(Z,\bar{Z}) \geq 0\}$ (where dZ is a short hand for the Lebesgue measure in the complex plane) as a direct (although a bit tedious) computation shows from a direct application of formula (5) (see Sect. 4 for a proof in a general context which applies in particular here).

This family of operators plays a rôle similar in this context to the one played by the family $\mathscr{J}_{\alpha,\beta}$ for Jacobi polynomials introduced in Sect. 3.2 or for the family $\mathscr{L}_{(\alpha)}$ introduced in Sect. 3.3 for Laguerre polynomials.

This density equation (5) indicates that, for any pair of smooth functions compactly supported in $\{D(Z,\bar{Z}) > 0\}$, the integration by parts (4) holds true. Indeed, we have a much stronger result, which extends this formula to any pair of smooth functions defined in a neighborhood of $\bar{\Omega}$. This relies of some miraculous property of $\partial\Omega$ itself, which has as boundary equation $\{D(Z,\bar{Z}) = 0\}$ and for which

$$\begin{cases} \Gamma(Z,Z)\partial_Z D + \Gamma(Z,\bar{Z})\partial_{\bar{Z}} D = -3ZD, \\ \Gamma(\bar{Z},Z)\partial_Z D + \Gamma(\bar{Z},\bar{Z})\partial_{\bar{Z}} D = -3\bar{Z}D. \end{cases} \tag{10}$$

In particular, $\Gamma(Z,D)$ and $\Gamma(\bar{Z},D)$ vanish on $\{D = 0\}$. This is a sufficient (and indeed necessary) for the integration by parts formula (4) to be valid for any pair smooth functions restricted on the set $\{D \geq 0\}$, in particular for any pair of polynomials (see [5]). Since on the other hand the operator $\mathscr{L}^{(\lambda)}$ maps polynomials in $(Z,\bar{Z})$ into polynomials, without increasing their total degrees, the restriction of $\mathscr{L}^{(\lambda)}$ on the finite dimensional space of polynomials with total degree less than k is a symmetric operator (with respect to the $\mathbb{L}^2(\mu_\lambda)$-Euclidean structure) on this linear space. We may therefore find an orthonormal basis of such polynomials which are eigenvectors for $\mathscr{L}^{(\lambda)}$, and therefore construct a full orthonormal basis of polynomials made of eigenvectors for $\mathscr{L}^{(\lambda)}$.

These polynomials are an example of Jack's polynomials associated with root systems (here the root system A_2), see [15, 24], generalized by MacDonald [23–25], see also [8, 17, 18], and for which the associated generators are Dunkl operators of various kinds, see [14, 20, 21, 28, 30].

For $\lambda = 4$, it turns out that this operator is, up to a scaling factor 8/3, the image of the Laplace (Casimir) operator on $SU(3)$ acting on the trace of the matrix. More precisely, on the compact semi-simple Lie group $SU(3)$, we associate to each element E in the Lie algebra $\mathscr{G}$ a (right) vector field X_E as follows

$$X_E(f)(g) = \partial_t(f(ge^{tE}))_{|t=0}.$$

Then, one chooses in the Lie algebra $\mathscr{G}$ an orthonormal basis E_i for the Killing form (which is negative definite), and we consider the operator $\mathscr{L} = \sum_i X_{E_i}^2$. This is the canonical Laplace operator on the Lie group, and it commutes with the group action, from left and right: if $L_g(f)(x) = f(xg)$, and $R_g(f)(x) = f(gx)$, then $\mathscr{L}L_g = L_g\mathscr{L}$ and $\mathscr{L}R_g = R_g\mathscr{L}$. For the Casimir operator acting on the entries (z_{ij}) of an $SU(d)$ matrix, one may compute explicitly this operator, and obtain, up to a factor 2, the following formulae

$$\begin{cases} \mathscr{L}^{SU(d)}(z_{kl}) = -2\frac{(d-1)(d+1)}{n} z_{kl}, \; \mathscr{L}^{SU(d)}(\bar{z}_{kl}) = -2\frac{(d-1)(d+1)}{n}\bar{z}_{kl} \\ \Gamma^{SU(d)}(z_{kl}, z_{rq}) = -2z_{kq}z_{rl} + \frac{2}{d}z_{kl}z_{rq}, \; \Gamma(z_{kl}, \bar{z}_{rq}) = 2(\delta_{kr}\delta_{lq} - \frac{1}{d}z_{kl}\bar{z}_{rq}). \end{cases} \tag{11}$$

A Brownian motion on $SU(d)$ is a diffusion process which has this Casimir operator as generator (there are of course many other equivalent definitions of this Brownian motion).

On $SU(3)$, if one considers the function $SU(3) \mapsto \mathbb{C}$ which to $g \in SU(3)$ associates $Z(g) = \frac{1}{3}\text{trace}\,(g)$, then one gets for this function Z and for this Casimir operator, an image operator which is the operator $\frac{8}{3}\mathscr{L}^{(4)}$, where $\mathscr{L}^{(\lambda)}$ is defined through Eq. (9). Of course, one may perform the computation directly, or use the method described in Sect. 5.2 to compute from the operator given of $SU(d)$ through formulas (11), the actions of the generator and the carré du champ on the characteristic polynomial $P(X) = \det(X\text{Id} - g)$ (see also [7] for another approach, together with [6] for nice connections with the Riemann-Zeta function).

It is worth to observe that functions on $SU(3)$ which depend only on this renormalized trace Z are nothing else but spectral functions. Indeed, if a matrix $g \in SU(3)$ have eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, with $|\lambda_i| = 1$ and $\lambda_1\lambda_2\lambda_3 = 1$, then a spectral function, that is a symmetric function of $(\lambda_1, \lambda_2, \lambda_3)$, depends only on $\lambda_1 + \lambda_2 + \lambda_3 = 3Z$ and, as observed in Remark 3, $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 3\bar{Z}$.

Then, using the function D which is the determinant of the metric involved in Eq. (10), one may check directly that

$$\mathscr{L}^{(\lambda)}\big(D(Z,\bar{Z})^{(5-2\lambda)/6}\big) = (2\lambda - 5)D(Z,\bar{Z})^{(5-2\lambda)/6},$$

so that one may use the function $h = D(Z, \bar{Z})^{(5-2\lambda)/6}$ to perform an h transform on $\mathcal{L}^{(\lambda)}$ and we obtain

$$(\mathcal{L}^{(\lambda)})^{(h)} = \mathcal{L}^{(5-\lambda)}.$$

Indeed, as we shall see in Sect. 4, this h-transform identity relies only on Eq. (10). In particular, moving back to the triangle through the inverse function Z^{-1}, for $\lambda = 1$, which corresponds to the Brownian motion reflected at the boundaries of the triangular lattice, the h transform is $\mathcal{L}^{(4)}$, which corresponds to the spectral measure on $SU(3)$. Then, for this particular case $\lambda = 1$, we get

Proposition 6 *A Brownian motion in the equilateral triangle T, conditioned to never reach the boundary of the triangle, has the law of the image under Z^{-1} of the spectrum of an $SU(3)$ Brownian matrix.*

3.5 An Example in the Unit Ball in $\mathbb{R}^d$

Another example comes from the spherical Brownian motion on the unit sphere

$$\mathbb{S}^d = \{(x_1, \cdots, x_{d+1}) \in \mathbb{R}^{d+1}, \sum_i x_i^2 = 1\}.$$

To describe the Brownian motion on $\mathbb{S}^d$, we look at its generator, that is this the spherical Laplace operator may. It may be described through its action on the restriction to the sphere of the coordinates x_i, seen as functions $\mathbb{S}^d \mapsto \mathbb{R}$. Then, for the Laplace operator $\Delta^{\mathbb{S}^d}$ and its associated carré du champ operator Γ, one has

$$\Delta^{\mathbb{S}^d}(x_i) = -dx_i, \quad \Gamma(x_i, x_j) = \delta_{ij} - x_i x_j. \tag{12}$$

This operator is invariant under the rotations of $\mathbb{R}^{d+1}$, and as a consequence its reversible probability measure is the uniform measure on the sphere (normalized to be a probability). A system of coordinates for the upper half sphere $\{x_{d+1} > 0\}$ is given by $(x_1, \cdots, x_d) \in \mathbb{B}_d$, where $\mathbb{B}_d = \{\sum_1^d x_i^2 = \|x\|^2 < 1\}$ is the unit ball in $\mathbb{R}^d$. In this system of coordinates, and thanks to formula (5), one checks easily that, up to a normalizing constant, the reversible measure is $(1 - \|x\|^2)^{-1/2} dx$, which is therefore the density of the uniform measure on the sphere in this system of coordinates (see [4]).

Now, one may consider some larger dimension $m > d$ and project the Brownian motion on $\mathbb{S}^m$ on the unit ball in $\mathbb{R}^d$ through $(x_1, \cdots, x_{m+1}) \mapsto (x_1, \cdots, x_d)$. Formula (12) provides immediately that this image is again a diffusion process with generator

$$\mathcal{L}^{(m)}(x_i) = -m\, x_i, \quad \Gamma(x_i, x_j) = \delta_{ij} - x_i x_j, \tag{13}$$

that is the same formula as (12) except that now m is no longer the dimension of the ball. Once again, formula (5) provides the reversible measure for this operator, which is, up to a normalizing constant, $(1-\|x\|^2)^{(m-1-d)/2}dx$, which is therefore the image measure of the uniform measure of the sphere through this projection.

As before, the boundary of the domain (the unit ball) has equation $\{1-\|x\|^2=0\}$, and we have a boundary equation

$$\Gamma\big(x_i, \log(1-\|x\|^2)\big) = -2x_i, \tag{14}$$

similar to Eq. (10).

Now, it is again easily checked that, for the function $h=(1-\|x\|^2)^{-(m-1-d)/2}$, one has

$$\mathscr{L}^{(m)}(h) = d(m-d-1)h,$$

so that one may perform the associated h-transform for which

$$(\mathscr{L}^{(m)})^{(h)} = \mathscr{L}^{(2d+2-m)}.$$

In the case where $m=d$, one sees that $\mathscr{L}^{(d)}$, which is the Laplace operator in this system of coordinates, is transformed into $\mathscr{L}^{(d+2)}$, which is the projection of the spherical Laplace operator in $\mathbb{S}^{d+2}$ onto the unit ball in $\mathbb{R}^d$.

As a consequence, we get

Proposition 7 *A spherical Brownian motion on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ conditioned to remain in a half sphere $\{x_{d+1}>0\}$, has the law of the projection of a spherical Brownian motion on $\mathbb{S}^{d+2}$ onto the unit ball in $\mathbb{R}^d$, lifted on the half upper sphere in $\mathbb{R}^{d+1}$.*

4 General *h*-Transform for Models Associated with Orthogonal Polynomials

We shall see in this section that all the above examples appear as particular examples, or limit examples, of a very generic one when orthogonal polynomials come into play. Everything relies on a boundary equation similar to (10) or (14), which appears as soon as one has a family of orthogonal polynomials which are eigenvectors of diffusion operators.

Let us recall some basic facts about diffusion associated with orthogonal polynomials, following [5]. We are interested in bounded open sets $\Omega \subset \mathbb{R}^d$, with piecewise $\mathscr{C}^1$ boundary. On Ω, we have a probability measure μ with smooth density ρ with respect to the Lebesgue measure, and an elliptic diffusion operator $\mathscr{L}$ which is symmetric in $\mathbb{L}^2(\mu)$. We suppose moreover that polynomials belong to the domain of $\mathscr{L}$, and that $\mathscr{L}$ maps the set $\mathscr{P}_k$ of polynomials with total degree less than k into itself. Then, we may find a $\mathbb{L}^2(\mu)$ orthonormal basis formed with polynomials which are eigenvectors for $\mathcal{L}$. Following [4], this is entirely described by the triple (Ω, Γ, μ), where Γ is the square field operator of $\mathcal{L}$.

We call such a system (Ω, Γ, μ) a polynomial system.

Then, one of the main results of [5] is the following

Theorem 1

1. *The boundary $\partial\Omega$ is included in an algebraic surface with reduced equation $\{P = 0\}$, where P is a polynomial which may we written as $P_1 \cdots P_k$, where the polynomials P_i are real, and complex irreducible.*
2. *If $\mathcal{L} = \sum_{ij} g^{ij}\partial^2_{ij} + \sum_i b^i\partial_i$, where the coefficients g^{ij} are polynomials with degree at most 2 and b^i are polynomials with degree at most 1.*
3. *The polynomial P divides $\det(g^{ij})$ (that we write $\det(\Gamma)$ in what follows, and which is a polynomial with degree at most $2d$).*
4. *For each irreducible polynomial P_r appearing in the equation of the boundary, there exist polynomials $L_{i,r}$ with degree at most 1 such that*

$$\forall i = 1, \cdots, d, \ \sum_j g^{ij}\partial \log P_r = L_{i,r}. \tag{15}$$

5. *Let Ω be a bounded set, with boundary described by a reduced polynomial equation $\{P_1 \cdots P_k = 0\}$, such that there exists a solution $(g^{ij}, L_{i,k})$ to Eq. (15) with (g^{ij}) positive definite in Ω. Call $\Gamma(f,f) = \sum_{ij} g^{ij}\partial_i f\partial_j f$ the associated squared field operator. Then for any choice of real numbers $\{\alpha_1, \cdots, \alpha_k\}$ such that $P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ is integrable over Ω for the Lebesgue measure, setting*

$$\mu_{\alpha_1,\cdots,\alpha_k}(dx) = C_{\alpha_1,\cdots,\alpha_k} P_1^{\alpha_1} \cdots P_k^{\alpha_k} dx,$$

   where $C_{\alpha_1,\cdots,\alpha_k}$ is a normalizing constant, then $(\Omega, \Gamma, \mu_{\alpha_1,\cdots,\alpha_k})$ is a polynomial system.
6. *When $P = C\det(\Gamma)$, that is when those 2 polynomials have the same degree, then there are no other measures μ for which (Ω, Γ, μ) is a polynomial system.*

Remark 4 Equation (15), that we shall call the boundary equation (not to be confused with the equation of the boundary), may be written in a more compact form $\Gamma(x_i, \log P_r) = L_{i,r}$. Thanks to the fact that each polynomial P_r is irreducible, this is also equivalent to the fact that $\Gamma(x_i, \log P) = L_i$, for a family L_i of polynomials with degree at most 1.

One must be a bit careful about the reduced equation of the boundary $\{P = 0\}$, when $P = P_1 \cdots P_k$. This means that each regular point of the boundary is contained in exactly one of the algebraic surfaces $\{P_i(x) = 0\}$, and that for each $i = 1, \cdots, k$, there is at least one regular point x of the boundary such that $P_i(x) = 0$. In particular, for a regular point $x \in \partial\Omega$ such that $P_i(x) = 0$, then for $j \neq i$, $P_j(x) \neq 0$ in a neighborhood $\mathcal{U}$ of such a point, and $P_i(x) = 0$ in $\mathcal{U} \cap \partial\Omega$. It is not too hard to see that such a polynomial P_i, if real irreducible, is also complex irreducible (if not, it would be written as $P^2 + Q^2$, and $P = Q = 0$ on $\mathscr{U} \cap \partial\Omega$). It is worth to observe that since P divides $\det(\Gamma)$ and that (g^{ij}) is positive definite on Ω, then none of the polynomials P_i appearing in the boundary equation may vanish in Ω. We may therefore chose them to be all positive on Ω.

The reader should also be aware that Eq. (15), or more precisely the compact form given in Remark 4, and which is the generalization of Eqs. (10) and (14), is a very strong constraint on the polynomial P. Indeed, given P, if one wants to determine the coefficients (g^{ij}) and L_i, this equation is a linear equation in terms of the coefficients of g^{ij} and L_i, for which we expect to find some non vanishing solution. But the number of equations is much bigger than the number of unknowns, and indeed very few polynomials P may satisfy those constraints. In dimension 2 for example, up to affine invariance, there are exactly 10 such polynomials, plus one parameter family (see [5]). The deltoid curve of Sect. 3.4 is just one of them.

Remark 5 We shall not use the full strength of this theorem in the examples developed here. The important fact is the boundary equation (15), which may be checked directly on many examples, and is the unique property required for the general h-transform described in Theorem 2.

Given a bounded set Ω and an operator Γ satisfying the conditions of Theorem 1, and for any choice of $\{\alpha_1, \cdots, \alpha_k\}$ such that $P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ is integrable over Ω for the Lebesgue measure, we have a corresponding symmetric operator $\mathscr{L}_{\alpha_1,\cdots,\alpha_k}$. For this operator, as was the case in Sects. 3.4 and 3.5, one may extend the integration by parts (4) to any pair of polynomials, and this provides a sequence of orthogonal polynomials which are eigenvectors of the operator $\mathscr{L}_{\alpha_1,\cdots,\alpha_k}$.

Conversely, the boundary equation (15) is automatic as soon as we have a generator on a bounded set with regular boundary, and a complete system of eigenvectors which are polynomials. But it may happen that those conditions are satisfied even on non bounded domains, and even when the associated measure is infinite (this appears in general in limits of such polynomial models, as in the Laguerre and Bessel cases). We may therefore give a statement in a quite general setting.

Theorem 2 *Assume that a symmetric positive definite matrix (g^{ij}) on some open set $\Omega \subset \mathbb{R}^d$, is such that for any (i,j), g^{ij} is a polynomial of degree at most 2. Let us call Γ the associated square field operator. Suppose moreover that we have some polynomials P_k, positive on Ω, such that, for any k,*

$$\forall i = 1, \cdots, d, \ \sum_i g^{ij} \partial_j \log P_r = \sum_i \Gamma(x^i, \log P_k) = L_{i,k}, \tag{16}$$

where $L_{i,k}$ are degree 1 polynomials. For any $(\alpha_1, \cdots, \alpha_k)$, let $\mu_{\alpha_1,\cdots,\alpha_k}$ be the measure with density $P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ with respect to the Lebesgue measure on Ω, and let $\mathscr{L}_{\alpha_1,\cdots,\alpha_k}$ be the generator associated with the Markov triple $(\Omega, \Gamma, \mu_{\alpha_1,\cdots,\alpha_k})$.

Then, there exist constants c_k such that, for any $(\alpha_1, \cdots, \alpha_k)$, the function $h = P_1^{-\alpha_1} \cdots P_k^{-\alpha_k}$ satisfies

$$\mathscr{L}_{\alpha_1,\cdots,\alpha_k}(h) = -(\sum_k \alpha_k c_k)h.$$

Moreover, $(\mathscr{L}_{\alpha_1,\cdots,\alpha_k})^{(h)} = \mathscr{L}_{-\alpha_1,\cdots,-\alpha_k}$.

Proof We shall prove the assertion with $c_k = \sum_i \partial_i L_{i,k}$.

With $\rho = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$, we write our operator $\mathscr{L}_{\alpha_1,\cdots,\alpha_k}$ as

$$\sum_{ij} g^{ij}\partial^2_{ij} + \sum_i b^i \partial_i,$$

where

$$b_i = \sum_j \partial_j g^{ij} + \sum_{r,j} \alpha_r g^{ij} \partial_j \log P_r = \sum_j \partial_j g^{ij} + \sum_r \alpha_r L_{i,r}. \tag{17}$$

With

$$\mathscr{L}_0 = \sum_{ij} g^{ij}\partial^2_{ij} + \sum_i \partial_j g^{ij} \partial_i,$$

then

$$\mathscr{L}_{\alpha_1,\cdots,\alpha_k}(f) = \mathscr{L}_0(f) + \sum_i \alpha_i \Gamma(\log P_i, f). \tag{18}$$

What we want to show is $\mathscr{L}_{\alpha_1,\cdots,\alpha_k}(h) = ch$, or

$$\mathscr{L}_{\alpha_1,\cdots,\alpha_k}(\log h) + \Gamma(\log h, \log h) = c.$$

With $\log h = -\sum_i \alpha_i \log P_i$, and comparing with Eq. (18), this amounts to

$$\mathscr{L}_0(\log h) = -\sum_r \alpha_q \mathscr{L}_0(\log P_r) = c.$$

We may first take derivative in Eq. (16) with respect to x_i and add the results in i to get

$$\sum_{ij} g^{ij}\partial_{ij} \log P_r + \sum_i \partial_i(g^{ij})\partial_j \log P_r = \sum_i \partial_i L_{i,r} = c_r,$$

that is $\mathscr{L}_0(\log P_r) = c_r$.

It remains to add these identities over r to get the required result.

Comparing the reversible measures, it is then immediate to check that $(\mathscr{L}_{\alpha_1,\cdots,\alpha_k})^{(h)} = \mathscr{L}_{-\alpha_1,\cdots,-\alpha_k}$, □

Remark 6 The function h is always the inverse of the density with respect to the Lebesgue measure, in the system of coordinates in which we have this polynomial structure. Of course, the choice of the coordinate system is related to the fact that, in those coordinates, we have orthogonal polynomials (at least when the measure is finite on a bounded set). In the Bessel case, for example, which is a limit of a Laguerre models, one has to change x to x^2 to get a simple correspondence between the h function and the density. The same is true in many natural examples, where one has to perform some change of variable to get the right representation (for example from the triangle to the deltoid in Sect. 3.4).

Remark 7 In many situations, there are natural geometric interpretations for these polynomial models when the parameters $(\alpha_1, \cdots, \alpha_k)$ are half integers, in general with $\alpha_i \geq -1/2$. The case $\alpha_i = -1/2$ often corresponds to Laplace operators, while the dual case $\alpha_i = 1/2$ often corresponds to the projection of a Laplace operator in larger dimension.

5 Further Examples

We shall provide two more examples, one which follows directly from Theorem 2, and another one on a non bounded domain with infinite measure. One may provide a lot of such examples, many of them arising from Lie group theory, Dunkl operators, random matrices, etc. However, we chose to present those two cases because they put forward some specific features of diffusion operators associated with orthogonal polynomials.

5.1 Matrix Jacobi Processes

This model had been introduced by Doumerc in his thesis [13], and had also been studied in the complex case, especially from the asymptotic point of view in [9, 10]. It plays a similar rôle than the one-dimensional Jacobi processes for matrices. One starts from the Brownian motion on the group $SO(d)$. Since $SO(d)$ is a semi-simple compact Lie group, it has a canonical Casimir operator similar to the one described in Eq. (11). If $O = (m_{ij})$ is an $SO(d)$ matrix, then the Casimir operator may be described through its action on the entries m_{ij}. One gets

$$\mathscr{L}(m_{ij}) = -(d-1)m_{ij}, \quad \Gamma(m_{kl}, m_{qp}) = \delta_{(kl)(qp)} - m_{kp}m_{ql}. \tag{19}$$

Observe that when restricted to a single line or column, one recovers the spherical Laplace operator on $\mathbb{S}^{d-1}$ described in Eq. (12).

An $SO(d)$-Brownian matrix is then a diffusion process with generator this Casimir operator on $SO(d)$.

It is again clear from the form of the operator $\mathscr{L}$ that it preserves for each $k \in \mathbb{N}$ the set of polynomials in the entries (m_{ij}) with total degree less that k. However, these "coordinates" (m_{ij}) are not independent, since they satisfy algebraic relations, encoded in the fact that $OO^* = \mathrm{Id}$. We may not apply directly our main result Theorem 2. We shall nevertheless look at some projected models on which the method applies.

One may extract some $p \times q$ submatrix N by selecting p lines and q columns, and we observe that the generator acting on the entries of this extracted matrix N depend only on the entries of N. Therefore, the operator projects on these extracted $p \times q$ matrices and the associated process is again a diffusion process: we call this the projection of the Brownian motion in $SO(d)$ onto the set $\mathscr{M}_{p,q}$ of $p \times q$ matrices. Thanks to formula (5), one may compute the density of the image measure, with respect to the Lebesgue measure in the entries of N. Whenever $p+q \leq d$, it happens to be, up to a normalizing constant $\det(\mathrm{Id} - NN^*)^{(d-1-p-q)/2}$, with support the set $\Omega = \{N, NN^* \leq \mathrm{Id}\}$. This formula is easy to check if we recall that, for a matrix M with entries (m_{ij}),

$$\partial_{m_{ij}} \log \det(M) = M^{-1}_{ji},$$

a consequence of Cramer's formula.

When $p + q \geq d + 1$, there are however algebraic relations between the entries of N and the image measure has no density with respect to the Lebesgue measure. For example, when $p + q = d + 1$, then the measure concentrates on the algebraic set $\{\det(\mathrm{Id} - NN^*) = 0\}$. It may be checked that it has a density with respect of the Lebesgue measure of this hypersurface. Indeed, one may fix p and q and consider d as a parameter. It is worth to observe that the function $\det(\mathrm{Id} - NN^*)^\alpha$ is not integrable on the domain Ω whenever $\alpha \leq -1$. Moreover, when $\alpha > -1$ and $\alpha \to -1$, the probability measure with density $C_\alpha \det(\mathrm{Id} - NN^*)^\alpha$ concentrates on the set $\{\det(\mathrm{Id} - NN^*) = 0\}$, and the limit is a measure supported by this surface with a density with respect of the surface measure. Things become even worse as the number $p + q$ increases, the measure being concentrated on manifolds with higher and higher co-dimensions.

We are in a situation different from the sphere case here, since we may not chose the parameters in which the operator has a nice polynomial expression as a local system of coordinates. Indeed, the Lie group $SO(d)$ is a $d(d-1)/2$ manifold. Since we want algebraically independent coordinates, we are limited to pq ones, with $p + q \leq d$, we may have at most $d^2/4$ algebraically independent such polynomial coordinates, which for $d > 2$ is less than the dimension of the manifold.

It is worth to observe that, again when $p + q \leq d$, one has pq variables, the determinant of the metric Γ is a degree $2pq$ polynomial, whereas $\det(\mathrm{Id} - NN^*) = \det(\mathrm{Id} - N^*N)$ is of degree at most $2\min(p, q)$. We are not in the case of maximal degree for the boundary equation. When $p + q = d$, the density measure is $\det(\mathrm{Id} - NN^*)^{-1/2}$, but the corresponding operator is not a Laplace operator (for which the density of the measure would be $\det(\Gamma)^{-1/2}$). Since we are in the situation of orthogonal polynomials as described in Sect. 4, we know that we may perform an h-transform.

For the particular case where $d = p + q$, we get

Proposition 8 *The matrix N projected from an $SO(d)$-Brownian matrix on $\mathcal{M}_{p,q}$ conditioned to remain in the set $\{NN^* < \mathrm{Id}\}$ has the law of the projection of a $SO(d+2)$-Brownian matrix on $\mathcal{M}_{p,q}$.*

5.2 *Brownian Motion in a Weyl Chamber*

This last example is again quite well known, but it happens to fit also with the general picture associated with orthogonal polynomials, although no orthogonal polynomials are associated with it. Indeed, it does not follow directly from the setting of Sect. 4, on the one side because it is non compact, on the other side because the reversible measure in this situation is infinite. But it satisfies all the algebraic properties described is Sect. 4, and we may then check that we may apply the result for the associated h-transforms. Indeed, one may replace in what follows Brownian motion by Ornstein-Uhlenbeck operators, which have as reversible measure a Gaussian measure with variance σ^2, and then let σ go to infinity. In the Ornstein-Uhlenbeck case, we are in the setting of orthogonal polynomials, however with a non bounded domain. But this would introduce further complication, since the Brownian case gives simpler formulas.

As described above, the h-transform is easy to compute in a system of coordinates which have some relevant polynomial structure. Here, one good choice for the coordinate system are the elementary symmetric functions in d variables. We shall perform mainly computations on these elementary symmetric functions of the components of the d-dimensional Brownian motion, following [3]. In $\mathbb{R}^d$, one may consider the Brownian $(B_t^1, \cdots, B_t^d)$ and reflect it around the hyperplanes which are the boundaries of the set $\{x_1 < \cdots < x_d\}$, which is usually called a Weyl chamber. To describe this reflected Brownian motion, it is easier to consider the elementary symmetric functions which are the coefficients of the polynomial

$$P(X) = \prod_{i=1}^{d}(X - x_i) = \sum_{i=0}^{d} a_i X^i,$$

where $a_d = 1$ and the functions $a_i, i = 0,\cdots d-1$ are, up to a sign, the elementary symmetric functions of the variables (x_i). The map $(x_i) \mapsto (a_i)$ is a diffeomorphism in the Weyl chamber $\{x_1 < \cdots < x_d\}$ onto its image. To understand the image, one has to consider the discriminant disc(P), a polynomial in the variables (a_i), which is, up to a sign $(-1)^{d(d-1)/2}$, the following $(2d-1)\times(2d-1)$ determinant

$$\begin{pmatrix}
1 & a_{d-1} & a_{d-2} & \cdots & a_0 & 0 & \cdots & 0\\
0 & 1 & a_{d-1} & \cdots & a_1 & a_0 & \cdots & 0\\
0 & 0 & 1 & \cdots & a_2 & a_1 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & a_{p-2} & \cdots & a_1 & a_0\\
1 & (d-1)a_{d-1} & (d-2)a_{d-2} & \cdots & a_1 & 0 & \cdots & 0\\
0 & 1 & (d-1)a_{d-1} & \cdots & 2a_2 & a_1 & \cdots & 0\\
0 & 0 & 1 & \cdots & 3a_3 & 2a_2 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 2a_2 & a_1
\end{pmatrix}$$

It turns out that this discriminant is $\prod_{i<j}(x_j-x_i)^2$. The image of the Weyl chamber is the connected component Ω of the set $\{\mathrm{disc}(P) \neq 0\}$ which contains the image of any polynomial with d real distinct roots, and the image of the boundary of the Weyl chamber is $\partial\Omega$, a subset of the algebraic surface $\{\mathrm{disc}(P) = 0\}$. It is not hard to observe (by induction on the dimension d) that the image of the Lebesgue measure $dx_1\cdots dx_d$ on the Weyl chamber is nothing else than $\mathbf{1}_\Omega \mathrm{disc}(P)^{-1/2}\prod da_i$.

Now, the Brownian motion in $\mathbb{R}^d$ may be described, up to a factor 2, through

$$\Gamma(x_i,x_j) = \delta_{ij},\ \Delta(x_i) = 0.$$

We want to describe this operator acting on the variables $(a_0,\cdots,a_{d-1})$. Since any of the functions a_j is a polynomial with degree 1 in the variables x_i, one has $\Delta(a_j) = 0, j = 0,\cdots,d$. To compute $\Gamma(a_i,a_j)$, it is simpler to compute

$$\Gamma(P(X),P(Y)) = \sum_{i,j} X^iY^j\Gamma(a_i,a_j).$$

We obtain

Proposition 9 *The image of the operator Δ in $\mathbb{R}^n$ on the coefficients of the polynomial $P(X) = \prod_i(X-x_i)$ is given by*

$$\Gamma(P(X),P(Y)) = \frac{1}{Y-X}\big(P'(X)P(Y)-P'(Y)P(X)\big),\ \Delta(P(X)) = 0. \tag{20}$$

Proof The second formula is a direct consequence of $\Delta(a_i) = 0$, while for the first, it is simpler to look at $\Gamma(\log P(X), \log P(Y))$.

$$\begin{aligned}\Gamma(\log P(X), \log P(Y)) &= \sum_{ij} \Gamma(\log(X - x_i), \log(Y - x_j))\\ &= \sum_{ij} \frac{1}{(X - x_i)(Y - x_j)} \Gamma(x_i, x_j)\\ &= \sum_{i} \frac{1}{(X - x_i)(Y - x_i)} = \frac{1}{Y - X}\Big(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\Big).\end{aligned}$$

□

Remark 8 From formula (20), it is clear that $\Gamma(a_i, a_j)$ are polynomials with degree 2 in the variables a_i.

The image of the Brownian motion B_t in the variables (a_i) is nothing else than the Brownian motion reflected through the walls of the Weyl chamber. Its generator is described through the Γ operator given in Eq. (20) and it is the image of the Laplace operator on the Weyl chamber. Since it is an Euclidean Laplace operator, the reversible measure is, up to a constant, $\det(\Gamma)^{-1/2}$, and this shows that the determinant $\det(\Gamma)$ of the metric is, up to a constant, $\mathrm{disc}(P)$.

Moreover, from the general representation of diffusion operators (3), and the Eq. (5) giving the reversible measure, we have, with $\rho = \mathrm{disc}(P)^{-1/2}$, $b_i = 0$,

$$\sum_{ij} \Gamma(a_i, a_j)\partial_{a_i} \log \rho = -\sum_{j} \partial_{a_j} \Gamma(a_i, a_j). \tag{21}$$

Since $\partial_{a_j}\Gamma(a_i, a_j)$ is a polynomial with degree at most 1 in the variables a_i, this is nothing else than the boundary equation (15) for general polynomial models. We may therefore apply the general result described in Sect. 4.

In order to identify the result of the *h*-transform, an important formula relating Γ and the discriminant function is the following

Proposition 10 *For the operator Γ defined in* (20)*, one has*

$$\Gamma\big(P(X), \log \mathrm{disc}(P)\big) = -P''(X). \tag{22}$$

Proof One may find a proof of this formula in [3], but the one we propose here is simpler. To check Eq. (22), it is enough to establish it in a Weyl chamber $\{x_1 < x_1 < \cdots < x_d\}$ where $P(X) = \prod(X - x_i)$ and $\mathrm{disc}(P) = \prod_{i<j}(x_i - x_j)^2$, since the map $(x_1, \cdots, x_k) \mapsto P(X)$ is a local diffeormorphism in this domain.

In those coordinates, $\Gamma(x_i, x_j) = \delta_{ij}$ and, from the change of variable formula (2), one has

$$\Gamma(\log P(X), \mathrm{disc}(P)) = 2\sum_{i,j<k} \Gamma(\log(X - x_i), \log(x_j - x_k))$$
$$= -2\sum_{i,j<k} \frac{1}{X - x_i}\frac{1}{x_j - x_k}(\delta_{ij} - \delta_{ik}).$$

From which one gets

$$\Gamma(\log P(X), \log \mathrm{disc}(P)) = -2\sum_{i\neq j} \frac{1}{X - x_i}\frac{1}{x_i - x_j}.$$

On the other hand,

$$\frac{P''}{P} = \Big(\frac{P'}{P}\Big)' + \Big(\frac{P'}{P}\Big)^2 = \sum_{i\neq j} \frac{1}{(X - x_i)(X - x_j)}$$
$$= \sum_{i\neq j} \Big(\frac{1}{X - x_i} - \frac{1}{X - x_j}\Big)\frac{1}{x_i - x_j} = 2\sum_{i\neq j} \frac{1}{X - x_i}\frac{1}{x_i - x_j}.$$

From this we get

$$\Gamma(\log P, \log \mathrm{disc}(P)) = -\frac{P''}{P},$$

which in turns gives (22). □

Proposition 10 is central in the identification of various processes with the same Γ given by (20). It turns out that the same process with this Γ operator and reversible measure $\mathrm{disc}(P)^{1/2}$ has a nice geometric interpretation: namely, it is the Dyson complex process, that is the law of the spectrum of Hermitian Brownian matrices, introduced by Dyson [16]. In the same way, the case where the reversible measure is the Lebesgue measure corresponds to Dyson process for real symmetric matrices, and $\rho = \mathrm{disc}(P)^{3/2}$ corresponds to Dyson process for symmetric quaternionic matrices, see [1, 3, 19].

Let us show a direct way to check this (first in the real symmetric case, where it is simpler). The Brownian motion on symmetric matrices is nothing else that the Brownian motion of the Euclidean space of symmetric matrices M, endowed with the Euclidean norm $\|M\|^2 = \mathrm{trace}\,(M^2)$. When $M = (m_{ij})$, this may be described as

$$\Gamma(m_{ij}, m_{kl}) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),\ \mathscr{L}(m_{ij}) = 0.$$

One may look at its action of the characteristic polynomial $P(X) = \det(X\mathrm{Id} - M)$. We get

Proposition 11 *For the characteristic polynomial associated with a Brownian symmetric matrix, one has*

$$\Gamma\big(\log P(X), \log P(Y)\big) = \frac{1}{Y-X}\Big(\frac{P'(X)}{P(X)} - \frac{P'(Y)}{P(Y)}\Big),\ \mathscr{L}P(X) = -\frac{1}{2}P''.$$

Proof To compute $\Gamma\big(P(X), P(Y)\big)$ and $\mathscr{L}\big(P(X)\big)$. In order to apply the change of variable formula (3), we may apply the general formulas for the determinant function

$$\partial_{m_{ij}} \log\det M = M^{-1}_{ji},\ \partial_{m_{ij}}\partial_{m_{kl}} \log\det M = -M^{-1}_{jk}M^{-1}_{li},$$

which are direct consequences of Cramer's formulas for the inverse matrix.

Then the formulas are direct applications of the chain rule formula. □

We may now compare with Eq. (22) to see that the reversible measure for the spectral measure for Brownian symmetric matrices, given by the general formula (5), in the system of coordinates which are the coefficients (a_i) of the characteristic polynomial, is the Lebesgue measure.

We may perform the same computation for Hermitian matrices. In this situation, one would consider a complex valued matrix M with entries (z_{ij}) and satisfying

$$\Gamma(z_{ij}, z_{kl}) = 0,\ \Gamma(z_{ij}, \bar{z}_{kl}) = \delta_{il}\delta_{jk},\ \mathscr{L}(z_{ij}) = 0.$$

One may again perform the same computation on $P(X) = \det(X\mathrm{Id} - M)$, and we get

Proposition 12 *For the characteristic polynomial associated with a Brownian Hermitian matrix, one has*

$$\Gamma(P(X), P(Y)) = \frac{1}{Y-X}\big(P'(X)P(Y) - P'(Y)P(X)\big),\ \mathscr{L}P(X) = -P''.$$

We do not give the proof, which follows along the same lines that the one of Proposition 11. More details may be found in [3].

As a consequence, comparing with Eqs. (22) and (5) in the system of coordinates given by the coefficients of $P(X)$, the density of the reversible measure for the Hermitian Dyson process is $\mathrm{disc}(P)^{1/2}$ whereas the density of the reversible measure of the Brownian motion in the Weyl chamber is $\mathrm{disc}(P)^{-1/2}$.

Transferring back to the Weyl Chamber through the local diffeomorphism between the coefficients of $P(X)$ and the roots $(x_1 < x_2 < \cdots < x_d)$ of $P(X)$. We obtain

Proposition 13 *The Brownian motion conditioned not to reach the boundary of the Weyl chamber $\{x_1 < \cdots < x_d\}$ has the law of the spectrum of an Hermitian $d \times d$ matrix.*

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On an Optional Semimartingale Decomposition and the Existence of a Deflator in an Enlarged Filtration

Anna Aksamit, Tahir Choulli, and Monique Jeanblanc

Abstract Given a reference filtration $\mathbb{F}$, we consider the cases where an enlarged filtration $\mathbb{G}$ is constructed from $\mathbb{F}$ in two different ways: progressively with a random time or initially with a random variable. In both situations, under suitable conditions, we present a $\mathbb{G}$-optional semimartingale decomposition for $\mathbb{F}$-local martingales. Our study is then applied to the question of how an arbitrage-free semimartingale model is affected when stopped at the random time in the case of progressive enlargement or when the random variable used for initial enlargement satisfies Jacod's hypothesis. More precisely, we focus on the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition, also called non arbitrages of the first kind in the literature. We provide alternative proofs of some results from Aksamit et al. (Non-arbitrage up to random horizon for semimartingale models, short version, preprint, 2014 [arXiv:1310.1142]), incorporating a different methodology based on our optional semimartingale decomposition.

1 Introduction

We are interested in some specific enlargements of a given filtration, namely the progressive one and the initial one. The progressive enlargement $\mathbb{G}$ of a filtration $\mathbb{F}$ with a random time (a positive random variable) τ, is the smallest filtration larger than $\mathbb{F}$ making τ a stopping time. It is known that any $\mathbb{F}$-martingale, stopped at time τ is a $\mathbb{G}$ semi-martingale. In this paper, we do not consider the behavior of $\mathbb{F}$-martingales after τ, which is presented in [3], and requires specific assumptions on the random time τ.

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Then, we study the case where the enlarged filtration $\mathbb{G}$ is constructed from $\mathbb{F}$ as an initial enlargement, that is, adding to all the elements $\mathscr{F}_t$ of the filtration $\mathbb{F}$ a random variable ξ. We focus on a specific situation where the hypothesis $(\mathscr{H}')$, i.e., the property that each $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale, is satisfied. More precisely, we shall assume that the $\mathbb{F}$-conditional law of ξ is absolutely continuous with respect to the unconditional law of ξ (Jacod's hypothesis, see Definition1 below).

The goal of the paper is to study the impact of the new information for arbitrage opportunities in a financial market: assuming that one deals with an arbitrage free financial market with $\mathbb{F}$-adapted prices, can an agent using $\mathbb{G}$-adapted strategies realize arbitrage opportunities? More precisely we study how the No-Unbounded-Profit-with-Bounded-Risk (NUPBR) condition (see Definition 3 below) will be preserved in the enlarged filtration. The NUPBR condition is related to other no arbitrage conditions like No Free Lunch with Vanishing Risk (NFLVR) or (classical) No Arbitrage, see [8, 21]; in particular the NUPBR condition is equivalent to both NFLVR condition and No Arbitrage condition. This condition is also closely related with the notion of log-optimal portfolios and optimal growth rate portfolio and is proved to be an appropriate condition to study some financial notions like numéraire portfolio, or market viability (see [7, 14, 23, 24, 26, 27, 30] and the references therein). A general study of the NUPBR condition, and a list of references on the subject can be found in Kabanov et al. [22].

The literature on arbitrage conditions in an enlarged filtration is important, even if the subject is not so popular in mathematical finance. Quite surprisingly, the hypothesis that all the investors have the same knowledge is usually done in the literature, even if this hypothesis is not satisfied in reality. The main difficulty is that it is not easy to compare stochastic processes in various filtration (the most common approach is filtering study). Here we are interested in the opposite direction: some investor has an information larger than the one generated by prices of asset he is willing to trade. For progressive enlargement, the case of classical arbitrages is presented in [11], and it is proved that, for a class of random times (called honest times) arbitrages can occur in the case where the market described in the filtration $\mathbb{F}$ is complete and arbitrage free (see also [15] for the Brownian case). However, to the best of our knowledge, no necessary and sufficient conditions are known in an incomplete model. The recent literature concerns a weaker notion of arbitrages, called No-Unbounded-Profit-with-Bounded-Risk (NUPBR), deeply related with optimization problems, see [7]. A first paper on that subject was [11], in which the authors are dealing with continuous processes. Many examples of progressive enlargement (in particular for discontinuous processes) are given in [2]. A general study, giving necessary and sufficient condition for the stability of NUPBR condition is presented in [5]. A different proof of some results of that paper (mainly sufficient conditions), based on another representation of the deflators (see Sect. 2.3 for definition), is given in [1]. We shall explain here how our results are linked with the ones in [1]. The recent paper of Song [33] contains also a study of deflator in a progressive enlargement setting.

The case of initial enlargement was studied under the name of insider trading. Many papers, including [12, 13] and the thesis [6] present results under an assumption stronger that the Jacod's absolute continuity hypothesis.

In the first section, we recall some basic definitions and results on enlargement of filtration and on arbitrage opportunities. Section 3 addresses the case of progressive enlargement with τ and $\mathbb{F}$-martingales stopped at τ. In Sect. 3.1, we introduce a particular optional semi-martingale decomposition, which will be useful in the sequel, and we give the link between this decomposition and the deflator exhibited in the literature (see [1, 5]). In Sect. 3.2, we provide alternative and shorter proofs of some results from [5], and give a condition so that the NUPBR condition is preserved, using a methodology different from the one used in [5] avoiding the introduction of optional integral, and based on our optional semimartingale decomposition.

Section 4 presents the case of initial enlargement. In Sect. 4.2, we give an optional decomposition result for the $\mathbb{F}$-martingales, when the added random variable satisfies Jacod's hypothesis.We also obtain a result concerning the relationship between the predictable brackets of semimartingale computed in both filtrations. Then, we address the question of stability of the NUPBR condition. The results presented in this last section were obtained in parallel and independently of [1].

The last Sect. 5 presents a link between our optional decomposition and absolutely continuous change of measures.

2 Preliminaries

Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions. We say that a filtration $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ is an enlargement of $\mathbb{F}$ if, for each $t \geq 0$, we have $\mathscr{F}_t \subset \mathscr{G}_t$.

We recall some standard definitions and set some notation. For a filtration $\mathbb{H}$, the optional σ-field on $\Omega \times \mathbb{R}_+$, denoted by $\mathscr{O}(\mathbb{H})$, is the σ-field generated by all càdlàg $\mathbb{H}$-adapted processes and the predictable σ-field on $\Omega \times \mathbb{R}_+$, denoted by $\mathscr{P}(\mathbb{H})$, is the σ-field generated by all left-continuous $\mathbb{H}$-adapted processes. A stochastic set or process is called $\mathbb{H}$-optional (respectively $\mathbb{H}$-predictable) if it is $\mathscr{O}(\mathbb{H})$-measurable (respectively $\mathscr{P}(\mathbb{H})$-measurable).

For an $\mathbb{H}$-semimartingale Y, the set of $\mathbb{H}$-predictable processes integrable with respect to Y is denoted by $L(Y, \mathbb{H})$ and for $H \in L(Y, \mathbb{H})$, we denote by $H \centerdot Y$ the stochastic integral $\int_0^\cdot H_s dY_s$.

As usual, for a process X and a random time ϑ, we denote by X^ϑ the stopped process. For a given semimartingale X, $\mathscr{E}(X)$ stands for the stochastic exponential of X. The continuous local martingale part and the jump process of a càdlàg semimartingale X are denoted respectively by X^c and ΔX.

2.1 Progressively Enlarged Filtration

Consider a random time τ, i.e., a positive random variable. Then, we define two $\mathbb{F}$-supermartingales, which are the cornerstone for the classical enlargement decomposition formulae (2) and (3), in a progressive enlargement framework (see [19]), given by

$$Z_t := \mathbb{P}(\tau > t|\mathscr{F}_t) \quad \text{and} \quad \tilde{Z}_t := \mathbb{P}(\tau \geq t|\mathscr{F}_t).$$

In other terms, Z is the optional projection of $\mathbb{1}_{[0,\tau[}$, whereas $\tilde{Z}$ is the optional projection of $\mathbb{1}_{[0,\tau]}$. Let A^o be the $\mathbb{F}$-dual optional projection of the increasing process $A := \mathbb{1}_{[\tau,\infty[}$; then (see [19]), the process

$$m := Z + A^o \tag{1}$$

is an $\mathbb{F}$-martingale. Furthermore, $\tilde{Z} = Z_- + \Delta m$.

We denote by $\mathbb{F}^\tau = (\mathscr{F}_t^\tau)_{t\geq 0}$ the right-continuous progressively enlarged filtration with the random time τ, i.e.,

$$\mathscr{F}_t^\tau := \bigcap_{s>t} (\mathscr{F}_s \vee \sigma(\tau \wedge s)) \, .$$

The following result from [20] states that any $\mathbb{F}$-local martingale stopped at τ is an $\mathbb{F}^\tau$-semimartingale.

Proposition 1 *Let X be an $\mathbb{F}$-local martingale. Then, X^τ is an $\mathbb{F}^\tau$-semimartingale which can be decomposed as*

$$X_t^\tau = \hat{X}_t + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}} \tag{2}$$

where $\hat{X}$ is an $\mathbb{F}^\tau$-local martingale.

In what follows, we will refer to the equality (2) as the predictable decomposition of the $\mathbb{F}^\tau$-semi-martingale X^τ.

Remark 1 The decomposition (2) contains a predictable bracket computed in $\mathbb{F}$. When working in a larger filtration $\mathbb{G}$, predictable brackets are computed in $\mathbb{G}$. As can be seen in [5], we face the problem of comparison of the two different brackets.

Remark 2 It is rather easy to check that Z_- does not vanish on the set $\{t \leq \tau\}$. However, the first time where this process vanishes will play an important rôle.

Remark 3 Using the $\mathbb{F}$-local martingale

$$N := \mathscr{E}\left(\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}} \cdot m\right) , \tag{3}$$

Kardaras [25] notes that the decomposition (2) can be written $X_t^\tau = \hat{X}_t + \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\langle X, N\rangle_s^{\mathbb{F}}$.

2.2 Initially Enlarged Filtration

Let ξ be a random variable valued[1] in $(\mathbb{R}, \mathscr{B})$.

Definition 1 The random variable ξ satisfies the *Jacod's absolute continuity hypothesis* if there exists a σ-finite positive measure η on $(\mathbb{R}, \mathscr{B})$ such that for every $t \geq 0$,

$$\mathbb{P}(\xi \in du|\mathscr{F}_t)(\omega) \ll \eta(du), \mathbb{P}\text{-a.s..}$$

As shown by Jacod [17], without loss of generality, η can be taken as the law of ξ. We do not impose any condition on η, in particular, it is not necessarily atomless.

The random variable ξ satisfies the *Jacod's equivalence hypothesis* if

$$\mathbb{P}(\xi \in du|\mathscr{F}_t)(\omega) \sim \eta(du), \mathbb{P}\text{-a.s..}$$

Let $\mathbb{F}^{\sigma(\xi)} = (\mathscr{F}^{\sigma(\xi)})_{t\geq 0}$ be the right-continuous initial enlargement of the filtration $\mathbb{F}$ with the random variable ξ, i.e.,

$$\mathscr{F}_t^{\sigma(\xi)} := \bigcap_{s>t} (\mathscr{F}_s \vee \sigma(\xi)) .$$

The following result is due to Jacod [17, Lemme 1.8]. We give here the formulation of Amendinger as it provides a nice measurability property (see [6, Remark 1, p. 17]).

Proposition 2 *For ξ satisfying the Jacod's absolute continuity hypothesis, there exists a non negative $\mathscr{O} \otimes \mathscr{B}$-measurable function $(t, \omega, u) \to q_t^u(\omega)$ càdlàg in t such that*

(i) for every u, η a.s., the process $(q_t^u, t \geq 0)$ is an $\mathbb{F}$-martingale, and if the stopping time R^u is defined as

$$R^u := \inf\{t : q_{t-}^u = 0\} \tag{4}$$

one has $q^u > 0$ and $q_-^u > 0$ on $[\![0, R^u[\![$ and $q^u = 0$ on $[\![R^u, \infty[\![$,

(ii) for every $t \geq 0$, the measure $q_t^u(\omega)\eta(du)$ is a version of $\mathbb{P}(\xi \in du|\mathscr{F}_t)(\omega)$.

[1]The random variable ξ can take values in a more general space without any difficulty.

It is rather clear that we shall have to deal with the family of processes $(q^u, u \in \mathbb{R})$, that we shall call parametrized processes.

Definition 2 Consider a mapping $X : (t, \omega, u) \to X^u_t(\omega)$ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}$ with values in $\mathbb{R}$. Let $\mathscr{J}$ be a class of $\mathbb{F}$-optional processes, for example the class of $\mathbb{F}$-(local) martingales or the class of $\mathbb{F}$-locally integrable variation processes. Then, $(X^u, u \in \mathbb{R})$ is called a *parametrized* $\mathscr{J}$-process if for each $u \in \mathbb{R}$ the process X^u belongs to $\mathscr{J}$ and if it is measurable with respect to $\mathscr{O}(\mathbb{F}) \otimes \mathscr{B}$.

By [34, Proposition 3], the second condition can be obtained by taking appropriate versions of processes X^u.

The next theorem gives, in the case of Jacod's equivalence hypothesis, a particular change of measure making the reference filtration $\mathbb{F}$ and the random variable ξ independent, see [32], [6, Proposition 1.6], [12].

Theorem 1 *Assume that the Jacod's equivalence hypothesis is satisfied, so that* $\mathbb{P}(\xi \in du|\mathscr{F}_t) = q^u_t \eta(du)$ *with* $q^u_t > 0$, $(\eta \otimes P)$*-a.s. Then*

(a) the process $\frac{1}{q^\xi}$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-martingale,*
(b) the probability measure $\mathbb{P}^*$*, defined as*

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\,|_{\mathscr{F}^{\sigma(\xi)}_t} = \frac{1}{q^\xi_t}$$

has the following properties:

(i) under $\mathbb{P}^*$, τ *is independent from* $\mathscr{F}_t$ *for any* $t \geq 0$,
(ii) $\mathbb{P}^*|_{\mathscr{F}_t} = \mathbb{P}|_{\mathscr{F}_t}$,
(iii) $\mathbb{P}^*|_{\sigma(\xi)} = \mathbb{P}|_{\sigma(\xi)}$.

Remark 4 Note that, under Jacod's equivalence hypothesis, if the price process S is such that there are no arbitrages in $\mathbb{F}$, then there are no arbitrages in $\mathbb{G}$. Indeed, taking $\mathbb{P}$ as an equivalent martingale measure in $\mathbb{F}$, the previous result proves that $\mathbb{P}^*$ is an equivalent martingale measure in $\mathbb{G}$.

We now recall the computation of $\mathbb{F}$-predictable and $\mathbb{F}$-optional projections of $\mathbb{F}^{\sigma(\xi)}$-adapted processes. The first part is due to Jacod [17, Lemme 1.10], the second part is found in Amendinger [6, Lemma 1.3].

Lemma 1 *Assume that the Jacod's absolute continuity hypothesis is satisfied.*

(i) Assume that the map $(t, \omega, u) \to Y^u_t(\omega)$ *is* $\mathscr{P}(\mathbb{F}) \otimes \mathscr{B}$*-measurable, positive or bounded. Then, the* $\mathbb{F}$*-predictable projection of the process* $(Y^\xi_t)_{t\geq 0}$ *is given by*

$$^{p,\mathbb{F}}\big(Y^\xi\big)_t = \int_{\mathbb{R}} Y^u_t q^u_{t-} \eta(du) \quad t \geq 0. \tag{5}$$

(ii) Assume that the map $(t, \omega, u) \to Y_t^u(\omega)$ *is* $\mathscr{O}(\mathbb{F}) \otimes \mathscr{B}$*-measurable, positive or bounded. Then, the* $\mathbb{F}$*-optional projection of the process* $(Y_t^\xi)_{t \geq 0}$ *is given by*

$$ {}^{o,\mathbb{F}}\left(Y^\xi\right)_t = \int_{\mathbb{R}} Y_t^u q_t^u \eta(du) \quad t \geq 0. \tag{6}$$

As noticed in Jacod [17, Corollary 1.11], Lemma 1 implies in particular that

$$R^\xi = \infty \quad \mathbb{P}\text{-a.s.} \tag{7}$$

where R^u is defined through (4), or equivalently $q_t^\xi > 0$ and $q_{t-}^\xi > 0$, for $t \geq 0$, $\mathbb{P}$-a.s. Therefore, the $\mathbb{F}^{\sigma(\xi)}$-optional process $(\frac{1}{q_t^\xi}, t \geq 0)$ is well-defined.

The $\mathbb{F}^{\sigma(\xi)}$-semimartingale predictable decomposition of an $\mathbb{F}$-local martingale is given in [17, Theorem 2.5] in the following way:

Proposition 3 *Let* X *be an* $\mathbb{F}$*-local martingale. Then, under Jacod's absolute continuity hypothesis*

$$X_t = \hat{X}_t + \int_0^t \frac{1}{q_{s-}^\xi} d\langle X, q^u \rangle_s^{\mathbb{F}}|_{u=\xi} \tag{8}$$

where $\hat{X}$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-local martingale.*

To ensure the existence of well measurable versions of dual projections of parametrized processes, we assume from now on that the space $L^1(\Omega, \mathscr{G}, \mathbb{P})$ is separable. Then, we apply [34, Proposition 4].

In the next proposition, we extend Proposition 3 to a class of parametrized $\mathbb{F}$-local martingales.

Proposition 4 *Assume Jacod's absolute continuity hypothesis. Let* $(X^u, u \in \mathbb{R})$ *be a parametrized* $\mathbb{F}$*-local martingale. Then*

$$X_t^\xi = \hat{X}_t^\xi + \int_0^t \frac{1}{q_{s-}^\xi} d\langle X^u, q^u \rangle_s^{\mathbb{F}}|_{u=\xi}$$

where $\hat{X}^\xi$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-local martingale.*

Proof Let X be of the form $X_t^u(\omega) = g(u)x_t(\omega)$ where x is an $\mathbb{F}$-martingale and g is a Borel function. Then, $X^\xi = g(\xi)x$ and, using Jacod's decomposition given in Proposition 3, for $t \geq s$, we have

$$\begin{aligned} \mathbb{E}(X_t^\xi - X_s^\xi | \mathscr{F}_s^{\sigma(\xi)}) &= g(\xi)\mathbb{E}(x_t - x_s | \mathscr{F}_s^{\sigma(\xi)}) \\ &= g(\xi)\mathbb{E}\left(\int_s^t \frac{1}{q_{v-}^\xi} d\langle x, q^u \rangle_v |_{u=\xi} \middle| \mathscr{F}_s^{\sigma(\xi)} \right) \end{aligned}$$

$$= \mathbb{E}\left(\int_s^t \frac{1}{q_{v-}^{\xi}} d\langle X^u, q^u \rangle_v |_{u=\xi} \middle| \mathscr{F}_s^{\sigma(\xi)} \right).$$

For a general X, we proceed by Monotone Class Theorem. □

2.3 Local Martingale Deflators and Related Notions

As announced before, we shall study the No Unbounded Profit with Bounded Risk (NUPBR) condition. We start with some definitions for a general filtration $\mathbb{H}$.

Definition 3

(a) Let $(X^u, u \in \mathbb{R})$ be a parametrized $\mathbb{H}$-semimartingale. We say that $(X^u, u \in \mathbb{R})$ satisfies No Unbounded Profit with Bounded Risk condition in the filtration $\mathbb{H}$ (we shall write NUPBR($\mathbb{H}$)) if for each $T < \infty$, the set $\mathscr{K}_T(X)$ defined as

$$\mathscr{K}_T(X) := \{(H \centerdot X^u)_T : H \in L(\mathbb{H}, X^u),\ H \centerdot X^u \geq -1 \text{ and } u \in \mathbb{R}\}$$

is bounded in probability.

(b) A process Y is called an $\mathbb{H}$-local martingale deflator for $(X^u, u \in \mathbb{R})$ if it is a strictly positive $\mathbb{H}$-local martingale such that $(YX^u, u \in \mathbb{R})$ is a parametrized $\mathbb{H}$-σ-martingale.

(c) A process $\tilde{Y}$ is called an $\mathbb{H}$-supermartingale deflator for $(X^u, u \in \mathbb{R})$ if it is a strictly positive $\mathbb{H}$-supermartingale such that for each $H \in L(\mathbb{H}, X^u)$ with $H \centerdot X^u \geq -1$, the process $(1 + H \centerdot X^u)\tilde{Y}$ is an $\mathbb{H}$-supermartingale.

As proved in [29] in full generality, condition (a) and the existence of a supermartingale deflator stated in Definition 3(c) are equivalent. Moreover, as shown in [31], for a process X (instead of a family of processes as above), condition (a) and the existence of a local martingale deflator are equivalent. So, we have the following theorem:

Theorem 2 *For a semimartingale X, the NUPBR condition is equivalent to the existence of a local martingale deflator which is equivalent to the existence of a supermartingale deflator.*

The following proposition is a parametrized version of [5, Proposition 2.5].

Proposition 5 *Let $(X^u, u \in \mathbb{R})$ be a parametrized $\mathbb{H}$-adapted semi-martingale. Then, the following assertions are equivalent.*

(a) The process $(X^u, u \in \mathbb{R})$ admits an $\mathbb{H}$-local martingale deflator.

(b) There exist a $\mathscr{P}(\mathbb{H}) \otimes \mathscr{B}$-measurable parametrized process $(\phi^u, u \in \mathbb{R})$ such that $0 < \phi^u \leq 1$ and an $\mathbb{H}$-local martingale deflator for $(\phi^u \centerdot X^u, u \in \mathbb{R})$.

(c) *There exists a sequence of* $\mathbb{H}$*-stopping times* $(T_n)_{n\geq 1}$ *that increases to* ∞ *such that for each* $n \geq 1$*, there exist a probability* $\mathbb{Q}_n$ *on* $(\Omega, \mathscr{F}_{T_n})$ *such that* $\mathbb{Q}_n \sim \mathbb{P}$ *and an* $\mathbb{H}$*-local martingale deflator for* $((X^u)^{T_n}, u \in \mathbb{R})$ *under* $\mathbb{Q}_n$*.*

3 Progressive Enlargement up to a Random Time

3.1 Optional Semimartingale Decomposition for Progressive Enlargement

In this section, we derive an $\mathbb{F}^\tau$-semimartingale decomposition of any $\mathbb{F}$-local martingale stopped at τ, different from the one given in Proposition 1. Let us start with the definition of an important $\mathbb{F}$-stopping time, namely

$$R := \inf\{t \geq 0 \ : \ Z_t = 0\},$$

and define the $\mathbb{F}$-stopping time $\tilde{R}$ as

$$\tilde{R} := R_{\{\tilde{Z}_R = 0 < Z_{R-}\}} = R 1\!\!1_{\{\tilde{Z}_R = 0 < Z_{R-}\}} + \infty 1\!\!1_{\{\tilde{Z}_R = 0 < Z_{R-}\}^c}$$

We establish an optional decomposition in the following theorem. By optional decomposition, we mean that we write a semi-martingale as a martingale plus an optional bounded variation process.

Theorem 3 *Let X be an* $\mathbb{F}$*-local martingale. Then the process*

$$\bar{X}_t := X^\tau_t - \int_0^{t\wedge\tau} \frac{1}{\tilde{Z}_s} d[X, m]_s + \left(\Delta X_{\tilde{R}}\ 1\!\!1_{[\![\tilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{t\wedge\tau} \tag{9}$$

is an $\mathbb{F}^\tau$*-local martingale.*

Proof First of all, let us recall that for any $\mathbb{F}$-local martingale, the stopped process X^τ is an $\mathbb{F}^\tau$-semimartingale as stated in Proposition 2. Let H be an $\mathbb{F}^\tau$-predictable bounded process. Then, there exists an $\mathbb{F}$-predictable bounded process J such that $H 1\!\!1_{[\![0,\tau]\!]} = J 1\!\!1_{[\![0,\tau]\!]}$. By [19, Sect. IV-3 and Lemme (5,17)], the $\mathbb{F}$-martingale m given in (1) satisfies that for each $H^1(\mathbb{F})$ martingale Y, one has $\mathbb{E}(Y_\tau) = \mathbb{E}([m, Y]_\infty)$. Thus, we have

$$\begin{aligned}
\mathbb{E}\left((H \centerdot X^\tau)_\infty\right) &= \mathbb{E}\left((J \centerdot X)_\tau)\right) = \mathbb{E}\left([J \centerdot X, m]_\infty\right) \\
&= \mathbb{E}\left(\int_0^\infty \frac{J_s \tilde{Z}_s}{\tilde{Z}_s} 1\!\!1_{\{\tilde{Z}_s > 0\}} d[X, m]_s\right) \\
&\quad + \mathbb{E}\left(\int_0^\infty J_s 1\!\!1_{\{\tilde{Z}_s = 0 < Z_{s-}\}} d[X, m]_s\right).
\end{aligned}$$

Since $\{\tilde{Z} = 0 < Z_-\} = [\![\tilde{R}]\!]$ and $\Delta m_{\tilde{R}} = -Z_{\tilde{R}-}$ on $\{\tilde{R} < \infty\}$, we can write

$$\mathbb{1}_{\{\tilde{Z}=0<Z_-\}} \cdot [X, m] = \Delta X_{\tilde{R}}\, \Delta m_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[} = -Z_{\tilde{R}-} \Delta X_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[}\,.$$

Furthermore, due to the fact that J and $[X, m]$ are $\mathbb{F}$-adapted, we obtain

$$\begin{aligned}\mathbb{E}\left((H \cdot X^\tau)_\infty\right) = \mathbb{E}&\left(\int_0^\tau \frac{J_s}{\tilde{Z}_s}\mathbb{1}_{\{\tilde{Z}_s>0\}} d[X, m]_s\right)\\ &-\mathbb{E}\left(\int_0^\infty J_s Z_{s-} d\left(\Delta X_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[}\right)_s\right).\end{aligned}$$

Then, as JZ_- is $\mathbb{F}$-predictable, it holds that

$$\mathbb{E}\left(\int_0^\infty J_s Z_{s-} d\left(\Delta X_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[}\right)_s\right) = \mathbb{E}\left(\int_0^\infty J_s Z_{s-} d\left(\Delta X_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[}\right)_s^{p,\mathbb{F}}\right)$$

and we obtain:

$$\begin{aligned}\mathbb{E}\left((H \cdot X^\tau)_\infty\right) &= \mathbb{E}\left(\int_0^\tau \frac{J_s}{\tilde{Z}_s}\mathbb{1}_{\{\tilde{Z}_s>0\}} d[X, m]_s\right)\\ &\quad-\mathbb{E}\left(\int_0^\infty J_s Z_{s-} d\left(\Delta X_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[}\right)_s^{p,\mathbb{F}}\right)\\ &= \mathbb{E}\left(\int_0^\tau \frac{H_s}{\tilde{Z}_s}\mathbb{1}_{\{\tilde{Z}_s>0\}} d[X, m]_s\right) - \mathbb{E}\left(\int_0^\tau H_s d\left(\Delta X_{\tilde{R}}\, \mathbb{1}_{[\tilde{R},\infty[}\right)_s^{p,\mathbb{F}}\right)\end{aligned}$$

where we have used the facts that J is predictable and that Z_- is the predictable projection of $\mathbb{1}_{[0,\tau]}$. That ends the proof. □

Remark 5 In [9, Sect. 77, Chap. XX] an optional semimartingale decomposition is mentioned (without any proof) in the form: given an $\mathbb{F}$-local martingale X, the process

$$\bar{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{\tilde{Z}_s} d[X, m]_s$$

is an $\mathbb{F}^\tau$-local martingale. This decomposition is valid for any $\mathbb{F}$-local martingale if and only if $\tilde{R} = \infty$ $\mathbb{P}$-a.s. In particular, if all $\mathbb{F}$-martingales are continuous, then $\tilde{R} = \infty$ $\mathbb{P}$-a.s. and the above formula is valid. The condition $\tilde{R} = \infty$ $\mathbb{P}$-a.s. will play an important rôle in the study of stability of NUPBR condition.

Remark 6 The $\mathbb{F}^\tau$-local martingale $\bar{X}$ which appears in (9) can be expressed in terms of the $\mathbb{F}$-local martingale N defined in (3). Indeed, from equalities $N =$

$N_-\left(\mathbb{1}_{\{Z_->0\}}\frac{\tilde{Z}}{Z_-}+\mathbb{1}_{\{Z_-=0\}}\right)$ and $N=1+N_-\frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}}\centerdot m$ and the fact that $Z_->0$ on $[\![0,\tau]\!]$, it follows that

$$\frac{1}{\tilde{Z}}\mathbb{1}_{[\![0,\tau]\!]}\centerdot[X,m]=\frac{1}{N}\mathbb{1}_{[\![0,\tau]\!]}\centerdot[X,N].$$

We will now study some particular martingales which will be important for the construction, under adequate conditions, of deflators for price processes and we will give the relation of our construction with previous works, in particular [5, Proposition 3.6]. The next lemma defines an $\mathbb{F}^\tau$-local martingale L^{pr} which is the corner stone in the construction of the deflator.[2] Due to this lemma, we avoid the use of optional integrals done in Eq. (3.9) in [5, Proposition 3.6].

Lemma 2

(a) The $\mathbb{F}^\tau$-predictable process $\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}$ is integrable with respect to $\bar{m}$, the $\mathbb{F}^\tau$-martingale part from the optional decomposition of m obtained in (9).

(b) Let

$$L^{\mathrm{pr}}=\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\centerdot\bar{m}\,.$$

Then

$$L^{\mathrm{pr}}=\frac{Z_-^2}{Z_-^2+\Delta\langle m\rangle^{\mathbb{F}}}\,\frac{1}{\tilde{Z}}\,\mathbb{1}_{[\![0,\tau]\!]}\odot\hat{m},$$

where $\hat{m}$ is the $\mathbb{F}^\tau$-local martingale part in the predictable decomposition of m (8) and $\odot$ stands for the optional stochastic integral.

Proof In the proof, we set $L=L^{\mathrm{pr}}$ for simplicity.

(a) Being càglàd, the process $\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}$ is locally bounded.

(b) The $\mathbb{F}^\tau$-continuous martingale part and the jump part of $\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\centerdot\bar{m}$ are given by

$$\left(\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\centerdot\bar{m}\right)^c=\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\centerdot\left(m^c-\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\centerdot\langle m^c\rangle^{\mathbb{F}}\right)$$

$$\Delta\left(\frac{1}{Z_-}\mathbb{1}_{[\![0,\tau]\!]}\centerdot\bar{m}\right)=\frac{\Delta m}{\tilde{Z}}\mathbb{1}_{[\![0,\tau]\!]}-{}^{p,\mathbb{F}}\left(\mathbb{1}_{[\![\tilde{R}]\!]}\right)\mathbb{1}_{[\![0,\tau]\!]},$$

where m^c is the $\mathbb{F}$-continuous martingale part of m. Let us now compute the $\mathbb{F}^\tau$-continuous martingale part and the jump part of L. By definition of the optional

[2]The upper script "pr" stands for progressive.

stochastic integral and Lemma 3.1(b) in [5], we have:

$$\begin{aligned} L^c &= \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} \; {}^{p,\mathbb{F}^\tau}\!\left(\frac{1}{\tilde{Z}}\right) \mathbb{1}_{[0,\tau]} \centerdot \hat{m}^c \\ &= \frac{1}{Z_-}\mathbb{1}_{[0,\tau]} \centerdot \hat{m}^c - \frac{{}^{p,\mathbb{F}}(\mathbb{1}_{\{\tilde{Z}=0<Z_-\}})}{Z_-}\mathbb{1}_{[0,\tau]} \centerdot \hat{m}^c. \end{aligned}$$

As $\{\tilde{Z} = 0 < Z_-\}$ is a thin set,[3] the set $\{{}^{p,\mathbb{F}}(\mathbb{1}_{\{\tilde{Z}=0<Z_-\}}) \neq 0\}$ is also thin, and from continuity of $\hat{m}^c$, we conclude that

$$L^c = \frac{1}{Z_-}\mathbb{1}_{[0,\tau]} \centerdot \hat{m}^c.$$

In the proof of Proposition 3.6 in [5], it is established that the jump process of L is given by

$$\Delta L = \frac{\Delta m}{\tilde{Z}}\mathbb{1}_{[0,\tau]} - {}^{p,\mathbb{F}}\!\left(\mathbb{1}_{\{\tilde{Z}=0<Z_-\}}\right)\mathbb{1}_{[0,\tau]}. \tag{10}$$

This completes the proof. □

The link between the $\mathbb{F}^\tau$-local martingale L^{pr} and the $\mathbb{F}^\tau$-adapted process $\frac{1}{N^\tau}$, where N is defined in (3), is made precise in the next lemma.

Proposition 6 *Let N be defined in* (3).

(a) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-supermartingale which can be written

$$\frac{1}{N^\tau} = \mathscr{E}\left(-(L^{\mathrm{pr}})^\tau - \left(\mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}_{\centerdot\wedge\tau}\right).$$

(b) The process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-local martingale if and only if $\tilde{R} = \infty$. In that case $\frac{1}{N^\tau} = \mathscr{E}\left(-(L^{\mathrm{pr}})^\tau\right)$.

Proof

(a) By Itô's formula and the obvious equality $dN = N_- \frac{1}{Z_-}\mathbb{1}_{\{Z_->0\}}dm$

$$\begin{aligned} \frac{1}{N_t^\tau} &= 1 - \int_0^{t\wedge\tau} \frac{1}{N_{s-}^2}dN_s + \int_0^{t\wedge\tau} \frac{1}{N_{s-}^3}d\langle N^c\rangle_s \\ &\quad + \sum_{s\le t\wedge\tau}\left(\frac{1}{N_s} - \frac{1}{N_{s-}} + \frac{1}{N_{s-}^2}\Delta N_s\right) \\ &= 1 - \int_0^{t\wedge\tau} \frac{1}{N_{s-}Z_{s-}}dm_s + \int_0^{t\wedge\tau} \frac{1}{N_{s-}Z_{s-}^2}d\langle m^c\rangle_s + \sum_{s\le t\wedge\tau} \frac{(\Delta m_s)^2}{N_{s-}Z_{s-}\tilde{Z}_s}, \end{aligned}$$

[3] A set $A \subset \Omega \times [0, \infty[$ is thin if, for all $\omega \in \Omega$ the set $A(\omega)$ is countable.

where we have used the fact that $\tilde{Z} = Z_- + \Delta m$. We continue with

$$\frac{1}{N_t^\tau} = 1 - \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\left(\frac{1}{Z_-} \cdot m - \frac{1}{Z_-^2} \cdot \langle m^c \rangle - \sum \frac{(\Delta m)^2}{Z_- \tilde{Z}}\right)_s$$
$$= 1 - \int_0^{t\wedge\tau} \frac{1}{N_{s-}} d\left(\frac{1}{Z_-} \cdot \bar{m} + \left(\mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}\right)_s$$

where the second equality comes from Theorem 3 applied to the $\mathbb{F}$-martingale m. Finally we conclude that

$$\frac{1}{N^\tau} = \mathscr{E}\left(-\frac{1}{Z_-}\mathbb{1}_{[0,\tau]} \cdot \bar{m} - \left(\mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau}\right).$$

(b) From the previous equality, we see that the process $\frac{1}{N^\tau}$ is an $\mathbb{F}^\tau$-local martingale if and only if $\left(\mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}_{\cdot\wedge\tau} = 0$. The last equality is equivalent to

$$0 = \mathbb{E}\left(\left(\mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}_\tau\right) = \mathbb{E}\left(\int_0^\infty Z_{s-} d\left(\mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}_s\right) = \mathbb{E}\left(Z_{\tilde{R}-}\mathbb{1}_{\{\tilde{R}<\infty\}}\right),$$

which in turn is equivalent to $\tilde{R} = \infty$, $\mathbb{P}$-a.s. since $Z_{\tilde{R}-} > 0$ on $\{\tilde{R} < \infty\}$. □

3.2 *Deflators for Progressive Enlargement up to τ*

In this section, we give alternative proofs, based on the optional semimartingale decomposition, to results in [1] and to Theorem 2.23 and Corollary 2.18(c) from [5] (or their general versions in [4]). In Proposition 7(a), we determine an $\mathbb{F}^\tau$-local martingale deflator for a large class of $\mathbb{F}$-local martingales. In Proposition 7(b), an $\mathbb{F}^\tau$-supermartingale deflator for $\mathbb{F}$-local martingales is studied.

We introduce an $\mathbb{F}^\tau$-predictable process V^{pr} which is crucial for proofs therein (also used in [5]). Denoting by $\tilde{R}^a$ the accessible part of the $\mathbb{F}$-stopping time $\tilde{R}$, we set

$$V_t^{\text{pr}} := \left(\mathbb{1}_{[\tilde{R}^a,\infty[}\right)^{p,\mathbb{F}}_{t\wedge\tau}.$$

Using the process V^{pr} we study, in the next proposition, a particular $\mathbb{F}^\tau$-supermartingale which will play the rôle of a deflator for some $\mathbb{F}$-local martingales.

Proposition 7 *Assume that X is an $\mathbb{F}$-local martingale such that*

$$\Delta X_{\tilde{R}} = 0 \quad \textit{on} \quad \{\tilde{R} < \infty\}.$$

(a) If X is quasi-left continuous, then $Y^{\text{pr}} := \mathscr{E}(-L^{\text{pr}})$ is an $\mathbb{F}^\tau$-local martingale deflator for X^τ.
(b) The process $\tilde{Y}^{\text{pr}} := \mathscr{E}(-L^{\text{pr}} - V^{\text{pr}})$ is an $\mathbb{F}^\tau$-supermartingale deflator for X^τ.

Proof

(a) In the proof, we set $Y = Y^{\text{pr}}$ and $L = L^{\text{pr}}$ for simplicity. Using integration by parts and the optional decomposition (9) given in Theorem 3 for X and then for m, we obtain:

$$\begin{aligned}
YX^\tau &= X^\tau_- \centerdot Y + Y_- \centerdot X^\tau + [Y, X^\tau] \\
&= X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + Y_- \frac{1}{\tilde{Z}} 1\!\!1_{[0,\tau]} \centerdot [m, X] \\
&\quad -Y_- 1\!\!1_{[0,\tau]} \centerdot (\Delta X_{\tilde{R}} 1\!\!1_{[\tilde{R},\infty[})^{p,\mathbb{F}} - Y_- 1\!\!1_{[0,\tau]} \centerdot [L, X] \\
&= X^\tau_- \centerdot Y + Y_- \centerdot \bar{X} + Y_- \frac{1}{\tilde{Z}} 1\!\!1_{[0,\tau]} \centerdot [\bar{m}, X] + Y_- \frac{1}{\tilde{Z}^2} 1\!\!1_{[0,\tau]} \centerdot [[m], X] \\
&\quad -Y_- \frac{1}{\tilde{Z}} 1\!\!1_{[0,\tau]} \centerdot [(\Delta m_{\tilde{R}} 1\!\!1_{[\tilde{R},\infty[})^{p,\mathbb{F}}, X] - Y_- 1\!\!1_{[0,\tau]} \centerdot (\Delta X_{\tilde{R}} 1\!\!1_{[\tilde{R},\infty[})^{p,\mathbb{F}} \\
&\quad -\frac{Y_-}{Z_-} 1\!\!1_{[0,\tau]} \centerdot [\bar{m}, X] \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

In a first step, we study the sum of third and seventh term of the last expression

$$\begin{aligned}
I_3 + I_7 &= Y_- \left(\frac{1}{\tilde{Z}} - \frac{1}{Z_-} \right) 1\!\!1_{[0,\tau]} \centerdot [\bar{m}, X] = -Y_- \frac{\Delta m}{\tilde{Z} Z_-} 1\!\!1_{[0,\tau]} \centerdot [\bar{m}, X] \\
&= -\sum Y_- \frac{\Delta m}{\tilde{Z} Z_-} 1\!\!1_{[0,\tau]} \Delta \bar{m} \Delta X,
\end{aligned}$$

where the third equality comes from the fact that $\{\Delta m \neq 0\}$ is a thin set. We add the term I_4 to the previous two

$$\begin{aligned}
I_4 + (I_3 + I_7) &:= \sum Y_- \frac{1}{\tilde{Z}^2} 1\!\!1_{[0,\tau]} (\Delta m)^2 \Delta X - \sum Y_- \frac{\Delta m}{\tilde{Z} Z_-} 1\!\!1_{[0,\tau]} \Delta \bar{m} \Delta X \\
&= -\sum Y_- \frac{\Delta m}{\tilde{Z}} \Delta X 1\!\!1_{[0,\tau]} \left(\frac{1}{Z_-} \Delta \bar{m} - \frac{1}{\tilde{Z}} \Delta m \right) \\
&= \sum Y_- \frac{\Delta m}{\tilde{Z}} \Delta X 1\!\!1_{[0,\tau]} {}^{p,\mathbb{F}} \left(1\!\!1_{[\tilde{R}]} \right),
\end{aligned}$$

where the last equality comes from (10). Note that, by Yoeurp's lemma (which states that, for a predictable bounded variation process V and a semimartingale Y, $[V, Y] = \Delta V \centerdot Y$, see, e.g., [18, Proposition 9.3.7.1]), properties of dual predictable projection, and the fact that ${}^p(\Delta V) = \Delta(V^p)$, the fifth term in the expression for YX^τ is equal to

$$I_5 = -Y_- \frac{1}{\tilde{Z}} \mathbb{1}_{[0,\tau]} \cdot [(\Delta m_{\tilde{R}} \mathbb{1}_{[\tilde{R},\infty[})^{p,\mathbb{F}}, X] = -Y_- \frac{1}{\tilde{Z}} \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}}(\Delta m_{\tilde{R}} \mathbb{1}_{[\tilde{R}]}) \cdot X$$
$$= \sum Y_- \frac{Z_-}{\tilde{Z}} \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]}) \Delta X,$$

where the last equality is due to $\Delta m_{\tilde{R}} = -Z_{\tilde{R}-}$ and the fact that the process ${}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]})$ is thin.

Finally, using the fact that $Z_- + \Delta m = \tilde{Z}$, we get

$$I_5 + (I_4 + I_3 + I_7) = \sum Y_- \frac{Z_-}{\tilde{Z}} \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]}) \Delta X$$
$$+ \sum Y_- \frac{\Delta m}{\tilde{Z}} \Delta X \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}} \left(\mathbb{1}_{[\tilde{R}]} \right)$$
$$= \sum Y_- \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]}) \Delta X.$$

Summing up we have that

$$YX^\tau = X_-^\tau \cdot Y + Y_- \cdot \bar{X}$$
$$+ \sum Y_- \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]}) \Delta X - Y_- \mathbb{1}_{[0,\tau]} \cdot \left(\Delta X_{\tilde{R}} \mathbb{1}_{[\tilde{R},\infty[} \right)^{p,\mathbb{F}}.$$

If X is an $\mathbb{F}$-quasi-left continuous local martingale, using the predictability of ${}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]})$ and $\Delta X_{\tilde{R}} = 0$ on $\{\tilde{R} < \infty\}$, then

$$YX^\tau = X_-^\tau \cdot Y + Y_- \cdot \bar{X}$$

which implies that YX^τ is a local martingale, hence Y is an $\mathbb{F}^\tau$-local martingale deflator for X^τ.

(b) In the proof, we set $\tilde{Y} = \tilde{Y}^{\mathrm{pr}}$, $L = L^{\mathrm{pr}}$ and $V = V^{\mathrm{pr}}$ for simplicity. Let H be an $\mathbb{F}^\tau$-predictable process such that $H \cdot X \geq -1$. By integration by parts, we get

$$(1 + H \cdot X^\tau)\, \tilde{Y} = (1 + H \cdot X^\tau)_- \cdot \tilde{Y} + H\, \tilde{Y}_- \cdot X^\tau$$
$$-H\, \tilde{Y}_- \cdot [X^\tau, L] - H\, \tilde{Y}_- \cdot [X^\tau, V].$$

Note that

$$H\tilde{Y}_- \cdot [X^\tau, V] = \sum H\, \tilde{Y}_-\, \mathbb{1}_{[0,\tau]}\ {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}]}) \Delta X.$$

Then, using the same arguments as in the proof of (a), we get

$$(1 + H \cdot X^\tau)\tilde{Y} = (1 + H \cdot X^\tau)_- \cdot \tilde{Y} + H\tilde{Y}_- \cdot \bar{X} - H\tilde{Y}_- \mathbb{1}_{[0,\tau]} \cdot \left(\Delta X_{\tilde{R}} \mathbb{1}_{[\tilde{R},\infty[} \right)^{p,\mathbb{F}}.$$

In particular, if $\Delta X_{\tilde{R}} = 0$ on $\{\tilde{R} < \infty\}$, then $\tilde{Y}$ is an $\mathbb{F}^\tau$-supermartingale deflator for X^τ and X^τ satisfies NUPBR($\mathbb{F}^\tau$). This ends the proof of the proposition. □

Proposition 8 *Let X be a process such that $\Delta X_{\tilde{R}} = 0$ on $\{\tilde{R} < \infty\}$ and admitting an $\mathbb{F}$-local martingale deflator. Then X^τ admits an $\mathbb{F}^\tau$-local martingale deflator.*

Proof There exist a real-valued $\mathbb{F}$-predictable process ϕ and a positive $\mathbb{F}$-local martingale K such that

$$0 < \phi \leq 1 \quad \text{and} \quad K(\phi \centerdot X) \quad \text{is an } \mathbb{F}\text{-local martingale.}$$

Then there exists a sequence of $\mathbb{F}$-stopping times $(\upsilon_n)_n$ that increases to infinity such that the stopped process K^{υ_n} is an $\mathbb{F}$-martingale. Put $\mathbb{Q}_n := K_{\upsilon_n} \centerdot \mathbb{P} \sim \mathbb{P}$. Then, by applying Proposition 7 to $(\phi \centerdot X)^{\upsilon_n}$ under $\mathbb{Q}_n$, we conclude that $(\phi \centerdot X)^{\upsilon_n \wedge \tau}$ satisfies NUPBR($\mathbb{F}^\tau$) under $\mathbb{Q}_n$. Thanks to Proposition 5, NUPBR($\mathbb{F}^\tau$) under $\mathbb{P}$ of X^τ follows immediately. □

The next result provides explicit $\mathbb{F}^\tau$-local martingale deflators for $\mathbb{F}$-local martingales. The proof differs from the one of Theorem [5, Theorem 2.23] and is based on the optional semimartingale decomposition and direct computations.

Theorem 4 *The following conditions are equivalent.*

(a) The thin set $\{\tilde{Z} = 0 < Z_-\}$ is evanescent.
(b) The $\mathbb{F}$-stopping time $\tilde{R}$ is infinite ($\tilde{R} = \infty$).
(c) For any $\mathbb{F}$-local martingale X, the process X^τ admits Y^{pr} as $\mathbb{F}^\tau$-local martingale deflator, hence, satisfies NUPBR($\mathbb{F}^\tau$).
(d) For any (bounded) process X satisfying NUPBR($\mathbb{F}$), the process X^τ satisfies NUPBR($\mathbb{F}^\tau$).

Proof The equivalence between (a) and (b) is obvious from definition of $\tilde{R}$.

The implication (b)⇒(c) follows from Proposition 7. To prove (c)⇒(b) (and (d)⇒(b)), we consider the $\mathbb{F}$-martingale

$$X = \mathbb{1}_{[\![\tilde{R},\infty[\![} - \left(\mathbb{1}_{[\![\tilde{R},\infty[\![}\right)^{p,\mathbb{F}}.$$

Note that $\mathbb{P}(\tau = \tilde{R}) = \mathbb{E}(\Delta A^o_{\tilde{R}}) = \mathbb{E}(\tilde{Z}_{\tilde{R}} - Z_{\tilde{R}}) = 0$ (since $0 = \tilde{Z}_{\tilde{R}} \geq Z_{\tilde{R}} \geq 0$). This implies that $\tau < \tilde{R}$ and

$$X^\tau = -\left(\mathbb{1}_{[\![\tilde{R},\infty[\![}\right)^{p,\mathbb{F}}_{\cdot \wedge \tau}$$

is a predictable decreasing process. Thus, from [5, Lemma 2.6], X^τ satisfies NUPBR($\mathbb{F}^\tau$) if and only if it is a null process. Then, we conclude that $\tilde{R}$ is infinite using the same argument as in the proof of Lemma 6(b). The implication (c) ⇒ (d) follows from Proposition 8. □

4 Initial Enlargement Under Jacod's Hypothesis

In this section, we study initial enlargement of filtration and NUPBR condition under Jacod's absolute continuity hypothesis. We extend some results of Amendinger [6] that require both Jacod's equivalence hypothesis and Theorem 1.

4.1 Optional Semimartingale Decomposition for Initial Enlargement

In this subsection, we develop our $\mathbb{F}^{\sigma(\xi)}$-optional semimartingale decomposition of parametrized $\mathbb{F}$-local martingales. We first decompose the $\mathbb{F}$-stopping time R^u, introduced in (4), as $R^u = \tilde{R}^u \wedge \bar{R}^u$ with

$$\tilde{R}^u = R^u_{\{q^u_{R^u-}>0\}} \quad \text{and} \quad \bar{R}^u = R^u_{\{q^u_{R^u-}=0\}}. \tag{11}$$

Clearly $\bar{R}^u$ is an $\mathbb{F}$-predictable stopping time and $\{R^u = \infty\} \subset \{\bar{R}^u = \infty\}$ so

$$\left(\mathbb{1}_{[\![R^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi} = \left(\mathbb{1}_{[\![\tilde{R}^u,\infty[\![}\right)^{p,\mathbb{F}}|_{u=\xi}.$$

In the following lemma, we express the $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection in terms of the $\mathbb{F}$-dual predictable projection. This is a result for initial enlargement case similar to the one given in [5, Lemmas 3.1(a) and 3.2] for progressive enlargement case.

Lemma 3 *Let $(V^u, u \in \mathbb{R})$ be a parametrized $\mathbb{F}$-adapted càdlàg process with locally integrable variation ($V \in \mathscr{A}_{loc}(\mathbb{F})$). Then the following properties hold:*

(a) The $\mathbb{F}^{\sigma(\xi)}$-dual predictable projection of V^ξ is

$$(V^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-} \centerdot \left(q^u \centerdot V^u\right)^{p,\mathbb{F}}|_{u=\xi}. \tag{12}$$

(b) If $(V^u, u \in \mathbb{R})$ belongs to $\mathscr{A}^+_{loc}(\mathbb{F})$ (respectively $V \in \mathscr{A}^+(\mathbb{F})$), then the process $(U^u, u \in \mathbb{R})$ with

$$U^u := \frac{1}{q^\xi} \centerdot V^u, \tag{13}$$

belongs to $\mathscr{A}^+_{loc}(\mathbb{F}^{\sigma(\xi)})$ (respectively to $\mathscr{A}^+(\mathbb{F}^{\sigma(\xi)})$).

(c) If $(V^u, u \in \mathbb{R})$ belongs to $\mathscr{A}_{loc}(\mathbb{F})$, the process $(U^u = \frac{1}{q^\xi} \centerdot V^u, u \in \mathbb{R})$ is well defined, its variation is $\mathbb{F}^{\sigma(\xi)}$-locally integrable, and the $\mathbb{F}^{\sigma(\xi)}$-dual predictable

projection of U^ξ is given by

$$(U^\xi)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q^\xi_-} \centerdot \left(\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}} |_{u=\xi}.$$

Proof

(a) We apply the predictable semimartingale decomposition given in Proposition 4 to the parametrized $\mathbb{F}$-local martingale $(X^u, u \in \mathbb{R}) = (V^u - (V^u)^{p,\mathbb{F}}, u \in \mathbb{R})$, obtaining

$$\begin{aligned} V^\xi &= \hat{X}^\xi + (V^u)^{p,\mathbb{F}}|_{u=\xi} + \frac{1}{q^\xi_-} \centerdot \langle V^u, q^u\rangle^{\mathbb{F}}|_{u=\xi} \\ &= \hat{X}^\xi + \left(\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}} |_{u=\xi} + \left(\frac{\Delta q^u}{q^u_-}\mathbb{1}_{\{q^u>0\}} \centerdot V^u\right)^{p,\mathbb{F}} |_{u=\xi} \\ &= \hat{X}^\xi + \frac{1}{q^\xi_-} \centerdot (q^u \centerdot V^u)^{p,\mathbb{F}}|_{u=\xi}, \end{aligned}$$

which proves assertion (a).

(b) Suppose that $(V^u, u \in \mathbb{R}) \in \mathscr{A}^+_{loc}(\mathbb{F})$. For fixed u, let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to infinity such that $\mathbb{E}\left(V^u_{\vartheta_n}\right) < \infty$. Then, $\mathbb{E}\left(U^u_{\vartheta_n}\right) < \infty$ since

$$\begin{aligned} \mathbb{E}\left(U^u_{\vartheta_n}\right) &= \mathbb{E}\left(\int_0^{\vartheta_n} \frac{1}{q^\xi_t} dV^u_t\right) = \mathbb{E}\left(\int_0^{\vartheta_n} \int_{\mathbb{R}} \mathbb{1}_{\{q^y_t>0\}}\eta(dy)\, dV^u_t\right) \\ &\leq \mathbb{E}\left(V^u_{\vartheta_n}\right) < \infty, \end{aligned}$$

where the last equality comes from (6) applied to $\frac{1}{q^u_t}\mathbb{1}_{\{q^u_t>0\}}$.

(c) Suppose that $(V^u, u \in \mathbb{R}) \in \mathscr{A}_{loc}(\mathbb{F})$, and denote by $W := V^+ + V^-$ its variation. Then $(W^u, u \in \mathbb{R}) \in \mathscr{A}^+_{loc}(\mathbb{F})$, and a direct application of (b) implies that

$$\left(\frac{1}{q^\xi} \centerdot W^u, u \in \mathbb{R}\right) \in \mathscr{A}^+_{loc}(\mathbb{F}^\tau).$$

As a result, we deduce that U given by (13) for the case of $V = V^+ - V^-$ is well defined and has variation equal to $\frac{1}{q^\xi} \centerdot W$ which is $\mathbb{F}^{\sigma(\xi)}$-locally integrable. For each $n \geq 1$, let us consider the parametrized process $(U^u_n, u \in \mathbb{R})$ with

$$U^u_n := \frac{1}{q^u}\mathbb{1}_{\{q^u\geq \frac{1}{n}\}} \centerdot V^u.$$

Due to (12), we derive

$$\left(U_n^{\xi}\right)^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q_-^{\xi}} \centerdot \left(\mathbb{1}_{\{q^u \geq 1/n\}} \centerdot V^u\right)^{p,\mathbb{F}} |_{u=\xi}.$$

Hence, since $(U^{\xi})^{p,\mathbb{F}^{\sigma(\xi)}} = \lim_{n\to\infty} \left(U_n^{\xi}\right)^{p,\mathbb{F}^{\sigma(\xi)}}$ by taking the limit in the above equality, we get

$$(U^{\xi})^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q_-^{\xi}} \centerdot \left(\mathbb{1}_{\{q^u > 0\}} \centerdot V^u\right)^{p,\mathbb{F}} |_{u=\xi}.$$

This ends the proof. □

Remark 7 The above lemma allows us to make precise the link between predictable brackets in $\mathbb{F}$ and in $\mathbb{G}$. Indeed, for two $\mathbb{F}$ martingales X and Y

$$\begin{aligned}\langle X, Y\rangle^{\mathbb{F}^{\sigma(\xi)}} &= ([X, Y])^{p,\mathbb{F}^{\sigma(\xi)}} = \frac{1}{q_-^{\xi}} \centerdot \left(q^u \centerdot [X, Y]\right)^{p,\mathbb{F}}|_{u=\xi} \\ &= \frac{1}{q_-^{\xi}} \centerdot \left(q_-^u \centerdot [X, Y]\right)^{p,\mathbb{F}}|_{u=\xi} + \frac{1}{q_-^{\xi}} \centerdot \left(\Delta q^u \centerdot [X, Y]\right)^{p,\mathbb{F}}|_{u=\xi} \\ &= \langle X, Y\rangle^{\mathbb{F}} + \left(\sum \frac{\Delta q^u}{q_-^u} \Delta X \Delta Y\right)^{p,\mathbb{F}} |_{u=\xi}.\end{aligned}$$

We are now ready to state, in the next theorem, the main result of this section with a proof based on Lemma 3.

Theorem 5 *Let* $(X^u, u \in \mathbb{R})$ *be a parametrized* $\mathbb{F}$*-local martingale. Then,*

$$\bar{X}_t^{\xi} := X_t^{\xi} - \int_0^t \frac{1}{q_s^{\xi}} d[X^{\xi}, q^{\xi}]_s + \left(\Delta X_{\tilde{R}^u}^u \mathbb{1}_{[\tilde{R}^u,\infty[}\right)_t^{p,\mathbb{F}} |_{u=\xi} \tag{14}$$

is an $\mathbb{F}^{\sigma(\xi)}$*-local martingale. Here,* $\tilde{R}^u$ *is defined in* (11).

Proof From the predictable decomposition given in Proposition 4, X^{ξ} can be written as

$$X^{\xi} = \hat{X}^{\xi} + \frac{1}{q_-^{\xi}} \centerdot \left(\mathbb{1}_{\{q^u>0\}} \centerdot [X^u, q^u]\right)^{p,\mathbb{F}} |_{u=\xi} + \frac{1}{q_-^{\xi}} \centerdot \left(\mathbb{1}_{\{q^u=0\}} \centerdot [X^u, q^u]\right)^{p,\mathbb{F}} |_{u=\xi}$$

Using Lemma 3(c) and the fact that $\Delta q^u_{\tilde{R}^u} = -q^u_{\tilde{R}^u -}$

$$X^\xi = \hat{X}^\xi + \left(\frac{1}{q^\xi} \centerdot [X^\xi, q^\xi]\right)^{p,\mathbb{F}^{\sigma(\xi)}} + \frac{1}{q^\xi_-} \centerdot \left(\Delta X^u_{\tilde{R}^u} \Delta q^u_{\tilde{R}^u} \mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}$$
$$= \bar{X}^\xi + \frac{1}{q^\xi} \centerdot [X^\xi, q^\xi] - \left(\Delta X^u_{\tilde{R}^u} \mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}$$

where

$$\bar{X}^\xi := \hat{X}^\xi - \frac{1}{q^\xi} \centerdot [X^\xi, q^\xi] + \left(\frac{1}{q^\xi} \centerdot [X^\xi, q^\xi]\right)^{p,\mathbb{F}^{\sigma(\xi)}}$$

is proved to be an $\mathbb{F}^{\sigma(\xi)}$-local martingale. □

In [6], the process $\frac{1}{q^\xi}$ was studied in the case of a random variable ξ satisfying Jacod's equivalence hypothesis, and was proved to be an $\mathbb{F}^{\sigma(\xi)}$-local martingale. Here we work under a weaker assumption, and we show that the martingale property established in [6] fails in the general case.

In the next two lemmas, we study the properties of the process q^ξ. In Lemma 4 we define particular $\mathbb{F}^{\sigma(\xi)}$-local martingales based on q^ξ. Then, in Lemma 5, we focus on the process $\frac{1}{q^\xi}$, which is proved to be an $\mathbb{F}^{\sigma(\xi)}$-supermartingale, and we give its semimartingale decomposition. We give a necessary and sufficient condition such that $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale.

Lemma 4 *Let $\bar{q}^\xi$ be the $\mathbb{F}^{\sigma(\xi)}$-local martingale part of q^ξ given by (14), i.e.,*

$$\bar{q}^\xi := q^\xi - \frac{1}{q^\xi} \centerdot [q^\xi] - q^\xi_- \centerdot \left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}.$$

Then, the $\mathbb{F}^{\sigma(\xi)}$-predictable process $\frac{1}{q^\xi_-}$ is integrable with respect to $\bar{q}^\xi$ and the $\mathbb{F}^{\sigma(\xi)}$-local martingale

$$L^{\mathrm{i}} := \frac{1}{q^\xi_-} \centerdot \bar{q}^\xi \qquad (15)$$

is such that $1 - \Delta L^{\mathrm{i}} > 0$.[4]

Proof The process $\frac{1}{q^\xi_-}$ is càglàd so it is locally bounded.

[4]The upper script "i" stands for initial.

The definition of $\bar{q}^\xi$ implies

$$1 - \Delta L^{\mathrm{i}} = 1 - \frac{1}{q^\xi_-}\left(\Delta q^\xi - \frac{1}{q^\xi}(\Delta q^\xi)^2\right) + \Delta\left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi}$$
$$= 1 - \frac{\Delta q^\xi}{q^\xi} + {}^{p,\mathbb{F}}\left(\mathbb{1}_{[\tilde{R}^u]}\right)|_{u=\xi}$$
$$= \frac{q^\xi_-}{q^\xi} + {}^{p,\mathbb{F}}\left(\mathbb{1}_{[\tilde{R}^u]}\right)|_{u=\xi} > 0,$$

which completes the proof. □

Under Jacod's equivalence hypothesis, as stated in Theorem 1, the process $\frac{1}{q^\xi}$ is true $\mathbb{F}^{\sigma(\xi)}$-martingale and provides an interesting change of probability.

Lemma 5

(a) The process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale with Doob-Meyer decomposition

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-}\cdot\bar{q}^\xi - \frac{1}{q^\xi_-}\cdot\left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi}. \tag{16}$$

Equivalently, it can be written as a stochastic exponential of the form

$$\frac{1}{q^\xi} = \mathscr{E}\left(-L^{\mathrm{i}} - \left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi}\right).$$

(b) The process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\tilde{R}^u = \infty$ $\mathbb{P}\otimes\eta$-a.s. Then $\frac{1}{q^\xi} = \mathscr{E}(-L^{\mathrm{i}})$.

(c) In particular, the process $\frac{1}{q^\xi}$ is a true $\mathbb{F}^{\sigma(\xi)}$-martingale if and only if $R^u = \infty$ $\mathbb{P}\otimes\eta$-a.s. (i.e., under Jacod's equivalence hypothesis).

Proof

(a) $(q^u, u \in \mathbb{R})$ is a parametrized $\mathbb{F}$-martingale, then by Proposition 4, q^ξ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale. By (7), q^ξ is strictly positive. Then, $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale, and by definition of the bracket, as

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-}\cdot q^\xi + \frac{1}{(q^\xi)^2_- q^\xi}\cdot[q^\xi].$$

Applying (14), we finally get that

$$\frac{1}{q^\xi} = 1 - \frac{1}{(q^\xi)^2_-}\cdot\bar{q}^\xi - \frac{1}{q^\xi_-}\cdot\left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi}.$$

The exponential form immediately follows.

(b) Since $\frac{1}{q^\xi_-} \centerdot \left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-predictable increasing process, the process $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi} = 0$. The last condition is equivalent to have that, for each t

$$0 = \mathbb{E}\left((\mathbb{1}_{[\tilde{R}^u,\infty[})^{p,\mathbb{F}}_t|_{u=\xi}\right) = \mathbb{E}\left({}^{o,\mathbb{F}}\left((\mathbb{1}_{[\tilde{R}^u,\infty[})^{p,\mathbb{F}}|_{u=\xi}\right)_t \right)$$
$$= \mathbb{E}\left(\int_{\mathbb{R}} (\mathbb{1}_{[\tilde{R}^u,\infty[})^{p,\mathbb{F}}_t q^u_t \eta(du)\right) = \int_{\mathbb{R}} \mathbb{E}\left((\mathbb{1}_{[\tilde{R}^u,\infty[})^{p,\mathbb{F}}_t q^u_t\right)\eta(du),$$

where the second equality comes from (6). Next, by Yoeurp's lemma we conclude that, for each t

$$0 = \int_{\mathbb{R}} \mathbb{E}\left(\int_0^t q^u_{s-} d\left((\mathbb{1}_{[\tilde{R}^u,\infty[})^{p,\mathbb{F}}\right)_s\right)\eta(du) = \int_{\mathbb{R}} \mathbb{E}\left(q^u_{\tilde{R}^u-}\mathbb{1}_{\{\tilde{R}^u\le t\}}\right)\eta(du)$$

which in turn is equivalent to $\tilde{R}^u > t$, $\mathbb{P}\otimes\eta$-a.s. for each t since $q^u_{\tilde{R}^u-} > 0$. Thus, $\frac{1}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale if and only if $\tilde{R}^u$ is infinite $\mathbb{P}\otimes\eta$-a.s.

(c) The "if" part is shown in Theorem 1. We show "only if" part here. Assume that the process $\frac{1}{q^\xi}$ is a true $\mathbb{F}^{\sigma(\xi)}$-martingale. Then, for each $t \ge 0$, we have $\mathbb{E}(\frac{1}{q^\xi_t}) = 1$. On the other hand, using Lemma 1(ii), we have that

$$\mathbb{E}\left(\frac{1}{q^\xi_t}\right) = \mathbb{E}\left(\int_{\mathbb{R}} \frac{1}{q^u_t}\mathbb{1}_{\{q^u_t>0\}} q^u_t \eta(du)\right) = \int_{\mathbb{R}} \mathbb{P}(q^u_t > 0)\eta(du)$$
$$= \int_{\mathbb{R}} \mathbb{P}(R^u > t)\eta(du),$$

which shows that $R^u = \infty$, $\mathbb{P}\otimes\eta$-a.s. □

In [6], Amendinger establishes that under Jacod's equivalence hypothesis, for any $\mathbb{F}$-martingale X, the process X/q^ξ is a $\mathbb{G}$ martingale. In the following proposition, we investigate the $\mathbb{F}^{\sigma(\xi)}$-semimartingale decomposition of a parametrized $\mathbb{F}$-local martingale X when ξ is plugged in and when multiplied by $\frac{1}{q^\xi}$ from previous lemma.

Proposition 9 *Let $(X^u, u \in \mathbb{R})$ be a parametrized $\mathbb{F}$-local martingale. Then $\frac{X^\xi}{q^\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-semimartingale with $\mathbb{F}^{\sigma(\xi)}$-local martingale part equal to*

$$X^\xi_0 - \frac{X^\xi_-}{(q^\xi)^2_-} \centerdot \bar{q}^\xi + \frac{1}{q^\xi_-} \centerdot \bar{X}^\xi,$$

and $\mathbb{F}^{\sigma(\xi)}$*-predictable finite variation part equal to*

$$-\frac{1}{q^{\xi}_{-}} \cdot \left(X^{u}_{\tilde{R}^u} 1\!\!1_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}.$$

Proof We compute, applying integration by parts formula:

$$\begin{aligned}
\frac{X^{\xi}}{q^{\xi}} &= X^{\xi}_0 + X^{\xi}_{-} \cdot \frac{1}{q^{\xi}} + \frac{1}{q^{\xi}_{-}} \cdot X^{\xi} + \left[X^{\xi}, \frac{1}{q^{\xi}}\right] \\
&= X^{\xi}_0 - \frac{X^{\xi}_{-}}{(q^{\xi})^2_{-}} \cdot \bar{q}^{\xi} - \frac{X^{\xi}_{-}}{q^{\xi}_{-}} \cdot \left(1\!\!1_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi} \\
&\quad + \frac{1}{q^{\xi}_{-}} \cdot \bar{X}^{\xi} + \frac{1}{q^{\xi}_{-} q^{\xi}} \cdot [X^{\xi}, q^{\xi}] - \frac{1}{q^{\xi}_{-}} \cdot \left(\Delta X^{u}_{\tilde{R}^u} 1\!\!1_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi} \\
&\quad - \frac{1}{(q^{\xi})^2_{-}} \cdot [X^{\xi}, q^{\xi}] + \frac{1}{(q^{\xi})^2_{-} q^{\xi}} \cdot \left[X^{\xi}, [q^{\xi}]\right],
\end{aligned}$$

where the second equality comes from (16). It follows that

$$\begin{aligned}
\frac{X^{\xi}}{q^{\xi}} &= X^{\xi}_0 - \frac{X^{\xi}_{-}}{(q^{\xi})^2_{-}} \cdot \bar{q}^{\xi} + \frac{1}{q^{\xi}_{-}} \cdot \bar{X}^{\xi} \\
&\quad - \frac{1}{q^{\xi}_{-}} \cdot \left(X^{u}_{\tilde{R}^u} 1\!\!1_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi} - \frac{\Delta q^{\xi}}{p_{(}\xi)^2 q^{\xi}} \cdot [X^{\xi}, q^{\xi}] + \frac{\Delta X^{\xi}}{(q^{\xi})^2_{-} q^{\xi}} \cdot [q^{\xi}] \\
&= X^{\xi}_0 - \frac{X^{\xi}_{-}}{(q^{\xi})^2_{-}} \cdot \bar{q}^{\xi} + \frac{1}{q^{\xi}_{-}} \cdot \bar{X}^{\xi} - \frac{1}{q^{\xi}_{-}} \cdot \left(X^{u}_{\tilde{R}^u} 1\!\!1_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}.
\end{aligned}$$

□

As a corollary, from Proposition 9, we recover [16, Proposition 5.2] on universal supermartingale density.

Corollary 1 *If* X *is a positive* $\mathbb{F}$*-supermartingale, then,* $\frac{X}{q^{\xi}}$ *is an* $\mathbb{F}^{\sigma(\xi)}$*-supermartingale.*

Proof Let X be decomposed as $X = M^X - V^X$ where M^X is a positive $\mathbb{F}$-local martingale and V^X is an increasing $\mathbb{F}$-predictable process. Then, $\frac{M^X}{q^{\xi}}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale since from the positiveness of M^X, by Proposition 9, we get that $\frac{1}{q^{\xi}_{-}} \cdot \left(M^X_{\tilde{R}^u} 1\!\!1_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-predictable increasing process. Moreover, as $\frac{1}{q^{\xi}}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale and V^X is predictable and increasing, the process $-\frac{V^X}{q^{\xi}}$ is as well an $\mathbb{F}^{\sigma(\xi)}$-supermartingale which ends the proof. □

4.2 NUPBR Condition for Initial Enlargement

In this section, we focus on the NUPBR condition in an initial enlargement framework. Using simple arguments based on our optional semimartingale decomposition, we prove the stability of the NUPBR condition with respect to an initial enlargement of filtration under Jacod's absolute continuity hypothesis. In Proposition 10, we give $\mathbb{F}^{\sigma(\xi)}$-local martingale deflators for quasi left-continuous parametrized $\mathbb{F}$-local martingales and $\mathbb{F}^{\sigma(\xi)}$-supermartingale deflators for parametrized $\mathbb{F}$-local martingales. In Theorem 6, we present a necessary and sufficient condition such that any parametrized $\mathbb{F}$-local martingale satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$). We close this section by giving two particular examples of initial enlargements under Jacod's hypothesis. We refer the reader to [1] for a study similar to the one contained in this section using fully different methodology.

We denote by $\tilde{R}^{u,a}$ the accessible part of the $\mathbb{F}$-stopping time $\tilde{R}^u$ and we define the process V^{i} as

$$V^{\mathrm{i}}_t := \sum_{0\le s\le t} {}^{p,\mathbb{F}}\left(\mathbb{1}_{[\![\tilde{R}^u]\!]}\right)_s |_{u=\xi} = \left(\mathbb{1}_{[\![\tilde{R}^{u,a},\infty[\![}\right)^{p,\mathbb{F}} |_{u=\xi}. \tag{17}$$

Proposition 10 *Let L^{i} be defined in (15), and let $(X^u, u\in\mathbb{R})$ be a parametrized $\mathbb{F}$-local martingale (see Definition 2) such that $\Delta X^u_{\tilde{R}^u} = 0$ on $\{\tilde{R}^u < \infty\}$ $\mathbb{P}\otimes\eta$-a.s.*

(a) If $(X^u, u\in\mathbb{R})$ is quasi-left continuous, then the process $Y^{\mathrm{i}} := \mathscr{E}\left(-L^{\mathrm{i}}\right)$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator for X^ξ.
(b) In general, the process $\tilde{Y}^{\mathrm{i}} := \mathscr{E}(-L^{\mathrm{i}} - V^{\mathrm{i}})$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale deflator for X^ξ.

Proof

(a) Using the optional decomposition (14) given in Theorem 5, firstly for X^ξ and then for q^ξ, we obtain

$$\begin{aligned}
Y^{\mathrm{i}}X^\xi &= X^\xi_- \cdot Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \cdot X^\xi + [Y^{\mathrm{i}}, X^\xi] \\
&= X^\xi_- \cdot Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \cdot \bar{X}^\xi + Y^{\mathrm{i}}_- \frac{1}{q^\xi} \cdot [X^\xi, q^\xi] \\
&\quad - Y^{\mathrm{i}}_- \cdot (\Delta X^u_{\tilde{R}^u})^{p,\mathbb{F}}|_{u=\xi} - Y^{\mathrm{i}}_- \cdot [L^{\mathrm{i}}, X^\xi] \\
&= X^\xi_- \cdot Y^{\mathrm{i}} + Y^{\mathrm{i}}_- \cdot \bar{X}^\xi + Y^{\mathrm{i}}_- \frac{1}{q^\xi} \cdot [X^\xi, \bar{q}^\xi] + Y^{\mathrm{i}}_- \frac{1}{(q^\xi)^2} \cdot [[q^\xi], X^\xi] \\
&\quad - Y^{\mathrm{i}}_- \frac{1}{q^\xi} \cdot [(\Delta q^u_{\tilde{R}^u -} \mathbb{1}_{[\![\tilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi}, X^\xi] \\
&\quad - Y^{\mathrm{i}}_- \cdot (\Delta X^u_{\tilde{R}^u} \mathbb{1}_{[\![\tilde{R}^u,\infty[\![})^{p,\mathbb{F}}|_{u=\xi} - \frac{Y^{\mathrm{i}}_-}{q^\xi_-} \cdot [X^\xi, \bar{q}^\xi].
\end{aligned}$$

We continue, computing the various brackets:

$$
\begin{aligned}
Y^{\mathrm{i}}X^{\xi} &= X^{\xi}_{-} \centerdot Y^{\mathrm{i}} + Y^{\mathrm{i}}_{-} \centerdot \bar{X}^{\xi} - \sum \frac{Y^{\mathrm{i}}_{-}\Delta q^{\xi}}{q^{\xi}q^{\xi}_{-}}\Delta X^{\xi}\Delta \bar{q}^{\xi} + \sum \frac{Y^{\mathrm{i}}_{-}}{(q^{\xi})^2}(\Delta q^{\xi})^2\Delta X^{\xi} \\
&\quad + \sum \frac{Y^{\mathrm{i}}_{-}q^{\xi}_{-}}{q^{\xi}}\, {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}^u]})|_{u=\xi}\Delta X^{\xi} - Y^{\mathrm{i}}_{-} \centerdot \left(\Delta X^{u}_{\tilde{R}^u}\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi} \\
&= X^{\xi}_{-} \centerdot Y^{\mathrm{i}} + Y^{\mathrm{i}}_{-} \centerdot \bar{X}^{\xi} + \sum Y^{\mathrm{i}}_{-}\, {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}^u]})|_{u=\xi}\Delta X^{\xi} \\
&\quad - Y^{\mathrm{i}}_{-} \centerdot \left(\Delta X^{u}_{\tilde{R}^u}\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi} \\
&= X^{\xi}_{-} \centerdot Y^{\mathrm{i}} + Y^{\mathrm{i}}_{-} \centerdot \bar{X}^{\xi} + \sum Y^{\mathrm{i}}_{-}\, {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}^u]})|_{u=\xi}\Delta X^{\xi} \\
&\quad + \frac{Y^{\mathrm{i}}_{-}}{q^{\xi}_{-}} \centerdot \left(\mathbb{1}_{[\tilde{R}^u]} \centerdot [X^u, q^u]\right)^{p,\mathbb{F}}|_{u=\xi}.
\end{aligned}
$$

where the last equality follows from $\Delta q^{u}_{\tilde{R}^u} = -q^{u}_{\tilde{R}^u-}$ on $\{\tilde{R}^u < \infty\}$.

Since $(X^u, u \in \mathbb{R})$ is an $\mathbb{F}$-quasi-left continuous local martingale and $\Delta X^{u}_{R^u} = 0$ on $\{R^u < \infty\}$, the two last terms are null, and $Y^{\mathrm{i}}X^{\xi}$ is an $\mathbb{F}^{\sigma(\xi)}$-local martingale. Therefore, Y^{i} is an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator for X^{ξ}.

(b) Let H be an $\mathbb{F}^{\sigma(\xi)}$-predictable process such that $H \centerdot X^{\xi} \geq -1$. Then, by integration by parts, we get

$$
\begin{aligned}
(1 + H \centerdot X^{\xi})\, \tilde{Y}^{\mathrm{i}} &= (1 + H \centerdot X^{\xi})_{-} \centerdot \tilde{Y}^{\mathrm{i}} + H\, \tilde{Y}^{\mathrm{i}}_{-} \centerdot X^{\xi} - H\, \tilde{Y}^{\mathrm{i}}_{-} \centerdot [X^{\xi}, L^{\mathrm{i}}] \\
&\quad - H\, \tilde{Y}^{\mathrm{i}}_{-} \centerdot [X^{\xi}, V^{\mathrm{i}}].
\end{aligned}
$$

Note that

$$
H\tilde{Y}^{\mathrm{i}}_{-} \centerdot [X^{\xi}, V^{\mathrm{i}}] = \sum H\tilde{Y}^{\mathrm{i}}_{-}\, {}^{p,\mathbb{F}}(\mathbb{1}_{[\tilde{R}^u]})|_{u=\xi}\Delta X^{\xi}.
$$

Then, using the same arguments as in the proof of Theorem 10, we get

$$
\begin{aligned}
(1 + H \centerdot X^{\xi})\tilde{Y}^{\mathrm{i}} &= (1 + H \centerdot X^{\xi})_{-} \centerdot \tilde{Y}^{\mathrm{i}} + H\, \tilde{Y}^{\mathrm{i}}_{-} \centerdot \bar{X}^{\xi} \\
&\quad - H\, \tilde{Y}^{\mathrm{i}}_{-} \left(\Delta X^{u}_{\tilde{R}^u}\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}}|_{u=\xi}
\end{aligned}
$$

and the assertion is proved. □

Proposition 11 *Let $(X^u, u \in \mathbb{R})$ be a parametrized process admitting an $\mathbb{F}$-local martingale deflator such that $\Delta X^{u}_{\tilde{R}^u} = 0$ on $\{\tilde{R}^u < \infty\}$, η-a.e. Then X^{ξ} admits an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator.*

Proof Let $(X^u, u \in \mathbb{R})$ be a parametrized $\mathbb{F}$-semimartingale admitting an $\mathbb{F}$-local martingale deflator, i.e., there exist a real-valued parametrized predictable process $(\phi^u, u \in \mathbb{R})$ and a positive $\mathbb{F}$-local martingale L such that

$$0 < \phi^u \leq 1 \quad \text{and} \quad (L(\phi^u \centerdot X^u), u \in \mathbb{R}) \quad \text{is a parametrized } \mathbb{F}\text{-local martingale.}$$

Then, there exists a sequence of $\mathbb{F}$-stopping times that increases to infinity $(T_n)_n$ such that L^{T_n} is a martingale. Put $\mathbb{Q}_n := L_{T_n} \centerdot P \sim P$. Then, by applying Proposition 10 to $((\phi^u \centerdot X^u)^{T_n}, u \in \mathbb{R})$ under $\mathbb{Q}_n$, we conclude that $(\phi^\xi \centerdot X^\xi)^{T_n}$ satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$) under $\mathbb{Q}_n$. Thanks to Proposition 5, NUPBR($\mathbb{F}^{\sigma(\xi)}$) property under $\mathbb{P}$ of X^ξ follows immediately. □

Theorem 6 *The following conditions are equivalent.*

(a) The thin set $\{q^u = 0 < q^u_-\}$ is evanescent η-a.e.
(b) The $\mathbb{F}$-stopping time $\tilde{R}^u$ is infinite $\mathbb{P} \otimes \eta$-a.s.
(c) For any parametrized $\mathbb{F}$-local martingale $(X^u, u \in \mathbb{R})$, the process X^ξ admits an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator $\frac{1}{q^\xi}$ (and satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$)).
(d) For any parametrized (bounded) process $(X^u, u \in \mathbb{R})$ admitting an $\mathbb{F}$-local martingale deflator, the process X^ξ admits an $\mathbb{F}^{\sigma(\xi)}$-local martingale deflator (and satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$)).

Proof The equivalence between (a) and (b) is obvious from the definition of $\tilde{R}^u$.

The implication (b) ⇒(c) follows from Proposition 10. To prove (c)⇒(b), we consider a parametrized $\mathbb{F}$-martingale $(M^u, u \in \mathbb{R})$ with

$$M^u := \mathbb{1}_{[\tilde{R}^u,\infty[} - (\mathbb{1}_{[\tilde{R}^u,\infty[})^{p,\mathbb{F}}.$$

Then, due to the equality $R^\xi = \infty$ established in (7), it is clear that

$$M^\xi = -\left(\mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}$$

is decreasing. Thus, M^ξ satisfies NUPBR($\mathbb{F}^{\sigma(\xi)}$) if and only if it is a null process. Then, we conclude that $\tilde{R}$ is infinite using the same argument as in the proof of Lemma 5(b). The implication (c) ⇒ (d) follows from Proposition 11. □

In the two following examples we look at two extreme situations.

Example 1 Let $\mathbb{F}$ be a filtration such that each $\mathbb{F}$-martingale is continuous. Then, the NUPBR condition is preserved in an initially enlarged filtration for any parametrized $\mathbb{F}$-local martingale from the reference filtration.

Example 2 Let B be a $\mathscr{G}$-measurable set such that $\mathbb{P}(B) = \frac{1}{2}$ and consider the filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ defined as

$$\mathscr{F}_t = \{\emptyset, \Omega\} \text{ for } t \in [0, 1[\quad \text{and} \quad \mathscr{F}_t := \{\emptyset, B, B^c, \Omega\} \text{ for } t \in [1, \infty[.$$

Define a random variable ξ as $\xi := \mathbb{1}_B + 2 \cdot \mathbb{1}_{B^c}$. The random variable ξ satisfies Jacod's hypothesis with density $(q^u, u \in \{1, 2\})$ equal to

$$
\begin{aligned}
q^1 &= \mathbb{1}_{[0,1[} + 2 \cdot \mathbb{1}_{\{\xi=1\}} \mathbb{1}_{[1,\infty[},\\
q^2 &= \mathbb{1}_{[0,1[} + 2 \cdot \mathbb{1}_{\{\xi=2\}} \mathbb{1}_{[1,\infty[}.
\end{aligned}
$$

Let the filtration $\mathbb{F}^{\sigma(\xi)} = (\mathscr{F}_t^{\sigma(\xi)})_{t\geq 0}$ be an initial enlargement of the filtration $\mathbb{F}$ with ξ, i.e.,

$$
\mathscr{F}_t^{\sigma(\xi)} := \{\emptyset, B, B^c, \Omega\} \quad \text{for} \quad t \in [0, \infty[.
$$

Let X be an $\mathbb{F}$-martingale defined as

$$
X := \left(\mathbb{1}_{\{\xi=1\}} - \frac{1}{2} \right) \mathbb{1}_{[1,\infty[}.
$$

Then, X is an $\mathbb{F}^{\sigma(\xi)}$-predictable process. Thus, by [5, Lemma 2.6] it does not satisfy NUPBR($\mathbb{F}^{\sigma(\xi)}$). Note that here, the set $\{q^u = 0 < q^u_-\}$ is not evanescent, and that $\tilde{R}^u$ is not equal to infinity.

5 Connection to Absolutely Continuous Change of Measure

In this section, we study the relationship between our optional semimartingale decompositions in progressive and initial enlargement of filtration cases and our optional semimartingale decomposition in an absolutely continuous change of measure set-up. First let us recall [28, Theorem 42, Chap. III].

Theorem 7 *Let X be a $\mathbb{P}$-local martingale with $X_0 = 0$. Let $\mathbb{Q}$ be a probability measure absolutely continuous with respect to $\mathbb{P}$, and define $\zeta_t := \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_t)$. Let $r := \inf\{t > 0 : \zeta_t = 0\}$ and $\tilde{r} := r_{\{\zeta_{r-}>0\}}$. Then*

$$
\bar{X} := X - \frac{1}{\zeta} \centerdot [X, \zeta] + \left(\Delta X_{\tilde{r}}\, \mathbb{1}_{[\tilde{r},\infty[}\right)^{p,\mathbb{P}} \tag{18}
$$

is a $\mathbb{Q}$-local martingale.

It is clear that Theorem 7 implies the same type of decompositions as the two decompositions stated in Sects. 3.1 and 4.1.

$$
\begin{aligned}
&\text{Up to random time } \tau: && X^\tau = \bar{X} + \tfrac{1}{N^\tau} \centerdot [X^\tau, N^\tau] - \left(\Delta X_{\tilde{R}} \mathbb{1}_{[\tilde{R},\infty[}\right)^{p,\mathbb{F}}_{\centerdot \wedge \tau}\\
&\text{Under Jacod's hypothesis:} && X^\xi = \bar{X} + \tfrac{1}{q^\xi} \centerdot [X^\xi, q^\xi] - \left(\Delta X^u_{\tilde{R}^u} \mathbb{1}_{[\tilde{R}^u,\infty[}\right)^{p,\mathbb{F}} |_{u=\xi}\\
&\text{Under measure } \mathbb{Q}: && X = \bar{X} + \tfrac{1}{\zeta} \centerdot [X, \zeta] - \left(\Delta X_{\tilde{r}} \mathbb{1}_{[\tilde{r},\infty[}\right)^{p,\mathbb{P}}
\end{aligned}
$$

In each of the three cases, there is a different mechanism to ensure the strict positivity of N^τ, q^ξ and ζ. In the case of progressive enlargement up to a random time, we stop at τ. In the case of initial enlargement with random variable satisfying Jacod's hypothesis, we plug ξ. In the case of absolutely continuous change of measure, the process ζ is strictly positive $\mathbb{Q}$-a.s.

The optional decomposition in the change of measure case can be used in the same way to obtain similar result on stability of the NUPBR condition with respect to absolutely continuous change of measure.

We remark here that the set introduced in Definition 3 may become bigger under absolutely continuous change of measure as under the new measure the condition $H \centerdot X \geq -1$ is more likely satisfied.

Let $\bar{\zeta}$ given by (18) in terms of the Radon Nikodym density ζ, and define a $\mathbb{Q}$-local martingale L^a by

$$L^a := \frac{1}{\zeta} \centerdot \bar{\zeta}.$$

Let us denote by $\tilde{r}^a$ the accessible part of the stopping time $\tilde{r}$, and set

$$V^a := \left(\mathbb{1}_{[\![\tilde{r}^a,\infty[\![}\right)^{p,\mathbb{P}}$$

Using the processes L^a and V^a we study, in the next lemma, the behaviour of particular $\mathbb{Q}$-deflators.

Proposition 12

(a) *Let $Y^a := \mathscr{E}(-L^a)$. If X is a quasi-left continuous local martingale and $\Delta X_{\tilde{r}} = 0$ on $\{\tilde{r} < \infty\}$ $\mathbb{P}$-a.s., then Y^a is a $\mathbb{Q}$-local martingale deflator for X.*

(b) *Let $\tilde{Y}^a := \mathscr{E}(-L^a - V^a)$. Let X be a $\mathbb{P}$-local martingale such that $\Delta X_{\tilde{r}} = 0$ on $\{\tilde{r} < \infty\}$ $\mathbb{P}$-a.s., then $\tilde{Y}^a$ is a $\mathbb{Q}$-supermartingale deflator for X.*

Proof

(a) Using integration by parts and the optional decomposition (18) given in Theorem 7 for X and then for ζ, we obtain

$$\begin{aligned}
Y^a X &= X_- \centerdot Y^a + Y^a_- \centerdot X + [Y^a, X] \\
&= X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + Y^a_- \frac{1}{\zeta} \centerdot [\zeta, X] - Y^a_- \centerdot (\Delta X_{\tilde{r}}\, \mathbb{1}_{[\![\tilde{r},\infty[\![})^{p,\mathbb{P}} - Y^a_- \centerdot [L^a, X] \\
&= X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + Y^a_- \frac{1}{\zeta} \centerdot [\bar{\zeta}, X] + Y^a_- \frac{1}{\zeta^2} [[\zeta], X] \\
&\quad - Y^a_- \frac{1}{\zeta} \centerdot [(\Delta \zeta_{\tilde{r}}\, \mathbb{1}_{[\![\tilde{r},\infty[\![})^{p,\mathbb{P}}, X] - Y^a_- \centerdot (\Delta X_{\tilde{r}}\, \mathbb{1}_{[\![\tilde{r},\infty[\![})^{p,\mathbb{P}} - \frac{Y^a_-}{\zeta_-} [\bar{\zeta}, X].
\end{aligned}$$

We continue, adding the two terms which contain $[\bar{\zeta}, X]$ and computing the brackets

$$Y^a X = X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} - \sum \frac{Y^a_- \Delta\zeta}{\zeta\zeta_-} \Delta X \Delta\bar{\zeta} + \sum \frac{Y^a_-}{\zeta^2} (\Delta\zeta)^2 \Delta X$$

$$+ \sum \frac{Y^a_- \zeta_-}{\zeta} \, {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\tilde{r}]\!]}) \Delta X - Y^a_- \centerdot (\Delta X_{\tilde{r}} \mathbb{1}_{[\![\tilde{r},\infty[\![})^{p,\mathbb{P}}$$

$$= X_- \centerdot Y^a + Y^a_- \centerdot \bar{X} + \sum Y^a_- \, {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\tilde{r}]\!]}) \Delta X - Y^a_- \centerdot \left(\Delta X_{\tilde{r}} \mathbb{1}_{[\![\tilde{r},\infty[\![}\right)^{p,\mathbb{P}} \quad (19)$$

Since for any $\mathbb{P}$-quasi-left continuous martingale X, the process ${}^{p,\mathbb{P}}(\mathbb{1}_{[\![\tilde{r}]\!]}) \Delta X$ is null and $\Delta X_{\tilde{r}} = 0$, Y^a is a $\mathbb{Q}$-local martingale deflator for X.

(b) Let H be a predictable process such that $H \centerdot X \geq -1$. Then, by integration by parts, we get

$$(1 + H \centerdot X)\, \tilde{Y}^a = (1 + H \centerdot X)_- \centerdot \tilde{Y}^a + H\, \tilde{Y}^a_- \centerdot X - H\, \tilde{Y}^a_- \centerdot [X, L^a] - H\, \tilde{Y}^a_- \centerdot [X, V^a].$$

Note that

$$H\tilde{Y}^a_- \centerdot [X, V^a] = \sum H\, \tilde{Y}^a_- \, {}^{p,\mathbb{P}}(\mathbb{1}_{[\![\tilde{r}]\!]}) \Delta X.$$

Then, using the same arguments as in the proof of Theorem 12 to derive (19), we get

$$(1 + H \centerdot X)\tilde{Y}^a = (1 + H \centerdot X)_- \centerdot \tilde{Y}^a + H\tilde{Y}^a_- \centerdot \bar{X} - H\tilde{Y}^a_- \centerdot \left(\Delta X_{\tilde{r}} \mathbb{1}_{[\![\tilde{S},\infty[\![}\right)^{p,\mathbb{P}}$$

and the assertion is proved. □

Proposition 13 *Let X be a process admitting a $\mathbb{P}$-local martingale deflator such that $\Delta X_{\tilde{\zeta}} = 0$ on $\{\tilde{\zeta} < \infty\}$. Then X admits a $\mathbb{Q}$-local martingale deflator.*

Proof Let X be an $\mathbb{P}$-semimartingale satisfying NUPBR($\mathbb{P}$). Thanks to Proposition 5 and Theorem 2, we deduce the existence of a real-valued predictable process ϕ and a positive $\mathbb{P}$-local martingale K such that

$$0 < \phi \leq 1 \quad \text{and} \quad K\,(\phi \centerdot X) \quad \text{is a } \mathbb{P}\text{-local martingale.}$$

Then there exists a sequence of stopping times $(v_n)_n$ that increases to infinity such that the stopped process K^{v_n} is a $\mathbb{P}$-martingale. Put $\mathbb{P}_n := K_{v_n} \centerdot \mathbb{P} \sim \mathbb{P}$ and

$$\mathbb{Q}_n := \frac{K_{v_n}}{\mathbb{E}_{\mathbb{P}}(\zeta_{v_n} K_{v_n})} \centerdot \mathbb{Q} = \frac{\zeta_{v_n}}{\mathbb{E}_{\mathbb{P}}(\zeta_{v_n} K_{v_n})} \centerdot \mathbb{P}_n \ll \mathbb{P}_n.$$

Define $\zeta^n_t := \mathbb{E}_{\mathbb{P}_n}\left(\frac{\zeta_{\upsilon_n}}{\mathbb{E}_{\mathbb{P}_n}(\zeta_{\upsilon_n})}|\mathscr{F}_t\right)$ and note that the condition that $\{\zeta = 0 < \zeta_-\}$ is $\mathbb{P}$-evanescent implies that $\{\zeta^n = 0 < \zeta^n_-\}$ is $\mathbb{P}_n$-evanescent. Then, by applying Proposition 12 to $(\phi \centerdot X)^{\upsilon_n}$ under $\mathbb{P}_n$, we conclude that $(\phi \centerdot X)^{\upsilon_n}$ satisfies NUPBR($\mathbb{Q}_n$). Thanks to Proposition 5, since $\mathbb{Q}_n \sim \mathbb{Q}$, NUPBR($\mathbb{Q}$) property of X immediately follows. ◻

We recover [10, Theorem 5.3] and [10, Proposition 5.7] with alternative proof in the next result.

Theorem 8 *The following conditions are equivalent.*

(a) The thin set $\{\zeta = 0 < \zeta_-\}$ is $\mathbb{P}$-evanescent.
(b) The stopping time $\tilde{r}$ is infinite $\mathbb{P}$-a.s.
(c) Any $\mathbb{P}$-local martingale X admits Y^a as a $\mathbb{Q}$-local martingale deflator, so X satisfies NUPBR($\mathbb{Q}$).
(d) Any (bounded) process X satisfying NUPBR($\mathbb{P}$) satisfies NUPBR($\mathbb{Q}$).

Proof The equivalence between (a) and (b) is obvious from the definition of $\tilde{r}$.

The implication (b)⇒(c) follows from Proposition 12. To prove (c)⇒(b) (and (d)⇒(b)), we consider the $\mathbb{P}$-martingale

$$X = \mathbb{1}_{[\![\tilde{r},\infty[\![} - \left(\mathbb{1}_{[\![\tilde{r},\infty[\![}\right)^{p,\mathbb{P}}.$$

Then, due to $\tilde{r} = \infty$ $\mathbb{Q}$-a.s. we have that, under $\mathbb{Q}$,

$$X = -\left(\mathbb{1}_{[\![\tilde{r},\infty[\![}\right)^{p,\mathbb{P}}$$

is a predictable decreasing process. Thus, X satisfies NUPBR($\mathbb{Q}$) if and only if it is a null process. Then, we conclude that $\tilde{S}$ is infinite using the same argument as in the proof of Lemma 6(b). The implication (c) ⇒ (d) follows from Proposition 13. ◻

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Martingale Marginals Do Not Always Determine Convergence

Jim Pitman

Abstract Baéz-Duarte (J. Math. Anal. Appl. **36**, 149–150, 1971, http://dx.doi.org/10.1016/0022-247X(71)90025-4 [ISSN 0022-247x]) and Gilat (Ann. Math. Stat. **43**, 1374–1379, 1972, http://dx.doi.org/10.1214/aoms/1177692494 [ISSN 0003-4851]) gave examples of martingales that converge in probability (and hence in distribution) but not almost surely. Here such a martingale is constructed with uniformly bounded increments, and a construction is provided of two martingales with the same marginals, one of which converges almost surely, while the other does not converge in probability.

1 Introduction

Recent work of Marc Yor and coauthors [4] has drawn attention to how properties of a martingale are related to its family of marginal distributions. A fundamental result of this kind is Doob's martingale convergence theorem:

- if the marginal distributions $(\mu_n, n \geq 0)$ of a discrete time martingale $(M_n, n \geq 0)$ are such that $\int |x|\mu_n(dx)$ is bounded, then M_n converges almost surely.

Other well known results relating the behavior of a discrete time martingale M_n to its marginal laws μ_n are:

- for each $p > 1$, the sequence $\int |x|^p \mu_n(dx)$ is bounded if and only if M_n converges in L^p;
- $\lim_{y\to\infty} \sup_n \int_{|x|>y} |x|\mu_n(dx) = 0$, that is $(M_n)_{n\geq 0}$ is uniformly integrable, if and only if M_n converges in L^1.

We know also from Lévy that if μ_n is the distribution of a partial sum S_n of independent random variables, and μ_n converges in distribution as $n \to \infty$, then

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S_n converges almost surely. These results can be found in most modern graduate textbooks in probability. See for instance Durrett [2].

What if the marginals of a martingale converge in distribution? Does that imply the martingale converges a.s? Báez-Duarte [1] and Gilat [3] gave examples of martingales that converge in probability but not almost surely. So the answer to this question is no. But worse than that, there is a sequence of martingale marginals converging in distribution, such that some martingales with these marginals converge almost surely, while others diverge almost surely. So by mixing, the probability of convergence of a martingale with these marginals can be any number in [0, 1]. Moreover, the same phenomenon can be exhibited for convergence in probability: there is a sequence of martingale marginals converging in distribution, such that some martingales with these marginals converge in probability, but others do not.

The purpose of this brief note is to record these examples, and to draw attention to the following problems which they raise:

1. What is a necessary and sufficient condition on martingale marginals for every martingale with these marginals to converge almost surely?
2. What is a necessary and sufficient condition on martingale marginals for every martingale with these marginals to converge in probability?

Perhaps the condition for almost sure convergence is Doob's L^1-bounded condition. But this does not seem at all obvious. Moreover, L^1-bounded is not the right condition for convergence in probability: convergence in distribution to a point mass is obviously sufficient, and this condition can hold for marginals that are not bounded in L^1. See also Rao [5] for treatment of some other problems related to non-L^1-bounded martingales.

2 Examples

2.1 Almost Sure Convergence

This construction extends and simplifies the construction by Gilat [3, §2] of a martingale which converges in probability but not almost surely, with increments in the set $\{-1, 0, 1\}$ See also Báez-Duarte [1] for an earlier construction with unbounded increments, based on the double or nothing game instead of a random walk.

Let $(S_n, n = 0, 1, 2, \ldots)$ be a simple symmetric random walk started at $S_0 = 0$, with $(S_{n+1} - S_n, n = 0, 1, 2, \ldots)$ a sequence of independent $U(\pm 1)$ random variables, where $U(\pm 1)$ is the uniform distribution on the set $\{\pm 1\} := \{-1, +1\}$. Let $0 = T_0 < T_1 < T_2 < \cdots$ be the successive times n that $S_n = 0$. By recurrence of the simple random walk, $P(T_n < \infty) = 1$ for every n. For each $k = 1, 2, \ldots$ let

$M^{(k)}$ be the process which follows the walk S_n on the random interval $[T_{k-1}, T_k]$ of its kth excursion away from 0, and is otherwise identically 0:

$$M_n^{(k)} := S_n 1(T_{k-1} \le n \le T_k)$$

where $1(\cdots)$ is an indicator random variable with value 1 if $\cdots$ and 0 otherwise. Each of these processes $M^{(k)}$ is a martingale relative to the filtration $(\mathcal{F}_n)$ generated by the walk (S_n), by Doob's optional sampling theorem. Now let (A_k) be a sequence of events such that the σ-field $\mathcal{G}_0$ generated by these events is independent of the walk $(S_n, n \ge 0)$, and set

$$M_n := \sum_{k=1}^{\infty} M_n^{(k)} 1(A_k)$$

So M_n follows the path of S_n on its kth excursion away from 0 if A_k occurs, and otherwise M_n is identically 0. Let $\mathcal{G}_n$ for $n \ge 0$ be the σ-field generated by $\mathcal{G}_0$ and $\mathcal{F}_n$. Then it is clear that $(M_n, \mathcal{G}_n)$ is a martingale, no matter what choice of the sequence of events (A_k) independent of (S_n). The distribution of M_n is determined by the formula

$$P(M_n = x) = \sum_{k=1}^{\infty} P(S_n = x, T_{k-1} \le n \le T_k) P(A_k)$$

for all integers $x \ne 0$. A family of martingales with the same marginals is thus obtained by varying the structure of dependence between the events A_k for a given sequence of probabilities $P(A_k)$. The only way that a path of M_n can converge is if M_n is eventually absorbed in state 0. So if $N := \sum_k 1(A_k)$ denotes the number of events A_k that occur,

$$P(M_n \text{ converges}) = P(N < \infty).$$

Now take $P(A_k) = p_k$ for a decreasing sequence p_k with limit 0 but $\sum_k p_k = \infty$, for instance $p_k = 1/k$. Then (A_k) can be constructed so that the A_k are mutually independent, and $P(N = \infty) = 1$ by the Borel-Cantelli lemma. Or these events can be nested:

$$A_1 \supseteq A_2 \supseteq A_3 \cdots$$

in which case

$$P(N \ge k) = P(A_k) \downarrow 0 \text{ as } k \to \infty,$$

so $P(N = \infty) = 0$ in this case. Thus we obtain a sequence of marginal distributions for a martingale, such that some martingales with these marginals converge almost surely, while others diverge almost surely.

2.2 Convergence in Probability

Let us construct a martingale M_n which converges in distribution, but not in probability, following indications of such a construction by Gilat [3, §1].

This will be an inhomogeneous Markov chain with integer values, starting from $M_0 = 0$. Its first step will be to M_1 with $U(\pm 1)$ distribution. Thereafter, the idea is to force M_n to alternate between the values ± 1, with probability increasing to 1 as $n \to \infty$. This achieves $U(\pm 1)$ as its limit in distribution, while preventing convergence in probability by the alternation. The transition probabilities of M_n are as follows:

$$P(M_{n+1} = M_n \pm 1 \,|\, M_n \text{ with } M_n \notin \{\pm 1\}) = 1/2 \tag{1}$$

$$P(M_{n+1} = -1 \,|\, M_n = 1) = 1 - 2^{-n} \tag{2}$$

$$P(M_{n+1} = 2^{n+1} - 1 \,|\, M_n = 1) = 2^{-n} \tag{3}$$

$$P(M_{n+1} = +1 \,|\, M_n = -1) = 1 - 2^{-n} \tag{4}$$

$$P(M_{n+1} = -2^{n+1} + 1 \,|\, M_n = -1) = 2^{-n}. \tag{5}$$

The first line indicates that whenever M_n is away from the two point set $\{\pm 1\}$, it moves according to a simple symmetric random walk, until it eventually gets back to $\{\pm 1\}$ with probability one. Once it is back in $\{\pm 1\}$, it is forced to alternate between these values, with probability $1 - 2^{-n}$ for an alternation at step n, compensated by moving to $\pm(2^{n+1} - 1)$ with probability 2^{-n}. Since the probabilities 2^{-n} are summable, the Borel-Cantelli Lemma ensures that with probability one only finitely many exits from $\{\pm 1\}$ ever occur. After the last of these exits, the martingale eventually returns to $\{\pm 1\}$ with probability one. From that time onwards, the martingale flips back and forth deterministically between $\{\pm 1\}$.

A slight modification of these transition probabilities, gives another martingale with the same marginal distributions which converges almost surely and hence in probability. With $M_0 = 0$ as before, the modified scheme is as follows:

$$P(M_{n+1} = M_n \pm 1 \,|\, M_n \text{ with } M_n \notin \{\pm 1\}) = 1/2 \tag{6}$$

$$P(M_{n+1} = 1 \,|\, M_n = 1) = 1 - 2^{-n} \tag{7}$$

$$P(M_{n+1} = 2^{n+1} - 1 \,|\, M_n = 1) = 2^{-n} p_n \tag{8}$$

$$P(M_{n+1} = -2^{n+1} + 1 \,|\, M_n = 1) = 2^{-n} q_n \tag{9}$$

$$P(M_{n+1} = -1 \mid M_n = -1) = 1 - 2^{-n} \tag{10}$$

$$P(M_{n+1} = -2^{n+1} + 1 \mid M_n = -1) = 2^{-n} p_n \tag{11}$$

$$P(M_{n+1} = 2^{n+1} - 1 \mid M_n = -1) = 2^{-n} q_n \tag{12}$$

where

$$p_n := 1/(2 - 2^{-n}) \text{ and } q_n := 1 - p_n$$

are chosen so that the distribution with probability p_n at $2^{n+1} - 1$ and q_n at $-2^{n+1} + 1$ has mean

$$p_n(2^{n+1} - 1) + q_n(-2^{n+1} + 1) = 1.$$

In this modified process, the alternating transition out of states ± 1 is replaced by holding in these states, while the previous compensating moves to $\pm(2^{n+1} - 1)$ are replaced by a nearly symmetric transitions from ± 1 to these values. This preserves the martingale property, and also preserves the marginal laws. But the previous argument for eventual alternation now shows that the modified martingale is eventually absorbed almost surely in one of the states ± 1. So the modified martingale converges almost surely to a limit which has $U(\pm 1)$ distribution.

These martingales (M_n) have jumps that are unbounded. Gilat [3, §2] left open the question of whether there exist martingales with uniformly bounded increments which converge in distribution but not in probability. But such martingales can be created by a variation of the first construction of (M_n) above, as follows.

Run a simple symmetric random walk starting from 0. Each time the random walk makes an alternation between the two states ± 1, make the walk delay for a random number of steps in its current state in ± 1 before continuing, for some rapidly increasing sequence of random delays. Call the resulting martingale M_n. So by construction, M_1 has $U(\pm)$ distribution,

$$M_n = (-1)^{k-1} M_1 \text{ for } S_k \le n \le T_k$$

for some increasing sequence of randomized stopping times

$$1 = S_1 < T_1 < S_2 < T_2 < \cdots,$$

and during the kth crossing interval $[T_k, S_{k+1}]$ the process M_n follows a simple random walk path starting in state $(-1)^{k-1} M_1$ and stopping when it first reaches state $(-1)^k M_1$.

The claim is that a suitable construction of the delays $T_k - S_k$ will ensure that the distribution of M_n converges to $U(\pm 1)$, while there is almost deterministic alternation for large k of the state M_{t_k} for some rapidly increasing deterministic sequence t_k. To achieve this end, let $t_1 = 1$ and suppose inductively for $k = 1, 2, \ldots$ that t_k has been chosen so that

$$P(M_{t_k} = (-1)^{k-1} M_1) > 1 - \epsilon_k \text{ for some } \epsilon_k \downarrow 0 \text{ as } k \to \infty. \tag{13}$$

Here $M_1 \in \{\pm 1\}$ is the first step of the simple random walk. The random number of steps required for random walk crossing between states ± 1 is a.s. finite. So having defined t_k, we can choose an even integer t_{k+1} so large, that $t_{k+1}/2 > t_k$ and all of the following events occur with probability at least $1 - \epsilon_{k+1}$:

- $M_{t_{k+1}/2} = (-1)^{k-1} M_1$, meaning that the $(k-1)$th crossing between ± 1 has been completed by time $S_k < t_{k+1}/2$;
- the kth crossing is started at time T_k that is uniform on $[t_{k+1}/2, t_{k+1})$ given $S_k < t_{k+1}/2$;
- the kth crossing is completed at time $S_{k+1} < t_{k+1}$, so $M_n = (-1)^k M_1$ for $S_{k+1} \leq n \leq t_{k+1}$.

Moreover, t_{k+1} can be chosen so large that the uniform random start time of the kth crossing given $S_k < t_{k+1}/2$ ensures that also

$$P(M_n \in \{\pm 1\}) \geq 1 - 2\epsilon_k \text{ for all } t_k \leq n \leq t_{k+1}$$

because with high probability the length $S_{k+1} - T_k$ of the kth crossing is negligible in comparison with the length $t_{k+1}/2$ of the interval $[t_{k+1}/2, t_{k+1}]$ in which this crossing is arranged to occur. It follows from this construction that M_n converges in distribution to $U(\pm 1)$, while the forced alternation (13) prevents M_n from having a limit in probability.

A feature of the previous example is that $\sup_n M_n = -\inf_n M_n = \infty$ almost surely, since in the end every step of the underlying simple symmetric random walk is made by the time-changed martingale M_n. A similar example can be created from a standard Brownian motion $(B_t, t \geq 0)$ using a predictable $\{0, 1\}$-valued process $(H_t, t \geq 0)$ to create successive switching between and holding in states ± 1 so that the martingale

$$M_t := \int_0^t H_t dB_t$$

converges in distribution to $U(\pm 1)$ while not converging in probability. In this example, $\int_0^\infty H_t dt = \sup_t M_t = -\inf_t M_t = \infty$ almost surely.

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Martingale Inequalities for the Maximum via Pathwise Arguments

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Abstract We study a class of martingale inequalities involving the running maximum process. They are derived from pathwise inequalities introduced by Henry-Labordère et al. (Ann. Appl. Probab., 2015 [arxiv:1203.6877v3]) and provide an upper bound on the expectation of a function of the running maximum in terms of marginal distributions at n intermediate time points. The class of inequalities is rich and we show that in general no inequality is *uniformly sharp*—for any two inequalities we specify martingales such that one or the other inequality is sharper. We use our inequalities to recover Doob's L^p inequalities. Further, for $p = 1$ we refine the known inequality and for $p < 1$ we obtain new inequalities.

1 Introduction

In this article we study certain martingale inequalities for the terminal maximum of a stochastic process. We thus contribute to a research area with a long and rich history. In seminal contributions, Blackwell and Dubins [7], Dubins and Gilat [14] and Azéma and Yor [2, 3] showed that the distribution of the maximum $\bar{X}_T := \sup_{t \leq T} X_t$ of a martingale (X_t) is bounded above, in stochastic order, by the so called Hardy-Littlewood transform of the distribution of X_T, and the bound is attained.

"This paper, alike the volume, is dedicated to the memory of Professor Marc Yor, who was my PhD supervisor and my mentor. I remain deeply indebted for all the help and advice he has given me over the years. I shall miss him greatly and will remember him as a brilliant but humble scholar, a passionate mathematician who was immensely generous in sharing his knowledge and ideas. Above all, he loved and enjoyed mathematics and I hope he would have liked this simple story about martingale inequalities." Jan Obloj

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This led to series of studies on the possible distributions of $(X_T, \bar{X}_T)$, see Carraro et al. [10] for a discussion and further references. More recently, such problems appeared very naturally within the field of mathematical finance. The original result was extended to the case of a non trivial starting law in Hobson [16] and to the case of a fixed intermediate law in Brown et al. [9].

The novelty of our study here, as compared with the works mentioned above, is that we look at inequalities which use the information about the process at n intermediate time points. One of our goals is to understand how the bound induced by these more elaborate inequalities compares to simpler inequalities which do not use information about the process at intermediate time points. We show that in our context these bounds can be both, better or worse. We also note that knowledge of intermediate moments does not induce a necessarily tighter bound in Doob's L^p-inequalities. Our main result is split into two Theorems. First, in Theorem 2.1, we present our class of inequalities, indexed with an n-tuple of functions ζ, and show that they are sharp: for a given ζ we find a martingale which attains equality. Second, in Theorem 3.1, we show that no inequality is universally better than another: for $\zeta \neq \tilde{\zeta}$ we find two processes X and $\tilde{X}$ which show that either of the inequalities can be strictly better than the other.

Throughout, we emphasise the simplicity of our arguments, which are all elementary. This is illustrated in Sects. 2.2–2.4 where we obtain amongst others the sharp versions of Doob's L^p-inequalities for all $p > 0$. While the case $p \geq 1$ is already known in the literature, our Doob's L^p-inequality in the case $p \in (0, 1)$ appears new.

The idea of deriving martingale inequalities from pathwise inequalities is already present in work on robust pricing and hedging by Hobson [16]. Other authors have used pathwise arguments to derive martingale inequalities, e.g. Doob's inequalities are considered by Acciaio et al. [1] and Obłój and Yor [19]. The Burkholder-Davis-Gundy inequality is rediscovered with pathwise arguments by Beiglböck and Siorpaes [6]. In this context we also refer to Cox and Wang [12] and Cox and Peskir [11] whose pathwise inequalities relate a process and time. In a similar spirit, bounds for local time are obtained by Cox et al. [13]. Beiglböck and Nutz [5] look at general martingale inequalities and explain how they can be obtained from deterministic inequalities. This approach builds on the so-called Burkholder's method, a classical tool in probability used to construct sharp martingale inequalities, see Osękowski [20, Chap. 2] for a detailed discussion.

In a discrete time and quasi-sure setup, the results of Bouchard and Nutz [8] can be seen as general theoretical underpinning of many ideas we present here in the special case of martingale inequalities involving the running maximum.

Organisation of the Article We first recall a remarkable pathwise inequality obtain by Henry-Labordère et al. [15] and some related results. The body of the paper is then split into two sections. In Sect. 2 we derive our class of submartingale inequalities and demonstrate how they can be used to derive, amongst others, Doob's inequalities. Then, in Sect. 3, we study if a given inequality can be universally better than another one for all submartingales.

1.1 Preliminaries

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is fixed which supports a standard real-valued Brownian motion B with some initial value $X_0 \in \mathbb{R}$. We will typically use $X = (X_t)$ to denote a (sub/super) martingale and, unless otherwise specified, we always mean this with respect to X's natural filtration. Throughout, we fix arbitrary times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n =: T$.

Before we proceed to the main result, we recall a remarkable pathwise inequality from Henry-Labordère et al. [15]. The version we give below appears in the proof of Proposition 3.1 in [15] and is best suited to our present context.

Proposition 1.1 (Proposition 3.1 of Henry-Labordère et al. [15]) *Let ω be a càdlàg path and denote $\bar{\omega}_t := \sup_{0\leq s\leq t} \omega_s$. Then, for $m > \omega_0$ and $\zeta_1 \leq \cdots \leq \zeta_n < m$:*

$$\mathbb{1}_{\{\bar{\omega}_{t_n} \geq m\}} \leq \Upsilon_n(\omega, m, \boldsymbol{\zeta}) := \sum_{i=1}^{n} \left(\frac{(\omega_{t_i} - \zeta_i)^+}{m - \zeta_i} + \mathbb{1}_{\{\bar{\omega}_{t_{i-1}} < m \leq \bar{\omega}_{t_i}\}} \frac{m - \omega_{t_i}}{m - \zeta_i} \right) \qquad (1)$$

$$- \sum_{i=1}^{n-1} \left(\frac{(\omega_{t_i} - \zeta_{i+1})^+}{m - \zeta_{i+1}} + \mathbb{1}_{\{m \leq \bar{\omega}_{t_i}, \zeta_{i+1} \leq \omega_{t_i}\}} \frac{\omega_{t_{i+1}} - \omega_{t_i}}{m - \zeta_{i+1}} \right).$$

Next, we recall a process with some special structure in view of (1). This process has been analysed in more detail by Obłój and Spoida [18].

Definition 1.2 (Iterated Azéma-Yor Type Embedding) Let $\xi_1, \ldots, \xi_n$ be non-decreasing functions on (X_0, ∞) and denote $\bar{B}_t := \sup_{u\leq t} B_u$. Set $\tau_0 \equiv 0$ and for $i = 1, \ldots, n$ define

$$\tau_i = \inf\left\{t \geq \tau_{i-1} : B_t \leq \xi_i(\bar{B}_t)\right\}. \qquad (2)$$

A continuous martingale X is called an iterated Azéma-Yor type embedding based on $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ if

$$(X_{t_i}, \bar{X}_{t_i}) = (B_{\tau_i}, \bar{B}_{\tau_i}) \; a.s. \qquad \text{for } i = 0, \ldots, n. \qquad (3)$$

Note from the non-decrease of the ξ_i's that $\tau_0 \leq \inf\{t \geq H_1 : B_t \leq \xi_1(1)\}$ for $H_1 = \inf\{t \geq 0 : B_t \geq 1\}$ and then $\tau_i \leq \inf\{t \geq \tau_{i-1} : B_t \leq \xi_i(\bar{B}_{\tau_{i-1}})\}$, $i = 2, \ldots, n$. It follows that $\tau_i < \infty$ a.s. for all $i = 1, \ldots, n$. Further, X being a martingale implies that B_{τ_i} are integrable and all have mean X_0. In particular, $\tau_n < \infty$ a.s. More importantly, it follows from the characterisation of uniform integrable martingales in Azéma et al. [4] that $(B_{t\wedge\tau_n}, t \geq 0)$ is uniformly integrable. Indeed, we have, with

$H_x = \inf\{t \geq 0 : B_t = x\}$,

$$\lim_{x\to\infty} x\mathbb{P}\left[\sup_{t\geq 0}|B_{t\wedge\tau_n}| > x\right] \leq \lim_{x\to\infty} x\mathbb{P}\left[H_x < H_{\max_i \xi_i^{-1}(-x)}\right] + x\mathbb{P}\left[\bar{B}_{t\wedge\tau_n} > x\right]$$

$$= \lim_{x\to\infty}\left(\frac{x(\max_i \xi_i^{-1}(-x) - X_0)}{\max_i \xi_i^{-1}(-x) + x} + x\mathbb{P}\left[\bar{X}_{t_n} > x\right]\right) = 0,$$

since $(X_t : t \leq t_n)$ is uniformly integrable and $\max_i \xi_i^{-1}(-x) - X_0 \searrow 0$. Conversely, if $(B_{t\wedge\tau_n} : t \geq 0)$ is uniformly integrable then an example of an iterated Azéma-Yor type embedding is obtained by taking

$$X_t := B_{\tau_i \wedge\left(\tau_{i-1}\vee\frac{t-t_{i-1}}{t_i - t}\right)}, \quad \text{for} \quad t_{i-1} < t \leq t_i,\ i = 1,\ldots,n. \tag{4}$$

Finally, we recall a version of Lemma 4.1 from Henry-Labordère et al. [15].

Proposition 1.3 (Pathwise Equality) *Let $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_n)$ be non-decreasing right-continuous functions and let X be an iterated Azéma-Yor embedding based on $\boldsymbol{\xi}$. Then, for any $m > X_0$ with $\xi_n(m) < m$, X achieves equality in (1), i.e.*

$$\mathbb{1}_{\{\bar{X}_{t_n}\geq m\}} = \Upsilon_n\big(X, m, \boldsymbol{\zeta}(m)\big) \quad a.s., \tag{5}$$

where

$$\zeta_i(m) = \min_{j\geq i}\xi_j(m), \qquad i = 1,\ldots,n. \tag{6}$$

We note that if we work on the canonical space of continuous functions then (5) holds pathwise and not only a.s. We also note that the assumption that X is an iterated Azéma-Yor type embedding, or that $(B_{\tau_n\wedge t})$ is a uniformly integrable martingale, may be relaxed as long as X satisfies (3).

2 (Sub)martingale Inequality and Its Applications

We present now an inequality on the expected value of a function of the running maximum of a submartingale which is obtained by taking expectations in the pathwise inequality of Proposition 1.1. We then demonstrate how this inequality can be used to derive and improve Doob's inequalities. Related work on pathwise interpretations of Doob's inequalities can be found in Acciaio et al. [1] and Obłój and Yor [19]. Peskir [21, Sect. 4] derives Doob's inequalities and shows that the constants he obtains are optimal. We give below an alternative proof of these statements and provide new sharp inequalities for the case $p < 1$.

2.1 Submartingale Inequality

We first deduce a general martingale inequality for $\mathbb{E}\left[\phi(\bar{X}_T)\right]$, similarly as in Proposition 3.2 in [15], and prove that it is attained under some conditions. Define

$$\mathscr{Z} := \Big\{ \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \ : \ \zeta_i : (X_0, \infty) \to \mathbb{R} \ \text{ is right-continuous,} \\ \zeta_1(m) \leq \cdots \leq \zeta_n(m) < m, \quad n \in \mathbb{N} \Big\}. \tag{7}$$

In order to ensure that the expectations we consider are finite we will occasionally need the technical condition that

$$\zeta_1^{\infty} := \liminf_{m\to\infty} \frac{\zeta_1(m)}{m} > 0 \ \text{ and } \ \limsup_{m\to\infty} \frac{\phi(m)}{m^{\gamma}} = 0 \text{ for some } \gamma < \frac{1}{1-\zeta_1^{\infty}}. \tag{8}$$

Theorem 2.1 *Let* $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathscr{Z}$. *Then,*

(i) *for any càdlàg submartingale X: for any $m > X_0$ we have*

$$\mathbb{P}\left[\bar{X}_T \geq m\right] \leq \mathbb{E}\left[\sum_{i=1}^{n} \frac{(X_{t_i} - \zeta_i(m))^+}{m - \zeta_i(m)} - \sum_{i=1}^{n-1} \frac{(X_{t_i} - \zeta_{i+1}(m))^+}{m - \zeta_{i+1}(m)}\right] \tag{9}$$

and, more generally, for a right-continuous non-decreasing function ϕ,

$$\mathbb{E}\left[\phi(\bar{X}_T)\right] \leq \mathrm{UB}\,(X, \phi, \boldsymbol{\zeta}) := \phi(X_0) + \int_{(X_0,\infty)} \sum_{i=1}^{n} \mathbb{E}\left[\lambda_i^{\boldsymbol{\zeta},m}(X_{t_i})\right] \mathrm{d}\phi(m) \tag{10}$$

where

$$\lambda_i^{\boldsymbol{\zeta},m}(x) := \frac{(x - \zeta_i(m))^+}{m - \zeta_i(m)} - \frac{(x - \zeta_{i+1}(m))^+}{m - \zeta_{i+1}(m)} \mathbb{1}_{\{i<n\}}, \tag{11}$$

(ii) *if ζ_1 is non-decreasing and satisfies, together with ϕ, the condition* (8)*, there exists a continuous martingale which achieves equality in* (10)*.*

Remark 2.2 (Optimization over $\boldsymbol{\zeta}$) If X and $t_1, \ldots, t_n$ are fixed we can optimize (10) over $\boldsymbol{\zeta} \in \mathscr{Z}$ to obtain a minimizer $\boldsymbol{\zeta}^\star$. Clearly, more intermediate points t_i in (10) can only improve the bound for this particular process X. However, only for very special processes (e.g. the iterated Azéma-Yor type embedding) there is hope that (10) will hold with equality. This is, loosely speaking, because a finite number of intermediate marginal law constraints does not, in general, determine uniquely the law of the maximum at terminal time t_n.

Proof of Theorem 2.1 Equation (9) follows from (1) by taking expectations. Then, (10) follows from (9) by integration and Fubini's theorem:

$$\mathbb{E}\left[\phi(\bar{X}_T)\right] = \mathbb{E}\left[\phi(X_0) + \int_{(X_0,\infty)} \mathbf{1}_{\{\bar{X}_T \geq m\}} \mathrm{d}\phi(m)\right].$$

Note that for a fixed m, $\mathbb{E}\left[|\lambda_i^{\zeta,m}(X_{t_i})|\right] < \infty$ for $i = 1, \ldots, n$, since $\mathbb{E}\left[X_{t_i}^+\right] < \infty$ by the submartingale property.

If ζ_1 is non-decreasing and $\zeta_1(m) \geq \alpha m$ for m large, $\alpha > 0$, we define X by

$$X_t = \begin{cases} B_{\frac{t}{t_1 - t} \wedge \tau_{\zeta_1}} & \text{if } t < t_1, \\ B_{\tau_{\zeta_1}} & \text{if } t \geq t_1, \end{cases}$$

where B is a Brownian motion, $B_0 = X_0$, and $\tau_{\zeta_1} := \inf\left\{u > 0 \,:\, B_u \leq \zeta_1(\bar{B}_u)\right\}$. Excursion theoretical considerations, cf. e.g. Rogers [22], combined with asymptotic bounds on ζ_1 in (8), allow us to compute

$$\begin{aligned}\mathbb{P}\left[\bar{X}_{t_n} \geq y\right] &= \exp\left(-\int_{(X_0,y]} \frac{1}{z - \zeta_1(z)} \mathrm{d}z\right) \leq \text{const} \cdot \exp\left(-\int_{(1,y]} \frac{1}{z - \alpha z} \mathrm{d}z\right) \\ &= \text{const} \cdot y^{-\frac{1}{1-\alpha}}\end{aligned}$$

for large y. We may take α such that $\gamma < 1/(1-\alpha)$ in (8) which then ensures that $\mathbb{E}\left[\phi(\bar{X}_{t_n})\right] < \infty$. Further, note that for large y, $\inf_{t \geq 0} X_t \leq -y$ implies $\bar{X}_\infty = \bar{X}_{t_n} \geq y/\alpha$ and hence it follows that

$$\lim_{y \to \infty} y \mathbb{P}\left[\sup_{t \geq 0} |X_t| \geq y\right] \leq \text{const} \cdot \lim_{y \to \infty} y^{1 - \frac{1}{1-\alpha}} = 0$$

which in turn implies that $(X_t \,:\, t \geq 0)$ is a uniformly integrable martingale, see Azéma et al. [4]. Finally, one readily verifies together with Proposition 1.3 that

$$\Upsilon_n(X, m, \boldsymbol{\zeta}) = \Upsilon_1(X, m, \boldsymbol{\zeta}) = \mathbb{1}_{\{\bar{X}_{t_1} \geq m\}} = \mathbb{1}_{\{\bar{X}_{t_n} \geq m\}}.$$

and then the claim follows from

$$\begin{aligned}\mathbb{E}\left[\phi(\bar{X}_{t_n})\right] &= \phi(X_0) + \int_{(X_0,\infty)} \mathbb{E}\left[\mathbb{1}_{\{\bar{X}_{t_n} \geq m\}}\right] \mathrm{d}\phi(m) \\ &= \phi(X_0) + \int_{(X_0,\infty)} \mathrm{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right) \mathrm{d}\phi(m) \\ &= \mathrm{UB}\left(X, \phi, \boldsymbol{\zeta}\right)\end{aligned}$$

where we applied Fubini's theorem. □

2.2 Doob's L^p-Inequalities, $p > 1$

Using a special case of Theorem 2.1 we obtain an improvement to Doob's inequalities. Denote $\mathrm{pow}^p(m) = m^p, \ \ \zeta_\alpha(m) := \alpha m.$

Proposition 2.3 (Doob's L^p-Inequalities, $p > 1$) *Let $(X_t)_{t\le T}$ be a non-negative càdlàg submartingale.*

(i) *Then,*

$$\mathbb{E}\left[\bar{X}_T^p\right] \le \mathrm{UB}\left(X, \mathrm{pow}^p, \zeta_{\frac{p-1}{p}}\right) \tag{12a}$$

$$\le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1} X_0^p. \tag{12b}$$

(ii) *For every $\epsilon > 0$, there exists a martingale X such that*

$$0 \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1} X_0^p - \mathbb{E}\left[\bar{X}_T^p\right] < \epsilon. \tag{13}$$

(iii) *The inequality in (12b) is strict if and only if either holds:*

$$\mathbb{E}\left[\bar{X}_T^p\right] < \infty \text{ and } X_T < \frac{p-1}{p} X_0 \text{ with positive probability.} \tag{14a}$$

$$\mathbb{E}\left[\bar{X}_T^p\right] < \infty \text{ and } X \text{ is a strict submartingale.} \tag{14b}$$

Proof Let us first prove (12a) and (12b). If $\mathbb{E}\left[X_T^p\right] = \infty$ there is nothing to show. In the other case, Eq. (12a) follows from Theorem 2.1 applied with $n = 1, \ \phi(y) = \mathrm{pow}^p(y) = y^p$ and $\zeta_1 = \zeta_{\frac{p-1}{p}}$. To justify this choice of ζ_1 and to simplify further the upper bound we start with a more general $\zeta_1 = \zeta_\alpha, \ \alpha < 1$ and compute

$$\begin{aligned}
\mathbb{E}\left[\bar{X}_T^p\right] - X_0^p \le \mathrm{UB}\left(X, \mathrm{pow}^p, \zeta_\alpha\right) - X_0^p &= \mathbb{E}\left[\int_{X_0}^{\infty} p y^{p-1} \frac{(X_T - \alpha y)^+}{y - \alpha y} \mathrm{d}y\right] \\
&= \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha} \vee X_0} p y^{p-1} \frac{X_T - \alpha y}{y - \alpha y} \mathrm{d}y\right] \\
&\le \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha}} p y^{p-1} \frac{X_T - \alpha y}{y - \alpha y} \mathrm{d}y\right] \\
&= \frac{p}{p-1} \frac{1}{1-\alpha} \mathbb{E}\left[\left\{\left(\frac{X_T}{\alpha}\right)^{p-1} - X_0^{p-1}\right\} X_T\right]
\end{aligned}$$

$$- \frac{\alpha}{1-\alpha}\mathbb{E}\left[\left(\frac{X_T}{\alpha}\right)^p - X_0^p\right]$$
$$\leq \frac{1}{p-1}\frac{1}{(1-\alpha)\alpha^{p-1}}\mathbb{E}\left[X_T^p\right] - \frac{p-\alpha(p-1)}{(p-1)(1-\alpha)}X_0^p, \tag{15}$$

where we used Fubini in the first equality and the submartingale property of X in the last inequality. We note that the function $\alpha \mapsto \frac{1}{(1-\alpha)\alpha^{p-1}}$ attains its minimum at $\alpha^\star = \frac{p-1}{p}$. Plugging $\alpha = \alpha^\star$ into the above yields (12b).

We turn to the proof that Doob's L^p-inequality is attained asymptotically in the sense of (13), a fact which was also proven by Peskir [21, Sect. 4]. Let $X_0 > 0$, otherwise the claim is trivial. Set $\alpha^\star = \frac{p-1}{p}$ and take $\alpha^\star < \alpha := \frac{p+\epsilon-1}{p+\epsilon} < 1$. Let $X_T = B_{\tau_\alpha}$ where B is a Brownian motion started at X_0 and $\tau_\alpha := \inf\{u > 0 : B_u \leq \alpha\bar{B}_u\}$. Then by using excursion theoretical results, cf. e.g. Rogers [22],

$$\mathbb{P}\left[\bar{X}_T \geq y\right] = \exp\left(-\int_{X_0}^{y}\frac{1}{z-\alpha z}\mathrm{d}z\right) = \left(\frac{y}{X_0}\right)^{-\frac{1}{1-\alpha}}$$

and then direct computation shows

$$\mathbb{E}\left[\bar{X}_T^p\right] = \frac{p+\epsilon}{\epsilon}X_0^p.$$

By Doob's L^p-inequality,

$$\mathbb{E}\left[\bar{X}_T^p\right] \leq \left(\frac{p}{p-1}\right)^p\mathbb{E}\left[X_T^p\right] - \frac{p}{p-1}X_0^p = \left(\frac{\alpha}{\alpha^\star}\right)^p\mathbb{E}\left[\bar{X}_T^p\right] - \frac{p}{p-1}X_0^p$$

and one verifies

$$\left\{\left(\frac{p}{p-1}\right)^p \cdot \left[\frac{p+\epsilon-1}{p+\epsilon}\right]^p - 1\right\} \cdot \frac{p+\epsilon}{\epsilon}X_0^p \quad \xrightarrow[\epsilon\downarrow 0]{} \quad \frac{p}{p-1}X_0^p.$$

This establishes the claim in (13).

Finally, we note that in the calculations (15) which led to (12b) there are three inequalities: the first one comes from Theorem 2.1 and does not concern the claim regarding (14a)–(14b). The second one is clearly strict if and only if (14a) holds. The third one is clearly strict if and only if (14b) holds. □

Remark 2.4 (Asymptotic Attainability) For the martingales in (ii) of Proposition 2.3 we have

$$\mathrm{UB}\left(X, \mathrm{pow}^p, \zeta_{\frac{p-1}{p}}\right) = \left(\frac{p}{p-1}\right)^p\mathbb{E}\left[X_T^p\right] - \frac{p}{p-1}X_0^p$$

and $\mathbb{E}\left[X_T^p\right] \to \infty$ as $\epsilon \to 0$.

2.3 Doob's L^1-Inequality

Using a special case of Theorem 2.1 we focus on Doob's $L \log L$ type inequalities. We recover here the classical constant $e/(e-1)$, see (17b) , with a refined structure on the inequality. A further improvement to the constant will be obtained in subsequent section in Corollary 2.7. Denote $\mathrm{id}(m) = m$, and

$$\underline{\zeta}_{\alpha}(m) := \begin{cases} -\infty & \text{if } m < 1, \\ \alpha m & \text{if } m \geq 1. \end{cases} \tag{16}$$

Proposition 2.5 (Doob's L^1-Inequality) *Let $(X_t)_{t \leq T}$ be a non-negative càdlàg submartingale. Then:*

(i) *with* $0 \log(0) := 0$ *and* $V(x) := x - x \log(x)$,

$$\begin{aligned} \mathbb{E}\left[\bar{X}_T\right] &\leq \mathrm{UB}\left(X, \mathrm{id}, \underline{\zeta}_{\frac{1}{e}}\right) && \text{(17a)} \\ &\leq \frac{e}{e-1}\Big(\mathbb{E}\left[X_T \log\left(X_T\right)\right] + V(1 \vee X_0)\Big). && \text{(17b)} \end{aligned}$$

(ii) *in the case* $X_0 \geq 1$ *there exists a martingale which achieves equality in both,* (17a) *and* (17b) *and in the case* $X_0 < 1$ *there exists a submartingale which achieves equality in both,* (17a) *and* (17b).

(iii) *the inequality in* (17b) *is strict if and only if either holds:*

$$\mathbb{E}\left[\bar{X}_T\right] < \infty \textit{ and } \bar{X}_T \geq 1, \;\; X_T < \frac{1}{e} X_0 \textit{ with positive probability,} \tag{18a}$$

$$\mathbb{E}\left[\bar{X}_T\right] < \infty \textit{ and } \bar{X}_T \geq 1, \;\; \mathbb{E}\left[X_T\right] > X_0 \vee 1, \tag{18b}$$

$$\mathbb{E}\left[\bar{X}_T\right] < \infty \textit{ and } \bar{X}_T < 1 \textit{ with positive probability.} \tag{18c}$$

Proof Let us first prove (17a) and (17b). If $\mathbb{E}\left[\bar{X}_T\right] = \infty$ there is nothing to show. In the other case, Eq. (17a) follows from Theorem 2.1 applied with $n = 1$, $\phi(y) = \mathrm{id}(y) = y$ and $\zeta_1 = \underline{\zeta}_{\frac{1}{e}}$.

In the case $X_0 \geq 1$ we further compute using $\zeta_1 = \underline{\zeta}_{\alpha}$, $\alpha < 1$,

$$\begin{aligned} \mathbb{E}\left[\bar{X}_T\right] - X_0 &\leq \mathrm{UB}\left(X, \mathrm{id}, \underline{\zeta}_{\alpha}\right) - X_0 \\ &= \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha} \vee X_0} \frac{X_T - \alpha y}{y - \alpha y} \mathrm{d}y\right] \leq \mathbb{E}\left[\int_{X_0}^{\frac{X_T}{\alpha}} \frac{X_T - \alpha y}{(1-\alpha) y} \mathrm{d}y\right] \\ &= \frac{\alpha}{1-\alpha}\mathbb{E}\left[\frac{X_T}{\alpha}\left\{\log\left(\frac{X_T}{\alpha}\right) - \log(X_0)\right\}\right] - \frac{\alpha}{1-\alpha}\mathbb{E}\left[\frac{X_T}{\alpha} - X_0\right] \end{aligned}$$

Choosing $\alpha = e^{-1}$ gives a convenient cancellation. Together with the submartingale property of X, this provides

$$\mathbb{E}\left[\bar{X}_T\right] - X_0 \le \frac{e}{e-1}\mathbb{E}\left[X_T \log\left(X_T\right)\right] - \frac{e}{e-1}\mathbb{E}\left[X_T\right]\log(X_0) + \frac{1}{e-1}X_0$$
$$\le \frac{e}{e-1}\mathbb{E}\left[(X_T)\log\left(X_T\right)\right] - \frac{eX_0\log(X_0)}{e-1} + \frac{X_0}{e-1}. \tag{19}$$

This is (17b) in the case $X_0 \ge 1$.

For the case $0 < X_0 < 1$ we obtain from Proposition 1.1 for $n = 1$,

$$\mathbb{P}\left[\bar{X}_T \ge y\right] \le \inf_{\zeta < y} \frac{\mathbb{E}\left[(X_T - \zeta)^+\right]}{y - \zeta} \le \frac{\mathbb{E}\left[(X_T - \alpha y)^+\right]}{y - \alpha y}$$

for $\alpha < 1$ and therefore

$$\mathbb{E}\left[\bar{X}_T\right] - X_0 = \int_{X_0}^{\infty} \mathbb{P}\left[\bar{X}_T \ge y\right] \mathrm{d}y$$
$$\le (1 - X_0) + \int_1^{\infty} \mathbb{P}\left[\bar{X}_T \ge y\right] \mathrm{d}y$$
$$\le (1 - X_0) + \frac{e}{e-1}\mathbb{E}\left[(X_T)\log\left(X_T\right)\right] + \frac{1}{e-1} \tag{20}$$

by (19). This is (17b) in the case $X_0 < 1$.

Now we prove that Doob's L^1-inequality is attained. This was also proven by Peskir [21, Sect. 4]. Firstly, let $X_0 \ge 1$. Then the martingale

$$X = \left(B_{\frac{t}{T-t} \wedge \tau_{\frac{1}{e}}}\right)_{t \le T}, \quad \text{where } \tau_{\frac{1}{e}} = \inf\{t : eB_t \le \overline{B}_t\}, \tag{21}$$

and B is a Brownian motion with $B_0 = X_0$, achieves equality in both (17a) and (17b). Secondly, let $X_0 < 1$. Then the submartingale X defined by

$$\begin{cases} X_0 & \text{if } t < T/2, \\ B_{\frac{t - T/2}{T/2 - (t - T/2)} \wedge \tau_{\frac{1}{e}}} & \text{if } t \ge T/2, \end{cases} \tag{22}$$

where B is a Brownian motion, $B_0 = 1$, achieves equality in both, (17a) and (17b).

Finally, we note that in the calculations (19) which led to (12b) there are three inequalities: the first one comes from Theorem 2.1 and does not concern the claim regarding (18a)–(18c). The second one is clearly strict if and only if (18a) holds. The third one is clearly strict if and only if (18b) holds. In addition, in the case $X_0 < 1$ there is an additional error coming from (20). Note that, in the case when

$\mathbb{E}\left[\bar{X}_T\right] < \infty$,

$$\left.\frac{\mathbb{E}\left[(X_T-\zeta)^+\right]}{y-\zeta}\right|_{\zeta=\infty} := \lim_{\zeta\to-\infty}\frac{\mathbb{E}\left[(X_T-\zeta)^+\right]}{y-\zeta} = 1.$$

Hence, the first inequality in (20) is strict if and only if (18c) holds. The second inequality in (20) is strict if and only if (18a) or (18b) holds. □

2.4 Doob Type Inequalities, $0 < p < 1$

It is well known that if X is a positive continuous local martingale converging a.s. to zero, then

$$\bar{X}_\infty \sim \frac{X_0}{U} \tag{23}$$

where U is a uniform random variable on $[0,1]$. More generally, for any non-negative supermartingale X, with a deterministic X_0, we have[1] $\mathbb{P}\left[\bar{X}_\infty \geq x\right] \leq X_0/x$, for all $x \geq X_0$. Hence, for any non-negative supermartingale X and $p > 1$

$$\mathbb{E}\left[\bar{X}_T^p\right] \leq \mathbb{E}\left[\left(\frac{X_0}{U}\right)^p\right] = \int_0^1\left(\frac{X_0}{u}\right)^p \mathrm{d}u = \frac{X_0^p}{1-p} \tag{24}$$

and (24) is attained. We now generalize (24) to a non-negative submartingale.

Proposition 2.6 (Doob Type Inequalities, $0 < p < 1$**)** *Let X be a non-negative càdlàg submartingale, $X_0 > 0$, and $p \in (0,1)$. Denote $m_r := X_0^{-r}\mathbb{E}\left[X_T^r\right]$ for $r \leq 1$. Then:*

(i) *there is a unique $\hat{\alpha} \in (0,1]$ which solves*

$$m_p\hat{\alpha}^{-p} = \frac{1-p+pm_1}{1-p+p\hat{\alpha}} \tag{25}$$

and for which we have

$$\mathbb{E}\left[\bar{X}_T^p\right] \leq X_0^p m_p\hat{\alpha}^{-p} = \frac{X_0^p}{1-p+p\hat{\alpha}} + X_0^{p-1}\frac{p}{1-p+p\hat{\alpha}}\Big(\mathbb{E}\left[X_T\right]-X_0\Big) \tag{26a}$$

$$< \frac{X_0^p}{1-p} + X_0^{p-1}\frac{p}{1-p}\Big(\mathbb{E}\left[X_T\right]-X_0\Big). \tag{26b}$$

[1] This follows by applying the optional sampling theorem at the stopping time $\inf\{t \geq 0 : X_t \notin (0,n)\}$ and using dominated convergence theorem when letting $n \to \infty$.

(ii) *there exists a martingale which attains equality in* (26a). *Further, for every* $\epsilon > 0$ *there exists a martingale such that*

$$0 \le \frac{X_0^p}{1-p} + X_0^{p-1}\frac{p}{1-p}\Big(\mathbb{E}\left[X_T\right] - X_0\Big) - \mathbb{E}\left[\bar{X}_T^p\right] < \epsilon. \tag{27}$$

Proof Following the calculations in (15), we see that

$$\mathbb{E}\left[\bar{X}_T^p\right] \le \frac{1}{1-\alpha}X_0^p + \frac{1}{(1-\alpha)(1-p)}\mathbb{E}\left[-\alpha^{1-p}X_T^p + pX_0^{p-1}X_T\right] = X_0^p f(\alpha),$$

where, with the notation m_r introduced in the statement of the Proposition,

$$f(\alpha) := \frac{1}{1-\alpha} + \frac{-\alpha^{1-p}m_p + pm_1}{(1-\alpha)(1-p)}, \qquad \alpha \in [0,1].$$

Next we prove the existence of a unique $\hat{\alpha} \in (0,1]$ such that $f(\hat{\alpha}) = \min_{\alpha\in[0,1]} f(\alpha)$. To do this, we first compute that

$$f'(\alpha) = \frac{h(\alpha)}{(1-p)(1-\alpha)^2}, \text{ where } h(\alpha) := 1-p+pm_1 - (1-p+p\alpha)m_p\alpha^{-p}.$$

By direct calculation, we see that h is continuous and strictly increasing on $(0,1]$, with $h(0+) = -\infty$ and $h(1) = 1-p+pm_1-m_p$. Moreover, it follows from the Jensen inequality and the submartingale property of X that $m_p \le m_1^p$ and $m_1 \ge 1$. This implies that $h(1) \ge 0$ since $1-p+px-x^p \ge 0$ for $x \ge 1$. In consequence, there exists $\hat{\alpha} \in (0,1]$ such that $h \le 0$ on $(0,\hat{\alpha}]$ and $h \ge 0$ on $[\hat{\alpha},1]$. This implies that f is decreasing on $[0,\hat{\alpha}]$ and increasing on $[\hat{\alpha},1]$, proving that $\hat{\alpha}$ is the unique minimizer of f.

Now the first inequality (26a) follows by plugging the equation $h(\hat{\alpha}) = 0$ into the expression for f. The bound in (26b) is then obtained by adding strictly positive terms. It also corresponds to taking $\alpha = 0$ in the expression for f. This completes the proof of the claim in (i).

As for (ii), the claim regarding a martingale attaining equality in (26a) follows precisely as in the proof of Proposition 2.3. Let $\alpha \in (0,1)$ and recall that $\tau_\alpha = \inf\{t : B_t \le \alpha\bar{B}_t\}$ for a standard Brownian motion B with $B_0 = X_0 > 0$. Then, similarly to the proof of Proposition 2.3, we compute directly

$$\mathbb{P}(\bar{B}_{\tau_\alpha} \ge y) = \mathbb{P}(B_{\tau_\alpha} \ge \alpha y) = \left(\frac{X_0}{y}\right)^{\frac{1}{1-\alpha}}, \quad y \ge X_0. \tag{28}$$

Computing and simplifying we obtain $\mathbb{E}\left[\bar{B}_{\tau_\alpha}^p\right] = \frac{1}{1-p+p\alpha}X_0^p$, and hence $\mathbb{E}\left[B_{\tau_\alpha}^p\right] = \frac{\alpha^p}{1-p+p\alpha}X_0^p$, while $\mathbb{E}\left[B_{\tau_\alpha}\right] = X_0$. It follows that $\hat{\alpha} = \alpha$ solves (25) and equality holds in (26a). Taking α arbitrarily small shows (27) holds true. □

We close this section with a new type of Doob's $L \ln L$ type of L^1 inequality obtained taking $p \nearrow 1$ in Proposition 2.6. Since $\hat{\alpha}(p)$ defined in (25) belongs to $[0,1]$ there is a converging subsequence. So without loss of generality, we may assume $\hat{\alpha}(p) \longrightarrow \hat{\alpha}(1)$ for some $\hat{\alpha}(1) \in [0,1]$. In order to compute $\hat{\alpha}(1)$, we re-write (25) into

$$\frac{g(p)-g(1)}{p-1} = m_p \text{ where } g(p) := pm_p\hat{\alpha}(p) - (1-p+pm_1)\hat{\alpha}(p)^p. \tag{29}$$

We see by a direct differentiation, invoking implicit functions theorem, that

$$g'(1) = \hat{\alpha}(1)\left(1+\mathbb{E}\left[\frac{X_T}{X_0}\ln\frac{X_T}{X_0}\right]\right) - \hat{\alpha}(1)\ln\hat{\alpha}(1)\mathbb{E}\left[\frac{X_T}{X_0}\right].$$

Then, sending $p \to 1$ in (29), we get the following equation for $\hat{\alpha}(1)$:

$$\hat{\alpha}(1)\left(1+\mathbb{E}\left[\frac{X_T}{X_0}\ln\frac{X_T}{X_0}\right]\right) = \mathbb{E}\left[\frac{X_T}{X_0}\right](1+\hat{\alpha}(1)\ln\hat{\alpha}(1)). \tag{30}$$

We note that this equation does not solve explicitly for $\hat{\alpha}(1)$. Sending $p \to 1$ in the inequality of Proposition 3.4 we obtain the following improvement to the classical Doob's $L\log L$ inequality presented in Proposition 2.5 above.

Corollary 2.7 (Improved Doob's L^1 Inequality) *Let X be a non-negative càdlàg submartingale, $X_0 > 0$. Then:*

$$\mathbb{E}\left[\bar{X}_T\right] \le \frac{\mathbb{E}[X_T]}{\hat{\alpha}} = \frac{\mathbb{E}\left[X_T \ln X_T\right] + X_0 - \mathbb{E}\left[X_T\right]\ln X_0}{1+\hat{\alpha}\ln\hat{\alpha}} \tag{31}$$

where $\hat{\alpha} \in (0,1)$ is uniquely defined by (31).

Note that the equality in (31) is a rewriting of (30). To the best of our knowledge the above inequality in (31) is new. It bounds $\mathbb{E}\left[\bar{X}_T\right]$ in terms of a function of $\mathbb{E}\left[X_T\right]$ and $\mathbb{E}\left[X_T \ln X_T\right]$, similarly to the classical inequality in (17b). However here the function depends on $\hat{\alpha}$ which is only given implicitly and not explicitly. In exchange, the bound refines and improves the classical inequality in (17b). This follows from the fact that

$$1+\alpha\ln\alpha \ge \frac{e-1}{e}, \quad \alpha \in (0,1).$$

We note also that for $X_t := B_{\frac{t}{T-t}\wedge\tau_\alpha}$, $\alpha \in (0,1)$, we have $\hat{\alpha} = \alpha$ and equality is attained in (31). This follows from the proof above or is verified directly using (28). The corresponding classical upper bound in (17b) is strictly greater expect for $\alpha = 1/e$ when the two bounds coincide.

3 Universally Best Submartingale Inequalities

As mentioned in the introduction, the novelty of our martingale inequality from Theorem 2.1 is that it uses information about the process at intermediate times. In the previous section we saw that careful choice of functions ζ in Theorem 2.1 allowed us to recover and improve the classical Doob's inequalities. In this section we study the finer structure of our class of inequalities and the question whether the information from the intermediate marginals gives us more accurate bounds than e.g. in the case when no information about the process at intermediate times is used. In short, the answer is negative, i.e. we demonstrate that for a large class of $\tilde{\boldsymbol{\zeta}}$'s there is no "universally better" choice of $\boldsymbol{\zeta}$ in the sense that it yields a tighter bound in the class of inequalities for $\mathbb{E}\left[\phi(\bar{X}_T)\right]$ from Theorem 2.1.

3.1 No Inequality is Universally Better than Other

To avoid elaborate technicalities, we impose additional conditions on $\boldsymbol{\zeta} \in \mathscr{Z}$ and ϕ below. Many of these conditions could be relaxed to obtain a slightly stronger, albeit more involved, statement in Theorem 3.1. We define

$$\mathscr{Z}^{\text{cts}} := \left\{\boldsymbol{\zeta} \in \mathscr{Z} \ : \ \boldsymbol{\zeta} \text{ are continuous}\right\} \tag{32}$$

and

$$\begin{aligned}\tilde{\mathscr{Z}} := \Big\{\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}} \ : \ &\boldsymbol{\zeta} \text{ are strictly increasing, } \liminf_{m\to\infty} \frac{\zeta_1(m)}{m} > 0, \\ &\text{and } \zeta_1 = \cdots = \zeta_n \text{ on } (X_0, X_0+\epsilon], \text{ for some } \epsilon > 0\Big\}.\end{aligned} \tag{33}$$

Before we proceed, we want to argue that the set $\tilde{\mathscr{Z}}$ arises quite naturally. In the setting of Remark 2.2, if X is a martingale such that its marginal laws

$$\mu_1 := \mathcal{L}\left(X_{t_1}\right), \quad \ldots, \quad \mu_n := \mathcal{L}\left(X_{t_n}\right)$$

satisfy Assumption ⊛ of Obłój and Spoida [18], $\int(x-\zeta)^+\mu_i(\mathrm{d}x) < \int(x-\zeta)^+\mu_{i+1}(\mathrm{d}x)$ for all ζ in the interior of the support of μ_{i+1} and their barycenter functions satisfy the mean residual value property of Madan and Yor [17] close to X_0 and have no atoms at the left end of support, then the optimization over $\boldsymbol{\zeta}$ as described in Remark 2.2 yields a unique $\tilde{\boldsymbol{\zeta}}^\star \in \tilde{\mathscr{Z}}$. Hence, the set of these $\tilde{\mathscr{Z}}$ seems to be a "good candidate set" for $\boldsymbol{\zeta}$'s to be used in Theorem 2.1.

The statement of the Theorem 3.1 concerns the negative orthant of $\mathscr{Z}^{\text{cts}}$,

$$\mathscr{Z}_{-}^{\text{cts}}(\phi, \tilde{\boldsymbol{\zeta}}) := \Big\{ \boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}} \ : \ \text{UB}\,(X, \phi, \boldsymbol{\zeta}) \leq \text{UB}\left(X, \phi, \tilde{\boldsymbol{\zeta}}\right) \text{ for all càdlàg submartingales } X \text{ and } < \text{for at least one } X \Big\}, \tag{34}$$

and hence it complements Theorem 2.1. Part (ii) in Theorem 2.1 studied sharpness of (10) for a fixed ζ with varying X while Theorem 3.1 studies (10) for a fixed X with varying ζ.

Theorem 3.1 *Let ϕ be a right-continuous, strictly increasing function. Then, for $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ such that (8) holds we have*

$$\mathscr{Z}_{-}^{\text{cts}}(\phi, \tilde{\boldsymbol{\zeta}}) = \emptyset. \tag{35}$$

The above result essentially says that no martingale inequality in (10) is universally better than another one. For any choice $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$, the corresponding martingale inequality (10) can not be strictly improved by some other choice of $\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}}$, i.e. no other $\boldsymbol{\zeta}$ would lead to a better upper bound for all submartingales and strictly better for some submartingale. The key ingredient to prove this statement is isolated in the following Proposition.

Proposition 3.2 (Positive Error) *Let $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ and $\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}}$ satisfy $\tilde{\boldsymbol{\zeta}} \neq \boldsymbol{\zeta}$. Then there exists a non-empty interval $(m_1, m_2) \subseteq (X_0, \infty)$ such that*

$$\text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \tilde{\boldsymbol{\zeta}}\right) < \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right) \textit{ for all } m \in (m_1, m_2),$$

where X is an iterated Azéma-Yor type embedding based on some $\tilde{\boldsymbol{\xi}}$.

Proof To each $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ we can associate non-decreasing and continuous stopping boundaries $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ which satisfies

$$\tilde{\zeta}_i(m) = \min_{j \geq i} \tilde{\xi}_j(m) \qquad \forall m > X_0. \tag{36}$$

Further, since $\tilde{\boldsymbol{\zeta}} \in \tilde{\mathscr{Z}}$ implies that $\tilde{\zeta}_i$ are all equal on some $(X_0, X_0 + \epsilon]$ we may take $\tilde{\boldsymbol{\xi}}$ such that

$$\tilde{\xi}_n(m) < \cdots < \tilde{\xi}_1(m) < m \qquad \forall m \in (X_0, X_0 + \epsilon), \tag{37a}$$

$$\tilde{\boldsymbol{\xi}}(m) = \tilde{\boldsymbol{\zeta}}(m) \qquad \forall m \geq X_0 + \epsilon, \tag{37b}$$

for some $\epsilon > 0$. A possible choice is given by

$$\tilde{\xi}_i(m) = \tilde{\zeta}_i(m) + (m - \tilde{\zeta}_i(m)) \frac{n-i}{n} \frac{(X_0 + \epsilon - m)^+}{\epsilon}, \quad m > X_0, \ i = 1, \ldots, n,$$

but we may take any $\tilde{\boldsymbol{\xi}}$ satisfying (36)–(37b). Let X be an iterated Azéma-Yor type embedding based on this $\tilde{\boldsymbol{\xi}}$, e.g. we may take X given by (4) since $(B_{t\wedge\tau_n} : t \geq 0)$ is uniformly integrable by the same argument as in the proof of Theorem 2.1. Let $j \geq 1$. Using the notation of Definition 1.2, it follows by monotonicity of $\tilde{\boldsymbol{\xi}}$, (37b) and (36) that on the set $\{B_{\tau_j} = \tilde{\xi}_j(\bar{B}_{\tau_j}),\ \bar{B}_{\tau_j} \geq X_0 + \epsilon\}$ we have $B_{\tau_j} = \tilde{\xi}_j(\bar{B}_{\tau_j}) \leq \tilde{\xi}_{j+1}(\bar{B}_{\tau_j})$. Therefore, the condition of (2) in the definition of the iterated Azéma-Yor type embedding is not satisfied and hence $\tau_{j+1} = \tau_j$. Consequently,

$$\begin{gathered} X_{t_j} = X_{t_{j+1}} = \cdots = X_{t_n} \quad \text{and} \quad \bar{X}_{t_j} = \bar{X}_{t_{j+1}} = \cdots = \bar{X}_{t_n} \\ \text{on the set } \left\{X_{t_j} = \tilde{\xi}_j(\bar{X}_{t_j}),\ \bar{X}_{t_j} \geq X_0 + \epsilon\right\} \end{gathered} \tag{38}$$

for all $j \geq 1$.

Take $1 \leq j \leq n$. Denote $\chi := \max\{k \leq n : \exists t \leq H_{X_0+\epsilon} \text{ s.t. } B_t \leq \tilde{\xi}_k(\bar{B}_t)\} \vee 0$, where $H_x := \inf\{u > 0 : B_u = x\}$ and $\mathcal{H} := \{\chi = j - 1,\ H_{X_0+\epsilon} < \infty\}$. By (37a) we have $\mathbb{P}[\mathcal{H}] > 0$. Further, by using $\tilde{\zeta}_1(m) \leq \cdots \leq \tilde{\zeta}_n(m) < m$ we conclude by the properties of Brownian motion that $\mathbb{P}\left[\mathcal{H} \cap \{\bar{B}_{\tau_j} \in \mathcal{O}\}\right] > 0$ for $\mathcal{O} \subseteq (X_0 + \epsilon, \infty)$ an open set. Relabelling and using (37b) yields

$$\mathbb{P}\left[X_{t_j} = \tilde{\zeta}_j(\bar{X}_{t_j}), \bar{X}_{t_j} \in \mathcal{O}, \bar{X}_{t_{j-1}} < X_0 + \epsilon\right] > 0 \text{ for all open } \mathcal{O} \subseteq (X_0 + \epsilon, \infty). \tag{39}$$

By $\tilde{\boldsymbol{\zeta}} \neq \boldsymbol{\zeta}$ either Case A or Case B below holds (possibly by changing ϵ above). In our arguments we refer to the proof of the pathwise inequality of Proposition 1.1 given by Henry-Labordère et al. [15] and argue that certain inequalities in this proof become strict.

Case A: There exist $m_2 > m_1 > X_0 + \epsilon$ and $j \leq n$ s.t. $\tilde{\zeta}_j(m_1) > \zeta_j(m_2)$. Set $\mathcal{O} := (m_1, m_2)$, and take $m > m_2$. Then, on $\left\{X_{t_j} = \tilde{\zeta}_j(\bar{X}_{t_j}),\ \bar{X}_{t_j} \in \mathcal{O}\right\}$, it follows from (38) and Proposition 1.3 that

$$\Upsilon_n(X, m, \boldsymbol{\zeta}) = \Upsilon_j(X, m, \boldsymbol{\zeta}) > 0 = \mathbb{1}_{\left\{m \leq \bar{X}_{t_j}\right\}} = \mathbb{1}_{\{m \leq \bar{X}_{t_n}\}} = \Upsilon_n(X, m, \tilde{\boldsymbol{\zeta}}), \text{ a.s.}$$

where the strict inequality holds by noting that $(X_{t_j} - \zeta_j(m))^+ > 0$ for all $m \in (m_1, m_2)$ on the above set and then directly verifying that the second inequality of Eq. (4.3) of Henry-Labordère et al. [15] applied with $\boldsymbol{\zeta}$ and X is strict.

Case B: There exist $m_2 > m_1 > X_0 + \epsilon$ and $j \leq n$ s.t. $\tilde{\zeta}_j(m_2) < \zeta_j(m_1)$. Take $m \in \mathcal{O} = (m_1, m_2)$. Then, on $\left\{X_{t_j} = \tilde{\zeta}_j(\bar{X}_{t_j}),\ \bar{X}_{t_j} \in \mathcal{O} \cap (m, \infty),\ \bar{X}_{t_{j-1}} < X_0 + \epsilon\right\}$, it follows again from (38) and Proposition 1.3 that

$$\Upsilon_n(X, m, \boldsymbol{\zeta}) = \Upsilon_j(X, m, \boldsymbol{\zeta}) > 1 = \mathbb{1}_{\left\{m \leq \bar{X}_{t_j}\right\}} = \mathbb{1}_{\{m \leq \bar{X}_{t_n}\}} = \Upsilon_n(X, m, \tilde{\boldsymbol{\zeta}}), \text{ a.s.}$$

where the strict inequality holds by observing that the last inequality in Eq. (4.3) of Henry-Labordère et al. [15] applied with $\boldsymbol{\zeta}$ and X is strict because $(X_j - \zeta_j(m))^+ = 0 > X_j - \zeta_j(m)$ for all $m \in \mathcal{O}$ on the above set.

Combining, in both cases A and B the claim (36) follows from (39). □

Proof of Theorem 3.1 Take $\boldsymbol{\zeta} \in \mathscr{Z}^{\text{cts}}$ such that strict inequality holds for one submartingale in the definition of $\mathscr{Z}_{-}^{\text{cts}}$, see (34). We must have $\boldsymbol{\zeta} \neq \tilde{\boldsymbol{\xi}}$.

As in the proof of Proposition 3.2 we choose a $\tilde{\boldsymbol{\xi}}$ such that (37a)–(37b), (36) hold and let X be an iterated Azéma-Yor type embedding based on this $\tilde{\boldsymbol{\xi}}$. Propositions 1.1 and 1.3 yield

$$\mathbb{E}\left[\mathbb{1}_{[m,\infty)}(\bar{X}_{t_n})\right] = \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \tilde{\boldsymbol{\xi}}\right) \leq \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right) \qquad \forall m > X_0$$

and by Proposition 3.2

$$\text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \tilde{\boldsymbol{\xi}}\right) < \text{UB}\left(X, \mathbb{1}_{[m,\infty)}, \boldsymbol{\zeta}\right)$$

for all $m \in \mathcal{O}$ where $\mathcal{O} \subseteq (X_0, \infty)$ is some open set. Now the claim follows as in the proof of Theorem 2.1. □

Remark 3.3 In the setting of Theorem 3.1 let $\tilde{\boldsymbol{\xi}}^1, \tilde{\boldsymbol{\xi}}^2 \in \tilde{\mathscr{Z}}$, $\tilde{\boldsymbol{\xi}}^1 \neq \tilde{\boldsymbol{\xi}}^2$, and assume that (8) holds for $(\phi, \tilde{\boldsymbol{\xi}}^1)$ and $(\phi, \tilde{\boldsymbol{\xi}}^2)$. Then there exist martingales X^1 and X^2 such that

$$\text{UB}\left(X^1, \phi, \tilde{\boldsymbol{\xi}}^1\right) < \text{UB}\left(X^1, \phi, \tilde{\boldsymbol{\xi}}^2\right),$$
$$\text{UB}\left(X^2, \phi, \tilde{\boldsymbol{\xi}}^1\right) > \text{UB}\left(X^2, \phi, \tilde{\boldsymbol{\xi}}^2\right).$$

This follows by essentially reversing the roles of $\tilde{\boldsymbol{\xi}}^1$ and $\tilde{\boldsymbol{\xi}}^2$ in the proof of Theorem 3.1.

3.2 *No Further Improvements with Intermediate Moments*

We now use the results of the previous section to show that beyond the improvement stated in Proposition 2.3 no sharper Doob's L^p bounds can be obtained from the inequalities of Theorem 2.1.

Proposition 3.4 (No Improvement of Doob's L^p-Inequality from Theorem 2.1) *Let $p > 1$ and $\tilde{\boldsymbol{\xi}} \in \tilde{\mathscr{Z}}$ be such that $\tilde{\xi}_j(m) \neq \zeta_{\frac{p-1}{p}}(m) = \frac{p-1}{p} m$ for some $m > X_0$ and* *some j. Then, there exists a martingale X such that*

$$\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] - \frac{p}{p-1}X_0^p < \mathrm{UB}\left(X, \mathrm{pow}^p, \tilde{\boldsymbol{\xi}}\right). \tag{40}$$

Proof Let $\alpha > \frac{p-1}{p} =: \alpha^\star$ and take X^α satisfying

$$0 = X^\alpha_{t_1} = \dots = X^\alpha_{t_{j-1}}, \qquad B_{\tau_\alpha} = X^\alpha_{t_j} = \dots = X^\alpha_{t_n}$$

where B is a Brownian motion started at X_0 and $\tau_\alpha = \inf\{u > 0 : B_u \le \zeta_\alpha(\bar{B}_u)\}$. It follows easily that for this process X^α,

$$\mathrm{UB}\left(X^\alpha, \mathrm{pow}^p, \tilde{\xi}_j\right) \le \mathrm{UB}\left(X^\alpha, \mathrm{pow}^p, \tilde{\boldsymbol{\xi}}\right)$$

and hence it is enough to prove the claim for $n = 1$ and $\tilde{\boldsymbol{\xi}} = \tilde{\xi}_j$.

For all $\alpha \in (\alpha^\star, \alpha^\star + \epsilon)$, $\epsilon > 0$, Proposition 3.2 yields existence of a non-empty open interval $\mathcal{I}_\alpha$ such that

$$\mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \zeta_\alpha\right) < \mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \tilde{\xi}_j\right) \text{ for all } m \in \mathcal{I}_\alpha.$$

In fact, taking $\epsilon > 0$ small enough, $\mathcal{I}_\alpha$ can be chosen such that

$$\bigcap_{\alpha\in(\alpha^\star,\alpha^\star+\epsilon)} \mathcal{I}_\alpha \quad \supseteq \quad (m_1, m_2), \qquad X_0 < m_1 < m_2. \tag{41}$$

We can further (recalling the arguments in Case A and Case B in the proof of Proposition 3.2) assume that for all $\alpha \in (\alpha^\star, \alpha^\star + \epsilon)$:

$$\mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \tilde{\xi}_j\right) - \mathrm{UB}\left(X^\alpha, \mathbb{1}_{[m,\infty)}, \zeta_\alpha\right) \ge \delta > 0 \text{ for all } m \in (m_1, m_2).$$

The claim follows by letting $\alpha \downarrow \alpha^\star$ and using the asymptotic optimality of $(X^\alpha)_\alpha$, see (13). □

In addition to the result of Proposition 3.4 we prove that there is no "intermediate moment refinement of Doob's L^p-inequalities" in the sense formalized in the next proposition. Intuitively, this could be explained by the fact that the pth moment of a continuous martingale is continuously non-decreasing and hence does not add relevant information about the pth moment of the maximum. Only the final pth moment matters in this context.

Proposition 3.5 (No Intermediate Moment Refinement of Doob's L^p-Inequality) *Let* $p > 1$, $0 = t_0 \le t_1 \le \ldots \le t_n = T$ *and* $a_1, \ldots, a_n \in \mathbb{R}$.

(i) *If* $\mathbb{E}\left[\bar{X}_T^p\right] \leq \sum_{i=1}^n a_i \mathbb{E}\left[X_{t_i}^p\right]$ *for every continuous non-negative submartingale* X *with* $X_0 = 0$*, then also*

$$\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right] \leq \sum_{i=1}^n a_i \mathbb{E}\left[X_{t_i}^p\right].$$

(i) *If* $\left(\mathbb{E}\left[\bar{X}_T^p\right]\right)^{1/p} \leq \sum_{i=1}^n a_i \left(\mathbb{E}\left[|X_{t_i} - X_{t_{i-1}}|^p\right]\right)^{1/p}$ *for every continuous non-negative submartingale* X *with* $X_0 = 0$*, then also*

$$\left(\frac{p}{p-1}\right) \left(\mathbb{E}\left[X_T^p\right]\right)^{1/p} \leq \sum_{i=1}^n a_i \left(\mathbb{E}\left[|X_{t_i} - X_{t_{i-1}}|^p\right]\right)^{1/p}.$$

Proof From Peskir [21, Example 4.1], or Proposition 2.3 above, we know that Doob's L^p-inequality given in (12b) is enforced by a sequence of continuous martingales (Y^ϵ) in the sense of (13), i.e.

$$\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|Y_T^\epsilon|^p\right] \leq \mathbb{E}\left[\max_{t \leq T} |Y_t^\epsilon|^p\right] + \frac{p}{p-1}|Y_0^\epsilon|^p + \epsilon, \quad \epsilon > 0.$$

Further, we may take Y^ϵ with $Y_0^\epsilon = 0$. We first prove (i). By scalability of the asymptotically optimal martingales (Y^ϵ) we can assume

$$\mathbb{E}\left[X_{t_n}^p\right] = \mathbb{E}\left[|Y_{t_n}^\epsilon|^p\right].$$

In addition we can find times $u_1 \leq \cdots \leq u_{n-1} \leq t_n$ such that

$$\mathbb{E}\left[X_{t_i}^p\right] = \mathbb{E}\left[|Y_{u_i}^\epsilon|^p\right], \quad 1 \leq i \leq n-1.$$

Furthermore, by a simple time-change argument, we may take $u_i = t_i$. Therefore, using asymptotic optimality of (Y^ϵ) and the assumed inequality, we have

$$\begin{aligned}\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_{t_n}^p\right] &= \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|Y_{t_n}^\epsilon|^p\right] \\ &\leq \mathbb{E}\left[\max_{t \leq T} |Y_t^\epsilon|^p\right] + \epsilon \\ &\leq \sum_{i=0}^n a_i \mathbb{E}\left[|Y_{t_i}^\epsilon|^p\right] + \epsilon \;=\; \sum_{i=}^n a_i \mathbb{E}\left[X_{t_i}^p\right] + \epsilon.\end{aligned}$$

We obtain the required inequality by sending $\epsilon \searrow 0$ in the above.

We next prove (ii). Taking a martingale which is constant until time t_{i-1} and constant after time t_i and using the fact that Doob's L^p inequality is sharp yields

$$\left(\frac{p}{p-1}\right) \leq a_i \text{ for all } i = 1, \ldots, n.$$

The required inequality follows using triangular inequality for the L^p norm. □

Remark 3.6 It follows from the above proof that we may also formulate Proposition 3.5(i) in terms of L^p norms instead of the expectations of the p-th moment. Also, analogous statements as in Proposition 3.5 hold for Doob's L^1 inequality. This can be argued in the same way by using that Doob's L^1 inequality is attained (cf. e.g. Peskir [21, Example 4.2] or Proposition 2.5 above), and observing that the function $x \mapsto x \log(x)$ is convex.

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Polynomials Associated with Finite Markov Chains

Philippe Biane

Abstract Given a finite Markov chain, we investigate the first minors of the transition matrix of a lifting of this Markov chain to covering trees. In a simple case we exhibit a nice factorisation of these minors, and we conjecture that it holds more generally.

1 Introduction

The famous matrix-tree theorem of Kirchhoff gives a combinatorial formula for the invariant measure of a finite Markov chain in terms of covering trees of the state space of the chain. One can provide a probabilistic interpretation of Kirchhoff's formula by lifting the Markov chain to the set of covering trees of its state space, see e.g. [1] or [2], Sect. 4.4. This yields a new Markov chain, whose transition matrix can be constructed from the transition matrix of the original Markov chain. In this paper, we investigate the first minors of this new matrix, which are polynomials in the entries of the original transition matrix. We will see that in a simple case, that of a Markov chain evolving on a ring, these polynomials exhibit a remarkable factorisation. We expect that such factorisations hold in a much more general context. This paper is organized as follows: we start in Sect. 2 by recalling some general facts about finite Markov chains and their invariant measure. In Sect. 3 we describe how to lift the Markov chain to its set of covering trees. In Sect. 4 we introduce a polynomial associated to the Markov chain, and show that in the case of a Markov chain with three states it has a nice factorisation. We generalize this observation to the case of Markov chains on a ring in Sect. 5, which contains the main result of the paper.

I would like to thank Jim Pitman for pointing out reference [2] to me.

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2 Finite Markov Chains and Invariant Measures

We start by recalling some well known facts about finite Markov chains.

2.1 Transition Matrix

We consider a continuous time Markov chain M on a finite set X. Let $Q = (q_{ij})_{i,j\in X}$ be its matrix of transition rates: $q_{ij} \geq 0$ if $i \neq j \in X$ and $\sum_j q_{ij} = 0$ for all i.

2.2 Invariant Measure

An invariant measure for M (more exactly, for Q) is a nonzero vector $\mu(i), i \in X$, with nonnegative entries such that $\sum_i \mu(i)q_{ij} = 0$ for all $j \in X$. An invariant measure always exists, it is unique up to a multiplicative constant if the chain is irreducible.

2.3 Projection of a Markov Chain

Let N be a Markov chain on a finite state space Y, with transition matrix $R = (r_{kl})_{k,l\in Y}$, and $p : Y \to X$ be a map such that, for all $i, j \in X$ and all $k \in Y$ such that $p(k) = i$, one has

$$q_{ij} = \sum_{l\in p^{-1}(j)} r_{kl} \tag{1}$$

then $p(N)$ is a Markov chain on X with transition rates q_{ij}. Furthermore, if ν is an invariant measure for R, then μ defined as

$$\mu(i) = \sum_{k\in p^{-1}(i)} \nu(k) \tag{2}$$

is an invariant measure for Q.

2.4 Oriented Graph and Covering Trees

To the matrix Q is associated a graph (X, E) with X as vertex set, and E as edge set, such that there is an edge from i to j if and only if $q_{ij} > 0$. This graph is oriented, has no multiple edges, and no loops (edges which begin and end at the same vertex). Let $i \in X$, a *covering tree of* (X, E)*, rooted at* i is an oriented subgraph of (X, E) which

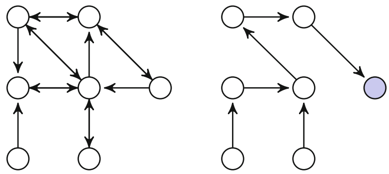


Fig. 1 An oriented graph, and a covering rooted tree

is a tree and such that, for every $j \in X$, there is a unique path from j to i in the graph (paths are oriented). The Markov chain is irreducible if and only if for all $i, j \in X$ there exists a path from i to j in the graph (X, E). If this is the case then for every vertex $i \in X$ there exists a covering tree rooted at i.

Figure 1 shows an oriented graph, together with a covering tree rooted at the shaded vertex (beware that a Markov chain corresponding to this graph is not irreducible).

2.5 Kirchhoff's Matrix Tree Theorem

We assume that the Markov chain is irreducible. For $i \in X$ let $Q^{(i)}$ be the matrix obtained from Q by deleting row and column i and let $\mu(i) = \det(-Q^{(i)})$, then it is well known, and easy to see that μ is an invariant measure for Q. Indeed, if $Q^{(ij)}$ is obtained by deleting row i and column j, then $\det(-Q^{(ij)}) = \det(-Q^{(ii)}) = \det(-Q^{(i)})$, since the sum of each line is 0, and $\det(-Q) = \sum_i q_{ij} \det(-Q^{(ij)}) = 0$ for all j, by expanding the determinant along columns. That μ has positive entries follows from irreducibility and Kirchhoff's formula:

$$\mu(i) = \sum_{t \in T_i} \pi(t) \tag{3}$$

where the sum is over the set T_i of oriented covering trees of X, rooted at i, and $\pi(t)$ is the product of the q_{kl} over all oriented edges (k, l) of the tree t. See [2], Sect. 4. More generally, if $\{i_1, \ldots, i_k\} \subset X$, then Kirchhoff's formula also applies to the determinant of the matrix obtained from Q by deleting columns and rows indexed by $i_1, \ldots, i_k$. This determinant is equal, up to a sign, to the sum over oriented covering forests, rooted at $i_1, \ldots, i_k$, of the product over edges of the forest.

3 Lifting the Markov Chain to Its Covering Trees

3.1 The Lift

Notations are as in the preceding section, furthermore we assume that Q is irreducible. The set of oriented covering rooted trees of (X, E) is $T = \cup_{i \in X} T_i$. Let the map $p : T \to X$ assign to each tree t its root (i.e. p maps T_i to i). There exists

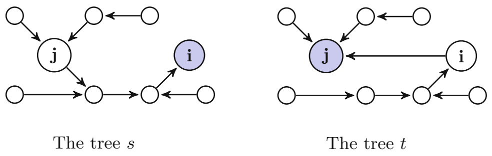


Fig. 2 Lifting a transition between i and j

an irreducible Markov chain on T whose image by p is a Markov chain on X with transition rates Q, and the vector $(\pi(t))_{t\in T}$ is an invariant measure for this Markov chain. In particular by (2) the invariant measure π projects by p to the invariant measure μ and this construction provides a probabilistic interpretation of Kirchhoff's formula (3). This Markov chain can be described by its transition rates $r_{st}, s, t \in T$. Let s be a covering tree of X, rooted at i, and let $j \in X$ be such that $q_{ij} > 0$. There is a unique edge of s coming out of j. Take out this edge from s and then add the edge (i, j). One obtains a new oriented tree t, rooted at j (see Fig. 2 for an example). One puts then $r_{st} = q_{ij}$. For all pairs $s \neq t$ which are not obtained by this construction, one puts $r_{st} = 0$. This defines a unique matrix of transition rates $(r_{st})_{s,t\in T}$.

It is clear that these transitions define a Markov chain which projects onto M by the map p.

Theorem 1 *The Markov chain with transition rates R is irreducible, and the vector π is an invariant measure for this Markov chain.*

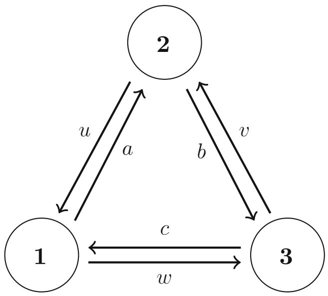
The proof can be found in [1].

3.2 An Example

Let $X = \{1, 2, 3\}$ and

$$Q = \begin{pmatrix} \lambda & a & w \\ u & \mu & b \\ c & v & \nu \end{pmatrix}$$

with $\lambda = -a - w$, $\mu = -b - u$, $\nu = -c - v$. We assume that $a, b, c, u, v, w > 0$. The graph (X, E) looks as follows:



Each covering rooted tree t can be indexed by the monomial $\pi(t)$. There are nine such covering trees: first cu, uv, bc rooted at 1, then av, ac, vw rooted at 2, and finally uw, bw, ab rooted at 3. With this ordering of T, the transition matrix for the lifted Markov chain is

$$R = \begin{pmatrix} \lambda & 0 & 0 & 0 & a & 0 & w & 0 & 0 \\ 0 & \lambda & 0 & a & 0 & 0 & w & 0 & 0 \\ 0 & 0 & \lambda & 0 & a & 0 & 0 & w & 0 \\ \\ 0 & u & 0 & \mu & 0 & 0 & 0 & 0 & b \\ u & 0 & 0 & 0 & \mu & 0 & 0 & 0 & b \\ 0 & u & 0 & 0 & 0 & \mu & 0 & b & 0 \\ \\ c & 0 & 0 & 0 & 0 & v & \nu & 0 & 0 \\ 0 & 0 & c & 0 & 0 & v & 0 & \nu & 0 \\ 0 & 0 & c & v & 0 & 0 & 0 & 0 & \nu \end{pmatrix}$$

Figure 3 shows the oriented graph. We have shown, for each vertex, its projection onto X (namely **1**, **2**, or **3**) and for each oriented edge, its weight (a, b, c, u, v or w).

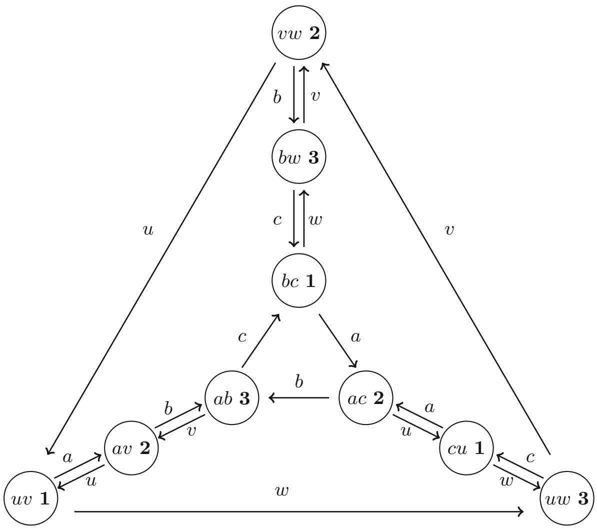


Fig. 3 The graph T

4 A Polynomial Associated to the Markov Chain

4.1 *The Polynomial*

We consider, as in the previous sections, an irreducible Markov chain on a finite set X with transition matrix Q and its canonical lift to T, with transition matrix R. For $t \in T$, consider the matrix $R^{(t)}$ obtained from R by taking out row and column t, and let $\rho(t) = \det(-R^{(t)})$, then ρ is an invariant measure for R, and gives a generating function for covering trees of the graph T. If we fix the graph (X, E), then $\rho(t)$ is a polynomial in the variables q_{ij}, where we keep only the pairs (i, j) forming an edge in E. Since π and ρ are invariant measures of the lifted Markov chain, they are proportional so that there exists there exists a function, $\Psi(q_{ij})$, independent of t, such that for all $t \in T$,

$$\rho(t) = \pi(t)\Psi$$

Actually it is not difficult to see that $\Psi(q_{ij})$ is a polynomial. Indeed one has $\Psi = \rho(t)/\pi(t)$, and $\pi(t)$ is a monomial so that, by reducing, $\Psi = P/m$ with P a polynomial and m a monomial prime with P. In particular, $\rho(t) = \pi(t)P/m$ is a polynomial for all t, hence m divides $\pi(t)$ for all t. But the $\pi(t)$ have no common divisor, since a variable q_{kl} cannot divide $\pi(t)$ is t is rooted at k, therefore $m = 1$.

4.2 *Some Examples*

If $|X| = 3$, with the notations of Sect. 3.2, one can compute

$$\begin{aligned}\Psi(a, b, c, u, v, w) &= (bc + cu + uv)(av + ac + vw)(ab + bw + uw)\\ &= \prod_{i\in X}\left(\sum_{t\in T_i}\pi(t)\right)\end{aligned}$$

so that Ψ is the product of all symmetric rank two minors of the matrix $-Q$ (a *symmetric minor of rank k* of a matrix of size n, is the determinant of a submatrix obtained by deleting $n-k$ rows and the $n-k$ columns with the same indices). I have computed the polynomial Ψ for various graphs with four vertices and found in many cases that Ψ can be written as a product of symmetric minors of the matrix $-Q$. I could not compute in the case of $|X| = 4$ and the graph (X, E) is a complete graph, but by putting some of the variables equal to 1 to make the determinant easier to compute, the results suggest that the formula for Ψ in this case should be

$$\Psi = m_2(Q)^3 m_3(Q)^2$$

where $m_k(Q)$ is the product of all symmetric minors of rank k of $-Q$.

Based on this small evidence it seems natural to conjecture that for any irreducible graph (X, E) the polynomial Ψ should be a product of symmetric minors of the matrix $-Q$. Which minors appear, and what are their exponents, should depend on the graph and encode some of its geometry. By symmetry, in the case of a complete graph on n vertices, the result should be a product $\prod_{k=1}^{n-1} m_k^{v_k^n}$ for some exponents v_k^n. Guillaume Chapuy (private communication, October 2014) has done some further computations for $n = 5$ and conjectured that $v_k^n = (k-1)(n-1)^{n-k-1}$. One can check that, at least, this gives the correct degree. In general the degree of Ψ is $|T| - n$, and in the case of a complete graph, $|T| = n^{n-1}$, moreover there are $\binom{n}{k}$ symmetric minors of rank k, which are polynomials of degree k, and

$$\sum_{k=2}^{n-1} \binom{n}{k} k(k-1)(n-1)^{n-k-1} = n^{n-1} - n$$

as follows easily from the binomial formula.

In the following I obtain a result for the case where the graph is a ring: $X = \{1, 2, \ldots, n\}$ and the edges are $(i, i \pm 1)$ (where $i \pm 1$ is taken modulo n).

Theorem 2 *If (X, E) is a ring of size $n \geq 3$, then Ψ is the product of the symmetric minors of size $n - 1$:*

$$\Psi = m_{n-1}(Q)$$

The proof of Theorem 2, which is the main result of this paper, occupies the next section.

5 Proof of Theorem 2

In this section, (X, E) denotes a ring, namely, $X = \{1, 2, \ldots, n\}$ and the edges are $(i, i \pm 1)$ (here and in the sequel $i \pm 1$ is always taken modulo n). I will illustrate this with $n = 4$, as in Fig. 4.

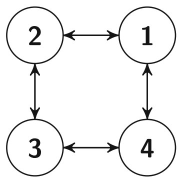
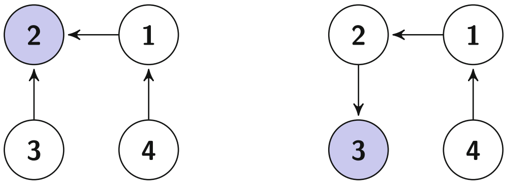


Fig. 4 The ring (X, E) with $n = 4$

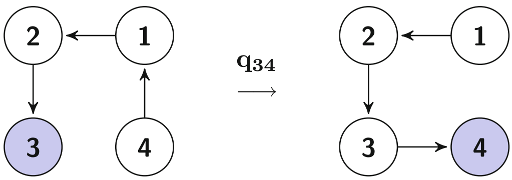
5.1 *Structure of the Graph T*

For each pair $(i,j) \in X^2$ there exists a unique covering tree of (X,E), rooted at i, which has no edge between j and $j+1$. Let us denote this covering rooted tree by $[i,j]$. For example, if $n = 4$ here are the trees denoted by, respectively, $[2,3]$ and $[3,3]$ (here and in the sequel the roots are shaded):

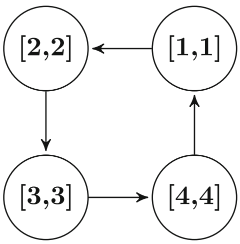


It is easy to check that these are all covering rooted trees of (X,E), in particular $|T| = n^2$. Let us now describe the structure of the graph on T induced by the lifting of the Markov chain.

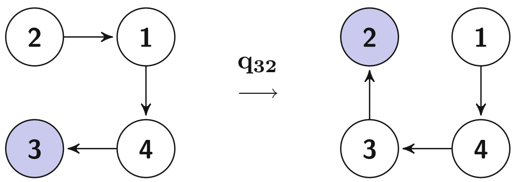
First consider the trees indexed by the pairs $[i,i]$. The trees $[i,i]$ and $[i+1,i+1]$ are connected by an edge labelled $q_{i,i+1}$ e.g.



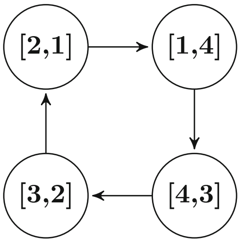
These trees form an oriented ring in T:



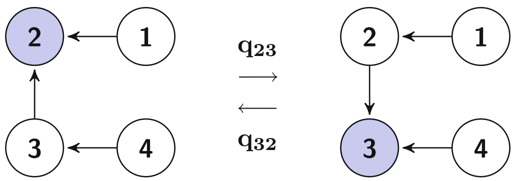
The trees indexed by pairs $[i, i-1]$ are connected by edges labelled $q_{i,i-1}$:



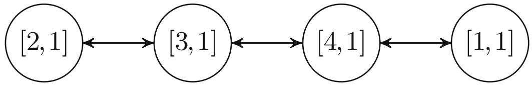
They form another oriented ring:



There are also edges in the two directions between $[i, j]$ and $[i+1, j]$, labelled by $q_{i,i+1}$ and $q_{i+1,i}$:



These form lines of length n:



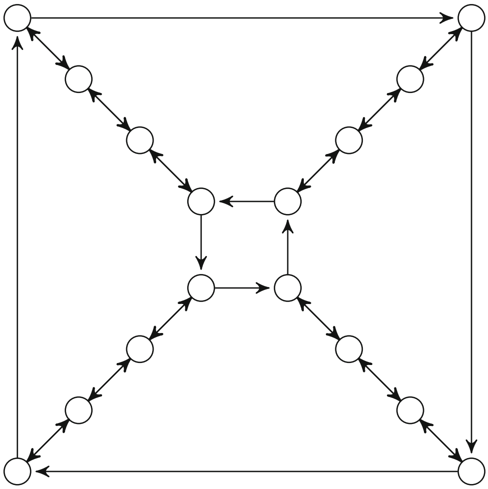


Fig. 5 The graph of T for $n = 4$

One can represent the graph T by putting two concentric oriented rings of size n, with opposite orientations, and joining the vertices of the rings by sequences of vertices connected by double edges, see Fig. 3 for $n = 3$ and Fig. 5 for $n = 4$:

5.2 The Symmetric $n - 1$ Minors of $-Q$

We will use the following lemma.

Lemma 1 *The symmetric $n - 1$ minors of $-Q$ are prime polynomials.*

Proof Let $i \in Q$, then $\det(-Q^{(i)})$, the symmetric $n - 1$ minor corresponding to i is a polynomial with degree at most one in each variable. More precisely, using Kirchhoff's formula this minor is the generating function of covering trees rooted at i and it can be written as $\alpha q_{i-1,i} + \beta$ where α and β are polynomials of degree 0 in $q_{i-1,i}$. Moreover β is a monomial since there exists a unique covering tree of (X, E) rooted at i which does not contain the edge $(i - 1, i)$. It follows that any nontrivial factorisation of this polynomial can be written as

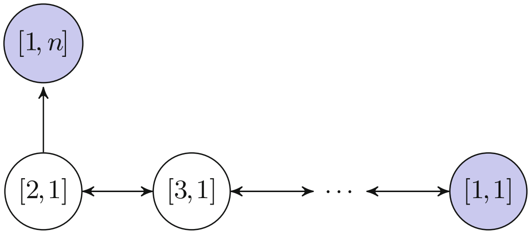
$$\alpha q_{i-1,i} + \beta = (\gamma q_{i-1,i} + \delta)\eta \tag{4}$$

where γ, δ, η have degree 0 in $q_{i-1,i}$ and $\eta\delta = \beta$. In particular, η is a nontrivial monomial, therefore there exists a variable q_{kl} which divides $\alpha q_{i-1,i} + \beta$, and this means that the edge (k, l) belongs to all covering trees rooted at i. Clearly this is not possible, therefore a nontrivial factorisation such as (4) does not exist, and the symmetric minor is a prime polynomial.

5.3 A Preliminary Lemma

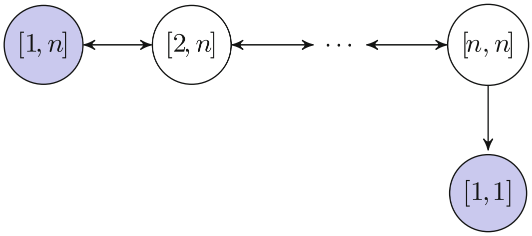
Consider the restriction of the graph T to the sets of vertices

$$G = \{[1,n],[2,1],[3,1],\ldots,[n,1],[1,1]\}$$



and

$$H = \{[1,n],[2,n],[3,n],\ldots,[n,n],[1,1]\}$$



We will need the following lemma.

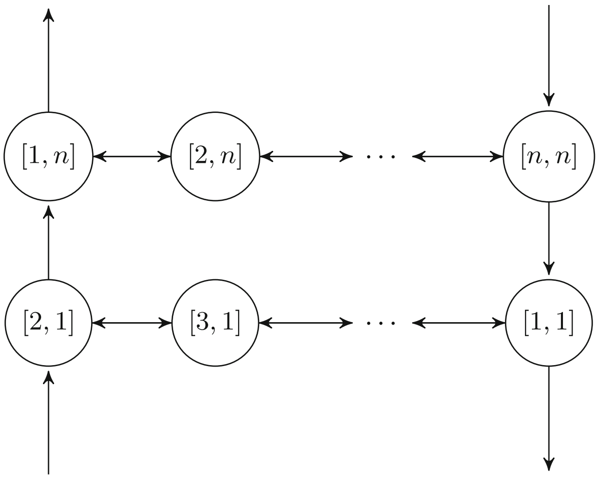
Lemma 2 *The generating function of the set of covering forests of G, rooted at* $[1,n]$ *and* $[1,1]$ *is equal to* $\det(-Q^{(1)})$*, the generating function for the set of covering trees of X, rooted at 1. The same is true with H instead of G.*

Proof One can check easily that the restriction of the projection p to G induces a bijection between the covering forests of G rooted at $[1,n]$ and $[1,1]$ and the covering trees of X rooted at 1 (observe that $[1,n]$ and $[1,1]$ both project to 1), and this bijection preserves the labels of the edges. The same is true for H and the lemma follows.

5.4

We will now prove that the symmetric minor $\det(-Q^{(i)})$ divides the symmetric minor $\det(-R^{([i,i])})$. By symmetry it is enough to prove this for $i = 1$. By Kirchhoff's formula, we know that the polynomial $\det(-R^{([1,1])})$ is the generating polynomial of the covering trees of T rooted at vertex $[1, 1]$.

Let $K = G \cup H$ and let $L = T \setminus K$. The part of the graph T containing K looks like



Observe that the only way one can enter the set K by a path coming from L is through the vertices $[2, 1]$ or $[n, n]$. Let now τ be a covering tree of T, rooted at $[1, 1]$. If we consider the set of vertices $L \cup \{[2, 1], [n, n]\}$ together with the edges of τ coming out of elements of L, we obtain two disjoint trees, rooted respectively at $[n, n]$ and $[2, 1]$. Let us now fix such a pair of trees A and B, and consider the set of covering trees τ of T, rooted at $[1, 1]$, which induce the pair (A, B). There are three possibilities for the edge coming out of $[1, n]$ in such a tree:

(i) it connects to $[2, n]$
(ii) it connects to $[n, n-1]$ which belongs to A
(iii) it connects to $[n, n-1]$ which belongs to B.

If we are in the first case then the restriction of the tree to G forms a covering forest of G, rooted at $[1, n]$ and $[1, 1]$. Furthermore any such forest can occur, independently of the trees A and B. It follows that the generating function of trees in case (i) is a multiple of the generating function of such covering forests, which is $\det(-Q^{(1)})$ by Lemma 2.

In case (ii) the same argument as in (i) can be applied, so we conclude again that the generating function of such trees is a multiple of $\det(-Q^{(1)})$.

Finally in case (iii) the edge $([2,1],[1,n])$ cannot belong to the tree, but a similar reasoning, this time with H instead of G, shows that the generating function of such trees is a again multiple of $\det(-Q^{(1)})$.

From this, summing over all three cases, and all pairs (A, B) we conclude that $\det(-R^{([1,1])})$, the generating function of the set of covering trees of T, rooted at $[1,1]$, is a multiple of $\det(-Q^{(1)})$. Since $\det(-R^{([1,1])}) = \pi([1,1])\Psi$ and $\pi([1,1])$ is a monomial which is prime with $\det(-Q^{(1)})$ it follows that $\det(-Q^{(1)})$ divides the polynomial Ψ. By symmetry, this is true of all the $\det(-Q^{(i)})$, for $i \in X$ and since these are distinct prime polynomials, we conclude that Ψ is a multiple of $m_{n-1} = \prod_i \det(-Q^{(i)})$. The degree of the polynomial $\det(-R^{([1,1])})$ is $n^2 - 1$, the degree of m_{n-1} is $n(n-1)$ and the degree of $\pi([1,1])$ is $n-1$. It follows that Ψ and m_{n-1} are proportional.

In order to find the constant of proportionality, we consider the generating function of the covering trees of T, rooted at $[n,n]$. This generating function is $\det(-R^{([n,n])}) = \pi([n,n])\Psi$. I claim that the coefficient of the monomial

$$q_{n1}^{n-1} \prod_{i=1}^{n-1} q_{i,i+1}^{n} \tag{5}$$

in $\det(-R^{([n,n])})$ is 1. Indeed for each $i \leq n$ there are exactly n edges in T which are labelled $q_{i,i+1}$, and one of the edges labelled q_{n1} goes out of $[n,n]$ so it cannot belong to a tree rooted at $[n,n]$, therefore there exists at most one covering tree rooted at $[n,n]$ whose product over labelled edges is equal to (5). On the other hand, one can check that, taking the graph formed with all these edges, one obtains a covering tree rooted at $[n,n]$, see e.g. Fig. 6 for the case of $n = 4$.

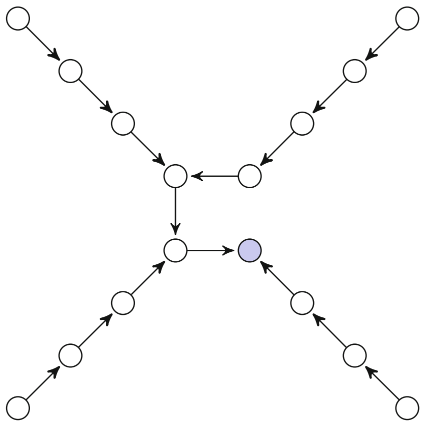


Fig. 6 The covering tree for $n = 4$

It remains now to check that the coefficient of $\pi([n,n]) \prod_i \det(-Q^{(i)})$ is 1. This follows from the fact that for each i there exists a unique covering tree of X rooted at i, whose labels are all of the form $q_{k,k+1}$. Taking the product over these trees one recovers the product (5).

This completes the proof of Theorem 2. □

5.5 Final Remark

If we look at formula

$$\det(-R^{([n,n])}) = \pi([n,n]) \prod_{i=1}^{n} \det(-Q^{(i)})$$

there is a combinatorial significance for both sides of the equality. The left hand sides is the generating function for covering trees of T rooted a $[n,n]$ whereas the right hand side is the generating function of the n-tuples of rooted covering trees of (X,E) rooted at $1,2,\ldots,n$. It would be interesting to transform our proof of this formula into a bijective proof by exhibiting a bijection between these two sets which respects the weights. This could shed some light on the general case.

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On σ-Finite Measures Related to the Martin Boundary of Recurrent Markov Chains

Joseph Najnudel

Abstract In our monograph with Roynette and Yor (Najnudel et al., *A Global View of Brownian Penalisations*, MSJ Memoirs, vol. 19, Mathematical Society of Japan, Tokyo, 2009), we construct a σ-finite measure related to penalisations of different stochastic processes, including the Brownian motion in dimension 1 or 2, and a large class of linear diffusions. In the last chapter of the monograph, we define similar measures from recurrent Markov chains satisfying some technical conditions. In the present paper, we give a classification of these measures, in function of the minimal Martin boundary of the Markov chain considered at the beginning. We apply this classification to the examples considered at the end of Najnudel et al. (*A Global View of Brownian Penalisations*, MSJ Memoirs, vol. 19, Mathematical Society of Japan, Tokyo, 2009).

1 Introduction

In a number of articles by Roynette, Vallois and Yor, summarized in [14], the authors study many examples of probability measures on the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$, which are obtained as weak limits of absolutely continuous measures, with respect to the law of the Brownian motion. More precisely, one considers the Wiener measure $\mathbb{W}$ on the space $\mathscr{C}(\mathbb{R}_+, \mathbb{R})$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$, and endowed with its canonical filtration $(\mathscr{F}_s)_{s\geq 0}$, and the following σ-algebra

$$\mathscr{F} := \bigvee_{s\geq 0} \mathscr{F}_s.$$

One then considers $(\Gamma_t)_{t\geq 0}$, a family of nonnegative random variables on $\mathscr{C}(\mathbb{R}_+, \mathbb{R})$, such that

$$0 < \mathbb{E}_{\mathbb{W}}[\Gamma_t] < \infty,$$

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and for $t \geq 0$, one defines the probability measure

$$\mathbb{Q}_t := \frac{\Gamma_t}{\mathbb{E}_{\mathbb{W}}[\Gamma_t]} .\mathbb{W}.$$

Under these assumptions, Roynette, Vallois and Yor have shown that for many examples of families of functionals $(\Gamma_t)_{t\geq 0}$, one can find a probability measure $\mathbb{Q}_\infty$ satisfying the following property: for all $s \geq 0$ and for all events $\Lambda_s \in \mathscr{F}_s$,

$$\mathbb{Q}_t[\Lambda_s] \underset{t\to\infty}{\longrightarrow} \mathbb{Q}_\infty[\Lambda_s].$$

In our monograph with Roynette and Yor [11], Chap. 1, we show that for a large class of functionals $(\Gamma_t)_{t\geq 0}$, the measure $\mathbb{Q}_\infty$ exists and is absolutely continuous with respect to a σ-finite measure **W**, which is explicitly described and which satisfies some remarkable properties. In Chaps. 2–4 of the monograph, we construct an analog of the measure **W**, respectively for the two-dimensional Brownian motion, for a large class of linear diffusions, and for a large class of recurrent Markov chains. In a series of papers with Nikeghbali (see [8, 10]), we generalize the construction to submartingales $(X_s)_{s\geq 0}$ satisfying some technical conditions we do not detail here, and such that $X_s = N_s + A_s$, where $(N_s)_{s\geq 0}$ is a càdlàg martingale, $(A_s)_{s\geq 0}$ is an increasing process, and the measure (dA_s) is carried by the set $\{s \geq 0, X_s = 0\}$. This class of submartingales, called (Σ), was first introduced by Yor in [18], and their main properties have been studied in detail by Nikeghbali in [12].

In the present paper, we focus on the setting of the recurrent Markov chains, stated in Chap. 4 of [11]. Our main goal is to classify the σ-finite measures which can be obtained by the construction given in the monograph. In Sect. 2, we summarize the most important ideas of this construction, and we state some of the main properties of the corresponding σ-finite measures. In Sect. 3, we show that these measures can be classified via the theory of Martin boundary, adapted to the case of recurrent Markov chains. In Sect. 4, we study the behavior of the canonical trajectory under some particular measures deduced from the classification given in Sect. 3. In Sect. 5, we apply our results to the examples considered at the end of our monograph [11].

2 The Main Setting

Let E be a countable set, $(X_n)_{n\geq 0}$ the canonical process on $E^{\mathbb{N}_0}$, $(\mathscr{F}_n)_{n\geq 0}$ its natural filtration, and $\mathscr{F}_\infty$ the σ-algebra generated by $(X_n)_{n\geq 0}$. We define $(\mathbb{P}_x)_{x\in E}$ as a family of probability measures on the filtered measurable space $(E^{\mathbb{N}_0}, (\mathscr{F}_n)_{n\geq 0}, \mathscr{F}_\infty)$ which corresponds to a Markov chain, i.e. there exists a family $(p_{y,z})_{y,z\in E}$ of elements in $[0, 1]$ such that for all $k \geq 0$, $x_0, \ldots, x_k \in E$,

$$\mathbb{P}_x(X_0 = x_0, X_1 = x_1, \ldots, X_k = x_k) = \mathbb{1}_{x_0=x} p_{x_0,x_1} p_{x_1,x_2} \ldots p_{x_{k-1},x_k}.$$

The expectation under $\mathbb{P}_x$ will be denoted $\mathbb{E}_x$. Moreover, we assume the following properties:

- For all $x \in E$, $p_{x,y} = 0$ for all but finitely many $y \in E$.
- The Markov chain is irreducible, i.e. for all $x, y \in E$, there exists $n \geq 0$ such that $\mathbb{P}_x(X_n = y) > 0$.
- The Markov chain is recurrent, i.e. for all $x \in E$, $\mathbb{P}_x\left(\sum_{n\geq 0} \mathbb{1}_{X_n=x} = \infty\right) = 1$.

Using the results in Chap. 4 of [11], the following proposition is not difficult to prove:

Proposition 1 *Let $x_0 \in E$, and let φ be a function from E to $\mathbb{R}_+$, such that $\varphi(x_0) = 0$, and φ is harmonic everywhere except at x_0, i.e. for all $x \neq x_0$,*

$$\mathbb{E}_x[\varphi(X_1)] = \sum_{y\in E} p_{x,y}\varphi(y) = \varphi(x).$$

Then, there exists a family of σ-finite measures $(\mathbb{Q}_x^{x_0,\varphi})_{x\in E}$ on $(E^{\mathbb{N}_0}, \mathscr{F}_\infty)$ satisfying the following properties:

- *For all $x \in E$, the canonical process starts at x under $\mathbb{Q}_x^{x_0,\varphi}$, i.e.*

$$\mathbb{Q}_x^{x_0,\varphi}\,(X_0 \neq x) = 0.$$

- *For all $x \in E$, the canonical process is transient under $\mathbb{Q}_x^{x_0,\varphi}$, i.e. for all $x, y \in E$,*

$$\mathbb{Q}_x^{x_0,\varphi}\left(\sum_{n\geq 1} \mathbb{1}_{X_n=y} = \infty\right) = 0.$$

- *For all $n \geq 0$, for all nonnegative, $\mathscr{F}_n$-measurable functionals F_n,*

$$\mathbb{Q}_x^{x_0,\varphi}\left(F_n\,\mathbb{1}_{\forall k\geq n, X_k\neq x_0}\right) = \mathbb{E}_x[F_n\varphi(X_n)],$$

  where $\mathbb{Q}_x^{x_0,\varphi}(H)$ denotes the integral of H with respect to $\mathbb{Q}_x^{x_0,\varphi}$.

Moreover, the two last items are sufficient to determine uniquely the measure $\mathbb{Q}_x^{x_0,\varphi}$.

Remark 1 If we refer to our joint work with Nikeghbali [9], we observe that under $\mathbb{P}_x$, $(\varphi(X_n))_{n\geq 0}$ is a the discrete-time submartingale of class (Σ), as stated in Theorem 3.5 of [9]. Moreover, one checks that $\mathbb{Q}_x^{x_0,\varphi}$ is the corresponding σ-finite measure, denoted $\mathscr{Q}$ in [9], as soon as $\varphi(y) > 0$ for all $y \neq x_0$.

Proof The second and third items are respectively Proposition 4.2.3 and Corollary 4.2.6 of [11]. The first item is a consequence of the second and the third. Indeed, by taking $F_n = \mathbb{1}_{X_0\neq x}$, we get for all $n \geq 0$,

$$\mathbb{Q}_x^{x_0,\varphi}\left(\mathbb{1}_{X_0\neq x, \forall k\geq n, X_k\neq x_0}\right) = \mathbb{E}_x[\varphi(X_n)\mathbb{1}_{X_0\neq x}] = 0,$$

and then, by taking the union for all $n \geq 0$,

$$\mathbb{Q}_x^{x_0,\varphi}\left(X_0 \neq x, \exists n \geq 0, \forall k \geq n, X_k \neq x_0\right)$$
$$= \mathbb{Q}_x^{x_0,\varphi}\left(X_0 \neq x, \sum_{k \geq 0} \mathbb{1}_{X_k = x_0} < \infty\right) = 0,$$

and then

$$\mathbb{Q}_x^{x_0,\varphi}(X_0 \neq x) = 0$$

by the transience of the canonical process under $\mathbb{Q}_x^{x_0,\varphi}$.

It remains to prove that the second and the third items uniquely determine $\mathbb{Q}_x^{x_0,\varphi}$. From the third item, we have for all $n \geq m \geq 0$,

$$\mathbb{Q}_x^{x_0,\varphi}\left(F_n \mathbb{1}_{\forall k \geq m, X_k \neq x_0}\right) = \mathbb{Q}_x^{x_0,\varphi}\left(F_n \mathbb{1}_{\forall k \in \{m,m+1,\ldots,n-1\}, X_k \neq x_0} \mathbb{1}_{\forall k \geq n, X_k \neq x_0}\right)$$
$$= \mathbb{E}_x[F_n \mathbb{1}_{\forall k \in \{m,m+1,\ldots,n-1\}, X_k \neq x_0} \varphi(X_n)].$$

Moreover,

$$\mathbb{Q}_x^{x_0,\varphi}\left(\forall k \geq m, X_k \neq x_0\right) = \mathbb{E}_x[\varphi(X_m)] < \infty,$$

since $\varphi(X_m)$ is almost surely in a finite subset of E. Indeed, by assumption, for all $y \in E$, there exist only finitely many $z \in E$ such that $p_{y,z} > 0$. Hence, the measure

$$\mathbb{1}_{\forall k \geq m, X_k \neq x_0} \cdot \mathbb{Q}_x^{x_0,\varphi}$$

is finite and uniquely determined for all sets in $\mathscr{F}_n$, for all $n \geq 0$. By monotone class theorem, this measure is uniquely determined. Taking the increasing limit for $m \to \infty$, the measure

$$\mathbb{1}_{\exists m \geq 0, \forall k \geq m, X_k \neq x_0} \cdot \mathbb{Q}_x^{x_0,\varphi}$$

is also uniquely determined. Now, the property of transience which is assumed implies that this measure is $\mathbb{Q}_x^{x_0,\varphi}$.

The homogeneity of the Markov chain can be stated as follows: for all $n \geq 1$, $x, y \in E$,

$$\mathbb{1}_{X_n = y} \cdot \mathbb{P}_x = \mathbb{1}_{X_n = y} \cdot (\mathbb{P}_x^{(n)} \circ \mathbb{P}_y),$$

where $\mathbb{P}^{(n)} \circ \mathbb{Q}$ denotes the image of $\mathbb{P} \otimes \mathbb{Q}$ by the map from $E^{\mathbb{N}_0} \times E^{\mathbb{N}_0}$ to $E^{\mathbb{N}_0}$, given by

$$((x_0, x_1, x_2, \ldots), (y_0, y_1, y_2 \ldots)) \mapsto (x_0, x_1, x_2, \ldots, x_n, y_1, y_2, y_3, \ldots).$$

In other words, conditionally on $X_n = y$, the canonical trajectory under $\mathbb{P}_x$ has the same law as the concatenation of the n first steps of the canonical trajectory under $\mathbb{P}_x$, and an independent trajectory following $\mathbb{P}_y$.

The following result shows that the family of measures $(\mathbb{Q}_x^{x_0,\varphi})_{x\in E}$ satisfies a similar property: informally, it can be obtained from $(\mathbb{P}_x)_{x\in E}$ and $(\mathbb{Q}_x^{x_0,\varphi})_{x\in E}$ itself by concatenation of the trajectories.

Proposition 2 *For all $n \geq 0$, $x, y \in E$,*

$$\mathbb{1}_{X_n=y} \cdot \mathbb{Q}_x^{x_0,\varphi} = \mathbb{1}_{X_n=y} \cdot (\mathbb{P}_x^{(n)} \circ \mathbb{Q}_y^{x_0,\varphi}).$$

Proof If $n = 0$, the two sides of the equality vanish for all $y \neq x$, since the canonical trajectory starts at x under $\mathbb{Q}_x^{x_0,\varphi}$ and $\mathbb{P}_x$. If $n = 0$ and $y = x$, the equality we want to show is also immediate. Hence, we can assume $n \geq 1$. Let $p \geq n \geq 1$, and let F_p be a $\mathscr{F}_p$-measurable, nonnegative functional. We have

$$\begin{aligned}
&\left(\mathbb{1}_{X_n=y} \cdot \mathbb{Q}_x^{x_0,\varphi}\right)\left(F_p \mathbb{1}_{\forall k\geq p, X_k\neq x_0}\right) \\
&\quad = \mathbb{Q}_x^{x_0,\varphi}\left(F_p \mathbb{1}_{X_n=y}\mathbb{1}_{\forall k\geq p, X_k\neq x_0}\right) = \mathbb{E}_x\left[F_p \mathbb{1}_{X_n=y}\varphi(X_p)\right] \\
&\quad = \sum_{x_1,x_2,\dots,x_{n-1}\in E} p_{x,x_1}p_{x_1,x_2}\dots,p_{x_{n-1},y}\mathbb{E}_x[F_p\varphi(X_p)|X_1 = x_1,\dots,X_{n-1} = x_{n-1}, X_n = y].
\end{aligned}$$

Now, the functional F_p is nonnegative and $\mathscr{F}_p$-measurable, so there exists a function Φ from E^{p+1} to $\mathbb{R}_+$ such that

$$F_p = \Phi(X_0,\dots,X_p).$$

We get, using the Markov property:

$$\begin{aligned}
&\left(\mathbb{1}_{X_n=y} \cdot \mathbb{Q}_x^{x_0,\varphi}\right)\left(F_p \mathbb{1}_{\forall k\geq p, X_k\neq x_0}\right) \\
&\quad = \sum_{x_1,x_2,\dots,x_{n-1}\in E} p_{x,x_1}p_{x_1,x_2}\dots \\
&\qquad \times p_{x_{n-1},y}\mathbb{E}_x[\Phi(X_0,\dots,X_p)\varphi(X_p)|X_1 = x_1,\dots,X_{n-1} = x_{n-1}, X_n = y] \\
&\quad = \sum_{x_1,x_2,\dots,x_{n-1}\in E} p_{x,x_1}p_{x_1,x_2}\dots,p_{x_{n-1},y}\mathbb{E}_y[\Phi(x_0,x_1,\dots,x_{n-1},y,X_1,\dots,X_{p-n})\varphi(X_{p-n})] \\
&\quad = \sum_{x_1,x_2,\dots,x_{n-1}\in E} p_{x,x_1}p_{x_1,x_2}\dots \\
&\qquad \times p_{x_{n-1},y}\mathbb{Q}_y^{y_0,\varphi}[\Phi(x_0,x_1,\dots,x_{n-1},y,X_1,\dots,X_{p-n})\mathbb{1}_{\forall k\geq p-n, X_k\neq x_0}]) \\
&\quad = \mathbb{P}_x \otimes \mathbb{Q}_y^{y_0,\varphi}[\mathbb{1}_{Y_n=y}\Phi(Y_0,\dots,Y_n,Z_1,Z_2,\dots,Z_{n-p})\mathbb{1}_{\forall k\geq p-n, Z_k\neq x_0}],
\end{aligned}$$

where in the last line, $(Y_n, Z_n)_{n\geq 0}$ denotes the canonical process on the space $E^{\mathbb{N}_0} \times E^{\mathbb{N}_0}$ where the measure $\mathbb{P}_x \otimes \mathbb{Q}_y^{x_0,\varphi}$ is defined. Hence

$$\begin{aligned}
&\left(\mathbb{1}_{X_n=y} \cdot \mathbb{Q}_x^{x_0,\varphi}\right)\left(F_p \mathbb{1}_{\forall k\geq p, X_k\neq x_0}\right) \\
&\quad = \mathbb{P}_x^{(n)} \circ \mathbb{Q}_y^{y_0,\varphi}[\mathbb{1}_{X_n=y}\Phi(X_0,\ldots,X_n,X_{n+1},X_{n+2},\ldots,X_p)\mathbb{1}_{\forall k\geq p, X_k\neq x_0}] \\
&\quad = (\mathbb{1}_{X_n=y} \cdot (\mathbb{P}_x^{(n)} \circ \mathbb{Q}_y^{x_0,\varphi}))\left(F_p \mathbb{1}_{\forall k\geq p, X_k\neq x_0}\right).
\end{aligned}$$

Hence, the two measures stated in the proposition coincide on all functionals of the form $F_p \mathbb{1}_{\forall k\geq p, X_k\neq x_0}$ if $p \geq n$, and then on all functionals of the form

$$F_q \mathbb{1}_{\forall k\geq p, X_k\neq x_0} = (F_q \mathbb{1}_{\forall k\in\{p,\ldots q-1\}, X_k\neq x_0})\mathbb{1}_{\forall k\geq q, X_k\neq x_0},$$

for all $q \geq p \geq n$. Using the finiteness of the measure $\mathbb{Q}_x^{x_0,\varphi}$ restricted to the set $\{\forall k \geq p, X_k \neq x_0\}$, and the monotone class theorem, one deduces that the two measures we are comparing coincide on all sets included in $\{\forall k \geq p, X_k \neq x_0\}$. Taking the union for $p \geq n$, and using the property of transience satisfied by the measures $\mathbb{Q}_y^{x_0,\varphi}$, $y \in E$, one deduces that the two measures we are comparing are equal.

Knowing the result we have just proven, it is natural to ask which families of σ-finite measures $(\mathbb{Q}_x)_{x\in E}$ on $(E^{\mathbb{N}_0}, \mathscr{F}_\infty)$ satisfy, for all $x, y \in E$, $n \geq 0$,

$$\mathbb{1}_{X_n=y} \cdot \mathbb{Q}_x = \mathbb{1}_{X_n=y} \cdot (\mathbb{P}_x^{(n)} \circ \mathbb{Q}_y). \tag{1}$$

We know that all linear combinations, with nonnegative coefficients, of families of the form $(\mathbb{P}_x)_{x\in E}$ and $(\mathbb{Q}_x^{x_0,\varphi})_{x\in E}$ satisfy this condition. It is natural to ask if there are other such families of measures. We do not know the complete answer. However, we have the following partial result:

Proposition 3 *Let $(\mathbb{Q}_x)_{x\in E}$ be a family of σ-finite measures such that (1) holds for all $x, y \in E$, $n \geq 0$. Then:*

- *If $\mathbb{Q}_x$ is a finite measure for at least one $x \in E$, then there exists $c \geq 0$ such that $\mathbb{Q}_x = c\mathbb{P}_x$ for all $x \in E$.*
- *If for some $x_0, x_1 \in E$,*

$$\varphi(x) := \mathbb{Q}_x\left(\forall k \geq 0, X_k \neq x_0\right) < \infty \tag{2}$$

for all $x \in E$, and

$$\mathbb{Q}_{x_1}\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) = 0, \tag{3}$$

then $\varphi(x_0) = 0$, $\mathbb{E}_x(\varphi(X_1)) = \varphi(x)$ for $x \neq x_0$ and $\mathbb{Q}_x = \mathbb{Q}_x^{x_0,\varphi}$ for all $x \in E$.

- *Moreover, if the conditions of the previous item are satisfied, then (2) and (3) are satisfied for all $x_0, x_1, x \in E$. The fact that (3) holds for all $x_0, x_1 \in E$ means that the canonical process is transient under $\mathbb{Q}_x$ for all $x \in E$.*

Proof Let us assume that $\mathbb{Q}_{x_0}$ has finite total mass for some $x_0 \in E$. For $y \in E$, let $\psi(y)$ be the total mass of $\mathbb{Q}_y$. For all $n \geq 1$, for any nonnegative, $\mathscr{F}_n$-measurable functional F_n, and for all $y \in E$, we deduce from (1):

$$\mathbb{Q}_{x_0}[F_n \mathbb{1}_{X_n=y}] = (\mathbb{P}_{x_0}^{(n)} \circ \mathbb{Q}_y)[F_n \mathbb{1}_{X_n=y}] = \mathbb{E}_{x_0}[F_n \mathbb{1}_{X_n=y}]\mathbb{Q}_y(1) = \psi(y)\mathbb{E}_{x_0}[F_n \mathbb{1}_{X_n=y}],$$

and then, by adding these expressions for all $y \in E$:

$$\mathbb{Q}_{x_0}[F_n] = \mathbb{E}_{x_0}[F_n \psi(X_n)].$$

For $F_n = 1$, we get

$$\mathbb{E}_{x_0}[\psi(X_n)] = \psi(x_0).$$

By assumption, $\psi(x_0) < \infty$, and one deduces that $\psi(X_n)$ is $\mathbb{P}_{x_0}$-almost surely finite for all $n \geq 1$. Since the Markov chain is assumed to be irreducible, one deduces that $\psi(y) < \infty$ for all $y \in E$. On the other hand, if F_n is nonnegative, $\mathscr{F}_n$-measurable, and then also $\mathscr{F}_{n+1}$-measurable, one has

$$\mathbb{E}_{x_0}[F_n \psi(X_{n+1})] = \mathbb{Q}_{x_0}[F_n] = \mathbb{E}_{x_0}[F_n \psi(X_n)],$$

and then

$$\mathbb{E}_{x_0}[\psi(X_{n+1})|\mathscr{F}_n] = \psi(X_n),$$

i.e. $(\psi(X_n))_{n\geq 1}$ is a $\mathbb{P}_{x_0}$-martingale. Since this martingale is nonnegative, it converges almost surely. On the other hand, since $(X_n)_{n\geq 0}$ is recurrent and irreducible, it visites all the states infinitely often. One easily deduces that ψ is a constant function, let $c \geq 0$ be this constant. One has

$$\mathbb{Q}_{x_0}[F_n] = c\mathbb{E}_{x_0}[F_n],$$

and then $\mathbb{Q}_{x_0}$ and $c\mathbb{P}_{x_0}$ are two finite measures which coincide on all $\mathscr{F}_n$, $n \geq 1$, and then on all $\mathscr{F}_\infty$. Moreover, since we have now proven that the total mass of $\mathbb{Q}_x$ is finite for all $x \in E$, we can replace, in the previous discussion, x_0 by any $x \in E$. We then deduce that $\mathbb{Q}_x = c\mathbb{P}_x$ for all $x \in E$.

Now, let us assume (2) and (3) for some $x_0, x_1 \in E$ and all $x \in E$. For all $y \neq x$,

$$\mathbb{1}_{X_0=y} \cdot \mathbb{Q}_x = \mathbb{1}_{X_0=y} \cdot (\mathbb{P}_x^{(0)} \circ \mathbb{Q}_y)$$

is the measure identically equal to zero, since $X_0 = x \neq y$ almost everywhere under $\mathbb{P}_x$, and then under $(\mathbb{P}_x^{(0)} \circ \mathbb{Q}_y)$. Hence, $X_0 = x$ almost everywhere under $\mathbb{Q}_x$. It is then obvious that

$$\varphi(x_0) = \mathbb{Q}_{x_0}(\forall k \geq 0, X_k \neq x_0) = 0.$$

Moreover, for all $x \neq x_0$,

$$\begin{aligned}\varphi(x) = \mathbb{Q}_x(\forall k \geq 0, X_k \neq x_0) &= \mathbb{Q}_x(\forall k \geq 1, X_k \neq x_0)\\ &= \sum_{y \in E} \mathbb{Q}_x(X_1 = y, \forall k \geq 1, X_k \neq x_0)\\ &= \sum_{y \in E} (\mathbb{P}_x^{(1)} \circ \mathbb{Q}_y)(X_1 = y, \forall k \geq 1, X_k \neq x_0)\\ &= \sum_{y \in E} \mathbb{P}_x[X_1 = y]\mathbb{Q}_y(\forall k \geq 0, X_k \neq x_0)\\ &= \sum_{y \in E} \mathbb{P}_x[X_1 = y]\varphi(y) = \mathbb{E}_x[\varphi(X_1)].\end{aligned}$$

Now, for all $\mathscr{F}_n$-measurable, nonnegative functional F_n, we get:

$$\begin{aligned}\mathbb{Q}_x[F_n \mathbb{1}_{\forall k \geq n, X_k \neq x_0}] &= \sum_{y \in E} (\mathbb{P}_x^{(n)} \circ \mathbb{Q}_y)[F_n \mathbb{1}_{X_n = y, \forall k \geq n, X_k \neq x_0}]\\ &= \sum_{y \in E} \mathbb{E}_x[F_n \mathbb{1}_{X_n = y}]\mathbb{Q}_y(\forall k \geq 0, X_k \neq x_0)\\ &= \sum_{y \in E} \varphi(y)\mathbb{E}_x[F_n \mathbb{1}_{X_n = y}]\\ &= \mathbb{E}_x[F_n \varphi(X_n)] = \mathbb{Q}_x^{x_0,\varphi}[F_n \mathbb{1}_{\forall k \geq n, X_k \neq x_0}].\end{aligned}$$

Hence, $\mathbb{Q}_x$ and $\mathbb{Q}_x^{x_0,\varphi}$ coincide for functionals of the form $F_n \mathbb{1}_{\forall k \geq n, X_k \neq x_0}$, then for functionals of the form

$$F_p \mathbb{1}_{\forall k \geq n, X_k \neq x_0} = F_p \mathbb{1}_{\forall k \in \{n, \ldots, p-1\}, X_k \neq x_0} \mathbb{1}_{\forall k \geq p, X_k \neq x_0}$$

for $p \geq n$, then for all events of the form $A \cap \{\forall k \geq n, X_k \neq x_0\}, A \in \mathscr{F}_\infty$, then for their union for $n \geq 1$, i.e. $A \cap \{\exists n \geq 1, \forall k \geq n, X_k \neq x_0\}$. This implies $\mathbb{Q}_x = \mathbb{Q}_x^{x_0,\varphi}$, provided that we check that under these two measures, the canonical process hits x_0 finitely many times, i.e.

$$\mathbb{Q}_x\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) = \mathbb{Q}_x^{x_0,\varphi}\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) = 0.$$

Now, for all $n \geq 0$, we have by assumption

$$\begin{aligned} 0 = \mathbb{Q}_{x_1}\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) &= \sum_{y\in E} \mathbb{Q}_{x_1}\left(X_n = y, \sum_{k=n}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) \\ &= \sum_{y\in E} (\mathbb{P}_{x_1}^{(n)} \circ \mathbb{Q}_y)\left(X_n = y, \sum_{k=n}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) \\ &= \sum_{y\in E} \mathbb{P}_{x_1}(X_n = y)\mathbb{Q}_y\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right), \end{aligned}$$

which implies that

$$\mathbb{Q}_y\left(\sum_{k=n}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) = 0$$

for all $y \in E$ such that $\mathbb{P}_{x_1}(X_n = y) > 0$. Since the Markov chain is irreducible, we deduce that

$$\mathbb{Q}_x\left(\sum_{k=n}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) = 0$$

for all $x \in E$. On the other hand, the transience of the canonical trajectory under $\mathbb{Q}_x^{x_0,\varphi}$, stated in Proposition 1, implies that

$$\mathbb{Q}_x^{x_0,\varphi}\left(\sum_{k=n}^{\infty} \mathbb{1}_{X_k=x_0} = \infty\right) = 0,$$

which completes the proof that $\mathbb{Q}_x = \mathbb{Q}_x^{x_0,\varphi}$. The transience of the canonical process under $\mathbb{Q}_x = \mathbb{Q}_x^{x_0,\varphi}$ for all $x \in E$ means that (3) is satisfied for all $x_0, x_1 \in E$. It only remains to check that (2) holds for all $x, x_0 \in E$, i.e. that for all $x, y \in E$,

$$\mathbb{Q}_x^{x_0,\varphi}(\forall k \geq 0, X_k \neq y) < \infty.$$

If g_{x_0} denotes the last hitting time of x_0 by the canonical process, which is finite almost everywhere since the process is transient, we get:

$$\begin{aligned}
&\mathbb{Q}_x^{x_0,\varphi}(\forall k \geq 0, X_k \neq y) \\
&\quad = \mathbb{Q}_x^{x_0,\varphi}(\forall k \geq 0, X_k \notin \{x_0, y\}) + \sum_{n \geq 0} \mathbb{Q}_x^{x_0,\varphi}(g_{x_0} = n, \forall k \geq 0, X_k \neq y) \\
&\quad \leq \mathbb{Q}_x^{x_0,\varphi}(\forall k \geq 0, X_k \neq x_0) \\
&\qquad + \sum_{n \geq 0} \mathbb{Q}_x^{x_0,\varphi}(X_n = x_0, \forall k \in \{0, 1, \dots, n\}, X_k \neq y, \forall k \geq n+1, X_k \neq x_0) \\
&\quad = \varphi(x) + \sum_{n \geq 0} \mathbb{E}_x[\mathbb{1}_{X_n = x_0, \forall k \in \{0,1,\dots,n\}, X_n \neq y} \varphi(X_{n+1})] \\
&\quad = \varphi(x) + \sum_{n \geq 0} \mathbb{E}_x[\mathbb{1}_{X_n = x_0, \forall k \in \{0,1,\dots,n\}, X_n \neq y} \mathbb{E}_x[\varphi(X_{n+1}) | \mathscr{F}_n]].
\end{aligned}$$

Now, on the event $X_n = x_0$, the conditional expectation of $\varphi(X_{n+1})$ given $\mathscr{F}_n$ is equal to

$$K := \mathbb{E}_{x_0}[\varphi(X_1)] = \sum_{y \in E} p_{x_0,y} \varphi(y),$$

where K is finite since $p_{x_0,y} = 0$ for all but finitely many $y \in E$. If T_y denotes the first hitting time of y by the canonical trajectory, we then get:

$$\mathbb{Q}_x^{x_0,\varphi}(\forall k \geq 0, X_k \neq y) \leq \varphi(x) + K\mathbb{E}_x\left[\sum_{n \geq 0} \mathbb{1}_{X_n = x_0, T_y > n}\right] = \varphi(x) + K\mathbb{E}_x[L_{T_y - 1}^{x_0}],$$

where L_n^x denotes the number of hitting times of x at or before time n. It is then sufficient to check that $\mathbb{E}_x[L_{T_y-1}^{x_0}]$ is finite. Now, if for $p \geq 1$, $\tau_p^{x_0}$ denotes the p-th hitting time of x_0, we get, using the strong Markov property:

$$\mathbb{P}_x[L_{T_y-1}^{x_0} \geq p] = \mathbb{P}_x[\tau_p^{x_0} < T_y] = \mathbb{P}_x[\tau_1^{x_0} < T_y] \left(\mathbb{P}_{x_0}[\tau_2^{x_0} < T_y]\right)^{p-1} \leq P^{p-1}$$

where

$$P = \mathbb{P}_{x_0}[\tau_2^{x_0} < T_y].$$

It is not possible that $P = 1$, otherwise, by the strong Markov property,

$$\mathbb{P}_{x_0}[n - 1 < T_y] \geq \mathbb{P}_{x_0}[\tau_n^{x_0} < T_y] = 1$$

for all $n \geq 1$, and then the canonical trajectory would never hit y under $\mathbb{P}_{x_0}$, which contradicts the fact that the Markov chain is irreducible and recurrent. Now, since $P < 1$, the tail of the law of $L^{x_0}_{T_y-1}$ under $\mathbb{P}_x$ is exponentially decreasing, which implies that

$$\mathbb{E}_x[L^{x_0}_{T_y-1}] < \infty$$

and then

$$\mathbb{Q}^{x_0,\varphi}_x(\forall k \geq 0, X_k \neq y) < \infty.$$

A corollary of Proposition 3 is the following result, already contained in Theorem 4.2.5 of [11]:

Corollary 1 *Let $x_0, x_1 \in E$, and let φ_{x_0} be a function from E to $\mathbb{R}_+$ such that $\varphi_{x_0}(x_0) = 0$ and $\mathbb{E}_x[\varphi_{x_0}(X_1)] = \varphi_{x_0}(x)$ for all $x \neq x_0$. Then, the function φ_{x_1} given by*

$$\varphi_{x_1}(x) := \mathbb{Q}^{x_0,\varphi_{x_0}}_x (\forall k \geq 0, X_k \neq x_1)$$

vanishes at x_1, takes finite values and is harmonic at any other point than x_1. Moreover, we have, for all $x \in E$, the equality of measures

$$\mathbb{Q}^{x_1,\varphi_{x_1}}_x = \mathbb{Q}^{x_0,\varphi_{x_0}}_x.$$

Proof We know that (1)–(3) are satisfied for $\mathbb{Q}_x = \mathbb{Q}^{x_0,\varphi_{x_0}}_x$ and $\varphi = \varphi_{x_0}$. By the last item of Proposition 3, (2) and (3) are still satisfied if we replace x_0 by x_1, i.e.

$$\varphi_{x_1}(x) := \mathbb{Q}_x (\forall k \geq 0, X_k \neq x_1) < \infty$$

and

$$\mathbb{Q}_{x_1}\left(\sum_{k=0}^{\infty} \mathbb{1}_{X_k = x_1} = \infty\right) = 0.$$

Now, from the second item of Proposition 3, φ_{x_1} vanishes at x_1 and is harmonic at any other point, and $\mathbb{Q}_x = \mathbb{Q}^{x_1,\varphi_{x_1}}_x$.

From this corollary, we see that in order to describe a family of measures of the form $(\mathbb{Q}^{x_0,\varphi_{x_0}}_x)_{x\in E}$, the role of x_0 can be taken by any point in E, so the choice of x_0 is not so important. In the next section, we will clarify this phenomenon, by studying the link between the measures of the form $\mathbb{Q}^{x_0,\varphi}_x$, and the Martin boundary of the Markov chain induced by $(\mathbb{P}_x)_{x\in E}$. We will use the following definition:

Definition 1 We will say that a family $(\mathbb{Q}_x)_{x\in E}$ of σ-finite measures on $(E^{\mathbb{N}_0}, \mathscr{F}_\infty)$ is in the *class* $\mathscr{Q}$, with respect to $(\mathbb{P}_x)_{x\in E}$, if and only if (1), (2) and (3) hold for all $x, x_0, x_1, y \in E$ and $n \geq 0$, or equivalently, iff it is of the form $(\mathbb{Q}_x^{x_0,\varphi})_{x \in E}$ for some $x_0 \in E$, and for some function φ which is nonnegative, equal to zero at x_0 and harmonic for $(\mathbb{P}_x)_{x \in E}$ at any point different from x_0.

3 Link with the Martin Boundary

In [7], Martin proves that one can describe all the nonnegative harmonic functions on a sufficiently regular domain of $\mathbb{R}^d$, by a formula which generalizes the Poisson integral formula, available for the harmonic functions on the unit disc. This construction has been adapted to the setting of transient Markov chains by Doob [1] and Hunt [3], and then to the setting of recurrent Markov chains by Kemeny and Snell in [4], and by Orey in [13]. The construction is also described in a survey by Woess (see [17], Sect. 7.H.).

Let us first recall a possible construction of the Martin boundary, for a transient Markov chain on the countable set E. For $x, y \in E$, let $q_{x,y}$ be the transition probability of the Markov chain from x to y, and let G be the Green function:

$$G(x, y) = \sum_{k=0}^{\infty} (q^k)_{x,y}$$

where q^k is defined inductively by

$$(q^0)_{x,y} = \mathbb{1}_{x=y}, \ (q^{k+1})_{x,y} = \sum_{z \in E} (q^k)_{x,z} q_{z,y}.$$

Let us fix $x_0 \in E$, and let us assume that $G(x_0, y) > 0$ for all $y \in E$, i.e. any state in E is accessible from x_0 by the Markov chain. Let K_{x_0} be the function, from E to $\mathbb{R}_+$, given by

$$K_{x_0}(x, y) = \frac{G(x, y)}{G(x_0, y)}.$$

One can prove that

$$C_{x_0}(x) := \sup_{y \in E} K_{x_0}(x, y) < \infty$$

and then, if $w = (w_x)_{x \in E}$ is a summable family of elements in $\mathbb{R}_+^*$, one can define a distance $\rho_{x_0,w}$ on E by

$$\rho_{x_0,w}(x, y) := \sum_{z \in E} w_z \frac{|K_{x_0}(z, x) - K_{x_0}(z, y)| + |\mathbb{1}_{z=x} - \mathbb{1}_{z=y}|}{C_{x_0}(z) + 1}.$$

The *Martin compactification* of E is the topological space $\hat{E}$, induced by the completion of the metric space $(E, \rho_{x_0,w})$: up to homeomorphism, $\hat{E}$ does not depend on the choice of w and the point x_0 such that $G(x_0, y) > 0$ for all $y \in E$. The space $\hat{E}$ is compact, its subspace $\partial E := \hat{E} \backslash E$ is a closed set in $\hat{E}$, called the *Martin boundary* of E.

If $G(x_0, y) > 0$ for all $y \in E$, and if $x \in E$, then the function $y \mapsto K_{x_0}(x, y)$ is Lipschitz (with a constant at most $[1 + C_{x_0}(x)]/w_x$), and then the function K_{x_0} from $E \times E$ to $\mathbb{R}_+$ can be uniquely extended by continuity to the set $E \times \hat{E}$. For all $\alpha \in \hat{E}$, the function $x \mapsto K_{x_0}(x, \alpha)$ is superharmonic for the transition probabilities $(q_{x,y})_{x,y \in E}$ i.e. for all $x \in E$,

$$K_{x_0}(x, \alpha) \geq \sum_{y \in E} q_{x,y} K_{x_0}(y, \alpha),$$

and it can be harmonic only for $\alpha \in \partial E$. We define the *minimal boundary* of E as the set $\partial_m E$ of points $\alpha \in \partial E$, such that the function $x \mapsto K_{x_0}(x, \alpha)$ is *minimal harmonic*, i.e. it is harmonic, and for any harmonic function $\psi : E \to \mathbb{R}$ such that $0 \leq \psi(x) \leq K_{x_0}(x, \alpha)$ for all $x \in E$, there exists $c \in [0, 1]$ such that $\psi(x) = c K_{x_0}(x, \alpha)$ for all $x \in E$. The following result holds:

Proposition 4 *The set $\partial_m E$ is a Borel subset of ∂E which, up to canonical homeomorphism, does not depend on the choice of x_0. Moreover, for any choice of x_0, a nonnegative function ψ from E to $\mathbb{R}$ is harmonic if and only if there exists a finite measure μ_{ψ,x_0} on $\partial_m E$, such that for all $x \in E$,*

$$\psi(x) = \int_{\partial_m E} K_{x_0}(x, \alpha) d\mu_{\psi,x_0}(\alpha).$$

If it exists, the measure μ_{ψ,x_0} is uniquely determined.

Let us now go back to the assumptions of Sect. 2. In this setting, the canonical process $(X_n)_{n \geq 0}$ is irreducible and recurrent under $\mathbb{P}_x$ for all $x \in E$, and all the nonnegative harmonic functions are constant. Indeed, if $\psi : E \to \mathbb{R}_+$ is harmonic, $(\psi(X_n))_{n \geq 1}$ is a nonnegative martingale, and then it converges a.s., which is only possible for ψ constant, since $(X_n)_{n \geq 1}$ hits all the points of E infinitely often. Then, the definition of the Martin boundary should be modified in order to give a non-trivial result. The idea is to kill the Markov chain at some time in order to get a finite Green function. The time which is chosen occurs just before the first strictly positive hitting time of some $x_0 \in E$. The Green function we obtain in this way is given by

$$G_{x_0}(x, y) := \mathbb{E}_x \left[L^y_{T'_{x_0} - 1} \right], \tag{4}$$

where

$$T'_{x_0} := \inf\{n \geq 1, X_n = x_0\}.$$

Recall that L_n^x denotes the number of hitting times of x at and before time n. It is easy to check, using the strong Markov property, that the tail of the distribution of $L^y_{T'_{x_0}}$ is exponentially decreasing, which implies that $G_{x_0}(x,y)$ is finite. Moreover, $G_{x_0}(x_0,y)$ is strictly positive, since all the states in E are accessible from x_0 (recall that the Markov chain is irreducible), and then they are also accessible without returning to x_0. Hence, one can define, similarly as $K_{x_0}(x,y)$ in the transient case:

$$L_{x_0}(x,y) := \frac{G_{x_0}(x,y)}{G_{x_0}(x_0,y)}.$$

The function L_{x_0} induces a distance $\delta_{x_0,w}$ on E, given by

$$\delta_{x_0,w}(x,y) := \sum_{z\in E} w_z \frac{|L_{x_0}(z,x) - L_{x_0}(z,y)| + |\mathbb{1}_{z=x} - \mathbb{1}_{z=y}|}{D_{x_0}(z)+1}.$$

where, as before, $w := (w_x)_{x\in E}$, and where

$$D_{x_0}(z) := \sup_{y\in E} L_{x_0}(z,y) < \infty.$$

The completion of $(E, \delta_{x_0,w})$ induces a topological space $\overline{E}$, called, as before, the Martin compactification of E: it is possible to prove that the topological structure of $\overline{E}$ does not depend on w and x_0.

The transitions of the Markov chain killed just before going to x_0 at of after time 1 are given by $(p_{x,y}\mathbb{1}_{y\neq x_0})_{x,y\in E}$. A function $\tilde{\varphi}$ from E to $\mathbb{R}_+$ is harmonic with respect to these transitions if an only if the function φ given by

$$\varphi(x) = \tilde{\varphi}(x)\mathbb{1}_{x\neq x_0}$$

is harmonic for the initial Markov chain at any point except x_0, and if

$$\tilde{\varphi}(x_0) = \sum_{y\in E} p_{x,y}\varphi(y) = \mathbb{E}_x[\varphi(X_1)].$$

The map going from $\tilde{\varphi}$ to φ is linear and bijective. By continuity, one can extend $L_{x_0}(x,\alpha)$ to all $x \in E$ and $\alpha \in \overline{E}$. For α fixed this function is, as in the transient case, superharmonic with respect to the transitions $(p_{x,y}\mathbb{1}_{y\neq x_0})_{x,y\in E}$, and it can only be harmonic for α in the boundary ∂E of $\overline{E}$, which is, as in the transient case, called the Martin boundary of the Markov chain. The minimal boundary $\partial_m E$ is the set of $\alpha \in \partial E$ such that $x \mapsto L_{x_0}(x,\alpha)$ is minimal harmonic for $(p_{x,y}\mathbb{1}_{y\neq x_0})_{x,y\in E}$. As in the transient case, one can show that all harmonic functions for $(p_{x,y}\mathbb{1}_{y\neq x_0})_{x,y\in E}$ can be written, in a unique way, as the integral of $x \mapsto L_{x_0}(x,\alpha)$ with respect to $d\mu(\alpha)$, μ being a measure on the minimal boundary $\partial_m E$. Stating this precisely, and writing this in terms of φ rather than $\tilde{\varphi}$ gives the following:

Proposition 5 *The set $\partial_m E$ is a Borel subset of ∂E which does not depend on the choice of x_0. Moreover, for any $x_0 \in E$, a nonnegative function φ from E to $\mathbb{R}$, such that $\varphi(x_0) = 0$, satisfies $\mathbb{E}_x[\varphi(X_1)] = \varphi(x)$ for all $x \neq x_0$ if and only if there exists a finite measure μ_{φ,x_0} on $\partial_m E$, such that for all $x \neq x_0$,*

$$\varphi(x) = \int_{\partial_m E} L_{x_0}(x, \alpha) d\mu_{\varphi,x_0}(\alpha).$$

If it exists, the measure μ_{φ,x_0} is uniquely determined, and has total mass equal to

$$\tilde{\varphi}(x_0) := \mathbb{E}_{x_0}[\varphi(X_1)].$$

Now, we can use this result in order to classify the families of measures $(\mathbb{Q}_x^{x_0,\varphi})_{x\in E}$ introduced in Sect. 2.

Since the Markov chain is irreducible and recurrent, it admits a nonnegative stationary measure, which is unique up to a multiplicative constant. If we fix the constant of normalization, one gets a function β from E to $\mathbb{R}_+$, such that for all $y \in E$,

$$\beta(y) = \sum_{y \in E} p_{x,y} \beta(x).$$

Moreover, the function β never vanishes. One then gets the following result:

Proposition 6 *For $\alpha \in \partial_m E$, and for all $x_0 \in E$, the function*

$$\varphi_{x_0,\alpha} : x \mapsto \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} \mathbb{1}_{x \neq x_0},$$

which vanishes at x_0, is harmonic at every point except x_0. Moreover, the family of σ-finite measures $(\mathbb{Q}_x^{x_0,\varphi_{x_0,\alpha}})_{x\in E}$ does not depend on x_0.

Proof The fact that $\varphi_{x_0,\alpha}$ is harmonic everywhere except at x_0 comes directly from the definition of the minimal Martin boundary. Moreover, if $x_0, x_1 \in E$, we have proven in Proposition 4.2.10 of [11] that $\mathbb{Q}_x^{x_0,\varphi_{x_0,\alpha}} = \mathbb{Q}_x^{x_1,\varphi_{x_1,\alpha}}$ for all $x \in E$, if and only if for all $\epsilon \in (0,1)$, there exists $A > 0$ such that for all $x \in E$, $\varphi_{x_0,\alpha}(x) + \varphi_{x_1,\alpha}(x) \geq A$ implies

$$(1-\epsilon)\varphi_{x_0,\alpha}(x) < \varphi_{x_1,\alpha}(x) < (1+\epsilon)\varphi_{x_0,\alpha}(x).$$

One easily checks that this condition is implied by:

$$\sup_{x \in E} |\varphi_{x_0,\alpha}(x) - \varphi_{x_1,\alpha}(x)| < \infty.$$

It is then sufficient to prove this bound for all $x_0, x_1 \in E$ such that $x_0 \neq x_1$. Now, a classical construction of the stationary measure β implies, using (4), that

$$G_{x_0}(x_0, y) = \frac{\beta(y)}{\beta(x_0)},$$

which implies, for all $x \neq x_0$,

$$\varphi_{x_0,\alpha}(x) = \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} = \lim_{y \to \alpha, y \in E} \frac{G_{x_0}(x,y)}{\beta(x_0) G_{x_0}(x_0,y)} = \lim_{y \to \alpha, y \in E} \frac{G_{x_0}(x,y)}{\beta(y)},$$

and for $x \neq x_1$,

$$\varphi_{x_1,\alpha}(x) = \lim_{y \to \alpha, y \in E} \frac{G_{x_1}(x,y)}{\beta(y)}.$$

It is then sufficient to prove

$$\sup_{x \in E \setminus \{x_0, x_1\}, y \in E} \frac{|G_{x_0}(x,y) - G_{x_1}(x,y)|}{\beta(y)} < \infty.$$

Let G_{x_0,x_1} be the Green function of the Markov chain corresponding to $(\mathbb{P}_x)_{x \in \mathbb{R}}$, killed just before its first strictly positive hitting time of the set $\{x_0, x_1\}$:

$$G_{x_0,x_1}(x,y) = \mathbb{E}_x[L^y_{(T'_{x_0} \wedge T'_{x_1})-1}],$$

where T'_z is the first strictly positive hitting time of z. It is sufficient to prove

$$\sup_{x \in E \setminus \{x_0, x_1\}, y \in E} \frac{|G_{x_0}(x,y) - G_{x_0,x_1}(x,y)|}{\beta(y)} < \infty$$

and

$$\sup_{x \in E \setminus \{x_0, x_1\}, y \in E} \frac{|G_{x_1}(x,y) - G_{x_0,x_1}(x,y)|}{\beta(y)} < \infty.$$

Let us show the first bound: the second is obtained by exchanging x_0 and x_1. If $\tau_p^{x_1}$ denotes the p-th hitting time of x_1, one gets for all $x \in E \setminus \{x_0, x_1\}, y \in E$,

$$\begin{aligned} G_{x_0}(x,y) &= \mathbb{E}_x\left[\sum_{n=0}^{\infty} \mathbb{1}_{X_n \in y, n < T'_{x_0}}\right] \\ &= \mathbb{E}_x\left[\sum_{n=0}^{\tau_1^{x_1}-1} \mathbb{1}_{X_n \in y, n < T'_{x_0}}\right] + \mathbb{E}_x\left[\sum_{p=1}^{\infty} \sum_{n=\tau_p^{x_1}}^{\tau_{p+1}^{x_1}-1} \mathbb{1}_{X_n \in y, n < T'_{x_0}}\right] \end{aligned}$$

Since $x \neq x_1$ and then $\tau_1^{x_1} = T'_{x_1}$, the first term of the last sum is exactly $G_{x_0,x_1}(x, y)$, and by the strong Markov property:

$$\mathbb{E}_x\left[\sum_{n=\tau_p^{x_1}}^{\tau_{p+1}^{x_1}-1} \mathbb{1}_{X_n\in y, n<T'_{x_0}}\right] = \mathbb{P}_x[\tau_p^{x_1} < T'_{x_0}]\mathbb{E}_{x_1}\left[\sum_{n=0}^{T'_{x_1}-1} \mathbb{1}_{X_n\in y, n<T'_{x_0}}\right]$$
$$= G_{x_0,x_1}(x_1, y)\mathbb{P}_x[\tau_p^{x_1} < T'_{x_0}].$$

Hence,

$$G_{x_0}(x, y) = G_{x_0,x_1}(x, y) + G_{x_0,x_1}(x_1, y)\sum_{p=1}^{\infty}\mathbb{P}_x[\tau_p^{x_1} < T'_{x_0}]$$
$$= G_{x_0,x_1}(x, y) + G_{x_0,x_1}(x_1, y)\mathbb{E}_x[L^{x_1}_{T'_{x_0}-1}]$$
$$= G_{x_0,x_1}(x, y) + G_{x_0,x_1}(x_1, y)G_{x_0}(x, x_1).$$

It is then sufficient to check

$$\sup_{x\in E\setminus\{x_0,x_1\}, y\in E} \frac{G_{x_0,x_1}(x_1, y)G_{x_0}(x, x_1)}{\beta(y)} < \infty.$$

Now,

$$G_{x_0,x_1}(x_1, y) \leq G_{x_1}(x_1, y) = \frac{\beta(y)}{\beta(x_1)}$$

and using the Markov property at the first hitting time of x_1,

$$G_{x_0}(x, x_1) \leq G_{x_0}(x_1, x_1),$$

which implies

$$\sup_{x\in E\setminus\{x_0,x_1\}, y\in E} \frac{G_{x_0,x_1}(x_1, y)G_{x_0}(x, x_1)}{\beta(y)} \leq \frac{G_{x_0}(x_1, x_1)}{\beta(x_1)} < \infty.$$

Since the normalization of β is supposed to be fixed, the result we have just proven allows to write, for all $\alpha \in \partial_m E$,

$$\mathbb{Q}_x^{\alpha} := \mathbb{Q}_x^{x_0,\varphi_{x_0,\alpha}},$$

since the right-hand side does not depend on $x_0 \in E$.

Using the minimal boundary, one deduces a complete classification of the families of σ-finite measures in the class $\mathscr{Q}$.

Proposition 7 *Let $(\mathbb{Q}_x)_{x\in E}$ be a family of σ-finite measures on $(E^{\mathbb{N}_0}, \mathscr{F}_\infty)$. Then $(\mathbb{Q}_x)_{x\in E}$ is in the class $\mathscr{Q}$ if and only if there exists a finite measure μ on $\partial_m E$ such that for all $A \in \mathscr{F}_\infty$, $x \in E$,*

$$\mathbb{Q}_x(A) = \int_{\partial_m E} \mathbb{Q}_x^\alpha(A) d\mu(\alpha).$$

In this case, μ is uniquely determined.

Proof Let us assume $\mathbb{Q}_x = \mathbb{Q}_x^{x_0,\varphi}$ for all $x \in E$. By Proposition 5, there exists a finite measure μ_{φ,x_0} on $\partial_m E$ such that for all $x \in E$,

$$\varphi(x) = \mathbb{1}_{x\neq x_0} \int_{\partial_m E} L_{x_0}(x,\alpha) d\mu_{\varphi,x_0}(\alpha).$$

For all $n \geq 1$, and for all nonnegative, $\mathscr{F}_n$-measurable functionals F_n, one has

$$\begin{aligned}
\mathbb{Q}_x[F_n \mathbb{1}_{\forall k\geq n, X_k\neq x_0}] &= \mathbb{E}_x[F_n \varphi(X_n)] \\
&= \mathbb{E}_x\left[F_n \mathbb{1}_{X_n\neq x_0} \int_{\partial_m E} L_{x_0}(X_n,\alpha) d\mu_{\varphi,x_0}(\alpha)\right] \\
&= \mathbb{E}_x\left[F_n \int_{\partial_m E} \beta(x_0)\varphi_{x_0,\alpha}(X_n) d\mu_{\varphi,x_0}(\alpha)\right] \\
&= \int_{\partial_m E} \mathbb{E}_x[F_n \varphi_{x_0,\alpha}(X_n)] d(\beta(x_0)\mu_{\varphi,x_0}(\alpha)) \\
&= \int_{\partial_m E} \mathbb{Q}^\alpha \left[F_n \mathbb{1}_{\forall k\geq n, X_k\neq x_0}\right] d(\beta(x_0)\mu_{\varphi,x_0}(\alpha)) \qquad (5)
\end{aligned}$$

Using the monotone class theorem and the fact that the canonical process is transient under $\mathbb{Q}_x$ and $\mathbb{Q}^\alpha$, one deduces, for all $A \in \mathscr{F}_\infty$,

$$\mathbb{Q}_x(A) = \int_{\partial_m E} \mathbb{Q}_x^\alpha(A) d\mu(\alpha),$$

where the measure

$$\mu := \beta(x_0)\mu_{\varphi,x_0}(\alpha)$$

is finite. Let us prove the uniqueness of μ. If for two finite measures μ and ν, and for all $A \in \mathscr{F}_\infty$,

$$\int_{\partial_m E} \mathbb{Q}_x^\alpha(A) d\mu(\alpha) = \int_{\partial_m E} \mathbb{Q}_x^\alpha(A) d\nu(\alpha),$$

then for $x_0 \in E$,

$$\int_{\partial_m E} \mathbb{Q}_x^{x_0,\varphi_{x_0,\alpha}}(A) d\mu(\alpha) = \int_{\partial_m E} \mathbb{Q}_x^{x_0,\varphi_{x_0,\alpha}}(A) d\nu(\alpha),$$

and for all $n \geq 1$, $B_n \in \mathscr{F}_n$, one gets, by taking $A = B_n \cap \{\forall k \geq n, X_k \neq x_0\}$,

$$\int_{\partial_m E} \mathbb{E}_x[\mathbb{1}_{B_n} \varphi_{x_0,\alpha}(X_n)] d\mu(\alpha) = \int_{\partial_m E} \mathbb{E}_x[\mathbb{1}_{B_n} \varphi_{x_0,\alpha}(X_n)] d\nu(\alpha),$$

i.e.

$$\mathbb{E}_x[\mathbb{1}_{B_n} \varphi_1(X_n)] = \mathbb{E}_x[\mathbb{1}_{B_n} \varphi_2(X_n)],$$

where

$$\varphi_1(x) = \int_{\partial_m E} \varphi_{x_0,\alpha}(x) d\mu(\alpha)$$

and

$$\varphi_2(x) = \int_{\partial_m E} \varphi_{x_0,\alpha}(x) d\nu(\alpha).$$

For $y \in E$, taking $B_n = \{X_n = y\}$ gives

$$\mathbb{E}_x[\mathbb{1}_{X_n = y} \varphi_1(X_n)] = \mathbb{E}_x[\mathbb{1}_{X_n = y} \varphi_2(X_n)],$$

i.e.

$$\varphi_1(y) \mathbb{P}_x[X_n = y] = \varphi_2(y) \mathbb{P}_x[X_n = y].$$

Since the Markov chain is irreducible, there exists $n \geq 1$ such that $\mathbb{P}_x[X_n = y] > 0$, which implies that $\varphi_1(y) = \varphi_2(y)$, i.e. for all $x \in E$,

$$\int_{\partial_m E} \varphi_{x_0,\alpha}(x) d\mu(\alpha) = \int_{\partial_m E} \varphi_{x_0,\alpha}(x) d\nu(\alpha),$$

and then for $x \neq x_0$,

$$\int_{\partial_m E} \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} d\mu(\alpha) = \int_{\partial_m E} \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} d\nu(\alpha).$$

The uniqueness given in Proposition 5 implies that $\mu = \nu$.

It remains to show that any family $(\mathbb{Q}_x)_{x\in E}$ of measures such that for all $A \in \mathscr{F}_\infty$,

$$\mathbb{Q}_x(A) = \int_{\partial_m E} \mathbb{Q}_x^\alpha(A) d\mu(\alpha)$$

has the form $(\mathbb{Q}_x^{x_0,\varphi})_{x\in E}$ if μ is a finite measure on $\partial_m E$. Indeed, by reversing the computation given in (5) and by replacing μ_{φ,x_0} by $\mu/(\beta(x_0))$, one deduces that for F_n nonnegative and $\mathscr{F}_n$-measurable,

$$\mathbb{Q}_x(F_n \mathbb{1}_{\forall k\geq n, X_k\neq x_0}) = \mathbb{Q}_x^{x_0,\varphi}(F_n \mathbb{1}_{\forall k\geq n, X_k\neq x_0}),$$

where

$$\varphi(x) := \mathbb{1}_{x\neq x_0} \int_{\partial_m E} \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} d\mu(\alpha).$$

Since the canonical process is transient under $\mathbb{Q}_x$ and $\mathbb{Q}_x^{x_0,\varphi}$, one deduces that $\mathbb{Q}_x = \mathbb{Q}_x^{x_0,\varphi}$.

The result we have just proven gives a disintegration of all families of measures in the class $\mathscr{Q}$, in terms of the families $(\mathbb{Q}_x^\alpha)_{x\in E}$ for $\alpha \in \partial_m E$.

4 Convergence of the Canonical Process Under $\mathbb{Q}_x^\alpha$

In this section, we study the canonical trajectory under $\mathbb{Q}_x^\alpha$, for α in the minimal boundary of the Markov chain corresponding to $(\mathbb{P}_x)_{x\in E}$. The main statement we will prove is the following result of convergence:

Proposition 8 *For all $x \in E$, and for all $\alpha \in \partial_m E$, $(\mathbb{Q}_x^\alpha)$-almost every trajectory tends to α at infinity.*

Proof The proof of this statement will be done in several steps. A difficulty in the study of $\mathbb{Q}_x^\alpha$ is the fact that this measure is not finite in general. Hopefully, $\mathbb{Q}_x^\alpha$ can be proven to be equivalent to probability measures, which can be explicitly described. Moreover, one can choose such a probability measure, in such a way that the corresponding random trajectory is a transient Markov chain.

Proposition 9 *For $r \in (0,1)$, $x, x_0 \in E$, $\alpha \in \partial_m E$, let*

$$\psi_{x_0,\alpha,r}(x) := \frac{1}{\beta(x_0)} \left[\frac{r}{1-r} + L_{x_0}(x,\alpha)\mathbb{1}_{x\neq x_0}\right].$$

Then, if $L_\infty^{x_0}$ denotes the total number of hitting time of x_0 by the canonical trajectory, the measure

$$\mathbb{P}_x^{x_0,\alpha,r} := \frac{r^{L_\infty^{x_0}}}{\psi_{x_0,\alpha,r}(x)} \cdot \mathbb{Q}_x^\alpha$$

is the probability distribution of a Markov chain, starting at x, with transition probabilities $(q^{x_0,\alpha,r}_{x,y})_{x,y\in E}$*, where*

$$q^{x_0,\alpha,r}_{x,y} = \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x)} p_{x,y}$$

if $x \neq x_0$*, and*

$$q^{x_0,\alpha,r}_{x_0,y} = r \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x_0)} p_{x_0,y}.$$

Proof The discussion at the beginning of Chap. 4 of [11] shows the following: if $\varphi(x_0) = 0$ and φ is harmonic at all points different from x_0, then for

$$\psi_r(x) := \frac{r}{1-r}\mathbb{E}_{x_0}[\varphi(X_1)] + \varphi(x),$$

the measure

$$\mu^{(r)}_x := r^{L^{x_0}_\infty} \cdot \mathbb{Q}^{x_0,\varphi}_x$$

is finite and satisfies, for all $n \geq 0$, and for all F_n nonnegative, $\mathscr{F}_n$-measurable,

$$\mu^{(r)}_x(F_n) := \mathbb{E}_x[\psi_r(X_n) r^{L^{x_0}_{n-1}} F_n],$$

where $L^{x_0}_{n-1}$ is the number of hitting times of x_0 at or before time $n-1$.

In the case we consider here, we have

$$\varphi(x) = \varphi_{x_0,\alpha}(x) = \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} \mathbb{1}_{x\neq x_0},$$

and by applying Proposition 5 to μ_{φ,x_0} equal to $1/\beta(x_0)$ times the Dirac mass at α,

$$\mathbb{E}_{x_0}[\varphi(X_1)] = \frac{1}{\beta(x_0)},$$

the total mass of μ_{φ,x_0}. Hence,

$$\psi_r(x) = \frac{r}{(1-r)\beta(x_0)} + \frac{L_{x_0}(x,\alpha)}{\beta(x_0)} \mathbb{1}_{x\neq x_0} = \psi_{x_0,\alpha,r}(x),$$

$$\mu^{(r)}_x = \psi_{x_0,\alpha,r}(x) \cdot \mathbb{P}^{x_0,\alpha,r}_x,$$

and then for $n \geq 0$, F_n nonnegative and $\mathscr{F}_n$-measurable,

$$\mathbb{P}^{x_0,\alpha,r}_x(F_n) = \mathbb{E}_x\left[\frac{\psi_{x_0,\alpha,r}(X_n)}{\psi_{x_0,\alpha,r}(x)} r^{L^{x_0}_{n-1}} F_n\right].$$

Taking $n = 0$ and $F_n = 1$, we deduce that $\mathbb{P}_x^{x_0,\alpha,r}$ is a probability measure. Moreover, for all $y_0, y_1, \dots, y_n \in E$,

$$\begin{aligned}
&\mathbb{P}_x^{x_0,\alpha,r}(X_0 = y_0, \dots, X_n = y_n) \\
&= \mathbb{1}_{y_0=x} \left(\prod_{j=0}^{n-1} p_{y_j,y_{j+1}} \right) \frac{\psi_{x_0,\alpha,r}(y_n)}{\psi_{x_0,\alpha,r}(x)} r^{\sum_{j=0}^{n-1} \mathbb{1}_{y_j=x_0}} \\
&= \mathbb{1}_{y_0=x} \left(\prod_{j=0}^{n-1} p_{y_j,y_{j+1}} \right) \left(\prod_{j=0}^{n-1} \frac{\psi_{x_0,\alpha,r}(y_{j+1})}{\psi_{x_0,\alpha,r}(y_j)} \right) \left(\prod_{j=0}^{n-1} r^{\mathbb{1}_{y_j=x_0}} \right) \\
&= \mathbb{1}_{y_0=x} \prod_{j=0}^{n-1} \left(p_{y_j,y_{j+1}} \frac{\psi_{x_0,\alpha,r}(y_{j+1})}{\psi_{x_0,\alpha,r}(y_j)} r^{\mathbb{1}_{y_j=x_0}} \right) \\
&= \mathbb{1}_{y_0=x} \prod_{j=0}^{n-1} q_{y_j,y_{j+1}}^{x_0,\alpha,r},
\end{aligned}$$

which proves the desired result.

Since the canonical trajectory is transient under $\mathbb{Q}_x^{\alpha}$, it is also transient under $\mathbb{P}_x^{x_0,\alpha,r}$, since the two measures are absolutely continuous with respect to each other. Hence, one can consider the Martin boundary of the corresponding transient Markov chain. If we take x_0 as the reference point, we need to consider the Green function $G_{x_0,\alpha,r}$ given by

$$G_{x_0,\alpha,r}(x, y) := \sum_{k=0}^{\infty} \mathbb{P}_x^{x_0,\alpha,r}(X_k = y),$$

and the function $K_{x_0,\alpha,r}$ given by

$$K_{x_0,\alpha,r}(x, y) := \frac{G_{x_0,\alpha,r}(x, y)}{G_{x_0,\alpha,r}(x_0, y)}.$$

One then gets the following result:

Proposition 10 *For all $x, y \in E$, one has:*

$$K_{x_0,\alpha,r}(x, y) = \frac{\psi_{x_0,\alpha,r}(x_0)}{\psi_{x_0,\alpha,r}(x)} \left(1 + \frac{1-r}{r} L_{x_0}(x, y) \mathbb{1}_{x \neq x_0} \right).$$

The Martin boundary associated to the transient Markov chain corresponding to $(\mathbb{P}_x^{x_0,\alpha,r})_{x\in E}$ is canonically homeomorphic to the Martin boundary associated to the recurrent Markov chain corresponding to $(\mathbb{P}_x)_{x\in E}$, and the analogous statement is true if we replace the Martin boundary by the minimal boundary. Moreover, *the function* 1 *is a minimal harmonic function, for the Markov chain given by* $(\mathbb{P}_x^{x_0,\alpha,r})_{x\in E}$*, which corresponds to the point* α *of the minimal boundary* $\partial_m E$*.*

Proof For $x, y \in E$, $n \geq 1$,

$$\mathbb{P}_x^{x_0,\alpha,r}(X_n = y) = \mathbb{E}_x\left[\frac{\psi_{x_0,\alpha,r}(X_n)}{\psi_{x_0,\alpha,r}(x)} r^{L_{n-1}^{x_0}} \mathbb{1}_{X_n=y}\right]$$
$$= \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x)} \mathbb{E}_x\left[r^{L_{n-1}^{x_0}} \mathbb{1}_{X_n=y}\right],$$

and then

$$G_{x_0,\alpha,r}(x,y) = \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x)} \mathbb{E}_x\left[\sum_{n=0}^{\tau_1^{x_0}} \mathbb{1}_{X_n=y} + \sum_{p=1}^{\infty} r^p \sum_{n=\tau_p^{x_0}+1}^{\tau_{p+1}^{x_0}} \mathbb{1}_{X_n=y}\right].$$

Using the strong Markov property, one deduces

$$G_{x_0,\alpha,r}(x,y) = \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x)} \left(\mathbb{E}_x\left[L_{T_{x_0}}^y\right] + \frac{r}{1-r}\mathbb{E}_{x_0}\left[L_{T'_{x_0}}^y - L_0^y\right]\right).$$

Now, under $\mathbb{P}_x$,

$$L_{T_{x_0}}^y = L_{T_{x_0}-1}^y + \mathbb{1}_{y=x_0} = L_{T'_{x_0}-1}^y \mathbb{1}_{x\neq x_0} + \mathbb{1}_{y=x_0}$$

and under $\mathbb{P}_{x_0}$,

$$L_{T'_{x_0}}^y - L_0^y = \left(L_{T'_{x_0}-1}^y + \mathbb{1}_{y=x_0}\right) - \mathbb{1}_{y=x_0} = L_{T'_{x_0}-1}^y.$$

Hence,

$$G_{x_0,\alpha,r}(x,y) = \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x)} \left(\mathbb{1}_{y=x_0} + G_{x_0}(x,y)\mathbb{1}_{x\neq x_0} + \frac{r}{1-r}G_{x_0}(x_0,y)\right).$$

In particular,

$$G_{x_0,\alpha,r}(x_0,y) = \frac{\psi_{x_0,\alpha,r}(y)}{\psi_{x_0,\alpha,r}(x_0)} \left(\mathbb{1}_{y=x_0} + \frac{r}{1-r}G_{x_0}(x_0,y)\right).$$

Taking the quotient of the two expressions, we get, after dividing the numerator and the denominator by $rG_{x_0}(x_0,y)/(1-r)$, and by checking separately the cases $y \neq x_0$ and $y = x_0$,

$$K_{x_0,\alpha,r}(x,y) = \frac{\psi_{x_0,\alpha,r}(x_0)}{\psi_{x_0,\alpha,r}(x)} \left(1 + \frac{1-r}{r}L_{x_0}(x,y)\mathbb{1}_{x\neq x_0}\right).$$

For $x, (y_n)_{n\geq 1}$ in E, it is then clear that $K_{x_0,\alpha,r}(x, y_n)$ converges if and only if $L_{x_0}(x, y_n)$ converges: this equivalence is also true for $x = x_0$, since $K_{x_0,\alpha,r}(x_0, y_n) = L_{x_0}(x_0, y_n) = 1$. This equivalence implies the equality, up to a canonical homeomorphism, of the Martin boundaries associated to $(\mathbb{P}_x)_{x\in E}$ and $(\mathbb{P}_x^{x_0,\alpha,r})_{x\in E}$.

Let us now check the equality of the minimal boundaries. It is straightforward to check that there is a bijective map $\mathscr{R}$ from the set of functions φ from E to $\mathbb{R}$ for which $\varphi(x_0) = 0$ and $\mathbb{E}_x[\varphi(X_1)] = \varphi(x)$ if $x \neq x_0$, to the set of functions h from E to $\mathbb{R}$ which are harmonic with respect to the Markov chain associated to $(\mathbb{P}_x^{x_0,\alpha,r})_{x\in E}$. This map is given as follows:

$$\mathscr{R}(\varphi)(x) = \frac{1}{\psi_{x_0,\alpha,r}(x)} \left(\varphi(x) + \frac{r}{1-r}\mathbb{E}_{x_0}[\varphi(X_1)]\right),$$

and one has

$$\mathscr{R}^{-1}(h)(x) = \psi_{x_0,\alpha,r}(x)h(x) - r\mathbb{E}_{x_0}[\psi_{x_0,\alpha,r}(X_1)h(X_1)].$$

It is obvious that $\mathscr{R}$ and $\mathscr{R}^{-1}$ are linear maps, and that $\mathscr{R}$ sends nonnegative functions to nonnegative functions. Moreover this last property is also true for $\mathscr{R}^{-1}$. Indeed, if h is nonnegative, then $\mathscr{R}^{-1}(h)$ is harmonic for $(\mathbb{P}_x)_{x\in E}$ at every point except x_0, vanishes at x_0 and is bounded from below by

$$-C := -r\mathbb{E}_{x_0}[\psi_{x_0,\alpha,r}(X_1)h(X_1)].$$

Hence, for all $x \in E$,

$$\mathscr{R}^{-1}(h)(x) = \mathbb{E}_x[\mathscr{R}^{-1}(h)(X_{n\wedge T_{x_0}})] = \mathbb{E}_x[\mathscr{R}^{-1}(h)(X_n)\mathbb{1}_{T_{x_0}>n}] \geq -C\mathbb{P}_x[T_{x_0} > n],$$

and letting $n \to \infty$,

$$\mathscr{R}^{-1}(h)(x) \geq 0,$$

since by the recurrence of the canonical process under $\mathbb{P}_x$,

$$\mathbb{P}_x[T_{x_0} > n] \underset{n\to\infty}{\longrightarrow} 0.$$

One deduces that $\mathscr{R}$ and $\mathscr{R}^{-1}$ preserve the minimality of the corresponding harmonic functions. Moreover, one has for all $\gamma \in \partial_m E$,

$$\begin{aligned}\mathscr{R}\left(\varphi_{x_0,\gamma}\right)(x) &= \frac{1}{\psi_{x_0,\alpha,r}(x)} \left(\frac{1}{\beta(x_0)}L_{x_0}(x,\gamma)\mathbb{1}_{x\neq x_0} + \frac{r}{1-r}\mathbb{E}_{x_0}[\varphi_{x_0,\gamma}(X_1)]\right)\\ &= \frac{1}{\beta(x_0)\psi_{x_0,\alpha,r}(x)}\left(L_{x_0}(x,\gamma)\mathbb{1}_{x\neq x_0} + \frac{r}{1-r}\right)\\ &= \frac{r}{(1-r)\beta(x_0)\psi_{x_0,\alpha,r}(x_0)}K_{x_0,\alpha,r}(x,\gamma).\end{aligned}$$

In the second equality, we use that

$$\mathbb{E}_{x_0}[\varphi_{x_0,\gamma}(X_1)] = \frac{1}{\beta(x_0)},$$

which is a consequence of Proposition 5 applied to $1/\beta(x_0)$ times the Dirac measure at γ. Since $\gamma \in \partial_m E$, $\varphi_{x_0,\gamma}$ is minimal as a nonnegative function vanishing at x_0 and $(\mathbb{P}_x)_{x\in E}$-harmonic outside x_0, $\mathscr{R}(\varphi_{x_0,\gamma})$ is then minimal as a $(\mathbb{P}_x^{x_0,\alpha,r})_{x\in E}$-harmonic function, and by the previous computation, $x \mapsto K_{x_0,\alpha,r}(x,\gamma)$ is also minimal, which implies that γ is also in the minimal boundary of E for the transient Markov chain $(\mathbb{P}_x^{x_0,\alpha,r})_{x\in E}$. Using the reverse map $\mathscr{R}^{-1}$, we deduce similarly that any point in the minimal boundary for $(\mathscr{P}_x^{x_0,\alpha,r})_{x\in E}$ is also in the minimal boundary for $(\mathbb{P}_x)_{x\in E}$. We have then the identity (up to canonical homeomorphism) between the two minimal boundaries.

Moreover, for $\gamma = \alpha$, we get the following:

$$\begin{aligned} K_{x_0,\alpha,r}(x,\alpha) &= \frac{\psi_{x_0,\alpha,r}(x_0)}{\psi_{x_0,\alpha,r}(x)}\left(1 + \frac{1-r}{r} L_{x_0}(x,\alpha)\mathbb{1}_{x\neq x_0}\right) \\ &= \left(\frac{1-r}{r}\right)\beta(x_0)\psi_{x_0,\alpha,r}(x_0). \end{aligned}$$

Hence, if we refer to Proposition 4, the constant function equal to 1 can be written as follows:

$$1 = \int_{\partial_m E} K_{x_0,\alpha,r}(x,\beta)d\mu(\beta)$$

where

$$\mu = \frac{r}{(1-r)\beta(x_0)\psi_{x_0,\alpha,r}(x_0)}\delta_\alpha,$$

δ_α denoting the Dirac measure at α. Since μ is carried by α, and $\alpha \in \partial_m E$ by assumption, the last statement of Proposition 10 is proven.

We can now easily finish the proof of Proposition 8. Applying Theorem 3.2. of Kemeny and Snell [4] to the transient Markov chain associated to $\mathbb{P}_x^{x_0,\alpha,r}$ and to the constant harmonic function $h = 1$, and using the last statement of Proposition 10, we deduce that $\mathbb{P}_x^{x_0,\alpha,r}$-almost surely, the canonical trajectory tends to α. Since $\mathbb{Q}_x^\alpha$ is absolutely continuous with respect to $\mathbb{P}_x^{x_0,\alpha,r}$ (with density $r^{-L_\infty^{x_0}}$), the canonical trajectory also tends to α under $\mathbb{Q}_x^\alpha$.

From the fact that $(\mathbb{Q}_x^\alpha)_{x\in E}$ satisfies the condition (34) and from Proposition 8, we deduce the following informal interpretation: under $\mathbb{Q}_x^\alpha$, the canonical process corresponds to the Markov chain given by $\mathbb{P}_x$, conditioned to tend to α at infinity. Of course, this interpretation is not rigorous since $\mathbb{Q}_x^\alpha$ is not a probability measure in general, and even not a finite measure. Moreover, under $\mathbb{P}_x$, the canonical process is recurrent, so it cannot converge to a point of the Martin boundary.

5 Some Examples

In this section, we look again at the examples given in Chap. 4 of [11].

5.1 The Simple Random Walk on $\mathbb{Z}$

The simple random walk on $\mathbb{Z}$ is the Markov chain given by the transition probabilities $(p_{x,y})_{x,y \in \mathbb{Z}}$ where $p_{x,x+1} = p_{x,x-1} = 1/2$ and $p_{x,y} = 0$ if $|y - x| \neq 1$. For all $x \in \mathbb{Z}$, $p_{x,y} = 0$ for all but finitely many $y \in \mathbb{Z}$, and the simple random walk is irreducible and recurrent. We can then do the construction given in Chap. 4 of [11] and in the present article. If we take $x_0 = 0$, we get, by using standard martingale arguments,

$$G_0(0, y) = 1$$

for all $y \in \mathbb{Z}$, and for all $x \in \mathbb{Z}\backslash\{0\}$ and $y \in \mathbb{Z}$,

$$G_0(x, y) = 2(|x| \wedge |y|)\mathbb{1}_{xy>0}.$$

Hence, $L_0(0, y) = 1$ and for $x \neq 0$,

$$L_0(x, y) = 2(|x| \wedge |y|)\mathbb{1}_{xy>0}.$$

We deduce that the Martin boundary of the standard random walk has exactly two points. We denote these points $-\infty$ and ∞, the distinction between them being given by the formulas:

$$L_0(x, \infty) = 2x_+ + \mathbb{1}_{x=0},\ L_0(x, -\infty) = 2x_- + \mathbb{1}_{x=0}.$$

This notation is justified by the following fact: a sequence of points in $\mathbb{Z}$ tends to ∞ in the Martin compactification of $\mathbb{Z}$ if and only if it tends to ∞ in the usual sense, and the similar statement is true for $-\infty$. If we normalize the stationary measure by taking $\beta(x) = 1$ for all $x \in \mathbb{Z}$, we get

$$\varphi_{0,\infty}(x) = 2x_+, \varphi_{0,-\infty}(x) = 2x_-.$$

The nonnegative functions φ such that $\varphi(0) = 0$ and φ is harmonic at all $x \neq 0$ are exactly the linear combinations of $\varphi_{0,\infty}$ and $\varphi_{0,-\infty}$ with nonnegative coefficients.

We deduce that the minimal boundary of $\mathbb{Z}$ is equal to its Martin boundary, i.e. has the two points $-\infty$ and ∞. The families of σ-finite measures in the class $\mathscr{Q}$ are then exactly the families of the form $(\alpha\mathbb{Q}_x^{\infty} + \beta\mathbb{Q}_x^{-\infty})_{x\in\mathbb{Z}}$ for $\alpha, \beta \geq 0$. Hence, we do not obtain other measures than those given in Sect. 4.3.1 of [11].

5.2 The Simple Random Walk in $\mathbb{Z}^2$

In this case, we have $E = \mathbb{Z}^2$ and the transition probabilities are given by $p_{x,y} = 1/4$ if $||x-y|| = 1$ and $p_{x,y} = 0$ otherwise. It has been shown that in this situation, there exists, up to a multiplicative constant, a unique nonnegative function which vanishes at $(0,0)$ and which is harmonic everywhere else. This property is, for example, stated in Sect. 31 of [15] (statement P3), in the case where we replace the simple random walk by a general irreducible, recurrent, aperiodic Markov chain in $\mathbb{Z}^2$, for which the increments are i.i.d. random variables. Since the simple random walk is not aperiodic, the result in [15] doesn't apply directly. However, it is easy to deal with this problem: if a is a nonnegative function, vanishing at $(0,0)$ and harmonic elsewhere for the transitions $(p_{x,y})_{x,y\in\mathbb{Z}^2}$, then for $E' := \{(a,b) \in \mathbb{Z}^2, a+b \text{ even}\}$, the restriction of a to E' is harmonic, except at $(0,0)$, for the transition p^2 obtained by iterating two steps of the Markov chain with transition p, i.e. $(p^2)_{x,y}$ is $1/16$ for $||x-y|| = 2$, $1/8$ for $||x-y|| = \sqrt{2}$ and $1/4$ for $x = y$. This Markov chain on E' is irreducible, recurrent, aperiodic, and then the restriction of a to E' is uniquely determined up to a multiplicative constant. Now for $x \in \mathbb{Z}^2\backslash E'$, $a(x)$ is the average of the four numbers $a(x \pm (0,1))$, $a(x \pm (1,0))$, where $x \pm (0,1)$ and $x \pm (1,0)$ are in E', so it is also uniquely determined. We have already written the expression of a in Sect. 4.3.5 of [11]: if the multiplicative constant is suitably chosen, then a is the so-called *potential kernel*, given by

$$a(x) = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\mathbb{P}_{(0,0)}(X_n = (0,0)) - \sum_{n=0}^{N}\mathbb{P}_x(X_n = (0,0))\right).$$

In Sect. 15 of [15], some explicit values of a are given. If $x = (0,0)$, then $a(x) = 0$, if $||x|| = 1$, then $a(x) = 1$, and for all $n \geq 1$,

$$a(\pm n, \pm n) = \frac{4}{\pi}\sum_{j=1}^{n}\frac{1}{2j-1}.$$

Knowing these values is sufficient to successively recover all the values of a, by only using the fact that a is harmonic, and that a has the same symmetries as the lattice $\mathbb{Z}^2$. For example, we get

$$4a(1,0) = a(2,0) + a(0,0) + a(1,1) + a(1,-1),$$

and then

$$a(2,0) = 4a(1,0) - a(0,0) - 2a(1,1) = 4 - \frac{8}{\pi}.$$

Similarly,

$$4a(1,1) = a(1,2) + a(2,1) + a(1,0) + a(0,1) = 2a(2,1) + 2a(1,0),$$
$$a(2,1) = 2a(1,1) - a(1,0) = \frac{8}{\pi} - 1$$

and so on. In particular, for all $x \in \mathbb{Z}^2$, $a(x) \in \mathbb{Q} + \frac{1}{\pi}\mathbb{Q}$. The following asymptotics has been given by Stöhr [16], then improved and generalized by Fukai and Uchiyama [2]:

$$a(x) = \frac{2}{\pi} \log ||x|| + \frac{2\gamma + \log 8}{\pi} + O(1/||x||^2),$$

where γ is the Euler-Mascheroni constant.

The uniqueness of a, up to a multiplicative constant, shows that the simple random walk in $\mathbb{Z}^2$ has a Martin boundary with only one point, which can naturally be denoted ∞. If we go back to the definition of the Martin boundary given here, we deduce that for all $x \in \mathbb{Z}^2 \backslash \{0\}$,

$$\frac{G_{(0,0)}(x, y)}{G_{(0,0)}((0,0), y)} \underset{||y|| \to \infty}{\longrightarrow} Ca(x)$$

for some constant $C > 0$. Since the counting measure is invariant for the simple random walk we deduce that $G_{(0,0)}((0,0), y) = 1$ for all $y \in \mathbb{Z}^2$, and then for $x \neq (0,0)$,

$$\mathbb{E}_x[L^y_{T_{(0,0)}}] \underset{||y|| \to \infty}{\longrightarrow} Ca(x).$$

Moreover, by the Markov property, for $y \neq (0,0)$,

$$\mathbb{E}_{(0,0)}[L^y_{T'_{(0,0)}}] = \frac{1}{4} \sum_{x \in \{(0,1),(0,-1),(1,0),(-1,0)\}} \mathbb{E}_x[L^y_{T_{(0,0)}}],$$

and then, by letting $||y|| \to \infty$,

$$1 = \frac{1}{4} \sum_{x \in \{(0,1),(0,-1),(1,0),(-1,0)\}} Ca(x) = Ca(0,1),$$

and then $C = 1$, and

$$\mathbb{E}_x[L^y_{T(0,0)}] \underset{||y||\to\infty}{\longrightarrow} a(x).$$

Since the Martin boundary of E has only one point in this example, the class $\mathscr{Q}$ contains only the nonnegative multiples of the family of measures $(\mathbb{Q}_x^\infty)_{x\in\mathbb{Z}^2}$.

The results given here on the simple random walk in $\mathbb{Z}$ or $\mathbb{Z}^2$, its potential kernel and its Martin boundary has been adapted to more general random walks on groups. For example, see Kesten [5] or Kesten and Spitzer [6].

5.3 The "Bang-Bang Random Walk"

This Markov chain is given in Sect. 4.3.2 of [11]. We have $E = \mathbb{N}_0$, the set of nonnegative integers, and the transition probabilities are given by $p_{0,1} = 1$, and for all $y \geq 1$, $p_{y,y+1} = q \in (0, 1/2)$, $p_{y,y-1} = 1 - q$ and $p_{x,y} = 0$ for $|y - x| \neq 1$ (in [11], only the case $q = 1/3$ is considered, but the generalization is straightforward). It is easy to check that the Markov chain is irreducible and recurrent. Moreover, for $\alpha := (1 - q)/q$, $(\alpha^{X_{n\wedge T_0}})_{n\geq 0}$ is a martingale under $\mathbb{P}_x$ for all $n \in \mathbb{N}_0$. By a standard martingale argument, one deduces that for $0 < x < y$,

$$\mathbb{P}_x(T_y < T_0) = \frac{\alpha^x - 1}{\alpha^y - 1}.$$

Now, for all $y > 0$, under $\mathbb{P}_y$:

- With probability q, $X_1 = y + 1$ and then the Markov chain goes almost surely back to y before hitting 0.
- With probability $1 - q$, $X_1 = y - 1$, and then the conditional probability that the Markov chain goes to 0 before returning to y is

$$\mathbb{P}_{y-1}(T_y > T_0) = 1 - \frac{\alpha^{y-1} - 1}{\alpha^y - 1}.$$

Hence, the probability that the Markov hits 0 before returning to y is

$$\mathbb{P}_y[T_0 < \tau_2^y] = (1 - q)\frac{\alpha^y - \alpha^{y-1}}{\alpha^y - 1} = (1 - q)(1 - (1/\alpha))\frac{\alpha^y}{\alpha^y - 1} = \frac{(1 - 2q)\alpha^y}{\alpha^y - 1}.$$

If we choose $x_0 = 0$, we deduce

$$G_0(y, y) = \frac{\alpha^y - 1}{(1 - 2q)\alpha^y} = \frac{1 - [q/(1 - q)]^y}{1 - 2q}.$$

Applying the Markov property to the first hitting time of y, we deduce that for $x \geq y > 0$,

$$G_0(x, y) = G_0(y, y)$$

and for $y > 0$, $x \leq y$,

$$G_0(x, y) = \frac{\alpha^x - 1}{(1 - 2q)\alpha^y}.$$

If $x > 0$, one obviously has

$$G_0(x, 0) = 0,$$

one has

$$G_0(0, 0) = 1,$$

and for all $y > 0$,

$$G_0(0, y) = G_0(1, y) = \frac{\alpha - 1}{(1 - 2q)\alpha^y} = \frac{1}{q\alpha^y}.$$

For $x, y > 0$, one gets

$$L_0(x, y) = \frac{q(\alpha^{x \wedge y} - 1)}{(1 - 2q)},$$

and

$$L_0(0, y) = 1.$$

Hence, for all $x \in \mathbb{N}_0$, $L_0(x, y)$ converges when y goes to infinity. The Martin boundary has then only one point denoted ∞, and

$$L_0(x, \infty) = \frac{q(\alpha^x - 1)}{(1 - 2q)}, L_0(0, \infty) = 1.$$

In this setting, the exists a unique stationary probability measure, given by

$$\beta(0) = \frac{1 - 2q}{2(1 - q)}$$

and for all $x > 0$,

$$\beta(x) = \frac{1 - 2q}{2q(1 - q)\alpha^x}.$$

With this normalization, we get for all $x \geq 1$:

$$\varphi_{0,\infty}(x) = \frac{2q(1-q)}{(1-2q)^2}(\alpha^x - 1).$$

We then get, up to a multiplicative constant, a unique family of σ-finite measures $(\mathbb{Q}_x^\infty)_{x\in\mathbb{N}_0}$, described in Sect. 4.3.2 of [11] in the case $q = 1/3$, and then $\alpha = 2$.

5.4 The Random Walk on a Tree

Here, we consider an infinite k-ary tree for $k \geq 2$. It can be represented by the set E of all finite (possibly empty) sequences of elements in $\{0, 1, \ldots, k-1\}$. The transition probabilities we consider are given by $p_{\emptyset,(j)} = 1/k$ for all $j \in \{0, \ldots, k-1\}$, $p_{x,y} = 1/2$, $p_{y,x} = 1/2k$ if x is a nonempty sequence and y is obtained from x by removing the last element: we will say that y is the father of x and x is a son of y. All the other transition probabilities are equal to zero. With these transitions, under $\mathbb{P}_x$ for any $x \in E$, $(L_n)_{n\geq 1}$ is a reflected standard random walk if L_n denotes the length of the sequence X_n. One deduces that the Markov chain is irreducible and recurrent. We choose $x_0 = \emptyset$. Let $x, y \in E$, $x, y \neq \emptyset$, and let z be the last common ancestor of x and y. It is clear that under $\mathbb{P}_x$, the canonical process almost surely hits z before y. Using the strong Markov property, we deduce that for $z = \emptyset$,

$$G_\emptyset(x, y) = 0,$$

and for $z \neq \emptyset$,

$$G_\emptyset(x, y) = G_\emptyset(z, y).$$

Let $z_0 = \emptyset, z_1, z_2, \ldots, z_p = y$ be the ancestors of y, the sequence z_j having j elements. Under $\mathbb{P}_{z_j}$, $2 \leq j \leq p-1$, $X_1 = z_{j-1}$ with probability $1/2$, $X_1 = z_{j+1}$ with probability $1/2k$ and X_1 is another son of z_j with probability $(k-1)/2k$. One deduces

$$G_\emptyset(z_j, y) = \frac{1}{2}G_\emptyset(z_{j-1}, y) + \frac{1}{2k}G_\emptyset(z_{j+1}, y) + \frac{k-1}{2k}G_\emptyset(z_j, y),$$

and then

$$G_\emptyset(z_j, y) = \frac{k}{k+1}G_\emptyset(z_{j-1}, y) + \frac{1}{k+1}G_\emptyset(z_{j+1}, y).$$

Similarly, for $p \geq 2$,

$$G_\emptyset(z_1, y) = \frac{1}{k+1}G_\emptyset(z_2, y)$$

and

$$G_{\emptyset}(y,y) = G_{\emptyset}(z_p,y) = 1 + \frac{1}{2}G_{\emptyset}(z_p,y) + \frac{1}{2}G_{\emptyset}(z_{p-1},y),$$

i.e.

$$G_{\emptyset}(y,y) = 2 + G_{\emptyset}(z_{p-1},y).$$

Finally, for $p = 1$,

$$G_{\emptyset}(y,y) = 1 + \frac{1}{2}G_{\emptyset}(y,y) = 2.$$

From these equations, we deduce for $1 \leq j \leq p$,

$$G_{\emptyset}(z_j,y) = \frac{2(k^j - 1)}{k^{p-1}(k-1)}.$$

Hence, for $x, y \neq \emptyset$,

$$G_{\emptyset}(x,y) = \frac{2(k^j - 1)}{k^{p-1}(k-1)},$$

where j denotes the number of elements of the last common ancestor of x and y. Moreover, one has $G_{\emptyset}(x,\emptyset) = 0$ for $x \neq \emptyset$, $G_{\emptyset}(\emptyset,\emptyset) = 1$, and for $y = z_p \neq \emptyset$,

$$G_{\emptyset}(\emptyset,y) = \frac{1}{k}G_{\emptyset}(z_1,y) = \frac{2}{k^p}.$$

We then get for $x, y \neq \emptyset$,

$$L_{\emptyset}(x,y) = \frac{k(k^j - 1)}{k-1},$$
$$L_{\emptyset}(x,\emptyset) = 0,\ L_{\emptyset}(\emptyset,y) = L_{\emptyset}(\emptyset,\emptyset) = 1.$$

Now, let $(x_n)_{n\geq 1}$ be a sequence in E which converges in its Martin compactification. If the length of x_n does not tend to infinity, then x_n takes the same value infinitely often, and then the limit of x_n is equal to this value. If the length of x_n tends to infinity, then for all $m \geq 1$, x_n has at least m elements for n large enough. Necessarily, there exists a sequence y_m of m elements such that x_n starts with y_m infinitely often. Now, let us assume that another sequence y'_m satisfies the same property. In this case, the last common ancestor of x_n and y_m is y_m itself infinitely often, and some strict ancestor of y_m infinitely often. Hence, $L_{\emptyset}(x_n, y_m)$ is $k(k^m - 1)/(k-1)$ infinitely often, and $k(k^j - 1)/(k-1)$ for $j < m$ infinitely often, which contradicts the convergence of $(x_n)_{n\geq 1}$ in the Martin compactification of E. Hence, for all $m \geq 1$, there exists a sequence y_m of length m such that x_n starts with y_m for all n large enough. Conversely, if $(x_n)_{n\geq 1}$ satisfies the property given in the previous sentence, then for all $x \in E$ different from $\emptyset$, and for n large enough, the last common ancestor of x_n and x is the same as the last common ancestor of y_m and x, if $m \geq 1$ is larger than or equal to the length of x. Hence, for n large enough,

$$L_{\emptyset}(x, x_n) = L_{\emptyset}(x, y_m),$$

which implies the convergence of $(x_n)_{n\geq 1}$ in the Martin compactification of E. If $(x_n)_{n\geq 1}$ converges to a limit which is not in E, then the limit α in the Martin boundary of E is defined by the function

$$x \mapsto L_{\emptyset}(x, \alpha) = \lim_{n\to\infty} L_{\emptyset}(x, x_n) = L_{\emptyset}(x, y_m),$$

if $m \geq 1$ is at least the length of x. The sequences $(y_m)_{m\geq 1}$ are compatible with each other, i.e. for all $m \geq 1$, y_m is the father of y_{m+1}. Hence, there exists an sequence y_∞ such that for all $m \geq 1$, y_m is the sequence of the m first elements of y_∞. We deduce that $\alpha \in \partial E$ is identified by the function:

$$x \mapsto L_{\emptyset}(x, \alpha) = \mathbb{1}_{x=\emptyset} + \frac{k(k^j - 1)}{k-1}\mathbb{1}_{x\neq\emptyset},$$

where j is the largest integer such that the j first elements of x and y_∞ are the same. The function $x \mapsto L_{\emptyset}(x, \alpha)$ depends only on the sequence y_∞, and one easily checks that we obtain different functions for different infinite sequences. Hence α can be identified with y_∞. We deduce that ∂E can be identified with the set of all infinite sequences of elements in $\{0, 1, \ldots, k-1\}$. Intuitively, these sequences correspond to the "leafs of the tree". The topology of ∂E is given by the following convergence: $(y_\infty^{(n)})_{n\geq 1}$ converges to y_∞ if and only if for all $k \geq 1$, the k-the element of $y_\infty^{(n)}$ and the k-th element of y_∞ coincide for all but finitely many $n \geq 1$. The topology of ∂E is also induced by the distance D such that $D(y_\infty, y'_\infty) = 2^{-k}$, where k is the first integer such that the k-th elements of y_∞ and y'_∞ are different. If we normalize the stationary measure in such a way that $\beta(\emptyset) = k/(k-1)$, we get

$$\varphi_{\emptyset, y_\infty}(x) = k^j - 1$$

where j is the largest integer such that the j first elements of y_∞ and x coincide. This function is nonnegative, zero at $\emptyset$ and harmonic at every other point. It is minimal among such functions. Indeed, let φ be a function satisfying the same properties, which is smaller than or equal to $\varphi_{\emptyset, y_\infty}$. For $m \geq 0$, let E_m be the subset of E consisting of sequences whose m first elements coincide with those of y_∞. Let $F_m := E_m \backslash E_{m+1}$. If y_m is the sequence of the m first elements of y_∞, then for

$x \in F_m$, under $\mathbb{P}_x$, X_n is in $F_m \backslash \{y_m\}$ for all $n < T_{y_m}$. Now, for all $z \in F_m$,

$$\varphi(z) \leq \varphi_{\emptyset, y_\infty}(z) = k^m - 1,$$

which implies, since φ is harmonic at all points in $F_m \backslash \{y_m\}$, that $(\varphi(X_{n \wedge T_{y_m}}))_{n \geq 0}$ is a bounded martingale. Hence, under $\mathbb{P}_x$,

$$\varphi(x) = \varphi(X_0) = \mathbb{E}_x[\varphi(X_{T_{y_m}})] = \varphi(y_m).$$

Hence, $\varphi(x)$ depends only on the largest m such that the m first elements of x and y_∞ coincide. It is then sufficient to check that there exists $C > 0$ such that $\varphi(y_m) = C(k^m - 1)$ for all $m \geq 0$. Now, this result is a consequence of the fact that $\varphi(y_0) = \varphi(\emptyset) = 0$ and the harmonic property of φ, which imply that for $m \geq 1$,

$$\varphi(y_m) = \frac{1}{2}\varphi(y_{m-1}) + \frac{k-1}{2k}\varphi(y_m) + \frac{1}{2k}\varphi(y_{m+1}).$$

We have then proven that $\varphi_{\emptyset, y_\infty}$ is minimal, and then the minimal boundary $\partial_m E$ is equal to the Martin boundary ∂E. We have then again a complete description of the families of measures $(\mathbb{Q}_x^\alpha)_{x \in E}$ for $\alpha \in \partial_m E$. In the case $k = 2$, all these measures were already described in Sect. 4.3.3 of [11]. Integrating with respect to a finite measure on $\partial_m E$ gives all the possible families of measures in the class $\mathscr{Q}$. Note that in this example, the Martin boundary is uncountable.

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Loop Measures Without Transition Probabilities

Pat Fitzsimmons, Yves Le Jan, and Jay Rosen

Abstract The goal of this paper is to define and study loop measures for Markov processes without transition densities. In particular, we prove the shift invariance of the based loop measure.

Subject Classifications: Primary 60K99, 60J55; Secondary 60G17

1 Introduction

To the best of our knowledge, Brownian loop measures first appeared in the work of Symanzik on Euclidean quantum field theory [12], where they are referred to as 'blob measures'. Then discrete random walk loops were used in statistical mechanics (see in particular Brydges et al. [2]). Brownian loop measures next appear in the work of Lawler and Werner [7]. In [8] loop measures associated with a large class of Markov processes are defined and studied. In all these cases it is assumed that the underlying Markov process has transition densities. The goal of this paper is to define and study loop measures for Markov processes without transition densities. In particular, we prove the shift invariance of the based loop measure. Loop measures for processes with finite potential densities but without transition densities are discussed in [5]. Such a situation can occur for non-symmetric processes, such as Poisson processes with drift. In the present paper we

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give an alternative construction, assuming only that the potential densities $u(x, y)$ are finite off the diagonal. We allow them to be infinite on the diagonal, and in this case the construction of [5] breaks down. A simple example is spelled out at the end of the introduction.

Let $X = (\Omega, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process [11] with state space S, which we assume to be locally compact with a countable base. We use the canonical representation of X in which Ω is the set of right continuous paths $\omega : [0, \infty) \to S_\Delta = S \cup \Delta$ with $\Delta \notin S$, and is such that $\omega(t) = \Delta$ for all $t \geq \zeta = \inf\{t > 0 \,|\omega(t) = \Delta\}$. Set $X_t(\omega) = \omega(t)$.

Let m be a Borel measure on S which is finite on compact sets. We assume that with respect to m, X has strictly positive potential densities $u^\alpha(x, y)$, $\alpha \geq 0$, which satisfy the resolvent equations. We set $u(x, y) = u^0(x, y)$, and assume that $u(x, y)$ is excessive in x for each fixed y.

Let $h_z(x) = u(x, z)$. If we assume that u is finite, then the h_z-transform of X is a right process on S, see [11, Sect. 62], with probabilities P^{x/h_z}. Let $Q^{z,z} = u(z, z)P^{z/h_z}$. We can then define the loop measure as

$$\mu(F) = \int Q^{z,z}\left(\frac{F}{\zeta}\right) dm(z), \tag{1}$$

for any $\mathscr{F}$ measurable function F. Loop measures for processes with finite potential densities but without transition densities are discussed in [5]. In the present paper we assume that the potential densities $u(x, y)$ can be infinite on the diagonal, but are finite off the diagonal. Assuming that all points are polar, we show how to construct a family of measures $Q^{z,z}, z \in S$, which generalize the measures $Q^{z,z} = u(z, z)P^{z/h_z}$ in the case of finite $u(x, y)$.

After constructing $Q^{z,z}, z \in S$ and defining the loop measure μ using (1), we show how to calculate some important moments. We assume that

$$\sup_x \int_K (u(x, y) + u(y, x))^2 \; dm(y) < \infty \tag{2}$$

for any compact $K \subseteq S$. For exponentially killed Brownian motion in R^d this means that $d \leq 3$.

Theorem 1 *For any $k \geq 2$, and bounded measurable functions $f_1, \ldots, f_k$ with compact support*

$$\mu\left(\prod_{j=1}^{k} \int_0^\infty f_j(X_{t_j})\, dt_j\right) \tag{3}$$
$$= \sum_{\pi \in \mathscr{P}_k^\odot} \int u(x_1, x_2)u(x_2, x_3)\cdots u(x_k, x_1) \prod_{j=1}^{k} f_{\pi_j}(x_j)\, dm(x_j),$$

where $\mathscr{P}_k^{\odot}$ *denotes the set of permutations of* $[1,k] = \{1,2,\ldots,k\}$ *on the circle. (For example,* $(1,2,3)$*,* $(3,1,2)$ *and* $(2,3,1)$ *are considered to be one permutation* $\pi \in \mathscr{P}_3^{\odot}$*.)*

Our assumption (2) will guarantee that the right hand side of (3) is finite. Note that if $k = 1$ our formula would give

$$\mu\left(\int_0^\infty f(X_t)\,dt_j\right) = \int u(x,x)f(x)\,dm(x) = \infty, \tag{4}$$

for any $f \geq 0$, by our assumption that the potentials $u(x,y)$ are infinite on the diagonal.

For $f_1,\ldots,f_k$ as above consider more generally the multiple integral

$$M_t^{f_1,\ldots,f_k} = \sum_{\pi\in\mathscr{T}_k^{\odot}} \int_{0\le r_1\le\cdots\le r_k\le t} f_{\pi(1)}(X_{r_1})\cdots f_{\pi(k)}(X_{r_k})\,dr_1\cdots dr_k, \tag{5}$$

where $\mathscr{T}_k^{\odot}$ denotes the set of translations π of $[1,k]$ which are cyclic mod k, that is, for some i, $\pi(j) = j+i, \mod k$, for all $j = 1,\ldots,k$. In the proof of Theorem 1 we first show that

$$\mu\left(M_\infty^{f_1,\ldots,f_k}\right) = \int u(x_1,x_2)u(x_2,x_3)\cdots u(x_k,x_1)\prod_{j=1}^k f_j(x_j)\,dm(x_j). \tag{6}$$

Equation (3) will then follow since

$$\prod_{j=1}^k \int_0^\infty f_j(X_{t_j})\,dt_j = \sum_{\pi\in\mathscr{P}_k^{\odot}} M_\infty^{f_{\pi_1},\ldots,f_{\pi_k}}. \tag{7}$$

There is a related measure which we shall use which gives finite values even for $k = 1$. Set

$$\nu(F) = \int Q^{z,z}(F)\;dm(z). \tag{8}$$

Assume that for any $\alpha > 0$, any compact $K \subseteq S$, and any $\tilde{K}$ which is a compact neighborhood of K

$$\sup_{z\in\tilde{K}^c,\,x\in K} u^\alpha(z,x) < \infty. \tag{9}$$

Theorem 2 *For any $k \geq 1$, $\alpha > 0$, and bounded measurable functions $f_1, \ldots, f_k$ with compact support*

$$\nu\left(\prod_{j=1}^{k}\int_0^\infty f_j(X_{t_j})\,dt_j e^{-\alpha\zeta}\right) \tag{10}$$
$$= \sum_{\pi\in\mathscr{P}_k}\int u^\alpha(z,x_1)u^\alpha(x_1,x_2)\cdots u^\alpha(x_k,z)\prod_{j=1}^{k} f_{\pi_j}(x_j)\,dm(x_j)\,dm(z),$$

where $\mathscr{P}_k$ denotes the set of permutations of $[1,k]$, and both sides are finite.

We call μ the loop measure of X because, when X has continuous paths, μ is concentrated on the set of continuous loops with a distinguished starting point (since $Q^{x,x}$ is carried by loops starting at x). Moreover, in the next Theorem we show that it is shift invariant. More precisely, let ρ_u denote the loop rotation defined by

$$\rho_u\omega(s) = \begin{cases} \omega(s+u \bmod \zeta(\omega)), & \text{if } 0 \leq s < \zeta(\omega) \\ \Delta, & \text{otherwise.} \end{cases}$$

Here, for two positive numbers a, b we define $a \bmod b = a - mb$ for the unique positive integer m such that $0 \leq a - mb < b$.

For the next Theorem we need an additional assumption: for any $\delta > 0$ and compact $K \subseteq S$

$$\int_K P_\delta(z,dx)u(x,z)\,dm(z) < \infty. \tag{11}$$

Theorem 3 *μ is invariant under ρ_u, for any u.*

Note that if we have transition densities $p_\delta(z,x)$ then

$$\int_K P_\delta(z,dx)u(x,z)\,dm(z) = \int\int_K p_\delta(z,x)u(x,z)\,dm(x)\,dm(z) \tag{12}$$
$$= \int_K\left(\int_\delta^\infty p_t(x,x)\,dt\right)dm(x).$$

In our work [5] on processes with transition densities, it was always assumed that $\sup_x \int_\delta^\infty p_t(x,x)\,dt < \infty$ for any $\delta > 0$, which indeed gives (11).

For the next Theorem we assume that the measure m is excessive. With this assumption there is always a dual process $\hat{X}$ (essentially uniquely determined), but in general it is a moderate Markov process. We assume that the measures $\hat{U}(\cdot, y)$ are absolutely continuous with respect to m for each $y \in S$.

For CAF's $L_t^{\nu_1}, \dots L_t^{\nu_k}$ with Revuz measures $\nu_1, \dots, \nu_k$, let

$$A_t^{\nu_1,\dots,\nu_k} = \sum_{\pi \in \mathscr{T}_k^{\odot}} \int_{0 \le r_1 \le \cdots \le r_k \le t} dL_{r_1}^{\nu_{\pi(1)}} \cdots dL_{r_k}^{\nu_{\pi(k)}}. \tag{13}$$

We refer to $A_t^{\nu_1,\dots,\nu_k}$ as a multiple CAF.

Theorem 4 *For any $k \ge 2$, and any CAF's $L_t^{\nu_1}, \dots L_t^{\nu_k}$ with Revuz measures $\nu_1, \dots, \nu_k$,*

$$\mu\left(A_\infty^{\nu_1,\dots,\nu_k}\right) = \int u(x_1, x_2)u(x_2, x_3) \cdots u(x_k, x_1) \prod_{j=1}^{k} d\nu_j(x_j). \tag{14}$$

and

$$\mu\left(\prod_{j=1}^{k} L_\infty^{\nu_j}\right) = \sum_{\pi \in \mathscr{P}_k^{\odot}} \int u(x_1, x_2)u(x_2, x_3) \cdots u(x_k, x_1) \prod_{j=1}^{k} d\nu_{\pi_j}(x_j). \tag{15}$$

The finiteness of the right hand side of (15) will depend on the potential densities $u(x, y)$ and the measures $\nu_1, \dots, \nu_k$. For a more thorough discussion see [9, (1.5)] and the paragraph there following (1.5).

We now describe some simple processes without transition densities but with potential densities which are finite off the diagonal and infinite on the diagonal. Consider the space-time process, see [11, Sect. 16], associated with Brownian motion or any other reasonable process $X \in S$ with finite transition densities $p_t(x, y)$ with respect to m. The forward space-time process $(X_t, r+t)$ does not have transition densities, but does have potential densities

$$u((x, r), (y, s)) = p_{s-r}(x, y) \cdot 1_{\{s>r\}}$$

which blow up as $(x, r) \to (y, s)$. This process, taking values in $S \times [0, \infty)$ has no loops but its projection on $S \times [0, 1)$ (the time coordinate being taken modulo 1) can have loops. We could also include negative Poissonian jumps in the time coordinate.

With the results of this paper, most of the results of [5, 9, 10] on loop measures, loop soups, CAF's and intersection local times will carry over to processes without transition densities.

2 Construction of $Q^{z,z}$

Let us fix $z \in S$ and consider the excessive function $h_z(x) := u(x, z)$, finite and strictly positive on the subspace $S_z := \{x \in S : x \neq z\}$. Doob's h-transform theory yields the existence of laws $P^{x,z}$, $x \in S_z$, on path space under which the coordinate process is Markov with transition semigroup

$$P_t^z(x, dy) := P_t(x, dy)\frac{h_z(y)}{h_z(x)}. \tag{16}$$

See, for example, [11, pp. 298–299]. Now consider the family of measures

$$\eta_t^z(dx) := P_t(z, dx)h_z(x). \tag{17}$$

Since we assume that the singleton $\{z\}$ is polar, the transition semigroup (P_t) will not charge $\{z\}$, so these may be viewed as measures on S_z or on S. Adopting the latter point of view, it is immediate that $(\eta_t^z)_{t>0}$ is an entrance law for (P_t^z). There is a general theorem guaranteeing the existence of a right process with one-dimensional distributions (η_t^z) and transition semigroup (P_t^z); see [6, Proposition (3.5)]. The law of this process is the desired $Q^{z,z}$. Aside from the entrance law identity $\eta_t^z P_s^z = \eta_{t+s}^z$, their result only requires that each of the measures η_t^z be σ-finite, which is clearly the case in the present discussion.

With this we immediately obtain, for $0 < t_1 < \cdots < t_k$,

$$\begin{aligned} Q^{z,z}\left(\prod_{j=1}^k f_j(X_{t_j})\right) &= \eta_{t_1}^z(dx_1)\prod_{j=2}^k P^z_{t_j-t_{j-1}}(x_{j-1}, dx_j)\prod_{j=1}^k f_j(x_j) \\ &= \prod_{j=1}^k P_{t_j-t_{j-1}}(x_{j-1}, dx_j)\prod_{j=1}^k f_j(x_j)u(x_k, z) \end{aligned} \tag{18}$$

with $t_0 = 0$ and $x_0 = z$. Hence

$$\begin{aligned} &Q^{z,z}\left(\int_{0<t_1<\cdots<t_k<\infty}\prod_{j=1}^k f_j(X_{t_j})\,dt_j\right) \\ &\quad = \int u(z, x_1)u(x_1, x_2)\cdots u(x_k, z)\prod_{j=1}^k f_j(x_j)\,dm(x_j), \end{aligned} \tag{19}$$

so that

$$\begin{aligned} &Q^{z,z}\left(\prod_{j=1}^k \int_0^\infty f_j(X_{t_j})\,dt_j\right) \\ &\quad = \sum_{\pi\in\mathscr{P}_k}\int u(z, x_1)u(x_1, x_2)\cdots u(x_k, z)\prod_{j=1}^k f_{\pi_j}(x_j)\,dm(x_j). \end{aligned} \tag{20}$$

Returning to (18) with $0 < t_1 < \cdots < t_k$ and using the fact that $\zeta > t_k$ implies that $\zeta = t_k + \zeta \circ \theta_{t_k}$ we have

$$
\begin{aligned}
&Q^{z,z}\left(\prod_{j=1}^{k} f_j(X_{t_j})e^{-\alpha\zeta}\right) \qquad (21)\\
&= Q^{z,z}\left(\prod_{j=1}^{k} f_j(X_{t_j})e^{-\alpha t_k}\left(e^{-\alpha\zeta}\circ\theta_{t_k}\right)\right)\\
&= \prod_{j=1}^{k} P^{\alpha}_{t_j-t_{j-1}}(x_{j-1},dx_j)\prod_{j=1}^{k} f_j(x_j)h_z(x_k)P^{x_k,z}\left(e^{-\alpha\zeta}\right).
\end{aligned}
$$

Note that by (16) and the fact that X has α-potential densities for all $\alpha \geq 0$

$$
\begin{aligned}
P^{x_k,z}\left(\int_0^\infty e^{-\alpha t}1_{\{S\}}(X_t)\,dt\right) &= \int_0^\infty e^{-\alpha t}P^{x_k,z}\left(1_{\{S\}}(X_t)\right)\,dt \qquad (22)\\
&= \int_0^\infty e^{-\alpha t}\int_S P_t(x_k,dy)\frac{h_z(y)}{h_z(x_k)}\,dt\\
&= \int u^\alpha(x_k,y)\frac{h_z(y)}{h_z(x_k)}\,dm(y).
\end{aligned}
$$

Combining this with our assumption that the α-potential densities satisfy the resolvent equation we see that

$$
\begin{aligned}
h_z(x_k)P^{x_k,z}\left(e^{-\alpha\zeta}\right) &= h_z(x_k)P^{x_k,z}\left(1-\alpha\int_0^\zeta e^{-\alpha t}\,dt\right) \qquad (23)\\
&= h_z(x_k)-\alpha h_z(x_k)P^{x_k,z}\left(\int_0^\infty e^{-\alpha t}1_{\{S\}}(X_t)\,dt\right)\\
&= u(x_k,z)-\alpha\int u^\alpha(x_k,y)u(y,z)\,dm(y) = u^\alpha(x_k,z).
\end{aligned}
$$

Using this in (21) we obtain

$$
\begin{aligned}
&Q^{z,z}\left(\prod_{j=1}^{k} f_j(X_{t_j})e^{-\alpha\zeta}\right) \qquad (24)\\
&= \prod_{j=1}^{k} P^{\alpha}_{t_j-t_{j-1}}(x_{j-1},dx_j)u^\alpha(x_k,z)\prod_{j=1}^{k} f_j(x_j).
\end{aligned}
$$

We then have

$$Q^{z,z}\left(e^{-\alpha\zeta}\int_{0<t_1<\cdots<t_k<\infty}\prod_{j=1}^{k} f_j(X_{t_j})\,dt_j\right) \tag{25}$$
$$=\int u^{\alpha}(z,x_1)u^{\alpha}(x_1,x_2)\cdots u^{\alpha}(x_k,z)\prod_{j=1}^{k} f_j(x_j)\,dm(x_j),$$

and consequently

$$Q^{z,z}\left(e^{-\alpha\zeta}\prod_{j=1}^{k}\int_0^{\infty} f_j(X_{t_j})\,dt_j\right) \tag{26}$$
$$=\sum_{\pi\in\mathscr{P}_k}\int u^{\alpha}(z,x_1)u^{\alpha}(x_1,x_2)\cdots u^{\alpha}(x_k,z)\prod_{j=1}^{k} f_{\pi_j}(x_j)\,dm(x_j).$$

3 The Loop Measure and Its Moments

Set

$$\mu(F)=\int Q^{z,z}\left(\frac{F}{\zeta}\right)\,dm(z). \tag{27}$$

Proof of Theorem 1 We use an argument from the proof of [5, Lemma 2.1], which is due to Symanzik [12].

It follows from the resolvent equation that the potential densities $u^{\beta}(x,y)$ are continuous and monotone decreasing in β, for $x\neq y$. Using this together with the resolvent equation and the monotone convergence theorem we obtain that for $x_k\neq x_1$

$$\int_S u^{\alpha}(x_k,z)u^{\alpha}(z,x_1)\,dm(z)=-\frac{d}{d\alpha}u^{\alpha}(x_k,x_1). \tag{28}$$

Hence using (25)

$$\int Q^{z,z}\left(e^{-\alpha\zeta}M_{\infty}^{f_1,\ldots,f_k}\right)\,dm(z) \tag{29}$$
$$=-\sum_{\pi\in\mathscr{T}_k^{\odot}}\int u^{\alpha}(x_1,x_2)u^{\alpha}(x_2,x_3)\cdots u^{\alpha}(x_{k-1},x_k)\frac{d}{d\alpha}u^{\alpha}(x_k,x_1)\prod_{j=1}^{k} f_{\pi_j}(x_j)\,dm(x_j)$$

$$= -\frac{d}{d\alpha} \int u^\alpha(x_1, x_2) u^\alpha(x_2, x_3) \cdots u^\alpha(x_{k-1}, x_k) u^\alpha(x_k, x_1) \prod_{j=1}^k f_j(x_j)\, dm(x_j).$$

For the last step we used the product rule for differentiation and the fact that in the middle line we are summing over all translations mod k.

Since, as mentioned, $u^\alpha(x, y)$ is monotone decreasing in α for $x \neq y$,

$$v(x, y) = \lim_{\alpha \to \infty} u^\alpha(x, y) \tag{30}$$

exists and

$$\int v(x, y) f(y)\, dm(y) = \lim_{\alpha \to \infty} \int_0^\infty e^{-\alpha t} \int P_t(x_k, dy) f(y)\, dt = 0. \tag{31}$$

Hence $v(x, y) = 0$ for m-a.e. y. Integrating (29) with respect to α from 0 to ∞ and using Fubini's theorem we then obtain (6). Equation (3) then follows by (7).

To show that the right hand side of (3) is finite we repeatedly use the Cauchy-Schwarz inequality and our assumption (2). See the proof of [9, Lemma 3.3]. □

Proof of Theorem 2 The formula (10) follows immediately from (26). When $k \geq 2$, the right hand side of (10) can be shown to be finite by repeatedly using the Cauchy-Schwarz inequality, our assumption (2) and the fact that $u^\alpha(x, z)$ is integrable in z for any $\alpha > 0$. When $k = 1$, if K is a compact set containing the support of f_1 and $\tilde{K}$ is a compact neighborhood of K, then

$$\begin{aligned} &\int \int u^\alpha(z, x) u^\alpha(x, z) f_1(x)\, dm(x)\, dm(z) \\ &= \int_{\tilde{K}} \int u^\alpha(z, x) u^\alpha(x, z) f_1(x)\, dm(x)\, dm(z) \\ &\quad + \int_{\tilde{K}^c} \int u^\alpha(z, x) u^\alpha(x, z) f_1(x)\, dm(x)\, dm(z). \end{aligned}$$

Using (2)

$$\begin{aligned} &\int_{\tilde{K}} \left(\int u^\alpha(z, x) u^\alpha(x, z) f_1(x)\, dm(x) \right) dm(z) \\ &\leq m(\tilde{K}) \sup_z \int u^\alpha(z, x) u^\alpha(x, z) f_1(x)\, dm(x) < \infty, \end{aligned} \tag{32}$$

and using (9)

$$\int_{\tilde{K}^c} \int u^\alpha(z,x) u^\alpha(x,z) f_1(x)\, dm(x)\, dm(z) \tag{33}$$
$$\leq C \int \left(\int u^\alpha(x,z)\, dm(z) \right) f_1(x)\, dm(x) < \infty.$$

□

4 Subordination

The basic idea in our proof that the loop measure is shift invariant is to show that the loop measure can be obtained as the 'limit' of loop measures for processes with transition densities. These processes will be obtained from the original process by subordination.

We consider a subordinator T_t which is a compound Poisson process with Levy measure $c\psi$ so that

$$E^x\left(f\left(X_{T_t}\right)\right) = \sum_{j=1}^{\infty} \frac{(ct)^j}{j!} e^{-ct} \int_0^\infty E^x\left(f\left(X_s\right)\right) \psi^{*j}(ds). \tag{34}$$

If we take ψ to be exponential with parameter θ, then $\psi^{*j}(ds) = \frac{s^{j-1}\theta^j}{\Gamma(j)} e^{-s\theta}\, ds$ so that we have

$$E^x\left(f\left(X_{T_t}\right)\right) = \sum_{j=1}^{\infty} \frac{(ct)^j}{j!} e^{-ct} \int_0^\infty E^x\left(f\left(X_s\right)\right) \frac{s^{j-1}\theta^j}{\Gamma(j)} e^{-s\theta}\, ds. \tag{35}$$

Hence the subordinated transition semigroup

$$\tilde{P}_t(x,\, dy) = \sum_{j=1}^{\infty} \frac{(ct)^j}{j!} e^{-ct} \int_0^\infty P_s(x,\, dy) \frac{s^{j-1}\theta^j}{\Gamma(j)} e^{-s\theta}\, ds. \tag{36}$$

Noting that

$$\int_0^\infty P_s(x,A) s^{j-1} e^{-s\theta}\, ds = \frac{d^{j-1}}{d\theta^{j-1}} \int_0^\infty P_s(x,A) e^{-s\theta}\, ds = \frac{d^{j-1}}{d\theta^{j-1}} U^\theta(x,A), \tag{37}$$

we see that $\tilde{P}_t(x,\, dy)$ is absolutely continuous with respect to the measure m on S, and we can choose transition densities $\tilde{p}_t(x,y)$.

From now on we take $\theta = c = n$, and use (n) as a superscript or subscript to denote objects with respect to the subordinated process, denoted by $X_t^{(n)}$.

Lemma 1

$$u_{(n)}^{\alpha}(x,y) = \frac{1}{(1+\alpha/n)^2} u^{\alpha/(1+\alpha/n)}(x,y). \tag{38}$$

In particular,

$$u_{(n)}^{\alpha}(x,y) \leq u_{(n)}(x,y) = u(x,y). \tag{39}$$

Proof

$$\begin{aligned} U_{(n)}^{\alpha}(x,\, dy) &= \int_0^\infty e^{-\alpha t} P_t^{(n)}(x,\, dy)\, dt \\ &= \frac{1}{\alpha+n} \sum_{j=1}^{\infty} \int_0^\infty \left(\frac{n}{\alpha+n}\right)^j P_s(x,\, dy) \frac{s^{j-1} n^j}{\Gamma(j)} e^{-sn}\, ds. \\ &= \frac{n^2}{(\alpha+n)^2} \int_0^\infty P_s(x,\, dy) \left(\sum_{j=1}^{\infty} \frac{s^{j-1}(n^2/(\alpha+n))^{(j-1)}}{\Gamma(j)} e^{-sn} \right) ds \\ &= \frac{n^2}{(\alpha+n)^2} \int_0^\infty \left(e^{sn^2/(\alpha+n)} e^{-sn} \right) P_s(x,\, dy)\, ds \\ &= \frac{n^2}{(\alpha+n)^2} \int_0^\infty e^{-s\left(n - \frac{n^2}{\alpha+n}\right)} P_s(x,\, dy)\, ds \\ &= \frac{n^2}{(\alpha+n)^2} u^{n\alpha/(\alpha+n)}(x,y)\, dm(y) \\ &= \frac{1}{(1+\alpha/n)^2} u^{\alpha/(1+\alpha/n)}(x,y)\, dm(y). \end{aligned} \tag{40}$$

□

Lemma 2 *For any $\alpha, \alpha_j, j = 1, \dots, k$ and continuous compactly supported $f_j, j = 1, \dots, k$*

$$\begin{aligned} \lim_{n\to\infty} \int_{R_+^k} \prod_{j=1}^{k} e^{-\alpha_j t_j} Q_{(n)}^{z,z} \left(\prod_{j=1}^{k} f_j(X_{t_j}^{(n)}) e^{-\alpha\zeta} \right) \prod_{j=1}^{k} dt_j \\ = \int_{R_+^k} \prod_{j=1}^{k} e^{-\alpha_j t_j} Q^{z,z} \left(\prod_{j=1}^{k} f_j(X_{t_j}) e^{-\alpha\zeta} \right) \prod_{j=1}^{k} dt_j. \end{aligned} \tag{41}$$

Proof Recall (24). For $0 < t_1 < \cdots < t_k$

$$Q^{z,z}\left(\prod_{j=1}^{k} f_j(X_{t_j})e^{-\alpha\zeta}\right) \tag{42}$$
$$= \prod_{j=1}^{k} P^{\alpha}_{t_j - t_{j-1}}(x_{j-1}, dx_j)u^{\alpha}(x_k, z)\prod_{j=1}^{k} f_j(x_j)$$

with the corresponding

$$Q^{z,z}_{(n)}\left(\prod_{j=1}^{k} f_j(X^{(n)}_{t_j})e^{-\alpha\zeta}\right) \tag{43}$$
$$= \prod_{j=1}^{k} P^{(n),\alpha}_{t_j - t_{j-1}}(x_{j-1}, dx_j)u^{\alpha}_{(n)}(x_k, z)\prod_{j=1}^{k} f_j(x_j).$$

Using (43) we see that

$$\int_{R^k_+} \prod_{j=1}^{k} e^{-\alpha_j t_j} Q^{z,z}_{(n)}\left(\prod_{j=1}^{k} f_j(X^{(n)}_{t_j})e^{-\alpha\zeta}\right)\prod_{j=1}^{k} dt_j \tag{44}$$
$$= \sum_{\pi\in\mathscr{P}_k}\int_{\{0\le t_1\le\cdots\le t_k\le\infty\}} \prod_{j=1}^{k} e^{-\alpha_j t_j} Q^{z,z}_{(n)}\left(\prod_{j=1}^{k} f_{\pi(j)}(X^{(n)}_{t_j})e^{-\alpha\zeta}\right)\prod_{j=1}^{k} dt_j$$
$$= \sum_{\pi\in\mathscr{P}_k}\int_{\{0\le t_1\le\cdots\le t_k\le\infty\}} \prod_{j=1}^{k} P^{(n),\alpha+\sum_{l=j}^{k}\alpha_l}_{t_j - t_{j-1}}(x_{j-1}, dx_j)$$
$$u^{\alpha}_{(n)}(x_k, z)\prod_{j=1}^{k} f_{\pi(j)}(x_j)\prod_{j=1}^{k} dt_j$$
$$= \sum_{\pi\in\mathscr{P}_k}\prod_{j=1}^{k} U^{\alpha+\sum_{l=j}^{k}\alpha_l}_{(n)}(x_{j-1}, dx_j)u^{\alpha}_{(n)}(x_k, z)\prod_{j=1}^{k} f_{\pi(j)}(x_j)$$
$$= \sum_{\pi\in\mathscr{P}_k}\int\prod_{j=1}^{k} u^{\alpha+\sum_{l=j}^{k}\alpha_l}_{(n)}(x_{j-1}, x_j)u^{\alpha}_{(n)}(x_k, z)\prod_{j=1}^{k} f_{\pi(j)}(x_j)\, dm(x_j).$$

Equation (41) now follows from (39), (2) and the dominated convergence theorem. □

It follows from (41) that for a.e. $t_1, \cdots, t_k$

$$\lim_{n\to\infty} Q^{z,z}_{(n)}\left(\prod_{j=1}^{k} f_j(X^{(n)}_{t_j})e^{-\alpha\zeta}\right) = Q^{z,z}\left(\prod_{j=1}^{k} f_j(X_{t_j})e^{-\alpha\zeta}\right). \tag{45}$$

Lemma 3 *For any $\alpha, \alpha_j, j = 1, \dots, k$ and continuous compactly supported $f_j, j = 1, \dots, k$*

$$\lim_{n\to\infty} \int_{R^k_+} \prod_{j=1}^{k} e^{-\alpha_j t_j} \nu_{(n)}\left(\prod_{j=1}^{k} f_j(X^{(n)}_{t_j})e^{-\alpha\zeta}\right)\prod_{j=1}^{k} dt_j \tag{46}$$
$$= \int_{R^k_+} \prod_{j=1}^{k} e^{-\alpha_j t_j} \nu\left(\prod_{j=1}^{k} f_j(X_{t_j})e^{-\alpha\zeta}\right)\prod_{j=1}^{k} dt_j.$$

Proof By (44)

$$\int_{R^k_+} \prod_{j=1}^{k} e^{-\alpha_j t_j} \nu_{(n)}\left(\prod_{j=1}^{k} f_j(X^{(n)}_{t_j})e^{-\alpha\zeta}\right)\prod_{j=1}^{k} dt_j \tag{47}$$
$$= \sum_{\pi\in\mathcal{P}_k} \int \prod_{j=1}^{k} u^{\alpha+\sum_{l=j}^{k}\alpha_l}_{(n)}(x_{j-1}, x_j) u^{\alpha}_{(n)}(x_k, z) \prod_{j=1}^{k} f_{\pi(j)}(x_j)\, dm(x_j)\, dm(z).$$

If $k \geq 2$, then using the resolvent equation we see that

$$\left(\sum_{l=1}^{k}\alpha_l\right)\int_{R^k_+} \prod_{j=1}^{k} e^{-\alpha_j t_j} \nu_{(n)}\left(\prod_{j=1}^{k} f_j(X^{(n)}_{t_j})e^{-\alpha\zeta}\right)\prod_{j=1}^{k} dt_j \tag{48}$$
$$= \sum_{\pi\in\mathcal{P}_k} \int \prod_{j=2}^{k} u^{\alpha+\sum_{l=j}^{k}\alpha_l}_{(n)}(x_{j-1}, x_j) u^{\alpha}_{(n)}(x_k, x_1) \prod_{j=1}^{k} f_{\pi(j)}(x_j)\, dm(x_j)$$
$$- \sum_{\pi\in\mathcal{P}_k} \int \prod_{j=2}^{k} u^{\alpha+\sum_{l=j}^{k}\alpha_l}_{(n)}(x_{j-1}, x_j) u^{\alpha+\sum_{l=1}^{k}\alpha_l}_{(n)}(x_k, x_1) \prod_{j=1}^{k} f_{\pi(j)}(x_j)\, dm(x_j),$$

and (46) for $k \geq 2$ then follows from (39), (2) and the dominated convergence theorem. Here we repeatedly use the Cauchy-Schwarz inequality. See the proof of [9, Lemma 3.3].

When $k = 1$,

$$\int_{R_+} e^{-\alpha_1 t_1} \nu_{(n)} \left(f_1(X_{t_1}^{(n)}) e^{-\alpha\zeta} \right) dt_1 \tag{49}$$

$$= \int \int u_{(n)}^{\alpha+\alpha_1}(z,x) u_{(n)}^{\alpha}(x,z) f_1(x)\, dm(x)\, dm(z).$$

Note that by (38) we have $u_{(n)}^{\alpha}(x,z) \leq u^{\alpha/2}(x,z)$ for n sufficiently large. We can then use the argument from the proof of Theorem 2 and the dominated convergence theorem to get (46) for $k = 1$. □

It follows from (46) that

$$\lim_{n\to\infty} \int_{R_+^k} \prod_{j=1}^k e^{-\alpha_j t_j} \nu_{(n)} \left(\prod_{j=1}^k f_j(X_{t_j}^{(n)}) g(\zeta) \right) \prod_{j=1}^k dt_j \tag{50}$$

$$= \int_{R_+^k} \prod_{j=1}^k e^{-\alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(X_{t_j}) g(\zeta) \right) \prod_{j=1}^k dt_j$$

for all continuous exponentially bounded functions g. We will be particularly interested in g of the form

$$g(\zeta) = \frac{\prod_{j=1}^k (1 - e^{-\beta_j \zeta})}{\prod_{j=1}^k (1 - e^{-\alpha_j \zeta})} e^{-\alpha\zeta} h_s(\zeta) \tag{51}$$

where $0 \leq h_s(\zeta) \leq 1$ is a continuous function with $h_s(\zeta) = 0$ for $\zeta \leq s$.

5 Invariance Under Loop Rotation

Proof of Theorem 3 Because the lifetime ζ is rotation invariant ($\zeta(\rho_v \omega) = \zeta(\omega)$ so long as $\zeta(\omega) < \infty$), the rotation invariance of the loop measure

$$\mu(F) = \nu \left(\frac{F}{\zeta} \right) \tag{52}$$

is equivalent to that of the measure ν.

We note that by (11)

$$\nu\left(f(X_\delta)\right) = \int P_\delta(z, dx) f(x) u(x,z)\, dm(z) < \infty \tag{53}$$

for any $\delta > 0$ and bounded measurable f with compact support.

We next recall some ideas from [5]. Let us define the process $\overline{X}$ to be the periodic extension of X; that is,

$$\overline{X}_t = \begin{cases} X_{t-q\zeta}, & \text{if } q\zeta \leq t < (q+1)\zeta, q = 0, 1, 2, \ldots \\ X_t, & \text{if } \zeta = \infty \end{cases} \tag{54}$$

It will be convenient to write

$$\bar{I}_\alpha(f) := \int_0^\infty e^{-\alpha t} f(\overline{X}_t)\, dt, \qquad I_\alpha(f) := \int_0^\infty e^{-\alpha t} f(X_t)\, dt. \tag{55}$$

The key observation is that

$$\bar{I}_\alpha(f) = \frac{I_\alpha(f)}{1 - e^{-\alpha\zeta}}, \tag{56}$$

for all $\alpha > 0$. This follows from

$$\begin{aligned} \bar{I}_\alpha(f) &:= \int_0^\infty e^{-\alpha t} f(\overline{X}_t)\, dt \\ &= \sum_{q=0}^\infty \int_{q\zeta}^{(q+1)\zeta} e^{-\alpha t} f(\overline{X}_t)\, dt \\ &= \sum_{q=0}^\infty e^{-\alpha q\zeta} \int_0^\zeta e^{-\alpha t} f(X_t)\, dt = \frac{I_\alpha(f)}{1 - e^{-\alpha\zeta}}\,. \end{aligned}$$

The rotation invariance of μ or ν is equivalent to the following Lemma.

Lemma 4

$$\nu\left(\prod_{j=1}^k f_j(\overline{X}_{t_j+r}) 1_{\{t_k<\zeta\}}\right) = \nu\left(\prod_{j=1}^k f_j(\overline{X}_{t_j}) 1_{\{t_k<\zeta\}}\right) \tag{57}$$

for all $0 < t_1 < \cdots < t_k$ *and* $r > 0$ *and all* $f_j \geq 0$ *continuous with compact support.*

Let $0 \leq h_s(\zeta) \leq 1$ be a continuous function with $h_s(\zeta) = 0$ for $\zeta \leq s$. To prove Lemma 4 we first prove the following.

Lemma 5 *For all* $k \geq 1$*, and* $0 \leq t_1, \cdots, t_k < s$ *and all* $f_j \geq 0$ *continuous with compact support*

$$\nu\left(\prod_{j=1}^k f_j(\overline{X}_{t_j+r}) h_s(\zeta)\right) = \nu\left(\prod_{j=1}^k f_j(\overline{X}_{t_j}) h_s(\zeta)\right). \tag{58}$$

Proof of Lemma 5 Using first (56) and then (50) we have that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{R_+^k} \prod_{j=1}^k e^{-\alpha_j t_j} \nu^{(n)} \left(\prod_{j=1}^k f_j\left(\overline{X^{(n)}}_{t_j}\right)(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right) \prod_{j=1}^k dt_j \\
&= \lim_{n\to\infty} \int_{R_+^k} \prod_{j=1}^k e^{-\alpha_j t_j} \nu^{(n)} \left(\prod_{j=1}^k f_j\left(X^{(n)}{}_{t_j}\right) \frac{(1-e^{-\beta_j \zeta})}{(1-e^{-\alpha_j \zeta})} e^{-\alpha\zeta} h_s(\zeta) \right) \prod_{j=1}^k dt_j \\
&= \int_{R_+^k} \prod_{j=1}^k e^{-\alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(X_{t_j}) \frac{(1-e^{-\beta_j \zeta})}{(1-e^{-\alpha_j \zeta})} e^{-\alpha\zeta} h_s(\zeta) \right) \prod_{j=1}^k dt_j \\
&= \int_{R_+^k} \prod_{j=1}^k e^{-\alpha_j t_j} \nu \left(\prod_{j=1}^k f_j(\overline{X}_{t_j})(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right) \prod_{j=1}^k dt_j. \qquad (59)
\end{aligned}$$

It follows from this that for a.e. $0 \le t_1, \cdots, t_k$

$$\begin{aligned}
&\lim_{n\to\infty} \nu^{(n)} \left(\prod_{j=1}^k f_j\left(\overline{X^{(n)}}_{t_j}\right)(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right) \qquad (60)\\
&= \nu \left(\prod_{j=1}^k f_j(\overline{X}_{t_j})(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right).
\end{aligned}$$

The same calculations show that for any $r > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \nu^{(n)} \left(\prod_{j=1}^k f_j\left(\overline{X^{(n)}}_{t_j+r}\right)(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right) \qquad (61)\\
&= \nu \left(\prod_{j=1}^k f_j(\overline{X}_{t_j+r})(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right)
\end{aligned}$$

for a.e. $0 \le t_1, \cdots, t_k$. Since $\nu^{(n)}$ is invariant under loop rotation, see [5, Lemma 2.4] for the simple proof, it follows from our last two displays that for a.e. $0 \le t_1, \cdots, t_k$

$$\begin{aligned}
&\nu \left(\prod_{j=1}^k f_j(\overline{X}_{t_j+r})(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right) \qquad (62)\\
&= \nu \left(\prod_{j=1}^k f_j(\overline{X}_{t_j})(1-e^{-\beta_j \zeta}) e^{-\alpha\zeta} h_s(\zeta) \right).
\end{aligned}$$

We now use an argument from [5] (see from (5.31) there until the end of the paragraph). By Fubini we can find a set $T \subseteq (0, s)$ with full measure such that for all $t_1 \in T$ we have that (62) holds for a.e. $t_2, \dots, t_k \in (0, s)$. Using (53) with $\delta = t_1$, the boundedness and continuity of the f_j and the right continuity of $\bar{X}_t$ it follows from the Dominated Convergence Theorem that (62) holds for all $(t_1, t_2, \dots, t_k) \in T \times [0, s)^{k-1}$. Let now $f_{1,n}$ be a sequence of continuous functions with compact support with the property that $f_{1,n} \uparrow 1$. By the above, (62) with f_1 replaced by $f_{1,n}$ holds for all $(t_1, t_2, \dots, t_k) \in T_n \times [0, s)^{k-1}$ for an appropriate $T_n \subseteq (0, s)$ with full measure. In particular $T_* = \cap_n T_n \neq \emptyset$, and we can apply the Monotone Convergence Theorem with $t_1 \in T_*$ to conclude that

$$\nu\left((1 - e^{-\beta_1 \zeta}) \prod_{j=2}^{k} f_j(\overline{X}_{t_j + r})(1 - e^{-\beta_j \zeta}) e^{-\alpha \zeta} h_s(\zeta)\right) \tag{63}$$

$$= \nu\left((1 - e^{-\beta_1 \zeta}) \prod_{j=2}^{k} f_j(\overline{X}_{t_j})(1 - e^{-\beta_j \zeta}) e^{-\alpha \zeta} h_s(\zeta)\right)$$

for all $t_2, \dots, t_k < s$. Applying once again the Monotone Convergence Theorem for $\beta_j \to \infty, \alpha \to 0$ we obtain

$$\nu\left(\prod_{j=2}^{k} f_j(\overline{X}_{t_j + r}) h_s(\zeta)\right) = \nu\left(\prod_{j=2}^{k} f_j(\overline{X}_{t_j}) h_s(\zeta)\right) \tag{64}$$

for all $t_2, \dots, t_k < s$. Since k is arbitrary, we obtain our Lemma. □

Proof of Lemma 4 Fix $0 < t_1 < \cdots < t_k$. Choose a sequence $s_n \downarrow t_k$. It is clear that we can choose h_{s_n} so that $h_{s_n}(\zeta) \uparrow 1_{\{t_k < \zeta\}}$. Lemma 4 then follows from Lemma 5 by the Monotone Convergence Theorem. □

6 The Loop Measure and Continuous Additive Functionals

Before proving Theorem 4 we will need two facts about continuous additive functionals (CAFs). The first says that to each CAF A of X is associated a measure ν_A on S such that for any measurable function f

$$U_A f(x) := E^x \int_0^\infty f(X_t)\, dA_t = \int_S u(x, y) f(y)\, \nu_A(dy), \qquad \forall x \in S. \tag{65}$$

ν_A is referred to as the Revuz measure of A. The second fact we need is that if a CAF has Revuz measure ν with respect to X, it has Revuz measure $h \cdot \nu$ with respect to the h-transform of X. Following the proof of Theorem 4 we will discuss these facts and provide references.

Proof of Theorem 4 To prove (15) it is enough to prove the additive functional version of (19). We consider first our Borel right process X. These considerations will then be applied to the h-transform of X using $h_z = u(\cdot, z)$ for fixed $z \in S$.

Let A^j $(j = 1, 2, \ldots)$ be CAFs of X with Revuz measures ν_j. Using the Markov property, see for example Theorems 28.7 and 22.8 of [11], and (65) at the last step

$$E^x \int_{\{0<t_1<t_2<\cdots<t_n<\infty\}} \prod_{j=1}^{n} dA^j_{t_j} \tag{66}$$

$$= E^x \left(\int_0^\infty \left(\int_{\{0<t_2<\cdots<t_n<\infty\}} \prod_{j=2}^{n} dA^j_{t_j} \right) \circ \theta_{t_1} \, dA^1_{t_1} \right)$$

$$= E^x \left(\int_0^\infty E^{X_{t_1}} \left(\int_{\{0<t_2<\cdots<t_n<\infty\}} \prod_{j=2}^{n} dA^j_{t_j} \right) dA^1_{t_1} \right)$$

$$= \int_S u(x, x_1) E^{x_1} \left(\int_{\{0<t_2<\cdots<t_n<\infty\}} \prod_{j=2}^{n} dA^j_{t_j} \right) \nu_1(dx_1),$$

and then by induction

$$E^x \int_{\{0<t_1<t_2<\cdots<t_n<\infty\}} \prod_{j=1}^{n} dA^j_{t_j} \tag{67}$$

$$= \int_{S^n} u(x, x_1) u(x_1, x_2) \cdots u(x_{n-1}, x_n) \prod_{j=1}^{n} \nu_j(dx_j).$$

Notice that by our assumption that $u(x, x_1)$ is excessive in x for each x_1, the expressions in (67) are excessive functions of x. Thus if $\eta = (\eta_t)$ is an entrance law, then writing E^η for the measure under which the one-dimensional distributions are given by the entrance law we have

$$E^\eta \int_{\{0<t_1<t_2<\cdots<t_n<\infty\}} \prod_{j=1}^{n} dA^j_{t_j} \tag{68}$$

$$= \uparrow \lim_{t \downarrow 0} \int_S \eta_t(dx) E^x \int_{\{0<t_1<t_2<\cdots<t_n<\infty\}} \prod_{j=1}^{n} dA^j_{t_j}$$

$$=\uparrow \lim_{t\downarrow 0}\int_S \eta_t(dx)\int_{S^n} u(x,x_1)u(x_1,x_2)\cdots u(x_{n-1},x_n)\prod_{j=1}^n \nu_j(dx_j)$$

$$=\int_{S^n} g(x_1)u(x_1,x_2)\cdots u(x_{n-1},x_n)\prod_{j=1}^n \nu_j(dx_j),$$

where $g(x_1):=\uparrow \lim_{t\downarrow 0}\int \eta_t(dx)u(x,x_1)$. Here the notation $\uparrow \lim_{t\downarrow 0} f(t)$ means that $f(t)$ increases as t decreases to 0.

Now apply the above to the h-transform of the original process X, with $h_z = u(\cdot,z)$ for a fixed $z\in S$, as described in Sect. 2. This process has potential density $u^{h_z}(x,y) = u(x,y)/h_z(x)$ with respect to the measure $h_z(y)\,m(dy)$. Also, if a CAF has Revuz measure ν with respect to X, it has Revuz measure $h_z\cdot\nu$ with respect to the h-transform process. Thus by (67)

$$E^{x,z}\int_{\{0<t_1<t_2<\cdots<t_n<\infty\}}\prod_{j=1}^n dA^j_{t_j} \tag{69}$$

$$=\int_{S^n}\frac{u(x,x_1)}{h_z(x)}\frac{u(x_1,x_2)}{h_z(x_1)}\cdots\frac{u(x_{n-1},x_n)}{h_z(x_{n-1})}\prod_{j=1}^n h_z(x_j)\,\nu_j(dx_j)$$

$$=\frac{1}{h_z(x)}\int_{S^n} u(x,x_1)u(x_1,x_2)\cdots u(x_{n-1},x_n)\,h_z(x_n)\prod_{j=1}^n \nu_j(dx_j)$$

$$=\frac{1}{h_z(x)}\int_{S^n} u(x,x_1)u(x_1,x_2)\cdots u(x_{n-1},x_n)\,u(x_n,z)\prod_{j=1}^n \nu_j(dx_j).$$

When we use the entrance law $\eta^z_t(dx) = P_t(z,dx)h_z(x)$, the function g of the preceding paragraph is

$$\uparrow \lim_{t\downarrow 0}\eta^z_t(dx)u^{h_z}(x,x_1) =\uparrow \lim_{t\downarrow 0} P_t(z,dx)u(x,x_1) = u(z,x_1). \tag{70}$$

Thus, using the definition of $Q^{z,z}$ from Sect. 2,

$$Q^{z,z}\int_{0<t_1<\cdots<t_n<\infty}\prod_{j=1}^n dA^t_{t_j} = \int_{S^n} u(z,x_1)u(x_1,x_2)\cdots u(x_n,z)\prod_{j=1}^n \nu_j(dx_j). \tag{71}$$

Similar considerations work for the α-potentials, and the argument given in the proof of Theorem 1 proves (15). □

We now discuss the facts mentioned at the beginning of this section.

Given a right-continuous strong Markov process X (more precisely, a Borel right Markov process) and an excessive measure m, there is always a dual process $\hat{X}$

(essentially uniquely determined), but in general it is a moderate Markov process: the Markov property holds only at predictable times.

In what follows f and g are non-negative Borel functions on S. By duality

$$\int_S f(x)Ug(x)m(dx) = \int_S \hat{U}f(y)g(y)m(dy), \tag{72}$$

where the kernel $\hat{U}$ is the potential kernel of the moderate Markov dual of X. Under our assumptions it follows from [1, VI, Theorem 1.4] that the potential density u can be chosen so that $x \mapsto u(x,y)$ is excessive for each y, and $y \mapsto u(x,y)$ is co-excessive (that is, excessive with respect to the moderate Markov dual process $\hat{X}$) for each x. Equation (72) implies that

$$\hat{U}f(y) = \int_S u(x,y)f(x)\, m(dx), \tag{73}$$

for m-a.e. y. Since both sides of (73) are co-excessive, they agree for all y.

By [4, (5.13)] we have the Revuz formula

$$\int_S f(x)U_A g(x)\, m(dx) = \int_S \hat{U}f(y)g(y)\nu_A(dy), \tag{74}$$

where ν_A is the Revuz measure of the CAF A with respect to m. Feeding (73) into (74) and varying f we find that

$$U_A g(x) = \int_S u(x,y)g(y)\, \nu_A(dy), \tag{75}$$

first for m-a.e. x, then for all x because both sides of (75) are excessive. This proves (65).

One subtlety: the laws $\hat{P}^x$ of $\hat{X}$ are only determined modulo a class of sets ("m-exceptional") defined in [4], see (3.4) for the definition of the term, and then Remark (5.14); but that class is not charged by ν, so the exception causes no problem.

To establish the second fact that we needed, let τ_t be the right continuous inverse of A_t, and let f be a positive measurable function. Using the change of variables formula, [3, Chap. 6, (55.1)] and then Fubini

$$\begin{aligned} E^{x/h}\left(\int_0^\infty f(X_t)\, dA_t\right) &= E^{x/h}\left(\int_0^\infty f(X_{\tau(u)})\, du\right) \\ &= \int_0^\infty E^{x/h}\left(f(X_{\tau(u)})\right)\, du. \end{aligned} \tag{76}$$

Using [11, (62.20)] and then Fubini we have

$$\int_0^\infty E^{x/h}\left(f(X_{\tau(u)})\right)\,du = \frac{1}{h(x)}\int_0^\infty E^x\left(f(X_{\tau(u)})h(X_{\tau(u)})\right)\,du \tag{77}$$
$$= \frac{1}{h(x)}E^x\left(\int_0^\infty f(X_{\tau(u)})h(X_{\tau(u)})\,du\right).$$

Using the change of variables formula once again, the last two formulas show that

$$E^{x/h}\left(\int_0^\infty f(X_t)\,dA_t\right) = \frac{1}{h(x)}E^x\left(\int_0^\infty f(X_t)h(X_t)\,dA_t\right). \tag{78}$$

Using (65) we see that

$$E^{x/h}\left(\int_0^\infty f(X_t)\,dA_t\right) = \frac{1}{h(x)}\int_S u(x,y)f(y)h(y)\,\nu_A(dy). \tag{79}$$

This shows that if a CAF has Revuz measure ν with respect to X, then it has Revuz measure $h\cdot\nu$ with respect to the h-transform of X. (Recall that we use $\frac{u(x,y)}{h(x)}$ for the potential densities of the h-transform process.)

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The Joint Law of the Extrema, Final Value and Signature of a Stopped Random Walk

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Abstract A complete characterization of the possible joint distributions of the maximum and terminal value of uniformly integrable martingale has been known for some time, and the aim of this paper is to establish a similar characterization for continuous martingales of the joint law of the minimum, final value, and maximum, along with the direction of the final excursion. We solve this problem completely for the discrete analogue, that of a simple symmetric random walk stopped at some almost-surely finite stopping time. This characterization leads to robust hedging strategies for derivatives whose value depends on the maximum, minimum and final values of the underlying asset.

1 Introduction

Suppose given $h > 0$, and suppose that $(\xi_t, \mathcal{F}_t)_{t \in h\mathbb{Z}^+}$ is a symmetric simple random walk on the grid $h\mathbb{Z}$, started at zero. Define $S_t \equiv \sup_{s \le t} \xi_s$, $I_t \equiv \inf_{s \le t} \xi_s$, $g_t^+ \equiv \inf\{u \le t : \xi_u = S_u\}$, $g_t^- \equiv \inf\{u \le t : \xi_u = I_u\}$, and let

$$\begin{aligned} \sigma_t &= +1 \qquad \text{if } g_t^+ > g_t^- \\ &= -1 \qquad \text{else.} \end{aligned} \tag{1}$$

The process S records the running maximum of the martingale, and the process σ records whether the martingale is currently on an excursion down from its running maximum ($\sigma = +1$) or on an excursion up from its running minimum ($\sigma = -1$). We refer to the process σ as the *signature* of the random walk.

Suppose that T is an almost-surely finite $(\mathcal{F}_t)$-stopping time, and write

$$X_t \equiv \xi_{t \wedge T}$$

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for the stopped process. The paper is concerned with the possible joint laws m of the quadruple $(I_T, X_T, S_T, \sigma_T)$, which we will abbreviate to (I, X, S, σ) where no confusion may arise.

Clearly the law m must be defined on the set $\mathcal{X} \equiv (-h\mathbb{Z}^+) \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$, and evidently we must have $m(I \leq X \leq S) = 1$; but beyond this, is it possible to state a set of *necessary and sufficient conditions* for a probability m on $\mathcal{X}$ to be the joint distribution of $(I_t, X_T, S_T, \sigma_T)$? The motivation for this attempt is twofold. Firstly, the joint law of (X, S) has been characterized completely (for general local martingales, not assumed to be continuous or uniformly integrable) in [8]; and developed by Roynette, Vallois and Yor [9] to characterize the joint laws of (B_T, L_T) for almost-surely finite stopping times T of the Brownian motion with local time L at zero. Can the methods of those papers be extended to deal with the running minimum also? The second reason to look at this problem is the interesting recent work of Cox and Obloj [3] which finds extremal martingales for various derivatives whose payoffs depend on the maximum, minimum and terminal value of the underlying asset. This builds to some extent on the earlier work of Hobson and others ([7][1][2]), which addresses similar questions for derivatives whose payoffs depend only on the maximum and terminal value of the underlying asset. Many of the results of this literature can be derived alternatively using the results of [8], by converting the problem into a linear program. This approach is more general, but leads to less explicit answers in the specific instances analyzed to date.

What we shall find here is that it is possible to generalize the results of [8] to cover the joint law of (I, X, S, σ), but that the statements are more involved. For this reason, we shall restrict our analysis to a symmetric simple random walk taking values in a grid $h\mathbb{Z}$ for some $h > 0$, stopped at an almost-surely finite stopping time. The main result is presented in Sect. 2. The proof of necessity is in Sect. 2.1, and requires only the judicious use of the Optional Sampling Theorem. The proof of sufficiency, in Sect. 2.2, is constructive, and requires suitable modification of some of the techniques of [8]. We then show in Sect. 3 how this characterization can lead to robust hedging schemes and extremal prices for derivatives whose payoff depends on the maximum, minimum, terminal value and signature.

2 The Main Result

We take a symmetric simple random walk $(\xi_t, \mathcal{F}_t)_{t \in h\mathbb{Z}^+}$ on $h\mathbb{Z}$ for some fixed $h > 0$; in general, the filtration $(\mathcal{F}_t)$ is larger than the filtration of the random walk, to allow for additional randomization. Stopping ξ at the almost-surely finite stopping time T creates the martingale $X_t = \xi_{t \wedge T}$. We use the notation of the Introduction, and notice that

$$g_t^+ \equiv \sup\{u \leq t : S_u > S_{u-h}\}, \qquad g_t^- \equiv \sup\{u \leq t : I_u < I_{u-h}\}, \tag{2}$$

emphasizing the fact that we are dealing with *strict* ascending/descending ladder epochs, to use the language of Feller [5]. The process σ is defined as before at (1).

Definition 2.1 We say that the probability measure m on $\mathcal{X} \equiv -h\mathbb{Z}^+ \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$ is *consistent* if there is some almost-surely finite $(\mathcal{F}_t)$-stopping time T such that m is the law of $(I_T, X_T, S_T, \sigma_T)$.

2.1 Necessity

For $x \in h\mathbb{Z}$ we define the hitting time

$$H_x = \inf\{u : \xi_u = x\}, \tag{3}$$

with the usual convention that the infimum of the empty set is $+\infty$. In what follows, we will let a, b stand for two generic members of $h\mathbb{Z}^+$, and will be studying the exit time $H_b \wedge H_{-a} \equiv \inf\{u : \xi_u \notin (-a, b)\}$ and related stopping times. The measure m says nothing directly about these stopping times, but by way of the Optional Sampling Theorem we are able to deduce quite a lot of information about them if the law m is consistent. Indeed, assuming that m is consistent, we are able to find the probability that $H_{-a} < H_b$ (for example) in terms of m-expectations of functions defined on $\mathcal{X}$. The expressions derived make perfectly good sense even if m is not consistent, but it may be that the expressions do not in general satisfy positivity or other properties which would hold if m were consistent. For this reason, we will denote by $\bar{m}(Y)$ the expression for the m-expectation of a random variable Y which would be correct if m were consistent; if m is not consistent, all we have is an algebraic expression without the desired probabilistic meaning, and the use of the symbol $\bar{m}$ warns us not to assume properties which need not hold.

The first result we need is the following, which illustrates the use of this notational convention.

Proposition 2.2 *For any $a, b \in h\mathbb{Z}^+$ we have*

$$\bar{m}(H_b < H_{-a}) = \frac{a - m(a + X; S < b, I > -a)}{a + b} \equiv \varphi(b, -a), \tag{4}$$

$$\bar{m}(H_{-a} < H_b) = \frac{b - m(b - X; S < b, I > -a)}{a + b} \equiv \varphi(-a, b). \tag{5}$$

Proof We use the Optional Sampling Theorem at the time $H_b \wedge H_{-a}$ to derive the two equations

$$1 = \bar{m}(H_{-a} < H_b) + \bar{m}(H_b < H_{-a}) + m(S < b, I > -a) \tag{6}$$

$$0 = -a\,\bar{m}(H_{-a} < H_b) + b\,\bar{m}(H_b < H_{-a}) + m(X; S < b, I > -a). \tag{7}$$

Solving this pair of linear equations leads to the conclusion that

$$\bar{m}(H_b < H_{-a}) = \left\{ a - m(a + X; S < b, I > -a) \right\}/(a + b) \;,$$
$$\bar{m}(H_{-a} < H_b) = \left\{ b - m(b - X; S < b, I > -a) \right\}/(a + b) \;,$$

as claimed. □

If m is consistent, then we would have for any $a, b \in h\mathbb{Z}^+$ not both zero that

$$\begin{aligned}\bar{m}(H_{-a} < H_b < H_{-a-h}) &= \bar{m}(H_{-a} \le H_b < H_{-a-h}) \\ &= \bar{m}(H_b < H_{-a-h}) - \bar{m}(H_b < H_{-a}) \\ &= \bar{m}(H_b < \infty, I(H_b) = -a).\end{aligned}$$

This is because on the event $\{H_{-a} < H_b < H_{-a-h}\}$ the hitting time H_b is finite, and so cannot be equal to H_{-a}; the second equality follows from the inclusion $\{H_b < H_{-a}\} \subseteq \{H_b < H_{-a-h}\}$. We will therefore introduce the notation

$$\psi_+(-a, b) = \varphi(b, -a - h) - \varphi(b, -a), \tag{8}$$
$$\psi_-(-a, b) = \varphi(-a, b + h) - \varphi(-a, b). \tag{9}$$

Notice that $\psi_+(-a, b)$ is *defined* as an algebraic expression in terms of m via (8) and (4); if m is *consistent*, then $\psi_+(-a, b)$ is equal to $\bar{m}(H_b < \infty, I(H_b) = -a)$, but no such interpretation holds in general.

The necessary condition we derive comes from considering what may happen if the event $B_+ = \{H_b < \infty, I(H_b) = -a\}$ occurs. When this event occurs, the martingale X does reach b before being stopped, and at that time H_b the minimum value is $-a$. Thereafter, one of three things will happen:

(i) X reaches $b + h$ before reaching $-a - h$ and before T;
(ii) T happens before X reaches either $-a - h$ or $b + h$;
(iii) X reaches $-a - h$ before reaching $b + h$ and before T.

The next result derives a necessary condition from the Optional Sampling Theorem applied at $H_{-a-h} \wedge H_{b+h} \wedge T$.

Proposition 2.3 *Define the events*

$$B_+ = \{H_b < \infty, I(H_b) = -a)\}, \qquad B_- = \{H_{-a} < \infty, S(H_{-a}) = b\}, \tag{10}$$

set $p_\pm = \bar{m}(B_\pm) = \psi_\pm(-a, b)$*, and set*

$$p_{+0} = m(S = b, I = -a, \sigma = +1), \qquad p_{-0} = m(S = b, I = -a, \sigma = -1). \tag{11}$$

If we denote

$$v_\pm \equiv \frac{m(X; S = b, I = -a, \sigma = \pm 1)}{p_{\pm 0}} \equiv m(X \,|\, S = b, I = -a, \sigma = \pm 1), \qquad (12)$$

then the conditions[1]

$$\frac{p_{+0}}{p_+} \leq \frac{h}{b + h - v_+} \qquad (13)$$

$$\frac{p_{-0}}{p_-} \leq \frac{h}{a + h + v_-} \qquad (14)$$

are necessary for m to be consistent.

Proof We introduce the notation

$$\begin{aligned} p_{++} &= \bar{m}(H_{-a} < H_b < H_{b+h} < H_{-a-h}), \\ p_{+-} &= \bar{m}(H_{-a} < H_b < H_{-a-h} < H_{b+h}), \\ p_{--} &= \bar{m}(H_b < H_{-a} < H_{-a-h} < H_{b+h}), \\ p_{-+} &= \bar{m}(H_b < H_{-a} < H_{b+h} < H_{-a-h}). \end{aligned}$$

Using the Optional Sampling Theorem, we have similarly to (6), (7) the equations

$$p_+ = p_{++} + p_{+0} + p_{+-} \qquad (15)$$

$$bp_+ = (b + h)p_{++} - (a + h)p_{+-} + m(X; S = b, I = -a, \sigma = +1). \qquad (16)$$

If we write $\tilde{p}_{xy} = p_{xy}/p_x$ for $x \in \{-, +\}$, $y \in \{-, 0, +\}$ the Eqs. (15), (16) are expressed more simply in conditional form:

$$1 = \tilde{p}_{++} + \tilde{p}_{+-} + \tilde{p}_{+0} \qquad (17)$$

$$b = (b + h)\tilde{p}_{++} - (a + h)\tilde{p}_{+-} + \tilde{p}_{+0}v_+. \qquad (18)$$

The value of p_{+0} is known from m, as is the value of v_+, and since we assume that m is consistent the values of $p_\pm = \psi_\pm(-a, b)$ are also known from m. Therefore we can solve the linear system (17), (18) to discover

$$\tilde{p}_{++} = \frac{b + a + h - (a + h + v_+)\,\tilde{p}_{+0}}{b + a + 2h} \qquad (19)$$

$$\tilde{p}_{+-} = \frac{h - (b + h - v_+)\,\tilde{p}_{+0}}{b + a + 2h}. \qquad (20)$$

[1] If either of $p_\pm$ is zero, then the inequalities (13), (14) have to be understood in cross-multiplied form, when they state vacuously that $0 \leq 0$.

In order that $\tilde{p}_{+-}$ as given by (20) should be non-negative, we require that

$$\tilde{p}_{+0} \equiv \frac{m(S=b, I=-a, \sigma=+1)}{p_+} \leq \frac{h}{b+h-v_+}, \tag{21}$$

which is condition (13). Necessity of (14) is derived similarly. □

Remarks

(i) The necessary conditions (13), (14) come from the requirement that $\tilde{p}_{+-}$ and $\tilde{p}_{-+}$ should be non-negative. Do we know for sure that $\tilde{p}_{++}$ and $\tilde{p}_{--}$ are non-negative? The definition (12) of $v_\pm$ guarantees that $-a \leq v_\pm \leq b$, so if (21) holds then we know that $\tilde{p}_{+0} \leq 1$. From (19) we see then that $\tilde{p}_{++} \geq 0$. Since all the summands on the right-hand side of (17) are non-negative, we learn that they are probabilities summing to 1.

(ii) Notice that we have two expressions for $\bar{m}(H_{b+h} < \infty, I(H_{b+h}) = -a)$, either as $p_{++} + p_{-+}$, or as $\psi_+(-a, b+h)$. Confirming that these are the same is an important step in the proof of sufficiency.

2.2 *Sufficiency*

We have now identified necessary conditions (13) and (14) for m to be consistent. The main result of this paper is that these conditions are also sufficient.

Theorem 2.4 *The probability measure m on $\mathcal{X} \equiv -h\mathbb{Z}^+ \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$ is consistent if and only if $m(I \leq X \leq S) = 1$ and necessary conditions (13) and (14) hold.*

Proof Necessity has been proved, so what remains is to show that conditions (13) and (14) are sufficient. Not surprisingly, the proof of this is constructive.

We require a probability space $(\Omega, \mathcal{F}, P)$ rich enough to carry an IID sequence $U_0, U_1, \ldots$ of $U[0,1]$ random variables, and an independent standard Brownian motion (B_t). Let $\mathcal{U} = \sigma(U_0, U_1, \ldots)$, and let $(\mathcal{G}_t)$ be the usual augmentation of the filtration $(\mathcal{U} \vee \sigma(B_s : s \leq t))$. Define $(\mathcal{G}_t)$-stopping times

$$\alpha_0 \equiv 0, \qquad \alpha_{n+1} \equiv \inf\{t > \alpha_n : |B_t - B_{\alpha_n}| > h\},$$

the process $\xi_{nh} \equiv B(\alpha_n)$ and the filtration $\mathcal{F}_{nh} \equiv \mathcal{G}_{\alpha_n}$, so that $(\xi_t, \mathcal{F}_t)_{t \in h\mathbb{Z}^+}$ is a symmetric simple random walk. As before, define $S_t \equiv \sup_{s \leq t} \xi_s$, $I_t \equiv \inf_{s \leq t} \xi_s$ for $t \in h\mathbb{Z}^+$.

The construction borrows the technique of [8], where we firstly modify the given law m so that the conditional distribution of X_T given $\{S_T = b, I_T = -a, \sigma_T = s\}$ is a unit mass on the expected value $m[X_T \,|\, S_T = b, I_T = -a, \sigma_T = s]$. If we can construct a martingale with this degenerate conditional law, then we can build the required distribution of X_T given $\{S_T = b, I_T = -a, \sigma_T = s\}$ by Skorokhod embedding in a Brownian motion. So we may and shall suppose that[2]

$$m[X_T = v \mid S_T = b, I_T = -a,\ \sigma_T = s] = 1, \tag{22}$$

where $v = m[X_T \mid S_T = b, I_T = -a,\ \sigma_T = s]$.

The construction is sequential, and the proof that it succeeds is inductive. Let $\tau_n \equiv \inf\{t : S_t - I_t = nh\}$, and set $\sigma_n = \alpha_{\tau_n}$, the corresponding stopping time for the Brownian motion. The construction of T begins by setting $T = 0$ if $U_0 < m(S = I = 0)$, otherwise $T \geq h = \tau_1$. The sequential construction supposes[3] we have found that $T \geq \tau_n$, and $S_{\tau_n} = \xi_{\tau_n} = b$, $I_{\tau_n} = -a$. Then we place a lower barrier $\ell \in [-a-h, b+h]$ by the recipe

$$\begin{aligned} \ell &= v_+ \qquad \text{if } U_n < \theta \\ &= -a-h \qquad \text{else} \end{aligned}$$

where v_+ is defined in terms of m by (12), and θ is defined by

$$\tilde{p}_{+0} \equiv \frac{m(S = b, I = -a, \sigma = +1)}{\psi_+(-a, b)} = \frac{m(S = b, I = -a, \sigma = +1)}{\bar{m}(H_b < \infty, I(S_b) = -a)} = \theta\ \frac{h}{b+h-v_+} \tag{23}$$

with the notation of Proposition 2.3; in view of the fact that we have assumed the necessary conditions (13) and (14), *we can assert*[4] *that* θ *so defined* is *a probability*: $0 \leq \theta \leq 1$. We now run the Brownian motion B forward from time σ_n until it first hits ℓ or $b+h$. If $\ell = v_+$ and B hits ℓ before $b+h$, then we will stop everything at that time, and declare that $X_T = v_+$; otherwise, we will reach either $-a-h$ or $b+h$ and declare that $T \geq \tau_{n+1}$. If we determine that $T \geq \tau_{n+1}$, we take a further step of the construction.

For each $n \geq 1$, let Q_n be the combined statement[5]

(i) for all $a, b \in h\mathbb{Z}^+$, $0 < a+b \leq nh$

$$P(H_b \leq T,\ I(H_b) = -a) = \psi_+(-a, b) \tag{24}$$

$$P(H_{-a} \leq T,\ S(H_{-a}) = b) = \psi_-(-a, b) \tag{25}$$

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[2]There is no reason why v need be a multiple of h, but this does not matter; if $s = +$, say, we shall use the Brownian motion living in the original probability space, starting at b and run until it first hits either the upper barrier $b+h$ or the lower barrier, which will be *randomized*, taking value v_+ with suitably-chosen probability θ, otherwise taking value $-a-h$.

[3]We provide details of what happens if $S_{\tau_n} = \xi_{\tau_n}$; the treatment of the case $I_{\tau_n} = \xi_{\tau_n}$ is analogous.

[4]We shall establish in the inductive proof that $\psi_\pm$ are non-negative.

[5]The functions $\psi_\pm$ are defined in terms of m by (4), (5), (8), (9).

(ii)

$$P(S = x, I = -y, X = z, \sigma = s) = m(S = x, I = -y, X = z, \sigma = s) \tag{26}$$

for all $s \in \{-1, 1\}$, $x, y, z, \in h\mathbb{Z}$, $x, y \geq 0$, $x + y < nh$.

We shall prove by induction that Q_n is true for all $n > 0$, establishing the statement first for $n = 1$. We prove (24), leaving the analogous proof of (25) to the diligent reader. Taking $b = 0,\ a = h$, (24) says that

$$P(H_0 \leq T,\ I(H_0) = -h) = \psi_+(-h, 0),$$

and both sides are readily seen to be equal to zero; taking $b = h,\ a = 0$, (24) says that

$$\begin{aligned} P(H_h \leq T,\ I(H_h) = 0) &= \psi_+(0, h) \\ &= \varphi(h, -h) - \varphi(h, 0) \\ &= \frac{h - m(h + X; S < h, I > -h)}{2h} - 0 \\ &= \tfrac{1}{2}\big[\, 1 - m(S = X = I = 0)\,\big] \end{aligned}$$

which is clearly true, because if the construction does not stop immediately at time 0 (an event of probability $m(I = X = S = 0)$) then with equal probability the process steps at time 1 to $\pm h$. The second statement (26) holds because we have constructed the probability of $I = X = S = 0$ correctly.

Now suppose that Q_k has been proved to hold for $k \leq n$; we have to prove (24), (25) and (26) for $n + 1$. To prove (26), suppose that $x,\ y \in h\mathbb{Z}^+$ and $x + y = nh$. By construction, the random walk will be stopped before the range $S - I$ increases to $(n + 1)h$ if and only if the barrier ℓ happens to be positioned at v_+ *and* that barrier is hit before the Brownian motion rises to $b + h$. Conditional on the event $B_+ = \{T \geq \tau_n,\ S_{\tau_n} = \xi_{\tau_n} = b,\ I_{\tau_n} = -a\}$, the probability of that joint event is

$$\theta \times \frac{h}{b + h - v_+}. \tag{27}$$

By the inductive hypothesis (24) we have that the probability of the conditioning event B_+ is $\psi_+(-a, b)$; so from the definition (23) of θ we learn that

$$P(S_T = b,\ I_T = -a,\ \sigma = +1) = m(S = b, I = -a,\ \sigma = +1).$$

Given that this event happens, the conditional distribution of X_T is correct, by the Skorohod embedding construction of X_T with mean v_+. Therefore (26) has been proven for any $x,\ y \in h\mathbb{Z}$ with $x + y = nh$, and for any $z \in h\mathbb{Z}$, $s \in \{-1, 1\}$.

It remains to prove assertion (i) of Q_{n+1}, and for this we recall some of the notation of the proof of Proposition 2.3. For $a,\ b \in h\mathbb{Z}^+$, $a+b=nh$, we write

$$p_+ = P(B_+) \equiv P(H_b \le T,\ I(H_b) = -a),$$
$$p_- = P(B_-) \equiv P(H_{-a} \le T,\ S(H_{-a}) = b)$$

which in view of the truth of Q_n we know are equal to $\psi_+(-a,b)$ and $\psi_-(-a,b)$ respectively. If we now define

$$p_{++} = P(B_+, H_{b+h} \le T \wedge H_{-a-h})$$
$$p_{+-} = P(B_+, H_{-a-h} \le T \wedge H_{b+h})$$
$$p_{+0} = P(B_+, T < \tau_{n+1})$$
$$p_{-+} = P(B_-, H_{b+h} \le T \wedge H_{-a-h})$$
$$p_{--} = P(B_-, H_{-a-h} \le T \wedge H_{b+h})$$
$$p_{-0} = P(B_-, T < \tau_{n+1})$$

then by exactly the same Optional Sampling argument which led to (19), (20), we conclude that

$$p_{++} = \frac{(b+a+h)p_+ - (a+h+v_+)\,p_{+0}}{b+a+2h} \tag{28}$$

$$p_{+-} = \frac{hp_+ - (b+h-v_+)\,p_{+0}}{b+a+2h} \tag{29}$$

$$p_{-+} = \frac{hp_- - (a+h+v_-)p_{-0}}{a+b+2h} \tag{30}$$

$$p_{--} = \frac{(a+b+h)p_- - (b+h-v_-)p_{-0}}{a+b+2h} \tag{31}$$

and now the task is to prove (after cross-multiplying by $a+b+2h$) that

$$(a+b+2h)\{\,p_{++} + p_{-+}\,\} = (a+b+2h)\psi_+(-a, b+h), \tag{32}$$

and the minus analogue, which is just the same argument *mutatis mutandis.* Firstly we develop the left-hand side using (28), (29) and their analogues for B_- to obtain

$$\begin{aligned} LHS &= (a+b+h)\psi_+(-a,b) - (a+h+v_+)p_{0+} \\ &\quad + h\psi_-(-a,b) - (a+h+v_-)p_{-0} \\ &= (a+b+h)\{\,\varphi(b,-a-h) - \varphi(b,-a)\,\} \\ &\quad + h\{\,\varphi(-a,b+h) - \varphi(-a,b)\,\} \end{aligned}$$

$$
\begin{aligned}
&-(a+h)m(S=b, I=-a) - m(X; S=b, I=-a)\\
&= a+h-m(a+h+X; S<b, I>-a-h)\\
&\quad -\{\, a - m(a+X; S<b, I>-a)\,\}\\
&\quad -h(\varphi(b-a)+\varphi(-a,b)) + h\varphi(-a,b+h)\\
&\quad -m(a+h+X; S=b, I=-a)\\
&= h - m(a+h+X; S<b, I>-a-h)\\
&\quad +m(a+X; S<b, I>-a)\\
&\quad -h\{1-m(S<b, I>-a)\} + h\varphi(-a,b+h)\\
&\quad -m(a+h+X; S=b, I=-a)\\
&= -m(a+h+X; S<b, I>-a-h)\\
&\quad +m(a+h+X; S<b, I>-a)\\
&\quad -m(a+h+X; S=b, I=-a) + h\varphi(-a,b+h)\\
&= -m(a+h+X : (A_2\cup A_3)\backslash A_1) + h\varphi(-a,b+h)
\end{aligned}
$$

where $A_1 = \{S<b, I>-a\}, A_2 = \{S<b, I>-a-h\}$ and $A_3 = \{S=b, I=-a\}$. Noticing that $A_1 \subseteq A_2$ and A_3 is disjoint from A_1, the region of integration is

$$(A_2\cup A_3)\backslash A_1 = \{S<b, I=-a\}\cup A_3 = \{S\le b, I=-a\} = \{S<b+h, I=-a\}.$$

Hence the left-hand side is equal to

$$LHS = -m(a+h+X; S<b+h, I=-a) + h\varphi(-a,b+h). \tag{33}$$

Turning now to the right-hand side of (32), we have

$$
\begin{aligned}
RHS &= (a+b+2h)\{\,\varphi(b+h,-a-h)-\varphi(b+h,-a)\,\}\\
&= a+h-m(a+h+X : S<b+h, I>-a-h) - h\varphi(b+h,-a)\\
&\quad -\{\, a - m(a+X : S<b+h, I>-a)\,\}\\
&= h - m(a+h+X : S<b+h, I>-a-h)\\
&\quad +m(a+h+X; S<b+h, I>-a)\\
&\quad -hm(S<b+h, I>-a) - h\varphi(b+h,-a)\\
&= h\{\,1-m(S<b+h, I>-a)-\varphi(b+h,-a)\,\}\\
&\quad -m(a+h+X; S<b+h, I=-a).
\end{aligned} \tag{34}
$$

Comparing (33) and (34), we see that we have to prove

$$\varphi(b+h,-a)+\varphi(-a,b+h)=1-m(S<b+h,I>-a), \tag{35}$$

which is evidently true from the definition (4), (5) of φ. □

3 Hedging

Theorem 2.4 provides us with necessary and sufficient conditions for a measure m on $\mathcal{X}$ to be consistent. In principle, this allows us to construct extremal martingales, and robust hedges for derivatives.

Let us firstly see how this works in the context of the joint law of (S,X) studied in [8]. We begin by recalling some of the results of that paper. We let $X_t=B_{t\wedge T}$ be a Brownian motion stopped as an almost-surely finite stopping time T, with $S_t=\sup_{u\le t}X_u$, and with $S\equiv S_\infty$, $X\equiv X_\infty$. With this terminology, Theorem 3.1 of [8] says the following.

Theorem 3.1 *The probability measure μ on $\mathbb{R}^+\times\mathbb{R}^+$ is the joint law of $(S,S-X)$ for some almost-surely finite stopping time T if and only if*

$$\left(\iint_{(t,\infty)\times\mathbb{R}^+}\mu(ds,dy)\right)dt\ge\int_{(0,\infty)}y\;\mu(dt,dy). \tag{36}$$

If $(X_t)_{t\ge0}$ is also uniformly integrable, then inequality (36) holds with equality:

$$\left(\iint_{(t,\infty)\times\mathbb{R}^+}\mu(ds,dy)\right)dt=\int_{(0,\infty)}y\;\mu(dt,dy). \tag{37}$$

Finally, if (37) holds, and if $X\in L^1$,

$$\iint|t-y|\;\mu(dt,dy)<\infty, \tag{38}$$

then μ is the joint law of $(S,S-X)$ for a uniformly integrable martingale $(X_t)_{t\ge0}$.

Proof See [8]. The final assertion is not in [8], but can easily be deduced. In view of the first assertion, there is some stopping time $T<\infty$ such that μ is the joint law of $(S,S-X)$. By multiplying (37) by some non-negative test function φ and integrating with respect to t we discover that

$$\mu(\Phi)=\mu(\,(S-X)\varphi(S)\,) \tag{39}$$

where $\Phi(t) = \int_0^t \varphi(y)\, dy$. Taking $\varphi(x) = I_{\{x>b\}}$ for some $b \geq 0$ we find that

$$b\mu(S > b) = \mu(X : S > b). \tag{40}$$

Using the fact that $X \in L^1$, we can let $b \uparrow \infty$ in (40) to prove that $\lim_{b\uparrow\infty} b\mu(S > b) = 0$. Lemma 2.3 of [8] gives the result. □

Remark Standard monotone class arguments show that (36) is equivalent to the statement that

$$\mu(\Phi) \geq \mu(\,(S - X)\varphi(S)\,) \tag{41}$$

for all non-negative test functions, which again is equivalent to the statement that

$$b\mu(S > b) \geq \mu(X : S > b) \tag{42}$$

for all $b \geq 0$. Likewise, (37) is equivalent to (39) for all non-negative test functions φ, which again is equivalent to the statement (40):

$$\mu(X - b : S > b) = 0 \qquad \forall b \geq 0. \tag{43}$$

An important and typical[6] use of this would be to try to find an *extremal* martingale, which would in turn lead to a maximum possible derivative price and a robust hedging strategy. So, for example, suppose that we observe call option prices $C(K)$ for every strike K at a common fixed expiry time[7] for some (discounted) asset, and suppose that the asset has continuous paths $(X_t)_{0\leq t\leq 1}$, and is a uniformly-integrable martingale in the pricing measure.

Suppose now that we are given some derivative whose payoff at time 1 is $G(S_1, X_1)$, where $S_1 = \sup_{0\leq t\leq 1} X_t$; *what is the most expensive the time-0 price of this derivative can be?*

The time-0 price of the derivative is given by

$$\iint G(s, x)\; q(ds, dx) \tag{44}$$

where q is the joint law[8] of (S, X). Now provided the law q satisfies the conditions

$$\iint (x - K)^+\; q(ds, dx) = C(K) \qquad \forall K \tag{45}$$

[6]The papers ([7][1][2]) give examples of this kind. The recent paper of Galichon, Henry-Labordère and Touzi [6] strengthens [1] to multiple time points.

[7]Let us suppose that the expiry is 1.

[8]As before, when the time subscript of a process is omitted, we understand it to be 1.

and (see (43))

$$\iint_{s>b} (x-b)\, q(ds,dx) = 0 \qquad \forall b > 0 \tag{46}$$

then q is the joint distribution of (S,X) for *some* continuous martingale whose law at time 1 agrees with the data contained in the call prices. The problem of finding the most expensive time-0 price is therefore the problem of maximizing the *linear* objective (44) over non-negative probability measures q subject to the *linear* constraints (45) and (46). Writing the problem in Lagrangian form,[9] we seek

$$L(\alpha,\eta,\lambda) = \sup_{q\geq 0}\Big[\iint \Big\{ G(s,x) - \alpha - \int (x-K)^+ \; \eta(dK) \\ + \int_0^\infty (x-b) I_{\{s>b\}} \; \lambda(db) \Big\} \; q(ds,dx) + \alpha + \int C(K) \; \eta(dK) \; \Big]. \tag{47}$$

From standard linear programming results, we would expect that for dual feasibility we must have

$$G(s,x) \leq \alpha + \int (x-K)^+ \; \eta(dK) - \int_0^\infty (x-b) I_{\{s>b\}} \; \lambda(db) \tag{48}$$

everywhere, with equality everywhere that the optimal q places mass; and that the dual problem will be

$$\inf \Big[\; \alpha + \int C(K) \; \eta(dK) \; \Big] \tag{49}$$

over (α,η,λ) satisfying (48). These equations have a simple and beautiful interpretation. The dual-feasibility relation (48) expresses a *robust hedge*; if we hold α in cash, $\eta(dK)$ calls of strike K, and *sell forward* $\lambda(db)$ *units of the underlying when* S *reaches the level* b, then we generate a contingent claim at the terminal time which will always dominate the claim G which we have to pay out. The dual form of the linear program (49) says that the cost of constructing such a hedge, which is of course $\alpha + \int C(K) \; \eta(dK)$, must be minimized.

The primal problem seeks to find the most expensive that the derivative $G(S,X)$ can be, given the market prices $C(K)$; and the dual problem seeks the cheapest super-replicating hedge. The characterization (43) of the possible joint laws of (S,X) *tells us what the form of the hedge* (48) *must be.*

[9] This linear programming approach to the problem is also used in [4].

Our goal now is to try to use Theorem 2.4 to similarly bound the price of, and to super-replicate, contingent claims which depend on the maximum, terminal value, *minimum, and direction of the final excursion* for a stopped symmetric simple random walk. To understand how this is to be done, we focus on the 'plus' versions of the necessary and sufficient conditions (13). We shall also suppose that the martingale X is *uniformly integrable*, to avoid having to bother about side issues.

The condition (13) can be restated in terms of the measure m as

$$\begin{aligned} m(b+h-X : S=b, I=-a, \sigma=+1) &\le h\psi_+(-a,b) \qquad (50)\\ &= h\{\,\varphi(b,-a-h)-\varphi(b,-a)\,\} \end{aligned}$$

in the notation of Sect. 2. From the definition (4) of $\varphi(b,-a)$, from the fact that $m(X)=0$, and the Optional Sampling Theorem result that $m(a+X : I\le -a)=0$, we have

$$\begin{aligned} (a+b)\varphi(b,-a) &= a - m(a+X : S<b, I>-a)\\ &= m(a+X :\ S\ge b \text{ or } I\le -a)\\ &= m(a+X : S\ge b, I>-a)\\ &= (a+b)m(S\ge b, I>-a) - m(b-X : S\ge b, I>-a). \end{aligned}$$

Thus the inequality (50) may be re-expressed after some simple rearrangement as

$$\begin{aligned} 0 \le hm(S\ge b, I=-a) &- \frac{h}{a+b+h}\, m(b-X : S\ge b, I>-a-h)\\ &+\frac{h}{a+b}\, m(b-X : S\ge b, I>-a)\\ &-m(b+h-X : S=b, I=-a, \sigma=+1). \end{aligned}$$

This inequality for all $a,\ b\ \in\ h\mathbb{Z}^+$ not both zero, together with the 'minus' analogues, is necessary and sufficient for a probability measure m to be the joint law of (I,X,S,σ). Just as we did at (47) for derivatives depending only on (X,S), we can construct the Lagrangian for this problem, which would give us terms of the form

$$\begin{aligned} \lambda^+_{ab}\,(Z-w) \equiv \lambda^+_{ab}\Big[\, & hI_{\{S\ge b, I=-a\}} - \frac{h}{a+b+h}\,(b-X)I_{\{S\ge b, I>-a-h\}}\\ &+\frac{h}{a+b}\,(b-X)I_{\{S\ge b, I>-a\}} - (b+h-X)I_{\{S=b, I=-a, \sigma=+1\}} - w\,\Big], \end{aligned} \qquad (51)$$

where $w \geq 0$ is a non-negative slack variable to handle the inequality constraint. Dual feasibility will therefore require that $\lambda^+_{ab} \geq 0$, and at optimality we will have the complementary slackness condition $\lambda^+_{ab} w = 0$.

In the situation of derivatives depending only on (X, S), we had terms of the form $\lambda_a(X-a)I_{\{S>a\}}$, which were interpreted as forward purchase of the underlying asset when the supremum process reaches a new level. This forward purchase interpretation determines a hedging strategy which *can be implemented in an adapted fashion*. However, it is very far from clear that the random variable Z defined at (51) can be realized by some adapted trading strategy. For example, the term involving $(b-X)I_{\{S\geq b, I>-a\}}$ could be interpreted as a forward sale of the underlying when the price first gets to b; but this trade should only be put on if $I > -a$, and it is not known at time H_b whether or not the ultimate infimum I will be greater than $-a$ or not.

Nevertheless, we can specify an adapted trading strategy which will subreplicate the random variable Z, as follows. We construct a random variable Y which is the final value of the adapted hedging strategy made up of three component positions:

1. At H_b, buy forward $h/(a+b+h)$ units of the underlying if $I(H_b) > -a-h$, and come out of the position at time H_{-a-h};
2. At H_b, buy forward $-h/(a+b)$ units of the underlying if $I(H_b) > -a$, and come out of the position at time H_{-a};
3. At H_b, buy forward 1 unit of the underlying if $I(H_b) = -a$, and come out of the position at time $H_{b+h} \wedge H_{-a-h}$.

Now clearly the random variable

$$Z \equiv hI_{\{S\geq b, I=-a\}} - \frac{h}{a+b+h}\,(b-X)I_{\{S\geq b, I>-a-h\}} + \frac{h}{a+b}\,(b-X)I_{\{S\geq b, I>-a\}} - (b+h-X)I_{\{S=b, I=-a, \sigma=+1\}} \qquad (52)$$

will be zero if $S < b$ or if $I \leq -a-h$, so to understand Z we may suppose that $H_b < \infty = H_{-a-h}$.

But before we narrow our attention down to the event $\{H_b < \infty = H_{-a-h}\}$, we should consider what happens off that event to Y. If $H_b = \infty$, then none of the component positions of Y is ever entered, so $Y = 0$ in that case. If $H_b < \infty$ and $H_{-a-h} < \infty$, then we have three cases to consider:

(i) When $I(H_b) > -a$, the strategy enters positions 1 and 2 at time H_b, and closes out both when the infimum falls to $-a$ and then to $-a-h$; position 1 loses h, position 2 gains h, so altogether $Y = 0$;

(ii) When $I(H_b) = -a$, the strategy enters positions 1 and 3. If $H_{b+h} < H_{-a-h}$, then position 3 makes a gain of h when it is closed out, but position 1 makes a loss of h when it is closed out, so overall zero gain. On the other hand, if $H_{-a-h} < H_{b+h}$, then position 1 makes a loss of h when it is closed out, and position 3 makes a loss of $(a+b+h)$ when it is closed out, so overall $Y =$

$-(a+b+h)-h<0$, and as we shall subsequently see, *this is the only situation in which Y is strictly less than Z*;

(iii) When $I(H_b) \le -a-h$, none of the positions is entered, and $Y=0$.

We now have to compare the values of Z and Y on the event $\{H_b < \infty = H_{-a-h}\}$, breaking the comparison down into seven cases as presented in the following table. In the first two rows, we see what happens if $I > -a$, and in the remaining rows, we are considering situations where $I=-a$. The reader is invited to check through each of the entries of the table, and confirm the findings reported there. The only entry that requires comment is the penultimate row, in the column for Z. In this row, we are in the situation where $S=b$ and $I=-a$, so we get a contribution to Z from the first term in (52), and from the second term, none from the third term, and *none from the fourth term*, because if $H_b < H_{-a} < H_{b+h} = \infty$ it must be that *the signature σ is -1 !* What we see from the table is that in every case the value of Z is equal to the value of Y.

$H_{-a-h}=\infty$	Z	Y
$H_b < H_{b+h} < \infty = H_{-a}$	$\frac{h(b-X)}{a+b} - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} - \frac{h(X-b)}{a+b}$
$H_b < H_{b+h} = \infty = H_{-a}$	$\frac{h(b-X)}{a+b} - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} - \frac{h(X-b)}{a+b}$
$H_{-a} < H_b < H_{b+h} < \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$
$H_{-a} < H_b < H_{b+h} = \infty$	$h - \frac{h(b-X)}{a+b+h} + (X-b-h)$	$\frac{h(X-b)}{a+b+h} + X - b$
$H_b < H_{-a} < H_{b+h} < \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$
$H_b < H_{-a} < H_{b+h} = \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$
$H_b < H_{b+h} < H_{-a} < \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$

Thus we may conclude that $Y \le Z$ in all instances, and the only situation in which the inequality is strict is when $H_{-a} < H_b < H_{-a-h} < H_{b+h}$.

Now we explain how these observations lead to a super-replicating hedging strategy. For this, let us denote by Z^+_{ab} then random variable we have been calling Z up till now; this is because in the Lagrangian we have to consider such random variables (and their 'minus' analogues) for all $a, b \in h\mathbb{Z}^+$ not both zero. Suppose that we have some derivative $G(I, X, S, \sigma)$ whose price we wish to maximize subject to the distribution of X matching call price data, just as we did for derivatives depending only on (X, S) in the first part of our discussion in this section. We would find ourselves with a Lagrangian form similar to (47):

$$L(\alpha,\lambda,\eta) = \sup_{m\ge 0}\Big[\int \Big\{ G(I,X,S,\sigma) - \alpha - \int (X-K)^+ \eta(dK) \\ + \sum_{a,b,\pm} \lambda^{\pm}_{ab}(Z^{\pm}_{ab} - w^{\pm}_{ab}) \Big\}\, dm(I,X,S,\sigma) + \alpha + \int C(K)\,\eta(dK) \Big\} \Big] \tag{53}$$

with obvious notation. Now dual feasibility imposes the condition

$$G(I, X, S, \sigma) \leq \alpha + \int (X-K)^+ \eta(dK) - \sum_{a,b,\pm} \lambda_{ab}^{\pm} Z_{ab}^{\pm} \tag{54}$$

$$\leq \alpha + \int (X-K)^+ \eta(dK) - \sum_{a,b,\pm} \lambda_{ab}^{\pm} Y_{ab}^{\pm} \tag{55}$$

in another obvious notation. The interpretation of (55) is that *the derivative G is super-replicated by the adaptively-realizable hedge given by a position in calls and a position in the Y-hedges.*

At optimality, complementary slackness tells us that if $\lambda_{ab}^+ > 0$ then $w_{ab}^+ = 0$, and therefore the inequality (50) must hold with equality. Tracing this back to the condition (13), and its derivation from (20), we find that equality in (50) is equivalent to the statement that $\tilde{p}_{+-} = 0$. What this means is that on the event $\{H_{-a} < H_b < H_{-a-h}\}$ we *cannot have $H_{-a-h} < H_{b+h}$*, and as we saw, this was *the only situation where $Y < Z$*. We may therefore conclude that for the optimal m^*, not only will (54) hold with equality m^*-a.e., but also (55) will hold with equality m^*-a.e. In other words, if the joint law m is the optimal joint law, the hedging strategy expressed by (55) is a perfect replication of the contingent claim—there is no slack.

Summary

The main result of the paper is a characterization of the possible joint laws of (I, X, S, σ) for a stopped symmetric simple random walk. A natural and intriguing question is whether it is necessary to include the signature σ in the random vector, or whether we can indeed find a characterization of all possible laws of (I, X, S). At the moment, we cannot provide an answer to this natural question, though it may be possible using some of the approach of [9] to make progress. Another interesting question is whether we can deduce the corresponding result for Brownian motion. At one level, we could formally pass to a limit, assuming some continuous joint density exists, but the resulting conditions are not particularly easy to interpret. This leads one to believe that there may be some other characterization that admits a clean interpretation, rather in the style of the results of [8, 9].

LCGR Remembers Marc Yor

Marc and I had met at numerous conferences before I first visited him at Jussieu in October 1987. As I made my way to Paris VI, I wondered what kind of office this stellar young professor would have; something with a big carpet and a view over the

Seine perhaps? But no. His office had no view or carpet, but contained several desks, one for himself, others for students and visitors, and contained also a tangible aura of scholarly intensity emanating from Marc himself and compelling those around to try to match his effort. Anyone who spent time with Marc will remember many things about him: how he stroked his moustache when thinking; his earnest tone, almost reverent, as he explained the things he was working on; his generous support of junior colleagues; how he seemed when listening to be partly exploring some background maths question—if so, he was nevertheless completely in touch with what was being said, because any humorous remark would draw from him a gentle chuckle and a flash of his bright eyes.

One event of that week in October stays in my mind. The visit had been sponsored by the British Council, and their representative in Paris wanted to arrange a photoshoot for a publicity magazine they were producing. So one afternoon, Marc and I (together with François Murat and John Ball, who was also visiting on the British Council scheme) went up the main tower at Jussieu, where the plan was to have some beautiful pictures of us 'doing mathematics' with marker pens on the windows, with a view of the Eiffel Tower through the window. Of course it took ages to set the photos up, of course it was rather artificial, and I could sense that Marc was eager to get finished and back to work, but he co-operated politely and patiently, because he was under an obligation to the British Council representative to help. And to me that event displayed two of Marc's most important characteristics: no-one was more committed to mathematical research, which is why we admired him, but he always gave priority to his obligations to others, whoever they may have been, which is why we loved him.

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Convergence Towards Linear Combinations of Chi-Squared Random Variables: A Malliavin-Based Approach

Ehsan Azmoodeh, Giovanni Peccati, and Guillaume Poly

Abstract We investigate the problem of finding necessary and sufficient conditions for convergence in distribution towards a general finite linear combination of independent chi-squared random variables, within the framework of random objects living on a fixed Gaussian space. Using a recent representation of cumulants in terms of the Malliavin calculus operators Γ_i (introduced by Nourdin and Peccati, J. Appl. Funct. Anal. **258**(11), 3775–3791, 2010), we provide conditions that apply to random variables living in a finite sum of Wiener chaoses. As an important by-product of our analysis, we shall derive a new proof and a new interpretation of a recent finding by Nourdin and Poly (Electron. Commun. Probab. **17**(36), 1–12, 2012), concerning the limiting behavior of random variables living in a Wiener chaos of order two. Our analysis contributes to a fertile line of research, that originates from questions raised by Marc Yor, in the framework of limit theorems for non-linear functionals of Brownian local times.

MSC 2010: 60F05; 60G15; 60H07

1 Introduction

The aim of this paper is to provide necessary and sufficient conditions (expressed in terms of Malliavin operators), ensuring that a sequence of random variables living in a finite sum of Wiener chaoses converges in distribution towards a

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finite linear combination of independent centered chi-squared random variables. As discussed below, we regard the results of the present paper as a first step towards the solution of an open and notoriously difficult problem, namely: *can one derive necessary and sufficient analytical conditions, ensuring that a given sequence of smooth functionals of a Gaussian field converge in distribution towards an element of the second Wiener chaos?* Finite linear combinations of independent chi-squared random variables represent indeed the most elementary instance of random objects living in the second Wiener chaos of a Gaussian field (see Sect. 2.4 below for a discussion of this point). More sophisticated examples—that are crucial for applications and lay at present largely outside the scope of Malliavin-type techniques—include the so-called *Rosenblatt distribution*; see e.g. [21] for a detailed discussion of these objects.

1.1 Overview

We refer the reader to [10], as well as Sect. 2 below, for any unexplained notion evoked in the present section. Let $W = \{W(h) : \mathfrak{H}\}$ be an isonormal Gaussian process over some real separable Hilbert space $\mathfrak{H}$ and let $q \geq 1$. For every deterministic symmetric kernel $f \in \mathfrak{H}^{\odot q}$, we denote by $I_q(f)$ the multiple stochastic Wiener-Itô integral of f with respect to W. Random variables of the form $I_q(f)$ compose the so-called qth *Wiener chaos* associated with W. The concept of Wiener chaos represents a rough infinite-dimensional analogous of the Hermite polynomials for the one-dimensional Gaussian distribution (see e.g. [10, 17] for a detailed discussion of these objects).

The following two results, proved respectively in [15, 16] and [7] contain an exhaustive characterization of normal and Gamma approximations on Wiener chaos. As in [7], we denote by $F(\nu)$ a centered random variable with the law of $2G(\nu/2) - \nu$, where $G(\nu/2)$ has a Gamma distribution with parameter $\nu/2$. In particular, when $\nu \geq 1$ is an integer, then $F(\nu)$ has a centered χ^2 distribution with ν degrees of freedom.

Theorem 1

(A) (See [15, 16]) *Denote by D the Malliavin derivative associated with W. Let $N \sim \mathscr{N}(0,1)$, fix $q \geq 2$ and let $I_q(f_n)$ be a sequence of multiple stochastic integrals with respect to W, with each f_n an element of $\mathfrak{H}^{\odot q}$ such that $E[I_q(f_n)^2] = 1$. Then, the following are equivalent, as $n \to \infty$:*

  (i) *$I_q(f_n)$ converges in distribution to N;*
  (ii) *$E[I_q(f_n)^4] \to E[N^4] = 3$;*
  (iii) *$q^{-1}\|DI_q(f_n)\|^2_{\mathfrak{H}} \to 1$ in $L^2(\Omega)$.*

(B) (See [7]) *Fix $\nu > 0$, and let $F(\nu)$ have the centered Gamma distribution described above. Let $q \geq 2$ be an even integer, and let $I_q(f_n)$ be a sequence of multiple integrals, with each $f_n \in \mathfrak{H}^{\odot q}$ verifying $E[I_q(f_n)^2] = 2\nu$. Then, the following are equivalent, as $n \to \infty$:*

(i) $I_q(f_n)$ *converges in distribution to* $F(\nu)$*;*
(ii) $E[I_q(f_n)^4] - 12E[I_q(f_n)^3] \to E[F(\nu)^4] - 12E[F(\nu)^3] = 12\nu^2 - 48\nu$*;*
(iii) $\|DI_q(f_n)\|^2_{\mathfrak{H}} - 2qI_q(f_n) - 2q\nu \to 0$*, in* $L^2(\Omega)$.

The line of research associated with the content of Theorem 1 originates from some deep questions asked by Marc Yor, about the asymptotic behavior of non-linear functionals of Brownian local times (partially addressed in [18, 19]). As demonstrated e.g. in [16], results of this type are intimately connected to the powerful technique of *Brownian time changes* and associated limit theorems (a beautiful discussion of these topics can be found in [20, Chaps. V and XIII]): as such, they provide a drastic simplification of the so-called *method of moments* for probabilistic approximations.

Theorem 1 has triggered a huge amount of applications and generalizations, involving e.g. Stein's method, stochastic geometry, free probability, power variations of Gaussian processes and analysis of isotropic fields of homogeneous spaces. See [7] for an introduction to this field of research. See [6] for a constantly updated web resource, with links to all available papers.

As anticipated, the aim of the present paper is to address the following question: *for a general q, is it possible to prove a statement similar to Part* (B) *of Theorem 1, when the target distribution* $F(\nu)$ *is replaced by an object of the type*

$$F_\infty = \sum_{i=1}^{k} \alpha_i (N_i^2 - 1), \tag{1}$$

where k is a finite integer, the α_i*,* $i = 1, \dots, k$*, are pairwise distinct real numbers, and* $\{N_i : i = 1, \dots, k\}$ *is a collection of i.i.d.* $\mathcal{N}(0,1)$ *random variables?*

The following remarks are in order

- In the case $q = 2$ (that is, when the involved sequence of stochastic integrals belongs to the second Wiener chaos of W), the question has been completely answered by Nourdin and Poly [11]. See also the subsequent Theorem 2.
- The case $k = \alpha_1 = 1$ corresponds to Part (B) of Theorem 1, in the special case $\nu = 1$.
- When $k = 2$ and $\alpha_1 = \frac{1}{2} = -\alpha_2$, then one has that F_∞ has the same law as the random variable $N_1 \times N_2$. It is a well-known fact that the law of this random variable belongs to the general class of *Variance-Gamma* distributions: it follows that, in this special case, convergence towards F_∞ could be studied by means of the general Malliavin-Stein techniques developed by Eichelsbacher and Thäle in the (independently written) paper [3] (see also [4] for some related estimates). We observe that, in contrast to the present paper, the techniques developed in [3] yield explicit rates of convergence in some probability metric. On the other hand, our approach allows one to deal with target probability distributions that fall outside the class of Variance-Gamma laws, as well as to deduce necessary conditions for the convergence to take place.

In order to deal with the previously stated problem, one cannot rely on techniques that have been used in the previous literature on related subjects. In particular:

(a) For a general choice of k and $\alpha_1, \cdots, \alpha_k$ there is no suitable version of Stein's method that can be applied to the random variable F_∞ in (1), so that the Malliavin-Stein approach for normal and Gamma approximations developed in [8] cannot be used.
(b) For a general choice of k and $\alpha_1, \cdots, \alpha_k$, it seems difficult to represent the characteristic function of F_∞ as the solution of an ordinary differential equation: it follows that the characteristic function approach exploited in [7, 15] is not adapted to the framework of the present paper.
(c) The analytical approach used in [11] (for the case $q = 2$) cannot be applied in the case of a general order $q \geq 3$ since, in this case, the characteristic function of a non-zero random variable of the type $I_q(f)$ is not analytically known.

The main contribution of the present paper (stated in Theorem 3) is a full generalisation of the double implication (iii) $\leftrightarrow$ (i) in the statement of Theorem 1-(B) to the case of a general target random variable of the form (1) and of a general sequence of random variables living in a finite sum of Wiener chaoses. Our approach is based on a suitable extension of the method of moments, that relies in turn on several extensions of the results proved in [11]. One should notice that our findings involve the operators Γ_i from Malliavin calculus, as introduced in [9] (see also [10, Chap. 8]).

Remark 1 For the time being (and for technical reasons that will clearly appear in the sections to follow), it seems very arduous to extend the double implication (ii) $\leftrightarrow$ (i) in the statement of Theorem 1-(B).

1.2 Plan

The paper is organized as follows. Section 2 contains some preliminary materials including basic facts on Gaussian analysis and Malliavin calculus. Section 3 is devoted to our main results on a general criterion for convergence in distribution towards chi-squared combinations, whereas Sect. 4 provides some examples.

2 Elements of Gaussian Analysis and Malliavin Calculus

This section contains the essential elements of Gaussian analysis and Malliavin calculus that are used in this paper. See for instance [10, 14] for further details.

2.1 Isonormal Processes and Multiple Integrals

Let $\mathfrak{H}$ be a real separable Hilbert space. For any $q \geq 1$, we write $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ to indicate, respectively, the qth tensor power and the qth symmetric tensor power of $\mathfrak{H}$; we also set by convention $\mathfrak{H}^{\otimes 0} = \mathfrak{H}^{\odot 0} = \mathbb{R}$. When $\mathfrak{H} = L^2(A, \mathscr{A}, \mu) =: L^2(\mu)$, where μ is a σ-finite and non-atomic measure on the measurable space $(A, \mathscr{A})$, then $\mathfrak{H}^{\otimes q} = L^2(A^q, \mathscr{A}^q, \mu^q) =: L^2(\mu^q)$, and $\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathscr{A}^q, \mu^q) := L^2_s(\mu^q)$, where $L^2_s(\mu^q)$ stands for the subspace of $L^2(\mu^q)$ composed of those functions that are μ^q-almost everywhere symmetric. We denote by $W = \{W(h) : h \in \mathfrak{H}\}$ an *isonormal Gaussian process* over $\mathfrak{H}$. This means that W is a centered Gaussian family, defined on some probability space $(\Omega, \mathscr{F}, P)$, with a covariance structure given by the relation $E[W(h)W(g)] = \langle h, g\rangle_{\mathfrak{H}}$. We also assume that $\mathscr{F} = \sigma(W)$, that is, $\mathscr{F}$ is generated by W, and use the shorthand notation $L^2(\Omega) := L^2(\Omega, \mathscr{F}, P)$.

For every $q \geq 1$, the symbol C_q stands for the qth *Wiener chaos* of W, defined as the closed linear subspace of $L^2(\Omega)$ generated by the family $\{H_q(W(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the qth Hermite polynomial, defined as follows:

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\big(e^{-\frac{x^2}{2}}\big). \tag{2}$$

We write by convention $C_0 = \mathbb{R}$. For any $q \geq 1$, the mapping $I_q(h^{\otimes q}) = H_q(W(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and the qth Wiener chaos C_q. For $q = 0$, we write by convention $I_0(c) = c$, $c \in \mathbb{R}$.

It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces C_q: this means that any square-integrable random variable $F \in L^2(\Omega)$ admits the following *Wiener-Itô chaotic expansion*

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{3}$$

where the series converges in $L^2(\Omega)$, $f_0 = E[F]$, and the kernels $f_q \in \mathfrak{H}^{\odot q}$, $q \geq 1$, are uniquely determined by F. For every $q \geq 0$, we denote by J_q the orthogonal projection operator on the qth Wiener chaos. In particular, if $F \in L^2(\Omega)$ has the form (3), then $J_q F = I_q(f_q)$ for every $q \geq 0$.

Let $\{e_k,\ k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, for every $r = 0, \ldots, p \wedge q$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r}\rangle_{\mathfrak{H}^{\otimes r}}. \tag{4}$$

Notice that the definition of $f \otimes_r g$ does not depend on the particular choice of $\{e_k,\ k \geq 1\}$, and that $f \otimes_r g$ is not necessarily symmetric; we denote its symmetrization by $f \tilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. When $\mathfrak{H} = L^2(A, \mathscr{A}, \mu)$ and $r = 1, \dots, p \wedge q$, the contraction $f \otimes_r g$ is the element of $L^2(\mu^{p+q-2r})$ given by

$$\begin{aligned} & f \otimes_r g(x_1, \dots, x_{p+q-2r}) \\ & = \int_{A^r} f(x_1, \dots, x_{p-r}, a_1, \dots, a_r) \\ & \qquad \times g(x_{p-r+1}, \dots, x_{p+q-2r}, a_1, \dots, a_r) d\mu(a_1) \dots d\mu(a_r). \end{aligned} \tag{5}$$

It is a standard fact of Gaussian analysis that the following *multiplication formula* holds: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g). \tag{6}$$

2.2 Malliavin Operators

We now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process W. Let $\mathscr{S}$ be the set of all cylindrical random variables of the form

$$F = g\left(W(\phi_1), \dots, W(\phi_n)\right), \tag{7}$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function such that its partial derivatives have polynomial growth, and $\phi_i \in \mathfrak{H}$, $i = 1, \dots, n$. The *Malliavin derivative* of F with respect to W is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF \; = \; \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \left(W(\phi_1), \dots, W(\phi_n)\right) \phi_i.$$

In particular, $DW(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the mth derivative $D^m F$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$, for every $m \geq 2$. For $m \geq 1$ and $p \geq 1$, $\mathbb{D}^{m,p}$ denotes the closure of $\mathscr{S}$ with respect to the norm $\| \cdot \|_{m,p}$, defined by the relation

$$\|F\|_{m,p}^p \; = \; E\left[|F|^p\right] + \sum_{i=1}^{m} E\left[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right].$$

We often use the (canonical) notation $\mathbb{D}^\infty := \bigcap_{m \geq 1} \bigcap_{p \geq 1} \mathbb{D}^{m,p}$.

Remark 2 It is a well-known fact that any random variable F that is a finite linear combination of multiple Wiener-Itô integrals is an element of $\mathbb{D}^{\infty}$.

The Malliavin derivative D obeys the following *chain rule*. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F) DF_i. \tag{8}$$

Note also that a random variable F as in (3) is in $\mathbb{D}^{1,2}$ if and only if $\sum_{q=1}^{\infty} q \|J_q F\|^2_{L^2(\Omega)} < \infty$ and in this case one has the following explicit relation:

$$E\left[\|DF\|^2_{\mathfrak{H}}\right] = \sum_{q=1}^{\infty} q \|J_q F\|^2_{L^2(\Omega)}.$$

If $\mathfrak{H} = L^2(A, \mathscr{A}, \mu)$ (with μ non-atomic), then the derivative of a random variable F as in (3) can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_t F = \sum_{q=1}^{\infty} q I_{q-1}\left(f_q(\cdot, t)\right), \quad t \in A. \tag{9}$$

The operator L, defined as $L = \sum_{q=0}^{\infty} -qJ_q$, is the *infinitesimal generator of the Ornstein-Uhlenbeck semigroup*. The domain of L is

$$\mathrm{Dom}L = \{F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \left\|J_q F\right\|^2_{L^2(\Omega)} < \infty\} = \mathbb{D}^{2,2}.$$

For any $F \in L^2(\Omega)$, we define $L^{-1}F = \sum_{q=1}^{\infty} -\frac{1}{q} J_q(F)$. The operator L^{-1} is called the *pseudo-inverse* of L. Indeed, for any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \mathrm{Dom}L = \mathbb{D}^{2,2}$, and

$$LL^{-1}F = F - E(F). \tag{10}$$

The following integration by parts formula is used throughout the paper.

Lemma 1 *Suppose that $F \in \mathbb{D}^{1,2}$ and $G \in L^2(\Omega)$. Then, $L^{-1}G \in \mathbb{D}^{2,2}$ and*

$$E[FG] = E[F]E[G] + E[\langle DF, -DL^{-1}G\rangle_{\mathfrak{H}}]. \tag{11}$$

2.3 On Cumulants

The notion of cumulant will be crucial throughout the paper. We refer the reader to the monograph [17] for an exhaustive discussion of such a notion.

Definition 1 (Cumulants) Let F be a real-valued random variable such that $E|F|^m < \infty$ for some integer $m \geq 1$, and write $\phi_F(t) = E[e^{itF}]$, $t \in \mathbb{R}$, for the characteristic function of F. Then, for $j = 1, \ldots, m$, the jth *cumulant* of F, denoted by $\kappa_j(F)$, is given by

$$\kappa_j(F) = (-i)^j \frac{d^j}{dt^j} \log \phi_F(t)|_{t=0}. \tag{12}$$

Remark 3 When $E(F) = 0$, then the first four cumulants of F are the following: $\kappa_1(F) = E[F] = 0$, $\kappa_2(F) = E[F^2] = \mathrm{Var}(F)$, $\kappa_3(F) = E[F^3]$, and

$$\kappa_4(F) = E[F^4] - 3E[F^2]^2.$$

The following standard relation shows that moments can be recursively defined in terms of cumulants (and vice versa): fix $m = 1, 2 \ldots$, and assume that $E|F|^{m+1} < \infty$, then

$$E[F^{m+1}] = \sum_{i=0}^{m} \binom{m}{i} \kappa_{i+1}(F) E[F^{m-i}]. \tag{13}$$

Our aim is now to provide an explicit representation of cumulants in terms of Malliavin operators. To this end, it is convenient to introduce the following definition (see e.g. [10, Chap. 8] for a full multidimensional version).

Definition 2 Let $F \in \mathbb{D}^\infty$. The sequence of random variables $\{\Gamma_i(F)\}_{i\geq 0} \subset \mathbb{D}^\infty$ is recursively defined as follows. Set $\Gamma_0(F) = F$ and, for every $i \geq 1$,

$$\Gamma_i(F) = \langle DF, -DL^{-1}\Gamma_{i-1}(F)\rangle_{\mathfrak{H}}.$$

For instance, one has that $\Gamma_1(F) = \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}$. The following statement provides an explicit expression for $\Gamma_s(F)$, $s \geq 1$, when F has the form of a multiple integral.

Proposition 1 (See e.g. Chap. 8 in [10]) *Let $q \geq 2$, and assume that $F = I_q(f)$ with $f \in \mathfrak{H}^{\odot q}$. Then, for any $i \geq 1$, we have*

$$\Gamma_i(F) = \sum_{r_1=1}^{q} \cdots \sum_{r_i=1}^{[iq-2r_1-\ldots-2r_{i-1}]\wedge q} c_q(r_1, \ldots, r_i) \mathbf{1}_{\{r_1<q\}} \cdots \mathbf{1}_{\{r_1+\ldots+r_{i-1}<\frac{iq}{2}\}}$$
$$\times I_{(i+1)q-2r_1-\ldots-2r_i}\big((\ldots(f\tilde{\otimes}_{r_1}f)\tilde{\otimes}_{r_2}f)\ldots f)\tilde{\otimes}_{r_i}f\big),$$

where the constants $c_q(r_1, \ldots, r_{i-2})$ are recursively defined as follows:

$$c_q(r) = q(r-1)!\binom{q-1}{r-1}^2,$$

and, for $a \geq 2$,

$$c_q(r_1, \ldots, r_a) = q(r_a - 1)!\binom{aq - 2r_1 - \ldots - 2r_{a-1} - 1}{r_a - 1}\binom{q-1}{r_a - 1}c_q(r_1, \ldots, r_{a-1}).$$

The following statement explicitly connects the expectation of the random variables $\Gamma_i(F)$ to the cumulants of F.

Proposition 2 (See again Chap. 8 in [10]) *Let $F \in \mathbb{D}^\infty$. Then F has finite moments of every order, and the following relation holds for every $i \geq 0$:*

$$\kappa_{i+1}(F) = i!E[\Gamma_i(F)]. \tag{14}$$

We also use the following result taken from [1] throughout the paper.

Lemma 2 *Let $X \in \mathbb{D}^\infty$. Then, the relation*

$$E(\phi^{(k)}(X)\Gamma_r(X)) = E(X\phi^{(k-r)}(X)) - \sum_{s=1}^{r} E(\phi^{(k-s)}(X))E(\Gamma_{r-s}(X)) \tag{15}$$

holds for every k-times continuously differentiable mapping $\phi : \mathbb{R} \to \mathbb{R}$.

The next section is devoted to the elements of the second Wiener chaos.

2.4 Some Relevant Properties of the Second Wiener Chaos

In this subsection, we gather together some properties of the elements of the second Wiener chaos of the isonormal process $W = \{W(h);\ h \in \mathfrak{H}\}$; recall that these are random variables having the general form $F = I_2(f)$, with $f \in \mathfrak{H}^{\odot 2}$. Notice that, if $f = h \otimes h$, where $h \in \mathfrak{H}$ is such that $\|h\|_{\mathfrak{H}} = 1$, then using the multiplication formula (6), one has $I_2(f) = W(h)^2 - 1 \overset{\text{law}}{=} N^2 - 1$, where $N \sim \mathscr{N}(0, 1)$. To any kernel $f \in \mathfrak{H}^{\odot 2}$, we associate the following *Hilbert-Schmidt* operator

$$A_f : \mathfrak{H} \mapsto \mathfrak{H}; \quad g \mapsto f \otimes_1 g.$$

It is also convenient to introduce the sequence of auxiliary kernels

$$\left\{f \otimes_1^{(p)} f : p \geq 1\right\} \subset \mathfrak{H}^{\odot 2} \tag{16}$$

defined as follows: $f \otimes_1^{(1)} f = f$, and, for $p \geq 2$,

$$f \otimes_1^{(p)} f = \left(f \otimes_1^{(p-1)} f\right) \otimes_1 f. \tag{17}$$

In particular, $f \otimes_1^{(2)} f = f \otimes_1 f$. Finally, we write $\{\alpha_{f,j}\}_{j\geq 1}$ and $\{e_{f,j}\}_{j\geq 1}$, respectively, to indicate the (not necessarily distinct) eigenvalues of A_f and the corresponding eigenvectors.

Proposition 3 (See e.g. Sect. 2.7.4 in [10]) *Fix $F = I_2(f)$ with $f \in \mathfrak{H}^{\odot 2}$.*

1. *The following equality holds: $F = \sum_{j\geq 1} \alpha_{f,j}\big(N_j^2 - 1\big)$, where $\{N_j\}_{j\geq 1}$ is a sequence of i.i.d. $\mathscr{N}(0,1)$ random variables that are elements of the isonormal process W, and the series converges in L^2 and almost surely.*
2. *For any $i \geq 2$,*

$$\kappa_i(F) = 2^{i-1}(i-1)! \sum_{j\geq 1} \alpha_{f,j}^i = 2^{i-1}(i-1)! \times \langle f \otimes_1^{(i-1)} f, f\rangle_{\mathfrak{H}^{\otimes 2}}.$$

3. *The law of the random variable F is completely determined by its moments or equivalently by its cumulants.*

3 Main Results

Throughout this section, we assume that $\{W(h) : h \in \mathfrak{H}\}$ is a centered isonormal Gaussian process on a separable Hilbert space $\mathfrak{H}$ having $\{e_i\}_{i\geq 1}$ as a complete orthonormal basis.

3.1 A New View of Reference [11]

We now fix a symmetric kernel $f_\infty \in \mathfrak{H}^{\odot 2}$ such that its corresponding Hilbert-Schmidt operator A_{f_∞} (see Sect. 2.4) has a finite number of non-zero eigenvalues, that we denote by $\{\alpha_i\}_{i=1}^k$. To simplify the discussion, we assume that the eigenvalues are all distinct. In order to deal with the case of possibly repeated eigenvalues, one has to modify the polynomials P and Q below, as explained in [11, page 8]. As anticipated, we want to study convergence in distribution towards the random variable

$$F_\infty := I_2(f_\infty) = \sum_{i=1}^{k} \alpha_i \left(N_i^2 - 1\right), \tag{18}$$

where $\{N_i\}_{i=1}^k$ is the family of i.i.d. $\mathscr{N}(0,1)$ random variables appearing at Point 1 of Proposition 3. Following Nourdin and Poly [11], we define the two crucial polynomials P and Q as follows:

$$Q(x) = \big(P(x)\big)^2 = \left(x\prod_{i=1}^{k}(x-\alpha_i)\right)^2. \tag{19}$$

Note that, by definition, the roots of Q and P correspond with the set $\{0, \alpha_1, \ldots, \alpha_k\}$.

The starting point of our discussion is the following result, proved in [11]: it provides necessary and sufficient conditions for a sequence in the second Wiener chaos of W to converge in distribution towards F_∞.

Theorem 2 (See [11]) *Consider a sequence $\{F_n\}_{n\geq 1} = \{I_2(f_n)\}_{n\geq 1}$ of double Wiener integrals with $f_n \in \mathfrak{H}^{\odot 2}$. Then, the following statements are equivalent, as $n \to \infty$:*

(i) $F_n \overset{\text{law}}{\to} F_\infty$;
(ii) the following two asymptotic relations are verified:

1. $\kappa_r(F_n) \to \kappa_r(F_\infty)$, *for all* $2 \leq r \leq k+1 = deg(P)$,
2. $\sum_{r=2}^{deg(Q)} \frac{Q^{(r)}(0)}{r!} \frac{\kappa_r(F_n)}{2^{r-1}(r-1)!} \to 0.$

The original proof of Theorem 2 is based on methods from complex analysis, and exploits the fact that (owing to the representation stated at Point 1 of Proposition 3) the Fourier transform of a random variable with the form $I_2(f)$ can be written down explicitly in terms of the eigenvalues $\{\alpha_{f,j}\}$. Our aim in this section is to prove that condition **(ii)** of Theorem 2 can be equivalently stated in terms of contractions and Malliavin operators. Such equivalent conditions naturally lead to the main findings of the paper, as stated in Theorem 3, that will also provide (as a by-product) an alternate proof of Theorem 2 that does not make use of complex analysis (see, in particular, Remark 5 below). We start with a crucial lemma, that is in some sense the linchpin of the whole paper.

Lemma 3 *Let $F = I_2(f)$, $f \in \mathfrak{H}^{\odot 2}$, be a generic element of the second Wiener chaos of W, and write $\{\alpha_{f,j}\}_{j\geq 1}$ for the set of the eigenvalues of the associated Hilbert-Schmidt operator A_f we have*

$$\sum_{r=2}^{deg(Q)} \frac{Q^{(r)}(0)}{r!} \frac{\kappa_r(F)}{2^{r-1}(r-1)!}$$

$$= \sum_{j\geq 1} Q(\alpha_{f,j}) \tag{20}$$

$$= \left\| \sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!} f \otimes_1^{(r)} f \right\|_{\mathfrak{H}^{\otimes 2}}^2 \tag{21}$$

$$= \frac{1}{2} E\left(\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!\, 2^{r-1}} \Big(\Gamma_{r-1}(F) - E(\Gamma_{r-1}(F)) \Big) \right)^2, \tag{22}$$

where the operators $\Gamma_r(\cdot)$ *have been introduced in Definition* 2*. In particular, for the target random variable* F_∞ *introduced at* (18) *one has that*

$$0 = \sum_{r=2}^{deg(Q)} \frac{Q^{(r)}(0)}{r!} \frac{\kappa_r(F_\infty)}{2^{r-1}(r-1)!}$$

$$= \frac{1}{2} E\left(\sum_{r=1}^{deg(P)} \frac{P^{(r)}(0)}{r!\, 2^{r-1}} \Big(\Gamma_{r-1}(F_\infty) - E(\Gamma_{r-1}(F_\infty)) \Big) \right)^2. \tag{23}$$

Proof In view of the second equality at Point 2 of Proposition 3, one has that $\frac{\kappa_r(F)}{2^{r-1}(r-1)!} = \sum_{j\geq 1} \alpha_{f,j}^r$, from which we deduce immediately (20). To prove (21), observe that Point 1 of Proposition 3, together with the product formula (6), implies that the kernel f admits a representation of the type $f = \sum_{j\geq 1} \alpha_{f,j} \eta_j \otimes \eta_j$, where $\{\eta_j\}$ is some orthonormal system in $\mathfrak{H}$. It follows that, for $r \geq 1$, one has the representation $f \otimes_1^{(r)} f = \sum_{j\geq 1} \alpha_{f,j}^r \eta_j \otimes \eta_j$, and therefore

$$\sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} f \otimes_1^{(r)} f = \sum_{j\geq 1} \eta_j \otimes \eta_j \sum_{r=1}^{\deg(P)} \frac{P^{(r)}(0)}{r!} \alpha_{f,j}^r.$$

Taking norms on both sides of the previous relation and exploiting the orthonormality of the η_j yields (21). Finally, in order to show (22), it is clearly enough to prove that, for any $r \geq 1$,

$$I_2(f \otimes_1^{(r)} f) = \frac{1}{2^{r-1}} \big\{ \Gamma_{r-1}(F) - E(\Gamma_{r-1}(F)) \big\}. \tag{24}$$

We proceed by induction on r. It is clear for $r = 1$, because $\Gamma_0(F) = F$ and $E(F) = 0$. Take $r \geq 2$ and assume that (24) holds true. Without loss of generality, we can assume that $\mathfrak{H} = L^2(A, \mathscr{A}, \mu)$, where μ is a σ-finite and non-atomic measure on the measurable space $(A, \mathscr{A})$. Notice that, by definition of $\Gamma_r(F)$ and the induction assumption, one has

$$\begin{aligned}\Gamma_r(F) &= \langle DF, -DL^{-1}\Gamma_{r-1}(F)\rangle_{\mathfrak{H}} = \left\langle 2I_1(f(t,.)), 2^{r-1}I_1(f\otimes_1^{(r)} f(t,.))\right\rangle_{\mathfrak{H}}\\ &= 2^r\int_A\left\{\langle f(t,.), f\otimes_1^{(r)} f(t,.)\rangle_{\mathfrak{H}} + I_2\big(f(t,.)\otimes(f\otimes_1^{(r)} f)(t,.)\big)\right\}\mathrm{d}\mu(t)\\ &= 2^r\langle f, f\otimes_1^{(r)} f\rangle_{\mathfrak{H}^{\odot 2}} + 2^r I_2(f\otimes_1^{(r+1)} f),\end{aligned}$$

where we have used a standard stochastic Fubini Theorem. This proves that (24) is verified for every $r \geq 1$. The last assertion in the statement follows from (20), as well as the fact that the eigenvalues α_i are all roots of Q.

The next proposition, which is an immediate consequence of Lemma 3, provides the announced extension of Theorem 2.

Proposition 4 *Assume* $\{F_n\}_{n\geq 1} = \{I_2(f_n)\}_{n\geq 1}$ *be a sequence of double Wiener integrals with* $f_n \in \mathfrak{H}^{\odot 2}$. *Then the following statements are equivalent to either Point* ***(i)*** *or* ***(ii)*** *of Theorem 2, as* $n \to \infty$.

(a) The following relations 1.–2. are in order:

1. $\kappa_r(F_n) \to \kappa_r(F_\infty)$, *for all* $2 \leq r \leq k+1 = deg(P)$, *and*
2. $E\left(\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!\,2^{r-1}}\Big(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\Big)\right)^2 \to 0.$

(b) The following relations 1.–2. are in order:

1. $\kappa_r(F_n) \to \kappa_r(F_\infty)$, *for all* $2 \leq r \leq k+1 = deg(P)$, *and*
2. $\left\|\sum_{r=1}^{deg(P)}\frac{P^{(r)}(0)}{r!}f_n\otimes_1^{(r)} f_n\right\|_{\mathfrak{H}^{\otimes 2}}^2 \to 0.$

As anticipated, our aim in the sections to follow is to show that the equivalence between Condition **(a)** in Proposition 4 and Condition **(i)** in Theorem 2 is indeed valid for sequence of random variables living in a finite sum of Wiener chaoses. The next statement provides a first, non dynamical version of this fact.

Proposition 5 *Let the polynomial* P *be defined as in* (19) *and consider again the random variable* $F_\infty = I_2(f_\infty)$ *defined in* (18). *Let* F *be a centered random variable living in a finite sum of Wiener chaoses, i.e.* $F \in \bigoplus_{i=1}^M C_i$. *Moreover, assume that*

(i) $\kappa_r(F) = \kappa_r(F_\infty)$, *for all* $2 \leq r \leq k+1 = deg(P)$, *and*
(ii)

$$E\left(\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!\,2^{r-1}}\Big(\Gamma_{r-1}(F) - E(\Gamma_{r-1}(F))\Big)\right)^2 = 0.$$

Then, $F \overset{\text{law}}{=} F_\infty$, *and* $F \in C_2$.

Proof Let ϕ be a smooth function. Using the integration by parts formula (Lemma 1), Assumption (**ii**) in the statement and Proposition 2, we obtain

$$\begin{aligned} E\big(F\phi(F)\big) &= \sum_{r=0}^{k-1} \frac{\kappa_{r+1}(F)}{r!} E(\phi^{(r)}(F)) + E(\phi^{(k)}(F)\Gamma_k(F)) \\ &= \sum_{r=0}^{k-1} \frac{\kappa_{r+1}(F)}{r!} E(\phi^{(r)}(F)) + \frac{\kappa_{k+1}(F)}{k!} E(\phi^{(k)}(F)) \\ &\quad + \sum_{r=1}^{k} \frac{2^{k-r+1}\kappa_r(F)}{(r-1)!r!} P^{(r)}(0) E(\phi^{(k)}(F)) \\ &\quad - \sum_{r=1}^{k} \frac{2^{k-r+1}}{r!} P^{(r)}(0) E(\phi^{(k)}(F)\Gamma_{r-1}(F)) \end{aligned} \tag{25}$$

On the other hand, using (15) we obtain that

$$\begin{aligned} E(\phi^{(k)}(F)\Gamma_{r-1}(F)) &= E(F\phi^{(k-(r-1))}(F)) \\ &\quad - \sum_{s=1}^{r-1} E(\phi^{(k-s)}(F))E(\Gamma_{r-1-s}(F)). \end{aligned} \tag{26}$$

Using the relation $E(\Gamma_{r-1-s}(F)) = \kappa_{r-s}(F)/(r-s-1)!$, and therefore deduce that, for every smooth test function ϕ

$$\begin{aligned} E\big(F\phi(F)\big) &= \sum_{r=0}^{k-1} \frac{\kappa_{r+1}(F)}{r!} E(\phi^{(r)}(F)) + \frac{\kappa_{k+1}(F)}{k!} E(\phi^{(k)}(F)) \\ &\quad + \sum_{r=1}^{k} \frac{2^{k-r+1}\kappa_r(F)}{(r-1)!r!} P^{(r)}(0) E(\phi^{(k)}(F)) \\ &\quad - \sum_{r=1}^{k} \frac{2^{k-r+1}}{r!} P^{(r)}(0) E[F\phi^{(k-(r-1))}(F)] \\ &\quad + \sum_{r=1}^{k} \frac{2^{k-r+1}}{r!} P^{(r)}(0) \sum_{s=1}^{r-1} E[\phi^{(k-s)}(F)] \frac{\kappa_{r-s}(F)}{(r-s-1)!}. \end{aligned}$$

Considering the test function $\phi(x) = x^n$ with $n > k$, we infer that $E(F^{n+1})$ can be expressed in a recursive way in terms of the quantities

$$E(F^n), E(F^{n-1}), \cdots, E(F^{n-k}), \; \kappa_2(F), \cdots, \kappa_{k+1}(F)$$

and $P^{(1)}(0), \cdots, P^{(k)}(0)$. Using Assumption **(i)** in the statement together with last assertion in Lemma 3, we see that the moments of the random variable F_∞ also satisfy the same recursive relation. These facts immediately imply that

$$E\left(F^n\right) = E\left(F_\infty^n\right), \quad n \geq 1,$$

and the claim follows at once from Point 3 in Proposition 3. To prove that, in fact, $F \in C_2$, we assume that M is the smallest natural number such that $F \in \bigoplus_{i=1}^{M} C_i$. Hence $F \notin \bigoplus_{i=1}^{M-1} C_i$. Therefore, by applying [5, Theorem 6.12] to F, F_∞ and the fact that $F \overset{\text{law}}{=} F_\infty$, we deduce that $M = 2$. Let assume that $F = I_1(g) + I_2(h)$ for some $g \in \mathfrak{H}$ and $h \in \mathfrak{H}^{\otimes 2}$. Considering the trivial sequence $\{F_n\}_{n\geq 1}$ such that $F_n = F_\infty$, $n \geq 1$, using the fact that $F \overset{\text{law}}{=} F_\infty$ and applying [11, Theorem 3.1], we deduce that $I_1(g)$ is independent of $I_2(h)$. Let $\{\lambda_{f_\infty,k}\}_{k\geq 1}$ and $\{\lambda_{h,k}\}_{k\geq 1}$ denote the eigenvalues corresponding to the Hilbert-Schmidt operator A_{f_∞} and A_h associated with the kernels f_∞ and h respectively (see Sect. 2.4). Exploiting the independence of $I_1(g)$ and $I_2(h)$ and Point 2 in Proposition 3, we infer that

$$\sum_{k\in\mathbb{N}} \lambda_{f_\infty,k}^{3p} = \sum_{k\in\mathbb{N}} \lambda_{h,k}^{3p} \quad \forall\, p \geq 1.$$

As a result, Lemma 6 in Appendix implies that for some permutation π on N we have $\lambda_{\infty,k} = \lambda_{h,\pi(k)}$ for $k \geq 1$, which in turn implies

$$\sum_{k\in\mathbb{N}} \lambda_{f_\infty,k}^{2} = \sum_{k\in\mathbb{N}} \lambda_{h,k}^{2}. \tag{27}$$

On the other hand, from $F = I_1(g) + I_2(h) \overset{\text{law}}{=} F_\infty$, and computing the second cumulant of both sides, one can easily deduce that if $\kappa_2(I_1(g)) = \mathbb{E}(I_1(g))^2 = \|g\|_{\mathfrak{H}}^2 \neq 0$, then the equality (27) cannot hold. It follows that $I_1(g) = 0$, and therefore $F \in C_2$.

One of the arguments used in the previous proof will be exploited again in the next section. For future reference, we shall explicitly state the needed double implication in the form of a lemma.

Lemma 4 *Let F be a centered random variable, with finite moments of all orders and such that $\kappa_r(F) = \kappa_r(F_\infty)$, for all $2 \leq r \leq k+1 = deg(P)$. Then, $F \overset{\text{law}}{=} F_\infty$ if* *and only if, for every polynomial mapping $\phi : \mathbb{R} \to \mathbb{R}$,*

$$E\big(F\phi(F)\big) = \Psi_\phi(F) := \sum_{r=0}^{k-1} \frac{\kappa_{r+1}(F)}{r!} E(\phi^{(r)}(F)) + \frac{\kappa_{k+1}(F)}{k!} E(\phi^{(k)}(F)) \quad (28)$$
$$+ \sum_{r=1}^{k} \frac{2^{k-r+1}\kappa_r(F)}{(r-1)!r!} P^{(r)}(0) E(\phi^{(k)}(F))$$
$$- \sum_{r=1}^{k} \frac{2^{k-r+1}}{r!} P^{(r)}(0) E[F\phi^{(k-(r-1))}(F)]$$
$$+ \sum_{r=1}^{k} \frac{2^{k-r+1}}{r!} P^{(r)}(0) \sum_{s=1}^{r-1} E[\phi^{(k-s)}(F)] \frac{\kappa_{r-s}(F)}{(r-s-1)!}.$$

In the next section, which contains the main findings of the paper, we shall show that a slight variation of Condition **(a)** in Proposition 4 is basically necessary and sufficient for convergence in distribution towards F_∞ for *any* sequence of random variables living in a finite sum of Wiener chaoses.

3.2 A General Criterion

We recall that the *total variation* distance d_{TV} between the laws of two real-valued random variables X and Y is defined as

$$d_{\mathrm{TV}}(F, G) = \sup_{A \in \mathscr{B}(\mathbb{R})} \Big| \mathbb{P}(F \in A) - \mathbb{P}(G \in A) \Big|, \quad (29)$$

where the supremum is taken over all Borel sets $A \subseteq \mathbb{R}$. We also write $F_n \xrightarrow{\mathrm{TV}} F$ to indicate the asymptotic relation $d_{\mathrm{TV}}(F_n, F) \to 0$.

The next theorem is the main finding of the paper. Recall that the random variable F_∞ has been defined in formula (18).

Theorem 3 *Let $\{F_n\}_{n\geq 1}$ be a sequence of random variables such that each F_n lives in a finite sum of chaoses, i.e. $F_n \in \bigoplus_{i=1}^M C_i$ for $n \geq 1$ and some $M \geq 2$ (not depending on n). Consider the following three asymptotic relations, as $n \to \infty$:*

(i)

$$F_n \xrightarrow{\mathrm{TV}} F_\infty; \quad (30)$$

(ii) The following relations 1.–2. are in order:

1. $\kappa_r(F_n) \to \kappa_r(F_\infty)$, *for all* $2 \le r \le k+1 = deg(P)$, *and*
2. $E\left(\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \Big(\Gamma_{r-1}(F_n) - E[\Gamma_{r-1}(F_n)] \Big) \Bigg| F_n \right) \xrightarrow{L^2} 0.$

(iii) The following relations 1.–2. are in order:

1. $\kappa_r(F_n) \to \kappa_r(F_\infty)$, *for all* $2 \le r \le k+1 = deg(P)$, *and*
2. $E\left(\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \Big(\Gamma_{r-1}(F_n) - E[\Gamma_{r-1}(F_n)] \Big) \Bigg| F_n \right) \xrightarrow{L^1} 0.$

Then, one has the implications ***(ii)*** $\to$ ***(i)*** *and* ***(i)*** $\to$ ***(iii)****.*

Remark 4 We remark that, in the special case $k = 1 = \alpha_k$, the condition appearing at Point 2 of item **(ii)** in Theorem 3 is implied by the relation

$$E\big(\Gamma_1(F_n) - F_n - 2\big)^2 \to 0. \tag{31}$$

When $F_n = I_q(f_n)$, this corresponds to the condition appearing at Point (iii) of Part (B) of Theorem 1, by taking into account the fact that, for a multiple integral $F = I_q(f)$ of order q, we have the relation $\Gamma_1(F) = \frac{1}{q}\|DF\|^2_{\mathfrak{H}}$. Note that, as explained in [7], the asymptotic relation (31) cannot be fulfilled by a sequence F_n such that $F_n = I_q(f_n)$ with q odd and $E[F_n^2] \to 2$.

In order to prove Theorem 3, we need an additional lemma.

Lemma 5 (See Theorem 3.1 in [12]) *Let* $\{F_n\}_{n\ge1}$ *be a sequence of non-zero random variables living in a finite sum of Wiener chaoses, i.e.* $F_n \in \bigoplus_{i=0}^M C_i,\ \forall n \ge 1$. *Assume that the sequence* $\{F_n\}_{n\ge1}$ *converges in distribution to some non-zero target random variable F, as n tends to infinity. Then,*

$$\sup_{n\ge1} E(|F_n|^r) < \infty, \quad \forall\, r \ge 1, \tag{32}$$

and $F_n \xrightarrow{\text{TV}} F$*. Moreover, the distribution of F is necessarily absolutely continuous with respect to the Lebesgue measure.*

Proof of Theorem 3 Proof of **(ii)** $\to$ **(i)** Assumption 1 in **(ii)** implies that $\sup_{n\ge1} E\left(F_n^2\right) < \infty$. Hence, the sequence $\{F_n\}_{n\ge1}$ is tight. This yields that, for any subsequence $\{F_{n_k}\}_{k\ge1}$, there exists a sub-subsequence $\{F_{n_{k_l}}\}_{l\ge1}$ and a random variable F such that $F_{n_{k_l}} \xrightarrow{\text{law}} F$, as l tends to infinity. In order to show the desired implication, we have now to show that, necessarily, F has the same distribution as F_∞. To simplify the discussion, we may assume that $\{F_{n_{k_l}}\}_{l\ge1} = \{F_n\}_{n\ge1}$. By exploiting (32) together with the fact that the sequence $\{F_n\}_{n\ge1}$ lives in a fixed finite sum of Wiener chaoses, we deduce that, for every polynomial ϕ,

$$E\left(F_n\phi(F_n)\right) \to E\left(F\phi(F)\right), \quad n \to \infty. \tag{33}$$

and

$$\Psi_\phi(F_n) \longrightarrow \Psi_\phi(F), \quad \text{as } n \to \infty, \tag{34}$$

where we have used the notation (28). By virtue of Lemma 4, in order to show the desired implication, it is then sufficient to prove the asymptotic relation

$$\left|E\left(F_n\phi(F_n)\right) - \Psi_\phi(F_n)\right| \to 0, \quad n \to \infty, \tag{35}$$

for every polynomial ϕ. To show (35), we can use several times integration by parts (see Lemma 1) to infer that

$$\begin{aligned}
&\left|E\left(F_n\phi(F_n)\right) - \Psi_\phi(F_n)\right| \\
&= 2^k E\left[\phi^{(k)}(F_n) E\left(\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!\,2^{r-1}}\left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\Big| F_n\right)\right] \\
&\le 2^k \sqrt{E\left(\phi^{(k)}(F_n)\right)^2} \\
&\quad \times \sqrt{E\left(E\left(\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!\,2^{r-1}}\left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\Big| F_n\right)^2\right)}.
\end{aligned}$$

Now, a standard application of Lemma 5 shows that

$$\sup_{n\ge 1} E\left(\phi^{(k)}(F_n)\right)^2 < \infty,$$

and (35) follows by exploiting Assumption 2 at Point **(ii)**.

Proof of **(i)** $\to$ **(iii)** The proof is divided into several steps. Take $\phi \in \mathscr{C}_c^\infty$ with support in $[-M, M]$ where $M > 0$ and $\|\phi^{(k)}\|_\infty \le 1$.

Step 1. We have:

$$\begin{aligned}
&E\left(\phi^{(k)}(F_n) \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\right) \\
&\qquad = \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} E(\phi^{(k)}(F_n)\Gamma_{r-1}(F_n))
\end{aligned}$$

$$-E[\phi^{(k)}(F_n)] \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} E(\Gamma_{r-1}(F_n))$$

$$= \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} E(F_n \phi^{(k-(r-1))}(F_n))$$

$$- \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \sum_{s=1}^{r-1} E(\phi^{(k-s)}(F_n)) E(\Gamma_{r-1-s}(F_n))$$

$$-E(\phi^{(k)}(F_n)) \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} E(\Gamma_{r-1}(F_n))$$

$$= \sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\,(a_{r,n} F_n + b_{r,n})\right).$$

In the last line we have used the Proposition 2 which relates the $E(\Gamma_r)$ to the cumulants, so that $\{a_{r,n}, b_{r,n}\}_{0 \le r \le k}$ denote constants which are linear combinations of the $k+1$ first cumulants of F_n. Since (30) holds and since $\{F_n\}_{n \ge 1}$ is bounded in every $L^p(\Omega)$ in virtue of Lemma 5, for each $r \in \{1, 2, \cdots, k+1\}$ the continuous mapping Theorem implies that

$$\kappa_r(F_n) \to 2^{r-1}(r-1)! \sum_{i=1}^{k} \alpha_i^r = \kappa_r(F_\infty).$$

This yields that

$$E\left(\phi^{(k)}(F_n) \textstyle\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\right)$$
$$\to E\left(\phi^{(k)}(F_\infty) \textstyle\sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_\infty) - E(\Gamma_{r-1}(F_\infty))\right)\right) = 0, \tag{36}$$

where we have used Lemma 3.

Step 2. The conclusion at Step 1 implies that, for each fixed $\phi \in \mathscr{C}_c^\infty$ with support in $[-M, M]$, such that $\|\phi^{(k)}\|_\infty \le 1$, we have:

$$E\left(\phi^{(k)}(F_n) \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\right) \to 0. \tag{37}$$

For convenience we set

$$\mathscr{E}_M = \left\{\phi \in \mathscr{C}_c^{\infty} \,\middle|\, \|\phi^{(k)}\|_\infty \leq 1,\ \mathrm{supp}(\phi) \subset [-M, M]\right\}.$$

Exploiting again the arguments used in Step 1 we infer that

$$\begin{aligned} &E\left(\phi^{(k)}(F_n) \sum_{r=1}^{k+1} \frac{P^{(r)}(0)}{r!2^{r-1}} \left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\right) \\ &\quad = \sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\,(a_{r,n}F_n + b_{r,n})\right). \end{aligned}$$

One has that

$$\begin{aligned} &\sup_{\phi\in\mathscr{E}_M} \left| \sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\,(a_{r,n}F_n + b_{r,n})\right) - \sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\,(a_{r,\infty}F_n + b_{r,\infty})\right) \right| \\ &\quad \leq \sup_{\phi\in\mathscr{E}_M} \sum_{r=0}^{k} \|\phi^{(r)}\|_\infty \left(|a_{r,n} - a_{r,\infty}| \sup_{n\geq 1} E(|F_n|) + |b_{r,n} - b_{r,\infty}|\right) \\ &\quad \leq M^k \left(\sup_{n\geq 1} E(|F_n|) + 1\right) \sum_{r=0}^{k} \left(|a_{r,n} - a_{r,\infty}| + |b_{r,n} - b_{r,\infty}|\right) \\ &\quad \to 0, \end{aligned}$$

where we have used the fact that for $\phi \in \mathscr{E}_M$, and for any $0 \leq r \leq k$, we have $\|\phi^{(r)}\|_\infty \leq M^k$. On the other hand, we know that $F_n \xrightarrow[n\to\infty]{TV} F_\infty$. The following equality holds

$$\begin{aligned} &\left| \sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\,(a_{r,\infty}F_n + b_{r,\infty})\right) \right| \\ &\quad = \left| \sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\,(a_{r,\infty}F_n + b_{r,\infty})\right) \right. \\ &\qquad \left. - \sum_{r=0}^{k} E\left(\phi^{(r)}(F_\infty)\,(a_{r,\infty}F_\infty + b_{r,\infty})\right) \right|. \end{aligned}$$

The expression on the right-hand side of the previous equality is bounded by

$$\sum_{r=0}^{k} a_{r,\infty} \left| E\left(\phi^{(r)}(F_n)F_n - \phi^{(r)}(F_\infty)F_\infty\right)\right|$$

$$+ \sum_{r=0}^{k} b_{r,\infty} \left| E\left(\phi^{(r)}(F_n) - \phi^{(r)}(F_\infty)\right)\right|$$

$$\leq M^{k+1}\left(\sum_{r=0}^{k} a_{r,\infty} + b_{r,\infty}\right) d_{TV}(F_n, F_\infty).$$

To obtain the previous estimate, we have used the facts that

$$\sup_{x\in[-M,M]} \|\phi^{(r)}(x)x\| \leq M^{k+1} \quad \text{and} \quad \|\phi^{(r)}\|_\infty \leq M^k.$$

Now, letting $n \to \infty$, we deduce that

$$\sup_{\phi\in\mathscr{E}_M} \left|\sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\left(a_{r,\infty}F_n + b_{r,\infty}\right)\right)\right| \to 0, \tag{38}$$

as well as

$$\sup_{\phi\in\mathscr{E}_M} \left|\sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\left(a_{r,n}F_n + b_{r,n}\right)\right)\right| \to 0. \tag{39}$$

Step 3. Let $\mathscr{F}_M$ be the set of Borel functions bounded by 1 and supported in $[-M, M]$. By density we have

$$\sup_{\phi\in\mathscr{E}_M} \left|\sum_{r=0}^{k} E\left(\phi^{(r)}(F_n)\left(a_{r,n}F_n + b_{r,n}\right)\right)\right|$$

$$= \sup_{\phi\in\mathscr{E}_M} \left|E\left(\phi^{(k)}(F_n)\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\right)\right|$$

$$= \sup_{\phi\in\mathscr{F}_M} \left|E\left(\phi(F_n)\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\left(\Gamma_{r-1}(F_n) - E(\Gamma_{r-1}(F_n))\right)\right)\right|$$

$$\xrightarrow[n\to\infty]{\text{(Step2)}} 0.$$

To achieve the proof, we notice that

$$
\begin{aligned}
&E\left(\left|E\left(\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\Big|F_n\right)\right|\right)\\
&=\sup_{\|\phi\|_\infty\le 1}\left|E\left(\phi(F_n)\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\right)\right|\\
&\le\sup_{\phi\in\mathscr{F}_M}\left|E\left(\phi(F_n)\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\right)\right|\\
&\quad+E\left(\mathbf{1}_{\{|F_n|>M\}}\left|\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\right|\right)\\
&\le\sup_{\phi\in\mathscr{F}_M}\left|E\left(\phi(F_n)\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\right)\right|\\
&\quad+\sqrt{\mathbb{P}(|F_n|>M)}\\
&\qquad\times\sup_n\sqrt{E\left(\sum_{r=1}^{k+1}\frac{P^{(r)}(0)}{r!\,2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\right)^2}.
\end{aligned}
$$

We get the desired result by letting first $n\to\infty$ and next $M\to\infty$. We recall that the sup over n in the latter inequality is well defined in virtue of the Lemma 5 which provides the boundedness of all moments.

□

Remark 5 (On Theorem 2) As anticipated, Theorem 3 allows one to deduce an alternate proof of the implication **(ii)** → **(i)** in Theorem 2. Indeed, if Assumption **(ii)** in Theorem 2 is verified, one can apply (22) to deduce that

$$
E\left(\sum_{r=1}^{\deg(P)}\frac{P^{(r)}(0)}{r!\,2^{r-1}}\Big(\Gamma_{r-1}(F_n)-E(\Gamma_{r-1}(F_n))\Big)\right)^2\to 0,
$$

and the conclusion follows immediately from Theorem 3, as well as a standard application of Jensen's inequality.

4 Example: Two Eigenvalues

We will now illustrate the main findings of the present paper by considering the case of a target random variable of the type $F_\infty = I_2(f_\infty)$, where the Hilbert-Schmidt operator A_{f_∞} associated the kernel f_∞ has only two non-zero eigenvalues $\alpha_1 \neq \alpha_2$, thus implying that

$$F_\infty = \alpha_1 \left(N_1^2 - 1\right) + \alpha_2 \left(N_2^2 - 1\right) \tag{40}$$

where N_1 and N_2 are independent $\mathscr{N}(0,1)$ (see Proposition 3).

Theorem 4 *Assume that $F_\infty = I_2(f_\infty)$ is given by (40). Let $q \geq 2$ and $\{F_n\}_{n\geq 1} = \{I_q(f_n)\}_{n\geq 1}$ be a sequence of multiple Wiener integrals of order q such that*

$$\lim_{n\to\infty} E(F_n^2) = 2 \lim_{n\to\infty} \|f_n\|^2_{\mathfrak{H}^{\otimes q}} = 1.$$

Assume that, as n tends to infinity, we have

(a) $\langle f_n \tilde{\otimes}_{\frac{q}{2}} f_n, f_n \rangle_{\mathfrak{H}^{\otimes q}} \to 0,$ *when q is even, and*
(b) *the following three asymptotic conditions **(b1)–(b3)** take place:*

(b1)

$$\Bigg\| \sum_{r=1}^{q} \sum_{\substack{s=1 \\ r+s=q}}^{(2q-2r)\wedge q} \frac{1}{4} q^2 (r-1)!(s-1)! \binom{q-1}{r-1}^2 \times \binom{q-1}{s-1} \binom{2q-2r-1}{s-1} \left(f_n \tilde{\otimes}_r f_n\right) \tilde{\otimes}_s f_n - \frac{\alpha_1 + \alpha_2}{2} q (\frac{q}{2} - 1)! \binom{q-1}{\frac{q}{2}-1} f_n \tilde{\otimes}_{\frac{q}{2}} f_n + \alpha_1 \alpha_2 f_n \Bigg\|^2_{\mathfrak{H}^{\otimes q}} \to 0$$

(when q is not even or $\alpha_1 = -\alpha_2$, then the term in the middle – involving the contraction of order $\frac{q}{2}$ – is removed automatically).

(b2) *for all $2 \leq k \leq 2q-2$, we have*

$$\Bigg\| \sum_{r=1}^{q} \sum_{\substack{s=1 \\ (r,s)\neq(\frac{q}{2},q) \\ r+s\neq q \\ 3q-2(r+s)=k}}^{(2q-2r)\wedge q} (r-1)!(s-1)! \binom{q-1}{r-1}^2$$

$$\times\binom{q-1}{s-1}\binom{2q-2r-1}{s-1}\left(f_n\tilde{\otimes}_r f_n\right)\tilde{\otimes}_s f_n$$

$$-\frac{\alpha_1+\alpha_2}{2}\sum_{\substack{\frac{q}{2}\neq r=1\\ 2q-2r=k}}^{q-1} q(r-1)!\binom{q-1}{r-1}^2 f_n\tilde{\otimes}_r f_n\Bigg\|^2_{\mathfrak{H}^{\otimes k}}\to 0.$$

(b3) *for all* $2q-1\le k\le 3q-4$*, we have*

$$\Bigg\|\sum_{r=1}^{q}\sum_{\substack{s=1\\(r,s)\neq(\frac{q}{2},q)\\ r+s\neq q\\ 3q-2(r+s)=k}}^{(2q-2r)\wedge q}(r-1)!(s-1)!\binom{q-1}{r-1}^2\binom{q-1}{s-1}\binom{2q-2r-1}{s-1}$$

$$\times\left(f_n\tilde{\otimes}_r f_n\right)\tilde{\otimes}_s f_n\Bigg\|^2_{\mathfrak{H}^{\otimes k}}\to 0.$$

Then,

$$F_n\overset{\text{law}}{\to}F_\infty.$$

Proof In this case, a simple application of Jensen's inequality shows that the second moment of the quantity appearing on the left-hand side of Point 2 of Theorem 3-(**ii**) is bounded from above by

$$E\left(\frac{\Gamma_2(F_n)-E(\Gamma_2(F_n))}{4}-\frac{\alpha_1+\alpha_2}{2}\Big(\Gamma_1(F_n)-E(\Gamma_1(F_n))\Big)+\alpha_1\alpha_2F_n\right)^2.$$

The claim follows immediately from Proposition 1, orthogonality of multiple Wiener integrals, Theorem 3 and the fact that when q is even

$$\kappa_3(F_n)=2E(\Gamma_2(F_n))=2qq!(\frac{q}{2}-1)!\binom{q-1}{\frac{q}{2}-1}^2\langle f_n\tilde{\otimes}_{\frac{q}{2}}f_n,f_n\rangle_{\mathfrak{H}^{\otimes q}}.$$

In the special case when $\alpha_1=-\alpha_2=\frac{1}{2}$, the target random variable $F_\infty=I_2(f_\infty)$ in the limit takes the form

$$F_\infty=\frac{1}{2}\left(N_1^2-1\right)-\frac{1}{2}\left(N_2^2-1\right)\overset{\text{law}}{=}N_1\times N_2 \tag{41}$$

where N_1 and N_2 are independent $\mathscr{N}(0,1)$. If the elements F_n of approximating sequence take the special form of multiple Wiener integrals of a fixed order, then we have the following result. One should notice that, in this special case, the result stated below can be alternatively deduced from the findings contained in [3]. For a *free* counterpart of the next result, see [2, Theorem 1.1].

Corollary 1 *Assume that $F_\infty = I_2(f_\infty)$ is given by (41). Let $q \geq 2$ and $\{F_n\}_{n\geq 1} = \{I_q(f_n)\}_{n\geq 1}$ be a sequence of multiple Wiener integrals of order q such that*

$$\lim_{n\to\infty} E(F_n^2) = 2 \lim_{n\to\infty} \|f_n\|^2_{\mathfrak{H}^{\otimes q}} = 1.$$

Assume that, as n tends to infinity, we have

(a) $\langle f_n \tilde{\otimes}_{\frac{q}{2}} f_n, f_n\rangle_{\mathfrak{H}^{\otimes q}} \to 0,$ *when q is even,*
(b) *and moreover*

(b1)

$$\Bigg\| \sum_{\substack{r=1 \\ r+s=q}}^{q} \sum_{s=1}^{(2q-2r)\wedge q} q^2 (r-1)!(s-1)! \binom{q-1}{r-1}^2 \times \binom{q-1}{s-1}\binom{2q-2r-1}{s-1} \left(f_n \tilde{\otimes}_r f_n\right) \tilde{\otimes}_s f_n - f_n \Bigg\|^2_{\mathfrak{H}^{\otimes q}} \to 0.$$

(b2) *for all $2 \leq k \leq 3q-4$, we have*

$$\Bigg\| \sum_{\substack{r=1 \\ (r,s)\neq(\frac{q}{2},q) \\ r+s\neq q \\ 3q-2(r+s)=k}}^{q} \sum_{s=1}^{(2q-2r)\wedge q} (r-1)!(s-1)! \binom{q-1}{r-1}^2 \binom{q-1}{s-1} \times \binom{2q-2r-1}{s-1} \times \left(f_n \tilde{\otimes}_r f_n\right) \tilde{\otimes}_s f_n \Bigg\|^2_{\mathfrak{H}^{\otimes k}} \to 0.$$

Then,

$$F_n \overset{\text{law}}{\to} F_\infty.$$

Remark 6 Notice that when q is odd, the assumption (**a**) of Theorem 1 and restriction $(r,s) \neq (\frac{q}{2}, q)$ in the sums of (**b2**) can be removed. In other words, it is known that for any multiple Wiener integral $F = I_q(f)$ of odd order, we have $\kappa_3(F) = 2E(\Gamma_2(F)) = 0$.

Example 1 Let $q \geq 2$ be an even integer. Consider two sequences $\{G_n\}_{n\geq 1} = \{I_q(g_n)\}_{n\geq 1}$ and $\{H_n\}_{n\geq 1} = \{I_q(h_n)\}_{n\geq 1}$ of multiple Wiener integrals of order q where $g_n, h_n \in \mathfrak{H}^{\odot q}$ for $n \geq 1$. We assume that as n tends to infinity we have

(a) $G_n \overset{\text{law}}{\to} G_\infty \overset{\text{law}}{=} \frac{1}{2}(N^2 - 1)$.
(b) $H_n \overset{\text{law}}{\to} H_\infty \overset{\text{law}}{=} \frac{1}{2}(N^2 - 1)$.
(c) $\mathrm{Cov}(G_n^2, H_n^2) \to 0$.

We consider the sequence $\{F_n\}_{n\geq 1}$, where

$$F_n = I_q(f_n) := G_n - H_n = I_q(g_n - h_n), \quad n \geq 1.$$

Then [13, Theorem 4.5.] implies that as n tends to infinity, we have

$$(G_n, H_n) \overset{\text{law}}{\to} (G_\infty, H_\infty),$$

where the random variables G_∞ and F_∞ are independent. Hence, in particular we obtain that $F_n \overset{\text{law}}{\to} F_\infty$ where F_∞ is given by (41). We can also justify the later convergence with the help of our result, namely Corollary 1. To this end, first notice that relation (3.6) of [13] implies that

$$\mathrm{Cov}(G_n^2, H_n^2) \geq E(G_n^2 H_n^2). \tag{42}$$

Therefore, using assumption (c) we obtain that $\kappa_2(F_n) \to \kappa_2(F_\infty) = 1$. According to [7, Theorem 1.2] point (iii), assumption (a) implies that for constant $c_q = 4\frac{(\frac{q}{2})!^3}{q!^2}$, we have $\|g_n \tilde{\otimes}_{\frac{q}{2}} g_n - c_q g_n\|_{\mathfrak{H}^{\otimes q}} \to 0$, and moreover $\|g_n \otimes_r g_n\|_{\mathfrak{H}^{\otimes(2q-2r)}} \to 0$, for all $r = 1, \cdots, q-1$ and $r \neq \frac{q}{2}$, and similarly for the kernels h_n by assumption (b). Hence

$$\begin{aligned}
\left|E(G_n^2 H_n)\right| &= \left|\langle g_n \tilde{\otimes}_{\frac{q}{2}} g_n, h_n\rangle_{\mathfrak{H}^{\otimes q}}\right| \\
&= \left|\langle g_n \tilde{\otimes}_{\frac{q}{2}} g_n - c_q g_n, h_n\rangle_{\mathfrak{H}^{\otimes q}} + c_q \langle g_n, h_n\rangle_{\mathfrak{H}^{\otimes q}}\right| \\
&\leq \|g_n \tilde{\otimes}_{\frac{q}{2}} g_n - c_q g_n\|_{\mathfrak{H}^{\otimes q}} \|h_n\|_{\mathfrak{H}^{\otimes q}} + c_q q! |E(G_n H_n)| \\
&\to 0,
\end{aligned}$$

by assumptions (a), (b), and (42). In a similar way, one can see that $E(G_n H_n^2) \to 0$. Hence

$$\kappa_3(F_n) = \kappa_3(G_n) - 3E(G_n^2 H_n) + 3E(G_n H_n^2) - \kappa_3(H_n) \to 0,$$

and therefore, the assumption **(a)** of Corollary 1 is verified. To check assumption **(b2)**, take indices r and s such that $(r, s) \neq (\frac{q}{2}, q)$. Now, it is enough to show that

$\|(f_n\tilde{\otimes}_r f_n)\otimes_s f_n\|_{\mathfrak{H}^{\otimes(3q-2r-2s)}} \to 0$. Using the identity $(f_n\tilde{\otimes}_r f_n)\otimes_s f_n = (f_n\tilde{\otimes}_r f_n)\otimes_s g_n - (f_n\tilde{\otimes}_r f_n)\otimes_s h_n$, it reduces to show that $\|(f_n\tilde{\otimes}_r f_n)\otimes_s g_n\|_{\mathfrak{H}^{\otimes(3q-2r-2s)}} \to 0$. On the other hand,

$$\begin{aligned}
\|(f_n\tilde{\otimes}_r f_n)\otimes_s g_n\|_{\mathfrak{H}^{\otimes(3q-2r-2s)}} &\le \|f_n\otimes_r f_n\|_{\mathfrak{H}^{\otimes(2q-2r)}}\ \|g_n\|_{\mathfrak{H}^q}\\
&\le \|g_n\|_{\mathfrak{H}^q}\Big\{\|g_n\otimes_r g_n\|_{\mathfrak{H}^{\otimes(2q-2r)}}\\
&\quad + 2\|g_n\otimes_r h_n\|_{\mathfrak{H}^{\otimes(2q-2r)}} + \|h_n\otimes_r h_n\|_{\mathfrak{H}^{\otimes(2q-2r)}}\Big\}\\
&\to 0.
\end{aligned}$$

To obtain the last convergence, notice that (see [13, Theorem 3.1]) assumption (c) tells us that

$$\|g_n\otimes_r h_n\|_{\mathfrak{H}^{\otimes(2q-2r)}} \to 0, \quad \forall r = 1, 2, \cdots, q.$$

Hence the assumption **(b2)** of Corollary 1 is also verified. In similar way, one can check that assumption **(b1)** also satisfies.

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Appendix

Lemma 6 *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be two sequences in $l^1(\mathbb{N})$ such that for all $p \ge 1$ we have*

$$\sum_{k\in\mathbb{N}} a_k^p = \sum_{k\in\mathbb{N}} b_k^p. \tag{43}$$

Then, there exists a permutation π on natural numbers $\mathbb{N}$ such that $a_k = b_{\pi(k)}$ for all $k \ge 1$.

Proof Let $\mathbb{R}[X]$ denote the ring of all polynomials over real line. Then, relation (43) implies that for any polynomial $P \in \mathbb{R}[X]$, we have

$$\sum_{k\in\mathbb{N}} a_k P(a_k) = \sum_{k\in\mathbb{N}} b_k P(b_k). \tag{44}$$

Let $M := \max\{\|a\|_{l^1(\mathbb{N})}, \|b\|_{l^1(\mathbb{N})}\} < \infty$. Then by a density argument, for any continuous function $\varphi \in C([-M, M])$, we obtain

$$\sum_{k\in\mathbb{N}} a_k\varphi(a_k) = \sum_{k\in\mathbb{N}} b_k\varphi(b_k). \quad (45)$$

For any $i \in \mathbb{N}$, we can now choose a continuous function φ such that $\varphi(a_i) = 1$ and $\varphi = 0$ on the set $\{a_j | a_j \neq a_i\} \cup \{b_j | b_j \neq a_i\}$. This implies that, for some integer k_i, we have $a_i = b_{k_i}$. It is now sufficient to take $\pi(i) = k_i$.

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Mod-Gaussian Convergence and Its Applications for Models of Statistical Mechanics

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In memoriam, Marc Yor

Abstract In this paper we complete our understanding of the role played by the limiting (or residue) function in the context of mod-Gaussian convergence. The question about the probabilistic interpretation of such functions was initially raised by Marc Yor. After recalling our recent result which interprets the limiting function as a measure of "breaking of symmetry" in the Gaussian approximation in the framework of general central limit theorems type results, we introduce the framework of L^1-mod-Gaussian convergence in which the residue function is obtained as (up to a normalizing factor) the probability density of some sequences of random variables converging in law after a change of probability measure. In particular we recover some celebrated results due to Ellis and Newman on the convergence in law of dependent random variables arising in statistical mechanics. We complete our results by giving an alternative approach to the Stein method to obtain the rate of convergence in the Ellis-Newman convergence theorem and by proving a new local limit theorem. More generally we illustrate our results with simple models from statistical mechanics.

1 Introduction

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables. In the series of papers [3, 8, 10, 12, 13], we introduced the notion of mod-Gaussian convergence (and more generally of mod-ϕ convergence, with respect to an arbitrary infinitely divisible law ϕ):

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Definition 1 The sequence $(X_n)_{n\in\mathbb{N}}$ is said to converge in the mod-Gaussian sense with parameters $t_n \to +\infty$ and limiting (or residue) function θ if, locally uniformly in $\mathbb{R}$,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n}]\,\mathrm{e}^{\frac{t_n t^2}{2}} = \theta(t)\,(1+o(1)),$$

where θ is a continuous function on $\mathbb{R}$ with $\theta(0) = 1$.

A trivial situation of mod-Gaussian convergence is when $X_n = G_n + Y_n$ is the sum of a Gaussian variable of variance t_n and of an independent random variable Y_n that converges in law to a variable Y with characteristic function θ. More generally X_n can be thought of as a Gaussian variable of variance t_n, plus a noise which is encoded by the multiplicative residue θ in the characteristic function. In this setting, θ is not necessarily the characteristic function of a random variable (the residual noise). For instance, consider

$$X_n = \frac{1}{n^{1/3}} \sum_{i=1}^{n} Y_i,$$

where the Y_i are centred, independent and identically distributed random variables with convergent moment generating function. Then a Taylor expansion of $\mathbb{E}[\mathrm{e}^{\mathrm{i}tY}]$ shows that $(X_n)_{n\in\mathbb{N}}$ converges in the mod-Gaussian sense with parameters $n^{1/3}\,\mathrm{Var}(Y)$ and limiting function

$$\theta(t) = \exp\left(\frac{\mathbb{E}[Y^3]\,(\mathrm{i}t)^3}{6}\right),$$

which is not the characteristic function of a random variable, since it does not go to zero as t goes to infinity. In 2008, during the workshop "*Random matrices, L-functions and primes*" held in Zürich, Marc Yor asked the second author A. N. about the role of the limiting function θ. In [14] it is proved that the set of possible limiting functions is the set of continuous functions θ from $\mathbb{R}$ to $\mathbb{C}$ such that $\theta(0) = 1$ and $\theta(-t) = \bar{\theta}(t)$ for $t \in \mathbb{R}$. But this characterization does not say anything on the probabilistic information encoded in θ. We now wish to develop more on probabilistic interpretations of the limiting function and the implications of mod-Gaussian convergence in terms of classical limit theorems of probability theory.

We first note that by looking at $\mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n/\sqrt{t_n}}]$, one immediately sees that mod-Gaussian convergence implies a central limit theorem for the sequence $(\frac{X_n}{\sqrt{t_n}})$:

$$\frac{X_n}{\sqrt{t_n}} \rightharpoonup_{n\to\infty} \mathcal{N}(0,1), \tag{1}$$

where the convergence above holds in law (see [10, § 2–3] for more details on this). On the other hand, with somewhat stronger hypotheses on the remainder $o(1)$ that appears in Definition 1, a local limit theorem also holds, see [13, Theorem 4] and [3, Theorem 5]. Thus, if G_n is a centred real Gaussian random variable with variance t_n, then

$$\mathbb{P}[X_n \in B] = \mathbb{P}[G_n \in B]\,(1 + o(1)) = \frac{m(B)}{\sqrt{2\pi t_n}}\,(1 + o(1))$$

for relatively compact sets B with $m(\partial B) = 0$, m denoting the Lebesgue measure.

In [8], it is then explained that by looking at Laplace transforms instead of characteristic functions, and by assuming the convergence holds on a whole band of the complex plane, one can obtain in the setting of mod-Gaussian convergence precise estimates of moderate or large deviations. In fact these results provide a new probabilistic interpretation of the limiting function as a measure of the "breaking of symmetry" in the Gaussian approximation of the tails of X_n (see Sect. 1.1 for more details).

The goal of this paper is threefold:

- to propose a new interpretation of the limiting function in the framework of mod-Gaussian convergence with Laplace transforms; these results allow us in particular to recover some well known exotic limit theorems from statistical mechanics due to Ellis and Newman [5] and similar one for other models or in higher dimensions.
- to show that once one is able to prove mod-Gaussian convergence, then one can expect to obtain finer results than merely convergence in law, such as speed of convergence and local limit theorems. Results on the rate of convergence in the Curie-Weiss model at critical temperature $\beta = 1$ were recently obtained using Stein's method (see e.g. [4]), while the local limit theorem, to the best of our knowledge, is new (at high temperature, with $\beta < 1$, a local limit theorem is stated in [18, §4.2]).
- to explore the applications of the results obtained in [8] on the "breaking of symmetry" in the central limit theorem to some classical models of statistical mechanics. In particular our approach determines the scale up to which the Gaussian approximation for the tails is valid and its breaking at this critical scale.

Our results are best illustrated with some classical one-dimensional models from statistical mechanics, such as the Curie-Weiss model or the Ising model. To illustrate the flexibility of our approach, we shall also prove similar results for weighted symmetric random walks in dimensions 2 and 3. The statistics of interest to us will be the total magnetization, which can be written as a sum of dependent random variables. These examples add to the already large class of examples of sums of dependent random variables we have already been able to deal with in the context of mod-ϕ convergence.

In the remaining of the introduction we recall the results obtained in [8] which led us to the "breaking of symmetry" interpretation, as well as an underlying method of cumulants that enabled us to establish the mod-Gaussian convergence for a large family of sums of dependent random variables. The important aspect of the cumulant method is that it provides a tool to prove mod-Gaussian convergence in situations where one cannot explicitly compute the characteristic function. We eventually give an outline of the paper.

1.1 Complex Convergence and Interpretation of the Residue

We consider again a sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$, but this time we assume that their Laplace transforms $\mathbb{E}[\mathrm{e}^{zX_n}]$ are convergent in an open disk of radius $c > 0$. In this case, they are automatically well-defined and holomorphic in a band of the complex plane $\mathcal{B}_c = \{z \in \mathbb{C},\ |\mathrm{Re}(z)| < c\}$ (see [16, Theorem 6], and [7] for a general survey of the properties of Laplace and Fourier transforms of probability measures).

Definition 2 The sequence $(X_n)_{n\in\mathbb{N}}$ is said to converge in the complex mod-Gaussian sense with parameters t_n and limiting function ψ if, locally uniformly on $\mathcal{B}_c$,

$$\mathbb{E}[\mathrm{e}^{zX_n}]\,\mathrm{e}^{-\frac{t_n z^2}{2}} = \psi(z)\,(1+o(1)),$$

where ψ is a continuous function on $\mathcal{B}_c$ with $\psi(0) = 1$. Then, one has in particular convergence in the sense of Definition 1, with $\theta(t) = \psi(\mathrm{i}t)$.

In this setting which is more restrictive than before, the residue ψ has a natural interpretation as a measure of "breaking of symmetry" when one tries to push the estimates of the central limit theorem from the scale $\sqrt{t_n}$ to the scale t_n. The previously mentioned central limit theorem (1) tells us that:

$$\mathbb{P}\left[X_n \geq a\sqrt{t_n}\right] = \left(\frac{1}{\sqrt{2\pi}}\int_a^\infty \mathrm{e}^{-\frac{x^2}{2}}\,dx\right)(1+o(1))$$

for any $a \in \mathbb{R}$. In the setting of complex mod-Gaussian convergence, this estimate remains true with $a = o(\sqrt{t_n})$, so that if $\varepsilon = o(1)$, then

$$\begin{aligned}\mathbb{P}\left[X_n \geq \varepsilon\, t_n\right] &= \left(\frac{1}{\sqrt{2\pi}}\int_{\varepsilon\sqrt{t_n}}^\infty \mathrm{e}^{-\frac{x^2}{2}}\,dx\right)(1+o(1)),\\ &= \frac{\mathrm{e}^{-\frac{t_n\varepsilon^2}{2}}}{\sqrt{2\pi t_n}\,\varepsilon}\,(1+o(1)) \quad \text{if } 1 \gg \varepsilon \gg \frac{1}{\sqrt{t_n}},\end{aligned}$$

where the notation $a_n \gg b_n$ stands for $b_n = o(a_n)$. Then, at scale t_n, the limiting residue ψ comes into play, with the following estimate that holds without additional hypotheses than those in Definition 2:

$$\forall x \in (0,c),\ \ \mathbb{P}[X_n \geq xt_n] = \frac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{\sqrt{2\pi t_n}\,x}\,\psi(x)\,(1+o(1)), \tag{2}$$

the remainder $o(1)$ being uniform when x stays in a compact set of $\mathbb{R}_+^* \cap (0,c)$. Notice that this formula does not follow directly from the calculation of $\mathbb{P}[X_n \geq \varepsilon t_n]$

with $\varepsilon = o(1)$; thus, it requires additional tools in order to be proven, see [8]. The estimate of positive large deviations has the following counterpart on the negative side:

$$\forall x \in (0,c),\ \ \mathbb{P}[X_n \leq -xt_n] = \frac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{\sqrt{2\pi t_n}\,x}\,\psi(-x)\,(1+o(1)).$$

So for instance, if $(Y_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables with convergent moment generating function, mean 0, variance 1 and third moment $\mathbb{E}[Y_i^3] > 0$, then $X_n = \frac{1}{n^{1/3}}\sum_{i=1}^n Y_i$ converges in the *complex* mod-Gaussian sense with parameters $n^{1/3}$ and limiting function $\psi(z) = \exp(\mathbb{E}[Y^3]\,z^3/6)$, and therefore for $x > 0$,

$$\mathbb{P}\left[\sum_{i=1}^n Y_i \geq xn^{2/3}\right] = \mathbb{P}\big[\mathcal{N}(0,1) \geq xn^{1/6}\big]\ \exp\left(\frac{\mathbb{E}[Y^3]\,x^3}{6}\right)\ (1+o(1)).$$

Thus, at scale $n^{2/3}$, the fluctuations of the sum of i.i.d. random variables are no more Gaussian, and the residue $\psi(x)$ measures this "breaking of symmetry": in the previous example, it makes moderate deviations on the positive side more likely than moderate deviations on the negative side, since $\psi(x) > 1 > \psi(-x)$ for $x > 0$.

Remark 3 The problem of finding the normality zone, i.e. the scale up to which the central limit theorem is valid, is a known problem in the case of i.i.d. random variables (see e.g. [9]). The description of the "symmetry breaking" is new and moreover the mod-Gaussian framework covers many examples with dependent random variables (see also [8] for more examples).

Thus, the observation of large deviations of the random variables X_n provides a first probabilistic interpretation of the residue ψ in the deconvolution of a sequence of characteristic functions of random variables by a sequence of large Gaussian variables. In Sect. 3, we shall provide another interpretation of ψ, which is inspired by some classical results from statistical mechanics (*cf.* [5, 6]).

1.2 The Method of Joint Cumulants

The appearance of an exponential of a monomial $Kx^{r\geq 3}$ as the limiting residue in mod-Gaussian convergence is a phenomenon that occurs not only for sums of i.i.d. random variables, but more generally for sums of possibly non identically distributed and/or dependent random variables. For instance,

1. the number of zeroes of a random Gaussian analytic function $\sum_{k=0}^{\infty}(\mathcal{N}_{\mathbb{C}})_k\,z^k$ in the disk of radius $1-\frac{1}{n}$, the variables $(\mathcal{N}_{\mathbb{C}})_k$ being independent standard complex Gaussian variables;
2. the number of triangles in a random Erdös-Rényi graph $G(n,p)$;

are both mod-Gaussian convergent after proper rescaling, and with limiting function of the form $\exp(Lz^3)$, with the constant L depending on the model (see again [8]). The reason behind these universal asymptotics lies in the following method of cumulants. If X is a random variable with convergent Laplace transform $\mathbb{E}[\mathrm{e}^{zX}]$ on a disk, we recall that its cumulant generating function is

$$\log \mathbb{E}[\mathrm{e}^{zX}] = \sum_{r \geq 1} \frac{\kappa^{(r)}(X)}{r!}\, z^r, \tag{3}$$

which is also well-defined and holomorphic on a disk around the origin. Its coefficients $\kappa^{(r)}(X)$ are the cumulants of the variable X, and they are homogenenous polynomials in the moments of X; for instance, $\kappa^{(1)}(X) = \mathbb{E}[X]$, $\kappa^{(2)}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$, and $\kappa^{(3)}(X) = \mathbb{E}[X^3] - 3\,\mathbb{E}[X^2]\,\mathbb{E}[X] + 2\,\mathbb{E}[X]^3$.

Consider now a sequence of random variables $(W_n)_{n\in\mathbb{N}}$ with $\kappa^{(1)}(W_n) = 0$, and for $r \geq 2$,

$$\kappa^{(r)}(W_n) = K_r\, \alpha_n\, (1 + o(1)), \tag{4}$$

with $\alpha_n \to +\infty$. This assumption is inspired by the case of a sum $W_n = \sum_{i=1}^n Y_i$ of centred i.i.d. random variables for which $\kappa^r(W_n) = n\,\kappa^{(r)}(Y)$. If Eq. (4) is satisfied, then one can formally write

$$\begin{aligned}
\log\, \mathbb{E}\left[\mathrm{e}^{z\,\frac{W_n}{(\alpha_n)^{1/3}}}\right] &= (\alpha_n)^{-2/3}\, \frac{\kappa^{(2)}(W_n)\, z^2}{2} + (\alpha_n)^{-1}\, \frac{\kappa^{(3)}(W_n)\, z^3}{6} \\
&\quad + \sum_{r\geq 4} \frac{\kappa^{(r)}(W_n)}{r!} ((\alpha_n)^{-1/3} z)^r \\
&\simeq (\alpha_n)^{1/3}\, \frac{K_2\, z^2}{2} + \frac{K_3\, z^3}{6} + \sum_{r\geq 4} \frac{K_r\, z^r}{r!} (\alpha_n)^{1-r/3} \\
&\simeq (\alpha_n)^{1/3}\, \frac{K_2\, z^2}{2} + \frac{K_3\, z^3}{6}
\end{aligned}$$

whence the mod-Gaussian convergence of $X_n = (\alpha_n)^{-1/3}\, W_n$ with parameters $K_2\,(\alpha_n)^{1/3}$ and limiting function $\exp(K_3\, z^3/6)$. The approximation is valid if the $o(1)$ in the asymptotics of $\kappa^{(2)}(W_n)$ is small enough (namely $o((\alpha_n)^{-1/3})$), and if the series $\sum_{r\geq 4}$ can be controlled, which is the case if

$$\forall r,\ \ |\kappa^{(r)}(W_n)| \leq (Cr)^r\, \alpha_n \tag{5}$$

for some constant C. The method of cumulants in the setting of mod-Gaussian convergence amounts to prove (4) for the first cumulants of the sequence $(X_n)_{n\in\mathbb{N}}$, and (5) for all the other cumulants. From such estimates one then obtains mod-Gaussian convergence for an appropriate renormalisation of $(W_n)_{r\geq 3}$, with limiting function $\exp(K_r\, z^r/r!)$, where r is the smallest integer greater or equal than 3 such that $K_r \neq 0$.

This method of cumulants works well with sequences $(W_n)_{n\in\mathbb{N}}$ that write as sums of (weakly) dependent random variables. Indeed, cumulants admit the following generalization to families of random variables, see [15]. Denote $\mathfrak{Q}_r$ the set of partitions of $[\![1,r]\!] = \{1,2,3,\dots,r\}$, and μ the Möbius function of this poset (see [19] for basic facts about Möbius functions of posets). If $\Pi \in \mathfrak{Q}_r$, then

$$\mu(\Pi) = (-1)^{\ell(\Pi)-1}\,(\ell(\Pi)-1)!$$

where $\ell(\Pi) = s$ if $\Pi = \pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_s$ has s parts. The joint cumulant of a family of r random variables with well defined moments of all order is

$$\kappa(X_1,\dots,X_r) = \sum_{\Pi\in\mathfrak{Q}_r} \mu(\Pi) \prod_{i=1}^{\ell(\Pi)} \mathbb{E}\left[\prod_{j\in\pi_i} X_j\right].$$

It is multilinear and generalizes Eq. (3), since

$$\kappa(X_1,\dots,X_r) = \left.\frac{\partial^r}{\partial z_1 \partial z_2 \cdots \partial z_r}\right|_{z_1=\cdots=z_r=0} \left(\log \mathbb{E}[\mathrm{e}^{z_1X_1+\cdots+z_rX_r}]\right)$$
$$\kappa\big(\underbrace{X,\dots,X}_{r\text{ times}}\big) = \kappa^{(r)}(X).$$

Suppose now that $W = W_n = \sum_{i=1}^n Y_i$ is a sum of dependent random variables. By multilinearity,

$$\kappa^{(r)}(W) = \sum_{i_1,\dots,i_r} \kappa(Y_{i_1},\dots,Y_{i_r}), \tag{6}$$

so in order to obtain the bound (5), it suffices to bound each "elementary" joint cumulant $\kappa(Y_{i_1},\dots,Y_{i_r})$. To this purpose, it is convenient to introduce the dependency graph of the family of random variables $(Y_1,\dots,Y_n)$, which is the smallest subgraph G of the complete graph on n vertices such that the following property holds: if $(Y_i)_{i\in I}$ and $(Y_j)_{j\in J}$ are disjoint subsets of random variables with no edge of G between a variable Y_i and a variable Y_j, then $(Y_i)_{i\in I}$ and $(Y_j)_{j\in J}$ are independent. Then, in many situations, one can write a bound on the elementary cumulant $\kappa(Y_{i_1},\dots,Y_{i_r})$ that only depends on the induced subgraph $G[i_1,\dots,i_r]$ obtained from the dependency graph by keeping only the vertices $i_1,\dots,i_r$ and the edges between them. In particular:

1. $\kappa(Y_{i_1},\dots,Y_{i_r}) = 0$ if the induced graph $G[i_1,\dots,i_r]$ is not connected.
2. if $|Y_i| \leq 1$ for all i, then $|\kappa(Y_{i_1},\dots,Y_{i_r})| \leq 2^{r-1}\,\mathrm{ST}(G[i_1,\dots,i_r])$, where $\mathrm{ST}(H)$ is the number of spanning trees on a (connected) graph H.

By gathering the contributions to the sum of Formula (6) according to the nature and position of the induced subgraph $G[i_1, \ldots, i_r]$ in G, one is able to prove efficient bounds on cumulants of sums of dependent variables, and to apply the method of cumulants to get their mod-Gaussian convergence. We refer to [8] for precise statements, in particular in the case where each vertex in G has less than D neighbors, with D independent of the vertex and of n. In Sect. 5, we shall apply this method to a case where G is the complete graph on n vertices, but where one can still find correct bounds (and in fact exact formulas) for the joint cumulants $\kappa(Y_{i_1}, \ldots, Y_{i_r})$: the one-dimensional Ising model.

1.3 Basic Models

As mentioned above, the goal of the paper is to study the phenomenon of mod-Gaussian convergence for probabilistic models stemming from statistical mechanics; this extends the already long list of models for which we were able to establish this asymptotic behavior of the Fourier or Laplace transforms [8, 10, 13]. More precisely, we shall focus on one-dimensional spin configurations, which already yield an interesting illustration of the theory and technics of mod-Gaussian convergence. Given two parameters $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$, we recall that the Curie-Weiss model and the one-dimensional Ising model are the probability laws on spin configurations $\sigma : [\![1, n]\!] \to \{\pm 1\}$ given by

$$\mathbb{CW}_{\alpha,\beta}(\sigma) = \frac{1}{Z_n(\mathbb{CW}, \alpha, \beta)} \exp\left(\alpha \sum_{i=1}^{n} \sigma(i) + \frac{\beta}{2n}\left(\sum_{i=1}^{n} \sigma(i)\right)^2\right); \tag{7}$$

$$\mathbb{I}_{\alpha,\beta}(\sigma) = \frac{1}{Z_n(\mathbb{I}, \alpha, \beta)} \exp\left(\alpha \sum_{i=1}^{n} \sigma(i) + \beta\left(\sum_{i=1}^{n-1} \sigma(i)\,\sigma(i+1)\right)\right). \tag{8}$$

The coefficient α measures the strength and direction of the exterior magnetic field, whereas β measures the strength of the interaction between spins, which tend to align in the same direction. This interaction is local for the Ising model, and global for the Curie-Weiss model. Set $M_n = \sum_{i=1}^{n} \sigma(i)$: this is the total magnetization of the system, and a random variable under the probabilities $\mathbb{CW}_{\alpha,\beta}$ and $\mathbb{I}_{\alpha,\beta}$.

In Sect. 2, we quickly establish the mod-Gaussian convergence of the magnetization for the Ising model, using the explicit form of the Laplace transform of the magnetization, which is given by the transfer matrix method. Alternatively, when $\alpha = 0$, in the appendix, we apply the cumulant method and give an explicit formula for each elementary cumulant of spins (see Sect. 5). This allows us to prove the analogue for joint cumulants of the well-known fact that covariances between spins decrease exponentially with distance in the one-dimensional Ising model. This second method is much less direct than the transfer matrix method, but we consider the Ising model to be a very good illustration of the method of joint cumulants. Moreover it illustrates the fact that one does not necessarily need to be able to compute precisely the moment generating function of the random variables.

In Sect. 3, we focus on the Curie-Weiss model, and we interpret the magnetization as a change of measure on a sum of i.i.d. random variables. Since these sums converge in the mod-Gaussian sense, it leads us to study the effect of a change of measure on a mod-Gaussian convergent sequence. We prove that in the setting of L^1-mod-Gaussian convergence, such changes of measures either conserve the mod-Gaussian convergence (with different parameters), or lead to a convergence in law, with a limiting distribution that involves the residue ψ. We thus recover some results of [5, 6] (in particular [5, Theorem 2.1]), and extend them to the setting of L^1-mod-Gaussian convergence. In Sect. 4, using Fourier analytic arguments, we quickly recover the optimal rate of convergence of the Ellis-Newman limit theorem for the Curie-Weiss model which was recently obtained in [4] using Stein's method, and then we establish a local limit theorem, thus completing the existing limit theorems for the Curie-Weiss model at critical temperature $\mathbb{CW}_{0,1}$.

2 Mod-Gaussian Convergence for the Ising Model: The Transfer Matrix Method

In this section, $(\sigma(i))_{i\in[\![1,n]\!]}$ is a random configuration of spins under the Ising measure (8), and $M_n = \sum_{i=1}^{n} \sigma(i)$ is its magnetization (Fig. 1). The mod-Gaussian convergence of M_n after appropriate rescaling can be obtained by two different methods: the *transfer matrix method*, which yields an explicit formula for $\mathbb{E}[\mathrm{e}^{zM_n}]$; and the *cumulant method*, which gives an explicit combinatorial formula for the coefficients of the series $\log \mathbb{E}[\mathrm{e}^{zM_n}]$. We use here the transfer matrix method, and refer to the appendix (Sect. 5) for the cumulant method.

The Laplace transform $\mathbb{E}[\mathrm{e}^{zM_n}]$ of the magnetization of the one-dimensional Ising model is well-known to be computable by the following transfer matrix method, see [1, Chap. 2]. Introduce the matrix

$$T = \begin{pmatrix} \mathrm{e}^{\alpha+\beta} & \mathrm{e}^{-\alpha-\beta} \\ \mathrm{e}^{\alpha-\beta} & \mathrm{e}^{-\alpha+\beta} \end{pmatrix},$$

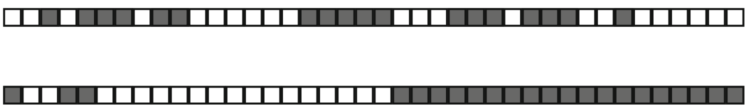


Fig. 1 Two configurations of spins under the Ising measures of parameters $(\alpha = 0, \beta = 0.3)$ and $(\alpha = 0, \beta = 1)$

and the two vectors $V = (e^{\alpha}, e^{-\alpha})$ and $W = \binom{1}{1}$. If the rows and columns of T correspond to the two signs $+1$ and -1, then any configuration of spins $\sigma = (\sigma(i))_{i\in[\![1,n]\!]}$ has under the Ising measure $\mathbb{I}_{\alpha,\beta}$ a probability proportional to $V_{\sigma(1)}T_{\sigma(1),\sigma(2)}T_{\sigma(2),\sigma(3)}\cdots T_{\sigma(n-1),\sigma(n)}$. Therefore, the partition function $Z_n(\mathbb{I},\alpha,\beta)$ is given by

$$\begin{aligned}Z_n(\mathbb{I},\alpha,\beta) &= \sum_{\sigma(1),\ldots,\sigma(n)} V_{\sigma(1)}T_{\sigma(1),\sigma(2)}T_{\sigma(2),\sigma(3)}\cdots T_{\sigma(n-1),\sigma(n)} \\ &= V(T)^{n-1}W = a_+(\lambda_+)^{n-1} + a_-(\lambda_-)^{n-1},\end{aligned}$$

where

$$a_+ = \cosh\alpha + \frac{e^{\beta}\sinh^2\alpha + e^{-\beta}}{\sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}};\quad a_- = \cosh\alpha - \frac{e^{\beta}\sinh^2\alpha + e^{-\beta}}{\sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}}$$

$$\lambda_+ = e^{\beta}\,\cosh\alpha + \sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}};\quad \lambda_- = e^{\beta}\,\cosh\alpha - \sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}.$$

Indeed, $\lambda_\pm$ are the two eigenvalues of T, and a_+ and a_- are obtained by identification of coefficients in the two formulas

$$\begin{aligned}Z_1(\mathbb{I},\alpha,\beta) &= e^{\alpha} + e^{-\alpha}\\ Z_2(\mathbb{I},\alpha,\beta) &= e^{2\alpha+\beta} + e^{-2\alpha+\beta} + 2e^{-\beta}.\end{aligned}$$

Then, the Laplace transform of M_n is given by

$$\mathbb{E}_{\alpha,\beta}[e^{zM_n}] = \frac{Z_n(\mathbb{I},\alpha+z,\beta)}{Z_n(\mathbb{I},\alpha,\beta)}.$$

In particular,

$$\mathbb{E}_{\alpha,\beta}[M_n] = \left.\frac{\partial\mathbb{E}_{\alpha,\beta}[e^{zM_n}]}{\partial z}\right|_{z=0} = \frac{\partial}{\partial\alpha}\log Z_n(\mathbb{I},\alpha,\beta) = n\,\frac{e^{\beta}\sinh\alpha}{\sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}} + O(1).$$

whence a formula for the (asymptotic) mean magnetization by spin:

$$\overline{m} = \frac{e^{\beta}\sinh\alpha}{\sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}}.$$

A more precise Taylor expansion of $Z_n(\mathbb{I},\alpha+z,\beta)$ leads to the following:

Theorem 4 *Under the Ising measure $\mathbb{I}_{\alpha,\beta}$, $\frac{M_n - n\overline{m}}{n^{1/3}}$ converges in the complex mod-Gaussian sense with parameters*

$$t_n = n^{1/3}\,\frac{e^{-\beta}\cosh\alpha}{(e^{2\beta}\sinh^2\alpha + e^{-2\beta})^{3/2}}$$

and limiting function

$$\psi(z) = \exp\left(-\frac{2e^{\beta}\sinh^3\alpha + (3e^{\beta} - e^{-3\beta})\sinh\alpha}{6(e^{2\beta}\sinh^2\alpha + e^{-2\beta})^{5/2}}\, z^3\right).$$

Proof In the following, we are dealing with square roots and logarithms of complex numbers, but each time in a neighborhood of $\mathbb{R}_+^*$, so there is no ambiguity in the choice of the branches of these functions. That said, it is easier to work with log-Laplace transforms:

$$\begin{aligned}
\log \mathbb{E}_{\alpha,\beta}\left[e^{z\,\frac{M_n - n\overline{m}}{n^{1/3}}}\right] &= \log Z_n\left(\mathbb{I}, \alpha + \frac{z}{n^{1/3}}, \beta\right) - \log Z_n(\mathbb{I},\alpha,\beta) - zn^{2/3}\overline{m} \\
\log Z_n(\mathbb{I},\alpha,\beta) &= \log a_+(\alpha,\beta) + (n-1)\log\lambda_+(\alpha,\beta) + o(1) \\
\log Z_n\left(\mathbb{I}, \alpha + \frac{z}{n^{1/3}}, \beta\right) &= \log a_+\left(\alpha + \frac{z}{n^{1/3}}, \beta\right) \\
&\quad + (n-1)\log\lambda_+\left(\alpha + \frac{z}{n^{1/3}}, \beta\right) + o(1) \\
&= \log a_+(\alpha,\beta) + (n-1)\log\lambda_+(\alpha,\beta) \\
&\quad + zn^{2/3}\frac{\partial}{\partial\alpha}(\log\lambda_+(\alpha,\beta)) \\
&\quad + \frac{z^2 n^{1/3}}{2}\frac{\partial^2}{\partial\alpha^2}(\log\lambda_+(\alpha,\beta)) \\
&\quad + \frac{z^3}{6}\frac{\partial^3}{\partial\alpha^3}(\log\lambda_+(\alpha,\beta)) + o(1).
\end{aligned}$$

Thus, it suffices to compute the first derivatives of $\log\lambda_+(\alpha,\beta)$ with respect to α:

$$\begin{aligned}
\log\lambda_+(\alpha,\beta) &= \log\left(e^{\beta}\cosh\alpha + \sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}\right) \\
\frac{\partial}{\partial\alpha}(\log\lambda_+(\alpha,\beta)) &= \frac{e^{\beta}\sinh\alpha}{\sqrt{e^{2\beta}\sinh^2\alpha + e^{-2\beta}}} = \overline{m} \\
\frac{\partial^2}{\partial\alpha^2}(\log\lambda_+(\alpha,\beta)) &= \frac{e^{-\beta}\cosh\alpha}{(e^{2\beta}\sinh^2\alpha + e^{-2\beta})^{3/2}} = \sigma^2 \\
\frac{\partial^3}{\partial\alpha^3}(\log\lambda_+(\alpha,\beta)) &= -\frac{2e^{\beta}\sinh^3\alpha + (3e^{\beta} - e^{-3\beta})\sinh\alpha}{(e^{2\beta}\sinh^2\alpha + e^{-2\beta})^{5/2}} = K_3.
\end{aligned}$$

We therefore get

$$\log \mathbb{E}_{\alpha,\beta}\left[e^{z\,\frac{M_n - n\overline{m}}{n^{1/3}}}\right] = n^{1/3}\,\frac{\sigma^2 z^2}{2} + \frac{K_3\, z^3}{6} + o(1). \qquad \square$$

By using Formula (2), this result leads to new estimates of moderate deviations for the probability $\mathbb{P}_{\alpha,\beta}[M_n \geq n\overline{m} + n^{1/3}x]$. In the special case when $\alpha = 0$, the limiting function $\psi(z)$ of Theorem 4 is equal to 1, and one has to push the expansion of $\log Z_n(\mathbb{I}, 0, \beta)$ to order 4 to get a meaningful mod-Gaussian convergence (the same phenomenon will occur in the case of the Curie-Weiss model):

Theorem 5 *Under the Ising measure $\mathbb{I}_{0,\beta}$, $\frac{M_n}{n^{1/4}}$ converges in the complex mod-Gaussian sense with parameters $t_n = n^{1/2}\,\mathrm{e}^{2\beta}$ and limiting function*

$$\psi(z) = \exp\left(-\frac{3\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta}}{24}\, z^4\right).$$

Proof This time one has to compute the fourth derivative of $\log \lambda_+(\alpha, \beta)$, which is

$$\begin{aligned}\frac{\partial^4}{\partial\alpha^4}(\log \lambda_+(\alpha,\beta)) = {} & (2\mathrm{e}^{\beta}\sinh^3\alpha + (3\mathrm{e}^{\beta} - \mathrm{e}^{-3\beta})\sinh\alpha) \\ & \times \frac{\partial}{\partial\alpha}\left(-\frac{1}{(\mathrm{e}^{2\beta}\sinh^2\alpha + \mathrm{e}^{-2\beta})^{5/2}}\right) \\ & - \frac{6\mathrm{e}^{\beta}\sinh^2\alpha\,\cosh\alpha + (3\mathrm{e}^{\beta} - \mathrm{e}^{-3\beta})\cosh\alpha}{(\mathrm{e}^{2\beta}\sinh^2\alpha + \mathrm{e}^{-2\beta})^{5/2}}.\end{aligned}$$

The second term is the only contribution when $\alpha = 0$, equal to $-(3\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta})$. Thus,

$$\log \mathbb{E}_{0,\beta}\left[\mathrm{e}^{z\,\frac{M_n - n\overline{m}}{n^{1/4}}}\right] = n^{1/2}\,\frac{\sigma^2 z^2}{2} - \frac{(3\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta})\,z^4}{24} + o(1).$$ ☐

3 Mod-Gaussian Convergence in L^1 and the Curie-Weiss Model

In this section, $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables with entire moment generating series $\mathbb{E}[\mathrm{e}^{zX_n}]$, and we assume the following:

(A) One has mod-Gaussian convergence of the Laplace transforms, i.e., there is a sequence $t_n \to +\infty$ and a function ψ continuous on $\mathbb{R}$ such that

$$\psi_n(t) = \mathbb{E}[\mathrm{e}^{tX_n}]\,\mathrm{e}^{-\frac{t_n t^2}{2}}$$

converges locally uniformly on $\mathbb{R}$ to $\psi(t)$.

(B) Each function ψ_n, and their limit ψ are in $\mathrm{L}^1(\mathbb{R})$.

We denote $\mathbb{P}_n$ the law of X_n,

$$\mathbb{Q}_n[dx] = \frac{e^{\frac{x^2}{2t_n}}}{\mathbb{E}\left[e^{\frac{(X_n)^2}{2t_n}}\right]}\,\mathbb{P}_n[dx], \tag{9}$$

and Y_n a random variable under the new law $\mathbb{Q}_n$. Note that hypothesis (3) implies that $Z_n = \mathbb{E}[e^{(X_n)^2/2t_n}]$ is finite for all $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}} \psi_n(t)\,dt &= \mathbb{E}\left[\int_{\mathbb{R}} e^{tX_n - \frac{t_n t^2}{2}}\,dt\right] = \mathbb{E}\left[e^{\frac{(X_n)^2}{2t_n}}\left(\int_{\mathbb{R}} e^{-\frac{(X_n - t_n t)^2}{2t_n}}\,dt\right)\right]\\ &= \sqrt{\frac{2\pi}{t_n}}\,\mathbb{E}\left[e^{\frac{(X_n)^2}{2t_n}}\right].\end{aligned}$$

Therefore the new probability measures $\mathbb{Q}_n$ are well defined. The goal of this section is to study the asymptotics of the new sequence $(Y_n)_{n\in\mathbb{N}}$. As we shall see in Sect. 3.3, the Curie-Weiss model defined by Eq. (7) is one of the main examples in this framework. However, it is more convenient to look at the general problem, and we shall introduce later other models concerned by our general results.

3.1 Ellis-Newman Lemma and Deconvolution of a Large Gaussian Noise

Suppose for a moment that hypothesis (A) is replaced by the stronger hypotheses of Definition 2, with $c = +\infty$ and therefore $\mathcal{B}_c = \mathbb{C}$. Fix then $0 < a < b$, and consider the partial integral $\mathbb{E}[e^{(X_n)^2/2t_n}\,\mathbb{1}_{t_n a \leq X_n \leq t_n b}]$. By integration by parts of Riemann-Stieltjes integrals, one has:

$$\begin{aligned}\int_{t_n a}^{t_n b} e^{\frac{x^2}{2t_n}}\,\mathbb{P}_n[dx] &= \left[-e^{\frac{x^2}{2t_n}}\,\mathbb{P}_n[X_n \geq x]\right]_{t_n a}^{t_n b} + \int_{t_n a}^{t_n b} \frac{x}{t_n}\,e^{\frac{x^2}{2t_n}}\,\mathbb{P}_n[X_n \geq x]\,dx\\ &= \left[-e^{\frac{t_n x^2}{2}}\,\mathbb{P}_n[X_n \geq t_n x]\right]_a^b + \int_a^b t_n x\,e^{\frac{t_n x^2}{2}}\,\mathbb{P}_n[X_n \geq t_n x]\,dx\\ &= \left(\left[-\frac{\psi(x)}{\sqrt{2\pi t_n}\,x}\right]_a^b + \sqrt{\frac{t_n}{2\pi}}\int_a^b \psi(x)\,dx\right)(1 + o_{a,b}(1))\\ &= \left(\sqrt{\frac{t_n}{2\pi}}\int_a^b \psi(x)\,dx\right)(1 + o_{a,b}(1))\end{aligned}$$

because of the estimates of precise deviations (2). In this computation, $o_{a,b}(1)$ is uniform for a, b in compact sets of $(0, +\infty)$. In fact this estimate remains true for a, b in a compact set of $\mathbb{R}$; hence, a and b can be possibly negative. If the estimate is also true with $a = -\infty$ and $b = +\infty$, then

$$\begin{aligned}\mathbb{Q}_n[t_n a \leq Y_n \leq t_n b] &= \frac{\mathbb{E}[\mathrm{e}^{(X_n)^2/2t_n}\, \mathbb{1}_{t_n a \leq X_n \leq t_n b}]}{\mathbb{E}[\mathrm{e}^{(X_n)^2/2t_n}]} \\ &= \frac{\sqrt{\frac{t_n}{2\pi}}\, \int_a^b \psi(x)\, dx}{\sqrt{\frac{t_n}{2\pi}}\, \int_{-\infty}^{+\infty} \psi(x)\, dx}\,(1 + o(1)) \\ &= \frac{\int_a^b \psi(x)\, dx}{\int_{-\infty}^{+\infty} \psi(x)\, dx}\,(1 + o(1)),\end{aligned}$$

so $(\frac{Y_n}{t_n})_{n\in\mathbb{N}}$ converges in law to the density $\psi(x)/\int_{\mathbb{R}} \psi(x)\, dx$.

We now wish to identify the most general conditions under which this convergence in law happens. To this purpose, it is useful to produce random variables with density $\psi_n(x)/\int_{\mathbb{R}} \psi_n(x)\, dx$. They are given by the following Proposition, which appeared in [5] as Lemma 3.3:

Proposition 6 *Let G_n be a centred Gaussian variable with variance $\frac{1}{t_n}$, and independent from Y_n. The law of $W_n = G_n + \frac{Y_n}{t_n}$ has density $\psi_n(x)/\int_{\mathbb{R}} \psi_n(x)\, dx$.*

Remark 7 This Proposition is related to the so-called Hubbard-Stratonovich transformation, which is commonly used in mean-field theory in order to replace a problem with interacting particles with a sum or integration over non-integrating systems. We refer to [2, p. 46] and references therein for precisions on this method coming from statistical mechanics.

Proof Denote $Z_n = \mathbb{E}[\mathrm{e}^{(X_n)^2/2t_n}]$, and $f_X(x)\, dx$ (respectively, $\mathbb{P}_X$) the density (respectively, the law) of a random variable X. One has

$$\begin{aligned}\mathbb{P}[W_n \leq w] &= \int_{-\infty}^{w} \left(\int_{\mathbb{R}} f_{G_n}(x-u)\, \mathbb{P}_{\frac{Y_n}{t_n}}[du]\right) dx \\ &= \sqrt{\frac{t_n}{2\pi}} \int_{-\infty}^{w} \left(\int_{\mathbb{R}} \mathrm{e}^{-\frac{t_n\,(x-\frac{y}{t_n})^2}{2}}\, \mathbb{P}_{Y_n}[dy]\right) dx \\ &= \sqrt{\frac{t_n}{2\pi}} \int_{-\infty}^{w} \left(\int_{\mathbb{R}} \mathrm{e}^{yx-\frac{y^2}{2t_n}}\, \mathbb{Q}_n[dy]\right) \mathrm{e}^{-\frac{t_n x^2}{2}}\, dx \\ &= \frac{1}{Z_n} \sqrt{\frac{t_n}{2\pi}} \int_{-\infty}^{w} \left(\int_{\mathbb{R}} \mathrm{e}^{yx}\, \mathbb{P}_n[dy]\right) \mathrm{e}^{-\frac{t_n x^2}{2}}\, dx \\ &= \frac{1}{Z_n} \sqrt{\frac{t_n}{2\pi}} \int_{-\infty}^{w} \psi_n(x)\, dx.\end{aligned}$$

Making w go to $+\infty$ gives an equation for $Z_n = \sqrt{\frac{t_n}{2\pi}} \int_{\mathbb{R}} \psi_n(x)\,dx$. One concludes that:

$$\mathbb{P}[W_n \leq w] = \frac{\int_{-\infty}^{w} \psi_n(x)\,dx}{\int_{-\infty}^{\infty} \psi_n(x)\,dx}. \qquad \square$$

This important property was not used in our previous works: to get the residue of deconvolution ψ_n of a random variable X_n by a large Gaussian variable of variance t_n (that is to say that one wants to *remove* a Gaussian variable of variance t_n from X_n), one can make the exponential change of measure (9), and *add* an independent Gaussian variable of variance t_n: the random variable thus obtained, which is $t_n W_n$ with the previous notation, has density proportional to $\psi_n(w/t_n)\,dw$.

3.2 The Residue of Mod-Gaussian Convergence as a Limiting Law

We can now state and prove the main result of this Section. We assume the hypotheses (A) and (B), and keep the same notation as before.

Theorem 8 *The following assertions are equivalent:*

(i) The sequence $(\frac{Y_n}{t_n})_{n\in\mathbb{N}}$ *is tight.*
(ii) The sequence $(\frac{Y_n}{t_n})_{n\in\mathbb{N}}$ *converges in law to a random variable with density* $\psi(x)/\int_{\mathbb{R}} \psi(x)\,dx$.
(iii) The convergence $\psi_n \to \psi$*, which is supposed locally uniform on* $\mathbb{R}$*, also occurs in* $\mathrm{L}^1(\mathbb{R})$.

We shall then say that $(X_n)_{n\in\mathbb{N}}$ *converges in the* L^1*-mod-Gaussian sense with parameters* t_n *and limiting function* ψ*. In this setting, the residue* ψ *can be interpreted as the limiting law of* $(X_n)_{n\in\mathbb{N}}$ *after an appropriate change of measure.*

Proof Since the Gaussian variable G_n of variance $\frac{1}{t_n}$ converges in probability to 0, $(\frac{Y_n}{t_n})_{n\in\mathbb{N}}$ converges to a law μ if and only if $(W_n)_{n\in\mathbb{N}}$ converges to the law μ. If (iii) is satisfied, then by Proposition 6,

$$\lim_{n\to\infty} \mathbb{P}[W_n \leq w] = \frac{\int_{-\infty}^{w} \psi(x)\,dx}{\int_{-\infty}^{\infty} \psi(x)\,dx},$$

so the cumulative distribution functions of the variables W_n converge to the cumulative distribution function of the law $\psi(x)/\int_{\mathbb{R}} \psi(x)\,dx$, and (ii) is established. Obviously, one also has (ii) $\Rightarrow$ (i). Finally, if (iii) is not satisfied, then by Scheffe's lemma one also has

$$\int_{\mathbb{R}} \psi_n(x)\,dx \not\to \int_{\mathbb{R}} \psi(x)\,dx.$$

However, by Fatou's lemma, $\int_{\mathbb{R}} \psi(x)\,dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} \psi_n(x)\,dx$. Therefore, the non-convergence in L^1 is only possible if $\int_{\mathbb{R}} \psi(x)\,dx < \limsup_{n\to\infty} \int_{\mathbb{R}} \psi_n(x)\,dx$. Thus, there is an $\varepsilon > 0$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N},\ \int_{\mathbb{R}} \psi_{n_k}(x)\,dx \geq \varepsilon + \int_{\mathbb{R}} \psi(x)\,dx.$$

Then, for all $a, b \in \mathbb{R}$,

$$\limsup_{k\to\infty} \mathbb{P}[a \leq W_{n_k} \leq b] = \limsup_{k\to\infty} \left(\frac{\int_a^b \psi_{n_k}(x)\,dx}{\int_{\mathbb{R}} \psi_{n_k}(x)\,dx}\right) = \frac{\int_a^b \psi(x)\,dx}{\liminf_{k\to\infty} \int_{\mathbb{R}} \psi_{n_k}(x)\,dx}$$
$$\leq \frac{\int_{\mathbb{R}} \psi(x)\,dx}{\varepsilon + \int_{\mathbb{R}} \psi(x)\,dx} < 1$$

which amounts to saying that $(W_n)_{n\in\mathbb{N}}$ (and therefore $(\frac{Y_n}{t_n})_{n\in\mathbb{N}}$) is not tight; hence, (i) implies (iii). □

To complete this result, it is important to compare the two notions of *complex* mod-Gaussian convergence and of *integral* L^1-mod-Gaussian convergence. Though there are no direct implication between these two assumptions, the following Proposition shows that the latter notion is a stronger type of convergence:

Proposition 9 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence that converges in the L^1-mod-Gaussian sense with parameters $t_n \to \infty$ and limiting function $\psi \in \mathrm{L}^1(\mathbb{R})$. The estimate of precise large deviations (2) is then satisfied.*

Proof Recall that $Z_n = \mathbb{E}[\mathrm{e}^{(X_n)^2/2t_n}] = \sqrt{\frac{t_n}{2\pi}} \int_{\mathbb{R}} \psi_n(x)\,dx$. We want to compute

$$\mathbb{P}[X_n \geq t_n x] = \int_{t_n x}^{\infty} \mathbb{P}_n[dy] = Z_n \int_{t_n x}^{\infty} \mathrm{e}^{-\frac{y^2}{2t_n}}\, \mathbb{Q}_n[dy] = Z_n \int_x^{\infty} \mathrm{e}^{-\frac{t_n u^2}{2}}\, \mathbb{P}_{\frac{Y_n}{t_n}}[du].$$

Suppose for a moment that we can replace the law of $\frac{Y_n}{t_n}$ by the one of $W_n = G_n + \frac{Y_n}{t_n}$ in the previous computation. Then, one obtains from Proposition 6

$$Z_n \int_x^{\infty} \mathrm{e}^{-\frac{t_n u^2}{2}}\, \mathbb{P}_{W_n}[du] = \sqrt{\frac{t_n}{2\pi}} \int_x^{\infty} \mathrm{e}^{-\frac{t_n u^2}{2}}\, \psi_n(u)\,du.$$

Fix $\varepsilon > 0$. Since ψ_n converges locally uniformly to the continuous function ψ, there is an interval $[x, x+\eta]$ such that for n large enough and $u \in [x, x+\eta]$,

$$\psi(x) - \varepsilon < \psi_n(u) < \psi(x) + \varepsilon.$$

Therefore, for n large enough,

$$
\begin{array}{ccccc}
(\psi(x)-\varepsilon)\displaystyle\int_x^{x+\eta} \mathrm{e}^{-\frac{t_n u^2}{2}}\,du & \leq & \displaystyle\int_x^{x+\eta} \mathrm{e}^{-\frac{t_n u^2}{2}}\,\psi_n(u)\,du & \leq & (\psi(x)+\varepsilon)\displaystyle\int_x^{x+\eta} \mathrm{e}^{-\frac{t_n u^2}{2}}\,du \\
\downarrow & & & & \downarrow \\
(\psi(x)-\varepsilon)\,\dfrac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{t_n x} & & & & (\psi(x)+\varepsilon)\,\dfrac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{t_n x}.
\end{array}
$$

Indeed, by integration by parts, $\int_x^{x+\eta} \mathrm{e}^{-\frac{t_n u^2}{2}}\,du$ is asymptotic to $\frac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{t_n x}$. On the other hand, since $\psi_n \to_{\mathrm{L}^1} \psi$, for the remaining part of the integral,

$$
\int_{x+\eta}^{\infty} \mathrm{e}^{-\frac{t_n u^2}{2}}\,\psi_n(u)\,du \leq \mathrm{e}^{-\frac{t_n(x+\eta)^2}{2}} \left(\int_{x+\eta}^{\infty} \psi_n(u)\,du\right) \simeq \mathrm{e}^{-\frac{t_n(x+\eta)^2}{2}} \left(\int_{x+\eta}^{\infty} \psi(u)\,du\right)
$$

which is much smaller than the previous quantities. Therefore, assuming that one can replace $\frac{Y_n}{t_n}$ by W_n, we obtain the asymptotics

$$
\mathbb{P}[X_n \geq t_n x] = \frac{\mathrm{e}^{-\frac{t_n x^2}{2}}}{\sqrt{2\pi t_n}\,x}\,\psi(x)\,(1+o(1))
$$

for all $x > 0$; this is what we wanted to prove. Finally, the replacement $\frac{Y_n}{t_n} \leftrightarrow W_n$ is indeed valid, because

$$
\begin{aligned}
\int_x^{\infty} \mathrm{e}^{-\frac{t_n u^2}{2}}\,\mathbb{P}_{W_n}[du] &= \left[\mathrm{e}^{-\frac{t_n u^2}{2}}\,\mathbb{P}[W_n \leq u]\right]_x^{\infty} + \int_x^{\infty} t_n u\,\mathrm{e}^{-\frac{t_n u^2}{2}}\,\mathbb{P}[W_n \leq u]\,du \\
&\simeq \left[\mathrm{e}^{-\frac{t_n u^2}{2}}\,\mathbb{P}[Y_n/t_n \leq u]\right]_x^{\infty} + \int_x^{\infty} t_n u\,\mathrm{e}^{-\frac{t_n u^2}{2}}\,\mathbb{P}[Y_n/t_n \leq u]\,du \\
&\simeq \int_x^{\infty} \mathrm{e}^{-\frac{t_n u^2}{2}}\,\mathbb{P}_{\frac{Y_n}{t_n}}[du]
\end{aligned}
$$

by using on the second line the fact that both $\frac{Y_n}{t_n}$ and W_n converge in law to the same limit, and therefore have equivalent cumulative distribution function on $\mathbb{R}_+$. □

In the same setting of L^1-mod-Gaussian convergence, one has similarly the estimates on the negative part of the real line, and around 0, as described on page 372 in the setting of complex mod-Gaussian convergence.

3.3 Application to the Curie-Weiss Model

Consider i.i.d. Bernoulli random variables $(\sigma(i))_{i\geq 1}$ with $\mathbb{P}[\sigma(i)=1]=1-\mathbb{P}[\sigma(i)=-1]=\frac{\mathrm{e}^{\alpha}}{2\cosh\alpha}$ for some $\alpha\in\mathbb{R}$. We set $U_n=\sum_{i=1}^n\sigma(i)$, so that

$$\mathbb{E}[\mathrm{e}^{zU_n}]=\left(\frac{\cosh(z+\alpha)}{\cosh\alpha}\right)^n=(\cosh z+\sinh z\,\tanh\alpha)^n$$

$$\mathbb{E}\left[\mathrm{e}^{z\,\frac{U_n-n\tanh\alpha}{n^{1/3}}}\right]=\left(\frac{\cosh(zn^{-1/3})+\sinh(zn^{-1/3})\,\tanh\alpha}{\mathrm{e}^{zn^{-1/3}\,\tanh\alpha}}\right)^n$$

$$\log\mathbb{E}\left[\mathrm{e}^{z\,\frac{U_n-n\tanh\alpha}{n^{1/3}}}\right]=\frac{n^{1/3}}{2\cosh^2\alpha}\,z^2-\frac{\sinh\alpha}{3\cosh^3\alpha}\,z^3+o(1)$$

so one has complex mod-Gaussian convergence of $\frac{U_n-n\tanh\alpha}{n^{1/3}}$ with parameters $\frac{n^{1/3}}{\cosh^2\alpha}$ and limiting function $\exp(-\frac{\sinh\alpha}{3\cosh^3\alpha}\,z^3)$.

If $\alpha=0$, then the term of order 3 disappears in the Taylor expansion of the characteristic function, and one obtains instead

$$\log\mathbb{E}\left[\mathrm{e}^{z\,\frac{U_n}{n^{1/4}}}\right]=\frac{n^{1/2}\,z^2}{2}-\frac{z^4}{12}+o(1),$$

hence a complex mod-Gaussian convergence of $X_n=\frac{U_n}{n^{1/4}}$ with parameters $n^{1/2}$ and limiting function $\exp(-z^4/12)$. Since this function restricted to $\mathbb{R}$ is integrable, this leads us to the following result, which originally appeared in [5] (without the mod-Gaussian interpretation):

Theorem 10 *Let $X_n=n^{-1/4}\sum_{i=1}^n\sigma(i)$ be a rescaled sum of centred ± 1 independent Bernoulli random variables. It converges in the L^1-mod-Gaussian sense, with parameters $n^{1/2}$ and limiting function $\exp(-\frac{z^4}{12})$. As a consequence, if $Y_n=n^{-1/4}M_n$ is the rescaled magnetization of a Curie-Weiss model $\mathbb{CW}_{0,1}$ of parameters $\alpha=0$ and $\beta=1$, then $Y_n/n^{1/2}$ converges in law to the distribution*

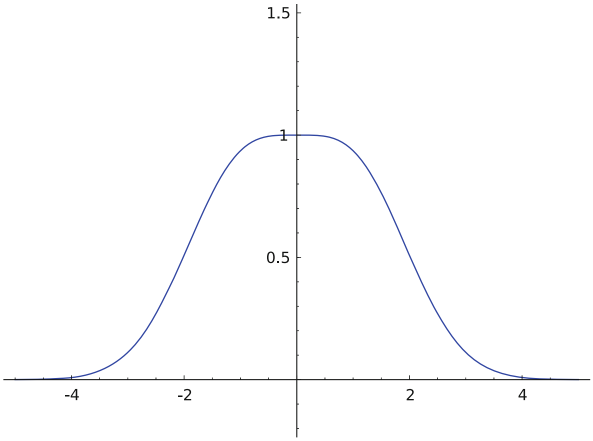
$$\frac{\exp(-\frac{x^4}{12})\,dx}{\int_{\mathbb{R}}\exp(-\frac{x^4}{12})\,dx}.$$

Proof The function $\psi_n(t)$ is in our case

$$\psi_n(t)=\mathrm{e}^{-\frac{t^2n^{1/2}}{2}}\left(\cosh\frac{t}{n^{1/4}}\right)^n,$$

and we have seen that it converges locally uniformly to $\psi(t)=\exp(-\frac{t^4}{12})$. By Scheffe's lemma, to obtain the L^1-mod-convergence, it is sufficient to prove that $\int_{\mathbb{R}}\psi_n(t)\,dt$ converges to $\int_{\mathbb{R}}\exp(-\frac{t^4}{12})\,dt$. This is a simple application of Laplace's

Fig. 2 The function $f(u) = \mathrm{e}^{-\frac{u^2}{2}} \cosh u$



method:

$$\int_{\mathbb{R}} \psi_n(t)\, dt = \int_{\mathbb{R}} \mathrm{e}^{-\frac{t^2 n^{1/2}}{2}} \left(\cosh \frac{t}{n^{1/4}}\right)^n dt = n^{1/4} \int_{\mathbb{R}} \left(\mathrm{e}^{-\frac{u^2}{2}} \cosh u\right)^n du$$

and the function $u \mapsto \mathrm{e}^{-\frac{u^2}{2}} \cosh u$ attains its global maximum at $u = 0$, with a Taylor expansion $1 - \frac{u^4}{12} + o(u^4)$, see Fig. 2.

Then, the exponential change of measure (9) gives a probability measure on spin configurations proportional to

$$\exp\left(\frac{(Y_n)^2}{2n^{1/2}}\right) = \exp\left(\frac{1}{2n}(M_n)^2\right),$$

so it is indeed the Curie-Weiss model $\mathbb{CW}_{0,1}$. □

Remark 11 The method of change of measures that was used so far has allowed us to treat the fluctuations of the Curie-Weiss model at critical temperature $\beta = 1$. One may ask what happens for other values of the temperature. The case of high temperature ($0 < \beta < 1$) is treated later in Theorem 13, see in particular the end of Example 14. For low temperatures ($\beta > 1$), there is no more a limiting law for M_n, though one can state a central limit theorem for conditioned versions of the magnetization. As far as we know, our results cannot be applied to this case.

It is easily seen that the proof of Theorem 10 adapts readily to the case where Bernoulli variables are replaced by so-called pure measures, so we recover most of the limit theorems stated in [5, 6]. However, by choosing the setting of mod-Gaussian convergence, we also obtain new limit theorems for models that do not fall in the Curie-Weiss setting. The following result explains how it would work to replace the Bernoulli distribution by more general ones; *cf.* [14, Proposition 2.2].

Proposition 12 *Let $k \geq 2$ be an integer, and let $(B_n)_{n\geq 1}$ be a sequence of i.i.d random variables in* L^r *for some $r > k+1$, such that the first k moments of B_1*

are the same as the corresponding moments of the Standard Gaussian distribution. Then the sequence of random variables

$$\left(\frac{1}{n^{1/(k+1)}}\sum_{k=1}^{n} B_k\right)_{n\geq 1}$$

converges in the mod-Gaussian sense with parameters

$$t_n = n^{(k-1)/(k+1)},$$

and limiting function

$$\theta(t) = \mathrm{e}^{(\mathrm{i}t)^{k+1}\frac{c_{k+1}}{(k+1)!}},$$

where c_{k+1} *denotes the* $(k+1)$*-th cumulant of* B_1.

When the random variables B_n have an entire moment generating function, then one can replace t with $-\mathrm{i}t$ to obtain mod-Gaussian convergence with the Laplace transforms. If B_1 is symmetric, then k is necessarily an odd number of the form $2s-1$ and hence

$$\psi(t) = \mathrm{e}^{(-1)^s t^{2s}\frac{c_{2s}}{(2s)!}}.$$

In the case of the Bernoulli random variables, $s = 2$ and $c_4 = -1/12$. In order to have our theorem of L^1-mod-Gaussian convergence to hold, we need to find conditions on the distribution of B_1 such that c_{2s} is negative and that $\int_{\mathbb{R}} \psi_n$ converges to $\int_{\mathbb{R}} \psi$. The conditions in [5, 6] precisely imply these. But within our more general framework, following the discussion in Sect. 1.2, we could well imagine a situation which fulfils the assumptions of Theorem 8 but where the initial symmetric random variables are not necessarily i.i.d but simply independent or even weakly dependent. The following paragraph yields an example of such a setting.

3.4 Mixed Curie-Weiss-Ising Model

Consider the one-dimensional Ising model of parameter $\alpha = 0$, and β arbitrary. We have shown in Sect. 2 the complex mod-Gaussian convergence of $(n^{-1/4} M_n)_{n\in\mathbb{N}}$ with parameters $n^{1/2}\,\mathrm{e}^{2\beta}$ and limiting function $\psi(z) = \exp(-(3\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta})\,z^4/24)$. Restricted to $\mathbb{R}$, this limiting function is integrable, and again one has L^1-mod-convergence. Indeed, recall that

$$\mathbb{E}[\mathrm{e}^{tM_n}] = \frac{Z_n(\mathbb{I},t,\beta)}{Z_n(\mathbb{I},0,\beta)} = \frac{1}{2}\left(a_+(t,\beta)\left(\frac{\lambda_+(t,\beta)}{2\cosh\beta}\right)^{n-1} + a_-(t,\beta)\left(\frac{\lambda_-(t,\beta)}{2\cosh\beta}\right)^{n-1}\right).$$

It will be convenient to work with $n^{-1/4} M_{n+1}$ instead of $n^{-1/4} M_n$ in order to work with n-th powers. Then,

$$\psi_n(t) = \mathbb{E}\left[\mathrm{e}^{t\frac{M_{n+1}}{n^{1/4}}}\right] \mathrm{e}^{-\frac{n^{1/2}\mathrm{e}^{2\beta} z^2}{2}}$$

$$\int_{\mathbb{R}} \psi_n(t)\, dt = \frac{n^{1/4}}{2} \int_{\mathbb{R}} a_+(u,\beta) \left(\frac{\lambda_+(u,\beta)}{2\cosh\beta}\,\mathrm{e}^{-\frac{\mathrm{e}^{2\beta}u^2}{2}}\right)^n + a_-(u,\beta) \left(\frac{\lambda_-(u,\beta)}{2\cosh\beta}\,\mathrm{e}^{-\frac{\mathrm{e}^{2\beta}u^2}{2}}\right)^n du$$

and for every parameter $\beta \geq 0$, the functions

$$u \mapsto \frac{\lambda_+(u,\beta)}{2\cosh\beta}\,\mathrm{e}^{-\frac{\mathrm{e}^{2\beta}u^2}{2}} \quad \text{and} \quad u \mapsto \frac{\lambda_-(u,\beta)}{2\cosh\beta}\,\mathrm{e}^{-\frac{\mathrm{e}^{2\beta}u^2}{2}}$$

attain their unique maximum at $u = 0$, see Fig. 3 for the graph of the first function. Their Taylor expansions at $u = 0$ are respectively

$$1 - \frac{3\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta}}{24}\, u^4 + o(u^4) \quad \text{and} \quad \tanh\beta + o(1),$$

so again by the Laplace method we get $\lim_{n\to\infty} \int_{\mathbb{R}} \psi_n(t)\, dt = \int_{\mathbb{R}} \psi(t)\, dt$ and the L^1-mod-convergence. As a consequence, consider the random configuration of spins σ on $[\![1, n]\!]$ with probability proportional to

$$\exp\left(\beta\left(\sum_{i=1}^{n-1} \sigma(i)\sigma(i+1)\right) + \frac{1}{2n\mathrm{e}^{2\beta}}\left(\sum_{i=1}^{n} \sigma(i)\right)^2\right).$$

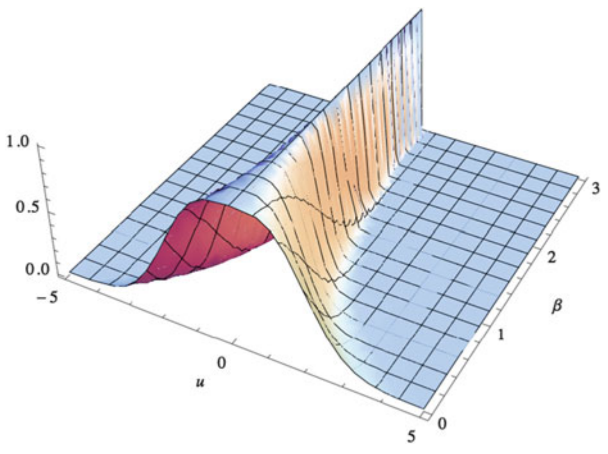


Fig. 3 The function $f(u,\beta) = \frac{\lambda_+(u,\beta)}{2\cosh\beta}\,\mathrm{e}^{-\frac{\mathrm{e}^{2\beta}u^2}{2}}$ (using MATHEMATICA)

This model has a local interaction with coefficient β and a global interaction with coefficient $\frac{1}{e^{2\beta}}$, so it is a mix of the Ising model and of the Curie-Weiss model. The previous discussion and Theorem 8 show that its magnetization satisfies the non standard limit theorem

$$\frac{M_n}{n^{3/4}} \rightharpoonup_{n\to\infty} \frac{\psi(x)\,dx}{\int_{\mathbb{R}} \psi(x)\,dx} \quad \text{with } \psi(x) = \exp\left(-\frac{3e^{6\beta} - e^{2\beta}}{24}\,x^4\right).$$

3.5 Sub-critical Changes of Measures

In the mixed Curie-Weiss-Ising model, one may ask what happens if instead of β and $\frac{1}{e^{2\beta}}$ one puts arbitrary coefficients for the local and the global interaction. More generally, given a sequence $(X_n)_{n\in\mathbb{N}}$ that converges in the L^1-mod-Gaussian sense with parameters t_n and limiting function ψ, one can look at the change of measure

$$\mathbb{Q}_n^{(\gamma)}[dx] = \frac{e^{\frac{\gamma x^2}{2t_n}}}{\mathbb{E}\left[e^{\frac{\gamma (X_n)^2}{2t_n}}\right]}\,\mathbb{P}_n[dx]$$

with $\gamma \in (0,1)$ (for $\gamma > 1$, the change of measure is not necessarily well-defined, since the hypotheses (A) and (B) do not ensure that $\mathbb{E}[e^{\gamma (X_n)^2/2t_n}] < +\infty$). These subcritical changes of measures do not modify the order of magnitude of the fluctuations of X_n, and more precisely:

Theorem 13 *Suppose that $(X_n)_{n\in\mathbb{N}}$ converges in the L^1-mod-Gaussian sense with parameters t_n and limiting function ψ. Then, if $(X_n^{(\gamma)})_{n\in\mathbb{N}}$ is a sequence of random variables under the new probability measures $\mathbb{Q}_n^{(\gamma)}$, it converges in the L^1-mod-Gaussian sense with parameters $\frac{t_n}{1-\gamma}$ and limit $t \mapsto \psi\left(\frac{t}{1-\gamma}\right)$.*

Example 14 Consider a random configuration of spins σ on $[\![1,n]\!]$ with probability proportional to

$$\exp\left(\beta\left(\sum_{i=1}^{n-1}\sigma(i)\sigma(i+1)\right) + \frac{\gamma}{2n}\left(\sum_{i=1}^{n}\sigma(i)\right)^2\right),$$

with $\gamma < e^{-2\beta}$. The total magnetization of the system has order of magnitude $n^{1/2}$, and more precisely, one has the central limit theorem

$$\frac{M_n}{n^{1/2}} \rightharpoonup_{n\to\infty} \mathcal{N}\left(0, \frac{e^{2\beta}}{1-\gamma e^{2\beta}}\right),$$

and in fact a L^1-mod-Gaussian convergence of $\frac{M_n}{n^{1/4}}$, with parameters $n^{1/2}\frac{e^{2\beta}}{1-\gamma e^{2\beta}}$ and limiting function

$$\psi(x) = \exp\left(-\frac{(3e^{6\beta} - e^{2\beta})\, x^4}{24(1-\gamma e^{2\beta})^4}\right).$$

In particular, if $\beta = 0$ and one considers the Curie-Weiss model at high temperatures ($\gamma < e^{-2\times 0} = 1$), then there is a L^1-mod-Gaussian convergence of $\frac{\sqrt{1-\gamma}\, M_n}{n^{1/4}}$ with parameters $n^{1/2}$ and limiting function

$$\psi(x) = \exp\left(-\frac{x^4}{12(1-\gamma)^2}\right).$$

Proof (of Theorem 13) We denote as before $(Y_n)_{n\in\mathbb{N}}$ a sequence of random variables under the laws $\mathbb{Q}_n = \mathbb{Q}_n^{(1)}$. We first compute the asymptotics of $Z_n^{(\gamma)} = \mathbb{E}[e^{\gamma (X_n)^2/2t_n}]$:

$$\begin{aligned} Z_n^{(\gamma)} &= Z_n\; \mathbb{E}[e^{-(1-\gamma)(Y_n)^2/2t_n}] \\ &= \sqrt{\frac{t_n}{2\pi}}\left(\int_{\mathbb{R}} \psi(x)\,dx\right)\, \mathbb{E}\left[e^{-\frac{t_n(1-\gamma)}{2}(\frac{Y_n}{t_n})^2}\right](1+o(1)) \\ &= \sqrt{\frac{t_n}{2\pi}}\left(\int_{\mathbb{R}} \psi(x)\,dx\right)\, \mathbb{E}\left[e^{-\frac{t_n(1-\gamma)}{2}(W_n)^2}\right](1+o(1)) \\ &= \sqrt{\frac{1}{1-\gamma}}\,(1+o(1)) \end{aligned}$$

by using on the third line an integration by parts as in the proof of Proposition 9 to replace $\frac{Y_n}{t_n}$ by W_n; and the Laplace method on the fourth line to compute $\int_{\mathbb{R}} e^{-t_n(1-\gamma)x^2/2}\,\psi_n(x)\,dx$. The same computations give the asymptotics of

$$\begin{aligned} \mathbb{E}[e^{tX_n+\gamma(X_n)^2/2t_n}] &= Z_n\; \mathbb{E}[e^{tY_n-(1-\gamma)(Y_n)^2/2t_n}] \\ &= \sqrt{\frac{t_n}{2\pi}}\left(\int_{\mathbb{R}} \psi(x)\,dx\right)\, \mathbb{E}\left[e^{t_n t\,(\frac{Y_n}{t_n})-\frac{t_n(1-\gamma)}{2}(\frac{Y_n}{t_n})^2}\right](1+o(1)) \\ &= \sqrt{\frac{t_n}{2\pi}}\left(\int_{\mathbb{R}} \psi(x)\,dx\right)\, \mathbb{E}\left[e^{t_n t\,W_n-\frac{t_n(1-\gamma)}{2}(W_n)^2}\right](1+o(1)) \\ &= e^{\frac{t_n t^2}{2(1-\gamma)}}\sqrt{\frac{1}{1-\gamma}}\;\psi\left(\frac{t}{1-\gamma}\right)\,(1+o(1)) \end{aligned}$$

with again a Laplace method on the fourth line. Since

$$\mathbb{E}[\mathrm{e}^{tX_n^{(\gamma)}}] = \frac{\mathbb{E}[\mathrm{e}^{tX_n+\gamma(X_n)^2/2t_n}]}{Z_n^{(\gamma)}},$$

this shows the hypotheses (A) and (B) for the sequence $(X_n^{(\gamma)})_{n\in\mathbb{N}}$, with parameters $\frac{t_n}{1-\gamma}$, and limiting function $\psi(\frac{t}{1-\gamma})$. Then, since $(Y_n/t_n)_{n\in\mathbb{N}}$ converges in law, by using the implication (ii) $\Rightarrow$ (iii) in Theorem 8 for the sequence $(X_n^{(\gamma)})_{n\in\mathbb{N}}$, we see that the mod-Gaussian convergence of Laplace transforms necessarily happens in $\mathrm{L}^1(\mathbb{R})$. □

3.6 Random Walks Changed in Measure

In this section, we shall make a brief excursion in the higher dimensions. Since we do not want to enter details on mod-Gaussian convergence for random vectors (for which we refer the reader to [8, 13]), we shall only consider the simple case $X = (X^{(1)}, \ldots, X^{(d)})$ is a random vector with values in $\mathbb{R}^d$ such that $\mathbb{E}[\exp(z_1 X^{(1)} + \cdots + z_d X^{(d)})]$ is entire in $\mathbb{C}^d$. We shall say that the sequence (X_n) of random vectors converges in the complex mod-Gaussian sense with parameter t_n and limiting function $\psi(z_1, \cdots, z_d)$ if the following convergence holds locally uniformly on compact subsets of $\mathbb{C}^d$:

$$\psi_n(t) = \mathbb{E}[\exp(z_1 X_n^{(1)} + \cdots + z_d X_n^{(d)})] \exp\left(-t_n \frac{(z_1)^2 + \cdots + (z_d)^2}{2}\right) \to \psi(z_1, \ldots, z_d).$$

In this vector setting, the assumptions (A) and (B) of Sect. 3 now simply amount to the fact that the convergence above holds locally uniformly for $t = (t^{(1)}, \cdots, t^{(d)}) \in \mathbb{R}^d$ and that ψ_n and ψ are both in $\mathrm{L}^1(\mathbb{R}^d)$.

Following the case $d = 1$ we denote $\mathbb{P}_n$ the law of X_n on $\mathbb{R}^d$,

$$\mathbb{Q}_n[dx] = \frac{\mathrm{e}^{\frac{\|x\|^2}{2t_n}}}{\mathbb{E}\left[\mathrm{e}^{\frac{\|X_n\|^2}{2t_n}}\right]} \, \mathbb{P}_n[dx],$$

and Y_n a random variable under the new law $\mathbb{Q}_n$. Note that here again hypothesis (3) implies that $Z_n = \mathbb{E}[\mathrm{e}^{\|X_n\|^2/2t_n}]$ is finite for all $n \in \mathbb{N}$. Indeed, with the notation $\langle u, v\rangle = u_1 v_1 + \cdots + u_d v_d$, we have

$$\int_{\mathbb{R}^d} \psi_n(t)\, dt = \mathbb{E}\left[\int_{\mathbb{R}^d} \mathrm{e}^{\langle t, X_n\rangle - \frac{t_n \|t\|^2}{2}}\, dt\right] = \mathbb{E}\left[\mathrm{e}^{\frac{(\|X_n\|^2}{2t_n}} \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\|X_n - t_n t\|^2}{2t_n}}\, dt\right)\right]$$
$$= \left(\frac{2\pi}{t_n}\right)^{d/2} \mathbb{E}\left[\mathrm{e}^{\frac{\|X_n\|^2}{2t_n}}\right].$$

Therefore, the new probabilities $\mathbb{Q}_n$ are well-defined and

$$Z_n = \mathbb{E}[\mathrm{e}^{\|X_n\|^2/2t_n}] = \left(\frac{t_n}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} \psi_n(t)\, dt.$$

Then it is clear that Proposition 6 holds with G_n being a Gaussian vector with covariance matrix $1/t_n\ I_d$ where I_d is the identity matrix of size d. Similarly one can establish an analogue of Theorem 8 in $\mathbb{R}^d$.

Let W_n be a simple random walk on the lattice $\mathbb{Z}^{d\geq 2}$: at each step, each of the $2d$ neighbors of the state that is occupied has the same probability of transition $(2d)^{-1}$. The d-dimensional characteristic function of $W_n = (W_n^{(1)}, \dots, W_n^{(d)})$ is

$$\mathbb{E}[\mathrm{e}^{z_1 W_n^{(1)} + \cdots + z_d W_n^{(d)}}] = \left(\frac{\cosh z_1 + \cdots + \cosh z_d}{d}\right)^n.$$

Therefore, one has the asymptotics

$$\begin{aligned}
&\log \mathbb{E}\left[\mathrm{e}^{\frac{z_1 W_n^{(1)} + \cdots + z_d W_n^{(d)}}{n^{1/4}}}\right] \\
&\quad = n \log\left(1 + \frac{(z_1)^2 + \cdots + (z_d)^2}{2dn^{1/2}} + \frac{(z_1)^4 + \cdots + (z_d)^4}{24dn} + o\left(\frac{1}{n}\right)\right) \\
&\quad = n^{1/2}\, \frac{(z_1)^2 + \cdots + (z_d)^2}{2d} - \frac{3((z_1)^2 + \cdots + (z_d)^2)^2 - d((z_1)^4 + \cdots + (z_d)^4)}{24d^2} \\
&\qquad + o(1).
\end{aligned}$$

One obtains a d-dimensional complex mod-Gaussian convergence of $X_n = n^{-1/4}\, W_n$ with parameters $\frac{n^{1/2}}{d}$ and limiting function

$$\psi(z_1, \dots, z_d) = \exp\left(-\frac{3\,((z_1)^2 + \cdots + (z_d)^2)^2 - d\,((z_1)^4 + \cdots + (z_d)^4)}{24d^2}\right).$$

In [8], we used this mod-convergence to prove quantitative estimates regarding the breaking of the radial symmetry when one considers random walks conditioned to be of large size (of order $n^{3/4}$ instead of the expected order $n^{1/2}$). With the notion of L^1-mod-Gaussian convergence, one can give another interpretation, but only for $d = 2$ or $d = 3$. Restricted to $\mathbb{R}^d$, the limiting function is indeed not integrable for $d \geq 4$: if $t_2, \dots, t_d \in [-1, 1]$, then

$$\begin{aligned}
3\,((t_1)^2 + \cdots + (t_d)^2)^2 - d\,((t_1)^4 + \cdots + (t_d)^4) &\leq 3\,((t_1)^2 + (d-1))^2 - d\,(t_1)^4 \\
&\leq (3-d)(t_1)^4 + 6(d-1)(t_1)^2 \\
&\quad + 3(d-1)^2.
\end{aligned}$$

So, restricted to the domain $\mathbb{R} \times [-1,1]^{d-1}$, $\psi(t_1,\ldots,t_d) \leq K\,\exp(a(t_1)^4 - b(t_1)^2)$ for some positive constants a, b and K; therefore, this function is not integrable.

On the other hand, if $d = 2$ or $d = 3$, then ψ is integrable on $\mathbb{R}^d$, and one has L^1-mod-Gaussian convergence. Indeed, when $d = 2$, the limiting function is

$$\psi(t_1,t_2) = \exp\left(-\frac{(t_1)^4 + (t_2)^4 + 6(t_1t_2)^2}{96}\right), \tag{10}$$

which is clearly integrable; and the residues

$$\psi_n(t_1,t_2) = \mathbb{E}\left[\mathrm{e}^{\frac{t_1 W_n^{(1)} + t_2 W_n^{(2)}}{n^{1/4}}}\right]\mathrm{e}^{-\frac{n^{1/2}((t_1)^2+(t_2)^2)}{4}}$$

converge locally uniformly on $\mathbb{R}^2$ to $\psi(t_1,t_2)$, but also in $\mathrm{L}^1(\mathbb{R}^2)$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^2} \psi_n(t_1,t_2)\,dt_1\,dt_2 &= \int_{\mathbb{R}^2}\left(\frac{\cosh\frac{t_1}{n^{1/4}} + \cosh\frac{t_2}{n^{1/4}}}{2}\right)^n \mathrm{e}^{-\frac{n^{1/2}((t_1)^2+(t_2)^2)}{4}}\,dt_1\,dt_2\\ &= n^{1/2}\int_{\mathbb{R}^2}\left(\frac{\cosh u_1 + \cosh u_2}{2}\,\mathrm{e}^{-\frac{(u_1)^2+(u_2)^2}{4}}\right)^n du_1\,du_2,\end{aligned}$$

and the function $(u_1,u_2) \mapsto \frac{\cosh u_1 + \cosh u_2}{2}\,\mathrm{e}^{-\frac{(u_1)^2+(u_2)^2}{4}}$ reaches its unique global maximum at $u_1 = u_2 = 0$, with Taylor expansion

$$1 - \frac{(u_1)^4 + (u_2)^4 + 6(u_1u_2)^2}{96} + o(\|u\|^4)$$

around this point (see Fig. 4).

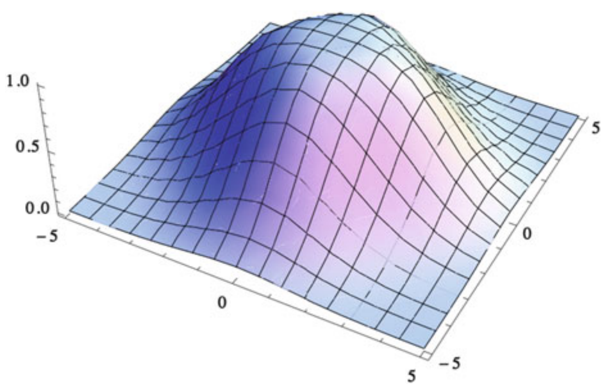


Fig. 4 The function $f(u_1,u_2) = \frac{\cosh u_1 + \cosh u_2}{2}\,\mathrm{e}^{-\frac{(u_1)^2+(u_2)^2}{4}}$

Thus, by using the multi-dimensional Laplace method, the limit of the integral $\int_{\mathbb{R}^2} \psi_n(t_1, t_2)\, dt_1\, dt_2$ is $\int_{\mathbb{R}^2} \psi(t_1, t_2)\, dt_1\, dt_2$, and the L^1 convergence is shown. Similarly, when $d = 3$, the limiting function is

$$\psi(t_1, t_2, t_3) = \exp\left(-\frac{(t_1t_2)^2 + (t_1t_3)^2 + (t_2t_3)^2}{36}\right), \tag{11}$$

and the following computation shows that it is integrable:

$$\begin{aligned}\int_{\mathbb{R}^3} \psi(x, y, z)\, dx\, dy\, dz &= \int_{\mathbb{R}^2} \mathrm{e}^{-\frac{(yz)^2}{36}} \left(\int_{\mathbb{R}} \mathrm{e}^{-\frac{y^2+z^2}{36}x^2} dx\right) dy\, dz \\ &= 6\sqrt{\pi} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{-\frac{(yz)^2}{36}}}{\sqrt{y^2+z^2}}\, dy\, dz \\ &= 3\sqrt{\pi} \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \mathrm{e}^{-\frac{r^4 \sin^2\theta}{144}}\, dr\, d\theta \\ &= 12\sqrt{3\pi} \int_{r=0}^{\infty} \mathrm{e}^{-r^4}\, dr \int_{\theta=0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} < +\infty\end{aligned}$$

since $\frac{1}{\sqrt{\sin\theta}}$ is integrable at 0. On the other hand, the residues

$$\psi_n(t_1, t_2, t_3) = \mathbb{E}\left[\mathrm{e}^{\frac{t_1 W_n^{(1)} + t_2 W_n^{(2)} + t_3 W_n^{(3)}}{n^{1/4}}}\right] \mathrm{e}^{-\frac{n^{1/2}((t_1)^2+(t_2)^2+(t_3)^2)}{6}}$$

converge to $\psi(t_1, t_2, t_3)$ locally uniformly on $\mathbb{R}^3$ and in $L^1(\mathbb{R}^3)$. Indeed, one has again

$$\int_{\mathbb{R}^3} \psi_n(t_1, t_2, t_3)\, dt = n^{1/2} \int_{\mathbb{R}^3} \left(\frac{\cosh u_1 + \cosh u_2 + \cosh u_3}{3}\, \mathrm{e}^{-\frac{(u_1)^2+(u_2)^2+(u_3)^2}{6}}\right)^n du$$

and the function in the brackets reaches its unique maximum at $u_1 = u_2 = u_3 = 0$, with Taylor expansion corresponding to the limiting function ψ after application of the Laplace method.

The multidimensional analogue of Theorem 8 thus yields the following multidimensional extension of the limit theorem for the Curie-Weiss model:

Theorem 15 *Let W_n be a simple random walk in dimension $d \leq 3$. If V_n is obtained from W_n by a change of measure by the factor* $\exp(d\, \|W_n\|^2/2n)$, *then*

$$\frac{V_n}{n^{3/4}} \rightharpoonup_{n\to\infty} \frac{\psi(x)\, dx}{\int_{\mathbb{R}^3} \psi(x)\, dx},$$

where $\psi(x) = \exp(-x^4/12)$ *in dimension* 1, *and* ψ *is given by Formulas* (10) *and* (11) *in dimension* 2 *and* 3.

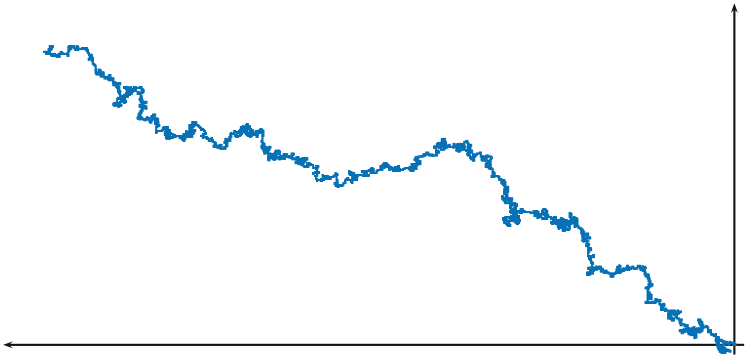


Fig. 5 A two-dimensional random walk changed in measure by $e^{\|W_n\|^2/n}$, here with $n = 10{,}000$

Remark 16 Suppose $d = 2$. Then, there is a limit in law not only for $\frac{V_n}{n^{3/4}}$, but in fact for the whole random walk $(\frac{V_k}{n^{3/4}})_{k\leq n}$, viewed as a random element of $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^2)$ or of the Skorohod space $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^2)$, see Fig. 5.

4 Local Limit Theorem and Rate of Convergence in the Ellis-Newman Limit Theorem

We keep the same notation as before and note $I_n = \int_{\mathbb{R}} \psi_n(x)\, dx$ and $I_\infty = \int_{\mathbb{R}} \psi(x)\, dx$.

In this section we wish to provide a quick approach based on Fourier analysis,

1. to compute the Kolmogorov distance between the rescaled magnetization $Y_n/n^{1/2} = M_n/n^{3/4}$ in the Curie-Weiss model and the random variable W_∞ with density $\psi(x)/I_\infty$, where $\psi(x) = \exp(-x^4/12)$. This problem was recently solved in [4] using Stein's method. As in [4], our method would cover many more general models as well: it is just a matter of specializing Lemmas 17 and 18 below which are stated in all generality.
2. to prove a new local limit theorem for the rescaled magnetization $n^{-1/4}M_n$ in the Curie-Weiss model. Here again we shall indicate how one can establish local limit theorems in more general situations.

4.1 Speed of Convergence

Getting back to our special case of the Curie-Weiss model, we denote $X_n = \frac{1}{n^{1/4}} \sum_{i=1}^{n} B_i$ a scaled sum of ± 1 independent Bernoulli random variables; Y_n the random variable with modified law

$$\mathbb{Q}_n[dy] = \frac{e^{\frac{y^2}{2n^{1/2}}}\,\mathbb{P}_n[dy]}{\mathbb{E}\left[e^{\frac{(X_n)^2}{2n^{1/2}}}\right]};$$

G_n an independent Gaussian random variable of variance $\frac{1}{n^{1/2}}$; and $W_n = \frac{Y_n}{n^{1/2}} + G_n$. It follows from the previous results that the law of W_n has density

$$\frac{\psi_n(x)}{I_n} = \frac{1}{I_n}\,e^{-\frac{n^{1/2}x^2}{2}}\,\left(\cosh\frac{x}{n^{1/4}}\right)^n,$$

which converges in L^1 towards the law $\frac{\psi(x)}{I_\infty} = \frac{1}{I_\infty}\,e^{-\frac{x^4}{12}}$. We hence wish for an upper bound for the Kolmogorov distance between $\frac{Y_n}{n^{1/2}}$ and W_∞. For this we shall need the following general lemmas.

Lemma 17 *Consider the two distributions* $W_n = \frac{\psi_n(x)\,dx}{I_n}$ *and* $W_\infty = \frac{\psi(x)\,dx}{I_\infty}$. *The Kolmogorov distance between them is smaller than*

$$\frac{\|\psi - \psi_n\|_{L^1}}{I_\infty}\,(1 + o(1)).$$

Proof Fix $a \in \mathbb{R}$, and suppose for instance that $\int_{\mathbb{R}} \psi(x)\,dx \geq \int_{\mathbb{R}} \psi_n(x)\,dx$. We have

$$\begin{aligned}
F_{W_n}(a) - F_{W_\infty}(a) &= \left(\frac{\int_{-\infty}^a \psi_n(x)\,dx}{I_n} - \frac{\int_{-\infty}^a \psi(x)\,dx}{I_n}\right) \\
&\quad + \left(\frac{\int_{-\infty}^a \psi(x)\,dx}{I_n} - \frac{\int_{-\infty}^a \psi(x)\,dx}{I_\infty}\right) \\
&= \frac{\int_{-\infty}^a (\psi_n(x) - \psi(x))\,dx}{I_n} \\
&\quad + \left(\int_{-\infty}^a \psi(x)\,dx\right)\frac{\int_{-\infty}^{\infty} (\psi(x) - \psi_n(x))\,dx}{I_\infty I_n} \\
&\leq -\frac{\int_{-\infty}^a (\psi(x) - \psi_n(x))\,dx}{I_n} + \frac{\int_{-\infty}^{\infty} (\psi(x) - \psi_n(x))\,dx}{I_n} \\
&\leq \frac{\int_a^{\infty} (\psi(x) - \psi_n(x))\,dx}{I_n} \leq \frac{\|\psi - \psi_n\|_{L^1}}{I_n}.
\end{aligned}$$

Writing $F_{W_\infty}(a) - F_{W_n}(a) = (1 - F_{W_n}(a)) - (1 - F_{W_\infty}(a))$, one sees that the inequality is in fact valid with an absolute value on the left-hand side. Since

$I_n = I_\infty(1 + o(1))$, this shows the claim. If $\int_{\mathbb{R}} \psi_n(x)\,dx \geq \int_{\mathbb{R}} \psi(x)\,dx$, it suffices to exchange the roles played by ψ_n and ψ to get the inequality. □

The asymptotics of the L^1-norm $\|\psi - \psi_n\|_{\mathrm{L}^1}$ in the Curie-Weiss model are computed as follows. Noting that one always has $\psi_n(x) \geq \psi(x)$, it suffices to compute

$$\int_{\mathbb{R}} \psi_n(x)\,dx = \int_{\mathbb{R}} \mathrm{e}^{-\frac{n^{1/2}x^2}{2}} \left(\cosh(x\,n^{-1/4})\right)^n dx = n^{1/4} \int_{\mathbb{R}} \left(\mathrm{e}^{-\frac{u^2}{2}} \cosh(u)\right)^n du.$$

By the Laplace method (see [22, Formula (19.17), pp. 624–625]), the asymptotics of the integral is

$$n^{-\frac{1}{4}} \left(\frac{12^{1/4}\,\Gamma(\frac{1}{4})}{2}\right) + n^{-\frac{3}{4}} \left(\frac{12^{3/4}\,\Gamma(\frac{3}{4})}{10}\right) + \text{smaller terms.}$$

The first term corresponds to $I_\infty = \int_{\mathbb{R}} \psi(x)\,dx = \int_{\mathbb{R}} \mathrm{e}^{-x^4/12}\,dx$. As a consequence,

$$\frac{\|\psi - \psi_n\|_{\mathrm{L}^1}}{I_\infty} = \frac{1}{n^{1/2}}\,\frac{\sqrt{12}\,\Gamma(\frac{3}{4})}{5\,\Gamma(\frac{1}{4})}\,(1 + o(1)).$$

The main work now consists in computing $d_{\mathrm{Kol}}(\frac{Y_n}{n^{1/2}}, W_n)$. We start by a Lemma which is a variation of arguments used for i.i.d. random variables in [21, p. 87]. In the following, given a function $f \in \mathrm{L}^1(\mathbb{R})$, we write its Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\,\mathrm{e}^{\mathrm{i}\xi x}\,dx$. Recall that the function

$$\upsilon(\xi) = \begin{cases} \mathrm{e}^{-\frac{1}{1-4\xi^2}} & \text{if } |\xi| < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

is even, of class $\mathcal{C}^\infty$ and with compact support $[-\frac{1}{2}, \frac{1}{2}]$. We set $\widehat{\rho_\star} = \upsilon$, so that

$$\rho_\star(x) = \frac{1}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \upsilon(\xi)\,\mathrm{e}^{-\mathrm{i}x\xi}\,d\xi$$

by the Fourier inversion theorem. By construction, the Fourier transform of $\rho_\star$ has support equal to $[-\frac{1}{2}, \frac{1}{2}]$. Set now

$$\rho(x) = \frac{(\rho_\star(x))^2}{\int_{\mathbb{R}} (\rho_*(y))^2\,dy}.$$

By construction, ρ is smooth, even, non-negative and with integral equal to 1. Moreover, $\widehat{\rho}$ is up to a constant equal to $\nu * \nu(\xi)$, so it has support included into $[-1, 1]$. The convolution of ρ with characteristic functions of intervals will allow us to transform estimates on test functions into estimates on cumulative distribution functions. More precisely, for $a \in \mathbb{R}$ and $\varepsilon > 0$, set $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\,\rho(\frac{x}{\varepsilon})$, and $\phi_{a,\varepsilon}(x) = \phi_\varepsilon(x-a)$, where ϕ_ε is the function $1_{(-\infty,0]} * \rho_\varepsilon$. One sees $\phi_{a,\varepsilon}$ as a smooth approximation of the characteristic function $1_{(-\infty,a]}$.

For all a, ε, $\phi_{a,\varepsilon}$ has Fourier transform compactly supported on $\left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right]$. Moreover, it has negative derivative, and decreases from 1 to 0. Later, we will use the identity

$$\phi_\varepsilon(\varepsilon x) = \phi_1(x) = \phi(x).$$

On the other hand, we have the following estimates for $K > 0$ (we used SAGE for numerical computations):

$$|\rho_*(K)| = \frac{1}{2\pi K^2}\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} \upsilon''(\xi)\,\mathrm{e}^{-\mathrm{i}K\xi}\,d\xi\right| \le \frac{1}{2\pi K^2}\int_0^{\frac{1}{2}} |\upsilon''(\xi)|\,d\xi = \frac{1.0166_-}{K^2};$$

$$\int_{\mathbb{R}} (\rho_*(y))^2\,dy = \frac{1}{2\pi}\int_0^{\frac{1}{2}} |\upsilon(\xi)|^2\,d\xi = 0.01059_+.$$

Therefore, for any $K > 0$,

$$\rho(K) = \rho(-K) = \frac{(\rho_*(K))^2}{\int_{\mathbb{R}}(\rho_*(y))^2\,dy} \le \frac{99}{K^4};$$

$$\phi(K) = 1 - \phi(-K) = \int_0^\infty \rho(K+y)\,dy \le \frac{33}{K^3}.$$

Lemma 18 *Let V and W be two random variables with cumulative distribution functions F_V and F_W. Assume that for some $\varepsilon > 0$*

$$|\mathbb{E}[\phi_{a,\varepsilon}(V)] - \mathbb{E}[\phi_{a,\varepsilon}(W)]| \le B\varepsilon,$$

where the positive constant B is independent of a. We also suppose that W has a density w.r.t. Lebesgue measure that is bounded by m. Then,

$$\sup_{a\in\mathbb{R}} |F_V(a) - F_W(a)| \le 2(B + 10m)\,\varepsilon.$$

Proof Fix a positive constant K, and denote $\delta = \sup_{a\in\mathbb{R}} |F_V(a) - F_W(a)|$ the Kolmogorov distance between V and W. One has

$$\begin{aligned} F_V(a) = \mathbb{E}[1_{V\le a}] &\le \mathbb{E}[\phi_{a+K\varepsilon,\varepsilon}(V)] + \mathbb{E}[(1-\phi_{a+K\varepsilon,\varepsilon}(V))\,1_{V\le a}] \\ &\le \mathbb{E}[\phi_{a+K\varepsilon,\varepsilon}(W)] + \mathbb{E}[(1-\phi_{a+K\varepsilon,\varepsilon}(V))\,1_{V\le a}] + B\varepsilon. \end{aligned}$$

The second expectation writes as

$$\begin{aligned}\mathbb{E}[(1-\phi_{a+K\varepsilon,\varepsilon}(V))\,1_{V\leq a}] &= \int_{\mathbb{R}} (1-\phi_{a+K\varepsilon,\varepsilon}(x))\,1_{(-\infty,a]}(x) f_V(x)\,dx \\ &= -\int_{\mathbb{R}} ((1-\phi_{a+K\varepsilon,\varepsilon}(x))\,1_{(-\infty,a]}(x))'\,F_V(x)\,dx \\ &= \int_{\mathbb{R}} \phi'_{a+K\varepsilon,\varepsilon}(x)\,1_{(-\infty,a]}(x)\,F_V(x)\,dx \\ &\quad + \int_{\mathbb{R}} (1-\phi_{a+K\varepsilon,\varepsilon}(x))\,1_a(x)\,F_V(x)\,dx.\end{aligned}$$

For the first integral, since $F_V(x) \geq F_W(x) - \delta$ and the derivative of $\phi_{a+K\varepsilon,\varepsilon}$ is negative, an upper bound on I_1 is

$$\begin{aligned}&\int_{\mathbb{R}} \phi'_{a+K\varepsilon,\varepsilon}(x)\,1_{(-\infty,a]}(x)\,F_W(x)\,dx - \delta \int_{\mathbb{R}} \phi'_{a+K\varepsilon,\varepsilon}(x)\,1_{V\leq a}(x) \\ &\quad = \int_{\mathbb{R}} \phi'_{a+K\varepsilon,\varepsilon}(x)\,1_{(-\infty,a]}(x)\,F_W(x)\,dx + (1-\phi_{a+K\varepsilon,\varepsilon}(a))\,\delta \\ &\quad = \int_{\mathbb{R}} \phi'_{a+K\varepsilon,\varepsilon}(x)\,1_{(-\infty,a]}(x)\,F_W(x)\,dx + (1-\phi(-K))\,\delta.\end{aligned}$$

As for the second integral, it is simply $(1-\phi_{a+K_\varepsilon,\varepsilon}(a))F_V(a)$, and by writing $F_V(a) \leq F_W(a) + \delta$, one gets the upper bound on I_2

$$\begin{aligned}&\int_{\mathbb{R}} (1-\phi_{a+K\varepsilon,\varepsilon}(x))\,1_a(x)\,F_W(x)\,dx + (1-\phi_{a+K\varepsilon,\varepsilon}(a))\,\delta \\ &\quad = \int_{\mathbb{R}} (1-\phi_{a+K\varepsilon,\varepsilon}(x))\,1_a(x)\,F_W(x)\,dx + (1-\phi(-K))\,\delta.\end{aligned}$$

One concludes that

$$\mathbb{E}[(1-\phi_{a+K\varepsilon,\varepsilon}(V))\,1_{V\leq a}] \leq \mathbb{E}[(1-\phi_{a+K\varepsilon,\varepsilon}(W))\,1_{W\leq a}] + 2(1-\phi(-K))\delta.$$

On the other hand, if m is a bound on the density f_W of W, then

$$\begin{aligned}\mathbb{E}[\phi_{a+K\varepsilon,\varepsilon}(W)\,1_{W\geq a}] &= \int_a^\infty \phi_{a+K\varepsilon,\varepsilon}(y) f_W(y)\,dy \\ &\leq m \int_a^\infty \phi_\varepsilon(y-a-K\varepsilon)\,dy = m\int_0^\infty \phi_\varepsilon(y-K\varepsilon)\,dy \\ &\leq m\varepsilon \int_0^\infty \phi(u-K)\,du \leq m\varepsilon\,(K+4.82)\,,\end{aligned}$$

by using on the last line the bound $\phi(x) \leq \frac{33}{x^3}$. As a consequence,

$$\mathbb{E}[\phi_{a+K\varepsilon,\varepsilon}(W)] \leq \mathbb{E}[\phi_{a+K\varepsilon,\varepsilon}(W)\, 1_{W\leq a}] + m\,(K+4.82)\,\varepsilon$$
$$F_V(a) \leq F_W(a) + (B + m\,(K+4.82))\,\varepsilon + 2\,\frac{33}{K^3}\,\delta.$$

Similarly, $F_V(a) \geq F_W(a) - (B + m(K+4.82))\varepsilon - 2\,\frac{33}{K^3}\,\delta$, so in the end

$$\delta = \sup_{a\in\mathbb{R}} |F_V(a) - F_W(a)| \leq (B + m\,(K+4.82))\,\varepsilon + \frac{66}{K^3}\,\delta.$$

As this is true for every K, one can for instance take $K = \sqrt[3]{132}$, which gives

$$\delta \leq \frac{1}{1-\frac{1}{2}}\left(B + m\left(\sqrt[3]{132} + 4.82\right)\right)\varepsilon \leq 2(B + 10m)\,\varepsilon. \qquad \square$$

We are going to apply Lemma 18 with $V = \frac{Y_n}{n^{1/2}}$ and $W = W_n$. First, notice that a bound on the density of W_n is

$$\frac{|\psi_n(x)|}{I_n} \leq \frac{1}{I_\infty} = \frac{2}{12^{1/4}\,\Gamma(\frac{1}{4})} = m.$$

On the other hand, using the Fourier transform of the Heaviside function

$$\hat{1}_{(-\infty,a]}(\xi) = \mathrm{e}^{\mathrm{i}a\xi}\left(\pi\delta_0(\xi) + \frac{\mathrm{i}}{\xi}\right),$$

we get

$$\mathbb{E}\left[\phi_{a,\varepsilon}\left(\frac{Y_n}{n^{1/2}}\right)\right] - \mathbb{E}[\phi_{a,\varepsilon}(W_n)] = \frac{1}{2\pi I_n}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \widehat{\phi_{a,\varepsilon}}(\xi)\,\widehat{\psi_n}(\xi)\left(\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}} - 1\right)d\xi$$
$$= \frac{1}{2\pi I_n}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \widehat{\rho_\varepsilon}(\xi)\,\mathrm{e}^{\mathrm{i}a\xi}\left(\frac{\mathrm{i}}{\xi}\right)\widehat{\psi_n}(\xi)\left(\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}} - 1\right)d\xi;$$
$$\left|\mathbb{E}\left[\phi_{a,\varepsilon}\left(\frac{Y_n}{n^{1/2}}\right)\right] - \mathbb{E}[\phi_{a,\varepsilon}(W_n)]\right| \leq \frac{1}{2\pi I_n\, n^{1/2}}\int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} |\hat{\rho}(\varepsilon\xi)|\,|\widehat{\psi_n}(\xi)|\,\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}}\,d\xi$$

by controlling $\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}} - 1$ by its first derivative (notice that we used the vanishing of this quantity at $\xi = 0$ in order to compensate the singularity of the Fourier transform of the Heavyside distribution). Since $\|\hat{\rho}\|_{\mathrm{L}^\infty} = \|\rho\|_{\mathrm{L}^1} = 1$, the previous bound can be rewritten as

$$\left|\mathbb{E}\left[\phi_{a,\varepsilon}\left(\frac{Y_n}{n^{1/2}}\right)\right] - \mathbb{E}[\phi_{a,\varepsilon}(W_n)]\right| \leq \frac{1}{2\pi I_n\, n^{1/2}} \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} |\widehat{\psi_n}(\xi)|\, \mathrm{e}^{\frac{\xi^2}{2n^{1/2}}}\, d\xi.$$

We then need estimates on the Fourier transform of $\widehat{\psi_n}$, and more precisely estimates of exponential decay. To this purpose, we use the following Lemma, which is related to [17, Theorem IX.13, p. 18]:

Lemma 19 *Let f be a function which is analytic on a band $\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < c\}$. For any $b \in (0, c)$,*

$$|\hat{f}(\xi)| \leq 2 \left(\sup_{-b\leq a\leq b} \|f(\cdot + \mathrm{i}a)\|_{\mathrm{L}^1} \right) \mathrm{e}^{-b|\xi|},$$

assuming that the supremum is finite.

Proof Notice that the Fourier transform of $\tau_a f(\cdot) = f(\cdot + \mathrm{i}a)$ is

$$\int_{\mathbb{R}} \tau_a f(x)\, \mathrm{e}^{\mathrm{i}x\xi}\, dx = \int_{\mathbb{R}} f(x+\mathrm{i}a)\, \mathrm{e}^{\mathrm{i}x\xi}\, dx = \left(\int_{\mathbb{R}} f(x+\mathrm{i}a)\, \mathrm{e}^{\mathrm{i}(x+\mathrm{i}a)\xi}\, dx \right) \mathrm{e}^{a\xi}.$$

By analyticity of the function in the integral, using Cauchy's integral formula, one sees that the last term is also

$$\left(\int_{\mathbb{R}} f(x)\, \mathrm{e}^{\mathrm{i}x\xi}\, dx \right) \mathrm{e}^{a\xi} = \hat{f}(\xi)\, \mathrm{e}^{a\xi},$$

(see the details on page 132 of the book by Reed and Simon). It follows that

$$|\hat{f}(\xi)|\, \mathrm{e}^{a|\xi|} \leq |\hat{f}(\xi)|\, (\mathrm{e}^{a\xi} + \mathrm{e}^{-a\xi}) \leq |\widehat{\tau_a f}(\xi)| + |\widehat{\tau_{-a} f}(\xi)| \leq \|\tau_a f\|_{\mathrm{L}^1} + \|\tau_{-a} f\|_{\mathrm{L}^1}. \qquad \square$$

Thus we need to compute for $a > 0$ the L^1-norm of $\psi_n(\cdot + \mathrm{i}a)$. We write

$$\begin{aligned} |\psi_n(x+\mathrm{i}a)| &= \mathrm{e}^{-\frac{n^{1/2}(x^2-a^2)}{2}} \left| \cosh\left(\frac{x+\mathrm{i}a}{n^{1/4}}\right)\right|^n \\ &= |\psi_n(x)|\, \mathrm{e}^{\frac{n^{1/2}a^2}{2}} \left| \cos^2\left(\frac{a}{n^{1/4}}\right) + \tanh^2\left(\frac{x}{n^{1/4}}\right) \sin^2\left(\frac{a}{n^{1/4}}\right)\right|^{\frac{n}{2}} \\ &= |\psi_n(x)|\, \mathrm{e}^{\frac{n^{1/2}a^2}{2}} \left| 1 - \left(1 - \tanh^2\left(\frac{x}{n^{1/4}}\right)\right) \sin^2\left(\frac{a}{n^{1/4}}\right)\right|^{\frac{n}{2}}. \end{aligned}$$

For n large enough, $\sin^2(\frac{a}{n^{1/4}}) \geq \frac{a^2}{n^{1/2}} - \frac{a^4}{3n}$, and on the other hand, $0 \leq \tanh^2\left(\frac{x}{n^{1/4}}\right) \leq \frac{x^2}{n^{1/2}}$, so

$$\left|\frac{\psi_n(x+\mathrm{i}a)}{\psi_n(x)}\right| \leq \mathrm{e}^{\frac{n^{1/2}a^2}{2}} \exp\left(-\frac{n^{1/2}a^2}{2} \left(1 - \tanh^2\left(\frac{x}{n^{1/4}}\right)\right) \left(1 - \frac{a^2}{3n^{1/2}}\right)\right) \leq \mathrm{e}^{\frac{a^4}{3}}\, \mathrm{e}^{\frac{a^2x^2}{2}}.$$

Since $\psi_n(x)$ behaves as $\mathrm{e}^{-x^4/12}$, the previous Lemma can be applied, with an asymptotic bound

$$\|\psi_n(\cdot + \mathrm{i}a)\|_{\mathrm{L}^1} \lesssim \mathrm{e}^{\frac{a^4}{3}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{x^4}{12}+\frac{a^2x^2}{2}}\,dx = \mathrm{e}^{\frac{13a^4}{12}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(x^2-3a^2)^2}{12}}\,dx$$
$$\lesssim \mathrm{e}^{\frac{13a^4}{12}} \left(2\sqrt{3}\,a + I_\infty\right)$$

by cutting the integral in two parts according to the sign of $x^2 - 3a^2$. We have therefore proven:

Proposition 20 *For any $b \geq 0$,*

$$|\hat{\psi}_n(\xi)| \lesssim K(b)\,\mathrm{e}^{-b|\xi|},$$

where $K(b) = 2\mathrm{e}^{\frac{13b^4}{12}}(2\sqrt{3}\,b + I_\infty)$ and where the symbol $\lesssim$ means that the inequality is true up to any multiplicative constant $1+\varepsilon$, for $\varepsilon > 0$ and n large enough.

We can now conclude. Fix $b > 0$, and $D < 2b$. On the interval $[-Dn^{1/2}, Dn^{1/2}]$, we have

$$\frac{\xi^2}{2n^{1/2}} - b|\xi| = -|\xi|\left(b - \frac{|\xi|}{2n^{1/2}}\right) \leq -|\xi|\left(b - \frac{D}{2}\right).$$

Therefore, with $\varepsilon = \frac{1}{Dn^{1/2}}$,

$$\left|\mathbb{E}\left[\phi_{a,\varepsilon}\left(\frac{Y_n}{n^{1/2}}\right)\right] - \mathbb{E}[\phi_{a,\varepsilon}(W_n)]\right| \lesssim \frac{K(b)}{2\pi I_\infty\, n^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-(b-\frac{D}{2})|\xi|}\,d\xi$$
$$= \frac{K(b)}{\pi I_\infty\, n^{1/2}\,\left(b - \frac{D}{2}\right)}$$
$$\lesssim \frac{K(b)\,D}{\pi I_\infty\,\left(b - \frac{D}{2}\right)}\,\varepsilon.$$

So, Lemma 18 applies to $V = \frac{Y_n}{n^{1/2}}$ and $W = W_n$, with

$$d_{\mathrm{Kol}}\left(\frac{Y_n}{n^{1/2}}, W_n\right) \lesssim 2\left(\frac{K(b)\,D}{\pi I_\infty\,\left(b - \frac{D}{2}\right)} + \frac{10}{I_\infty}\right)\varepsilon = \frac{2}{I_\infty\, n^{1/2}}\left(\frac{K(b)}{\pi\,\left(b - \frac{D}{2}\right)} + \frac{10}{D}\right).$$

Taking $b = D = 0.77$, we get finally

$$d_{\mathrm{Kol}}\left(\frac{Y_n}{n^{1/2}}, W_n\right) \lesssim \frac{2}{I_\infty\, n^{1/2}}\left(\frac{2\,K(b)}{\pi\, b} + \frac{10}{b}\right) \leq \frac{10.27}{n^{1/2}}.$$

Adding the bound on $d_{\mathrm{Kol}}(W_n, W_\infty)$ yields then:

Theorem 21 *For n large enough,*

$$d_{\mathrm{Kol}}\left(\frac{Y_n}{n^{1/2}}, W_\infty\right) \leq 11\, n^{-1/2}.$$

Notice that we have only used arguments of Fourier analysis and the language of mod-Gaussian convergence in order to get this bound.

4.2 Local Limit Theorem

Combining Proposition 20 with Theorem 5 in [3] on local limit theorems for mod-ϕ convergence, we obtain the following local limit theorem for the magnetization in the Curie-Weiss model:

Theorem 22 *In the Curie-Weiss model, if we note M_n for the total magnetization, then we have:*

$$\lim_{n\to\infty} n^{1/2}\,\mathbb{P}[n^{-1/4}M_n \in B] = \frac{2}{12^{1/4}\Gamma(\frac{1}{4})}\, m(B),$$

for relatively compact sets B with $m(\partial B) = 0$, m denoting the Lebesgue measure.

Proof With the notation of Sect. 4.1, $Y_n = n^{-1/4}M_n$ and we need to check assumptions **H1**, **H2** and **H3** of [3] for $(Y_n)_{n\in\mathbb{N}}$ in order to apply Theorem 5 in *loc. cit.*

- **H1.** The Fourier transform of the limiting law $\mu(dx) = \frac{\psi(x)\,dx}{I_\infty}$ of $\frac{Y_n}{n^{1/2}}$ is in the Schwartz space, hence is integrable.
- **H2.** The Fourier transforms $\frac{\widehat{\psi_n}(\xi)}{I_n}\,\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}}$ of $\frac{Y_n}{n^{1/2}}$ converge locally uniformly in ξ towards the Fourier transform $\frac{\hat{\psi}(\xi)}{I_\infty}$. Indeed, by Theorem 10,

$$\frac{\psi_n(x)}{I_n} \to_{\mathrm{L}^1(\mathbb{R})} \frac{\psi(x)}{I_\infty},$$

  so $\frac{\widehat{\psi_n}(\xi)}{I_n} \to \frac{\hat{\psi}(\xi)}{I_\infty}$, and the term $\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}}$ converges locally uniformly to 1.
- **H3.** Finally, we have to prove that for all $k \geq 0$,

$$f_{n,k}(\xi) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi\frac{Y_n}{n^{1/2}}}\right] 1_{|\xi|\leq kn^{1/2}}$$

is uniformly integrable. Following Remark 2 in [3], it is enough to show that

$$\left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi \frac{Y_n}{n^{1/2}}}\right]\right| \leq h(\xi)$$

for ξ such that $|\xi| \leq kn^{1/2}$ for some non-negative and integrable function h on $\mathbb{R}$. This is a consequence of Proposition 20: since $|\widehat{\psi_n}(\xi)| \leq C(k)\,\mathrm{e}^{-k|\xi|}$ for any $k > 0$, one can write

$$\begin{aligned}\left|\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi \frac{Y_n}{n^{1/2}}}\right]\right| &= \frac{|\widehat{\psi_n}(\xi)|}{I_n}\,\mathrm{e}^{\frac{\xi^2}{2n^{1/2}}} \\ &\leq \frac{C(k)}{I_\infty}\,\mathrm{e}^{-k|\xi|+\frac{\xi^2}{2n^{1/2}}} \\ &\leq \frac{C(k)}{I_\infty}\,\mathrm{e}^{-\frac{k}{2}|\xi|}\end{aligned}$$

for any $|\xi| < k\,n^{1/2}$. We can hence apply Theorem 5 of [3] with $\frac{d\mu}{dm}(0) = 1/I_\infty$,and the value of I_∞ was computed in the proof of Lemma 18. □

Remark 23 A similar result would more generally hold for Y_n whenever one has some estimates of exponential decay on $\widehat{\psi_n}(\xi)$ similar to the one given in Lemma 20:

$$\lim_{n\to\infty} t_n\,\mathbb{P}[Y_n \in B] = \frac{1}{I_\infty}\,m(B).$$

In particular, the result holds for the random walks changed in measure studied in Sect. 3.6.

Remark 24 The idea behind the proof the local limit theorem above and which is found in [3] is the following: thanks to approximation arguments, one can show that it is enough to prove the local limit theorem for functions whose Fourier transforms have compact support (instead of indicator functions 1_B). Then, one uses Parseval's relation for such functions f to write:

$$\mathbb{E}[f(Y_n)] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\widehat{\psi_n}(\xi)}{I_n}\,\mathrm{e}^{\frac{\xi^2}{2t_n}}\hat{f}\left(-\frac{\xi}{t_n}\right)\,d\xi$$

and then use the assumptions to conclude.

5 Mod-Gaussian Convergence for the Ising Model: The Cumulant Method

In this appendix, we give another combinatorial proof of the mod-Gaussian convergence of the magnetization in the Ising model, without ever computing the Laplace transform of M_n. This serves as an illustration of the cumulant method developed in [8].

5.1 *Joint Cumulants of the Spins*

When $\alpha = 0$, one can realize the Ising model by choosing $\sigma(1)$ according to a Bernoulli random variable of parameter $\frac{1}{2}$, and then each sign $X_i = \sigma(i)\sigma(i+1)$ according to independent Bernoulli random variables with

$$\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = -1] = \frac{e^{\beta}}{2\cosh\beta}.$$

In particular, one recovers immediately the value of the partition function $Z_n(\mathbb{I}, 0, \beta) = 2^n(\cosh\beta)^{n-1}$. We then want to compute the joint cumulants of the magnetization M_n; by parity, the odd cumulants and moments vanish. By multilinearity, one can expand

$$\kappa^{(2r)}(M_n) = \sum_{i_1,\ldots,i_{2r}=1}^{n} \kappa(\sigma(i_1), \ldots, \sigma(i_{2r})),$$

so the problem reduces to the computation of the joint cumulants of the individual spins, and to the gathering of these quantities. Notice that the joint moments of the spins can be computed easily. Indeed, fix $i_1 \leq i_2 \leq \cdots \leq i_{2r}$, and let us calculate $\mathbb{E}[\sigma(i_1)\cdots\sigma(i_{2r})]$. If $i_{2r-1} = i_{2r}$, then the two last terms cancel and one is reduced to the computation of a joint moment of smaller order. Otherwise, notice that

$$\begin{aligned}\mathbb{E}[\sigma(i_1)\cdots\sigma(i_{2r-2})\sigma(i_{2r-1})\sigma(i_{2r})] &= \mathbb{E}[\sigma(i_1)\cdots\sigma(i_{2r-2})X_{i_{2r-1}}X_{i_{2r-1}+1}\cdots X_{i_{2r}-1}] \\ &= \mathbb{E}[\sigma(i_1)\cdots\sigma(i_{2r-2})]\,x^{i_{2r}-i_{2r-1}} \quad \text{where } x = \tanh\beta.\end{aligned}$$

By induction, we thus get $\mathbb{E}[\sigma(i_1)\cdots\sigma(i_{2r})] = x^{(i_2-i_1)+(i_4-i_3)+\cdots+(i_{2r}-i_{2r-1})}$. Let us then go to the joint cumulants. We fix $i_1 \leq i_2 \leq \cdots \leq i_{2r}$, and to simplify a bit the notations, we denote $i_1 = \mathbf{1}$, $i_2 = \mathbf{2}$, *etc*. We recall that the joint cumulants write as

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r})) = \sum_{\Pi \in \mathfrak{Q}_{2r}} \mu(\Pi) \prod_{A \in \Pi} \mathbb{E}\left[\prod_{a \in A} \sigma(\mathbf{a})\right],$$

where the sum runs over set partitions of $[\![1, 2r]\!]$. By parity, the set partitions with odd parts do not contribute to the sum, so one can restrict oneself to the set $\mathfrak{Q}_{2r,\text{even}}$ of even set partitions. If $A = \{a_1 < \cdots < a_{2s}\}$ is an even part of $[\![1, 2r]\!]$, we write $x^{p(A)} = x^{(\mathbf{a}_2 - \mathbf{a}_1) + \cdots + (\mathbf{a}_{2s} - \mathbf{a}_{2s-1})}$. Thus,

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r})) = \sum_{\Pi \in \mathfrak{Q}_{2r,\text{even}}} \mu(\Pi) \prod_{A \in \Pi} x^{p(A)}.$$

In this polynomial in x, several set partitions give the same power of x; for instance, with $2r = 4$, the set partitions $\{1, 2, 3, 4\}$ and $\{1, 2\} \sqcup \{3, 4\}$ both give $x^{(\mathbf{2}-\mathbf{1})+(\mathbf{3}+\mathbf{4})}$. Denote $\mathfrak{P}_{2r}$ the set of set partitions of $[\![1, 2r]\!]$ whose parts are all of cardinality 2 (pair set partitions, or pairings). To every even set partition Π, one can associate a pairing $p(\Pi)$ by cutting all the even parts $\{a_1 < a_2 < \cdots < a_{2s-1} < a_{2s}\}$ into the pairs $\{a_1 < a_2\}, \ldots, \{a_{2s-1} < a_{2s}\}$. For instance, the even set partition $\Pi = \{1, 3, 4, 5\} \sqcup \{2, 6\}$ gives the pairing $(1, 3)(4, 5)(2, 6)$. Then, with obvious notations,

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r})) = \sum_{\Pi \in \mathfrak{Q}_{2r,\text{even}}} \mu(\Pi)\, x^{p(\Pi)}. \tag{12}$$

In Eq. (12), two important simplifications can be made:

1. One can gather the even set partitions Π according to the pairing $\rho = p(\Pi) \in \mathfrak{P}_{2r}$ that they produce. It turns out that the corresponding sum of Möbius functions $F(\rho)$ has a simple expression in terms of the pairing, see Sect. 5.1.3.
2. Some pairings ρ yield the same monomial x^{ρ} and the same functional $F(\rho)$. By gathering these contributions, one can reduce further the complexity of the sum, see Sect. 5.1.2.

In the end, we shall obtain an exact formula for $\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r}))$ that writes as a sum over Dyck paths of length $2r - 2$, with simple coefficients; see Theorem 28.

5.1.1 Pairings, Labelled Dyck Paths and Labelled Planar Trees

Before we start the reduction of Formula (12), it is convenient to recall some facts about the combinatorial class of pairings. We have defined a pairing ρ of size $2r$ to be a set partition of $[\![1, 2r]\!]$ in r pairs $(a_1, b_1), \ldots, (a_r, b_r)$. There are

$$\operatorname{card} \mathfrak{P}_{2r} = (2r - 1)!! = (2r - 1)(2r - 3) \cdots 3\, 1$$

pairings of size $2r$, and it is convenient to represent them by diagrams given in Fig. 6.

On the other hand, a labelled Dyck path of size $2r$ is a path $\delta : [\![0, 2r]\!] \to \mathbb{N}$ with $2r$ steps either ascending or descending, such that:

- the path δ starts from 0, ends at 0 and stays non-negative;
- each descending step $\delta(k) > \delta(k+1)$ is labelled by an integer i with $i \in [\![1, \delta(k)]\!]$.

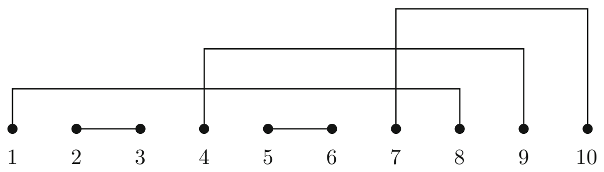


Fig. 6 The diagram of a pairing of size $2r = 10$

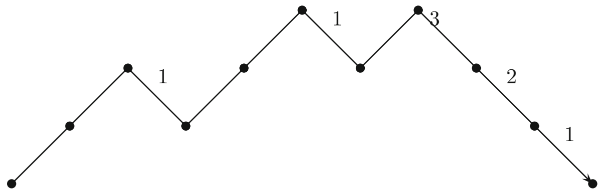


Fig. 7 The labelled Dyck path corresponding to the pairing of Fig. 6

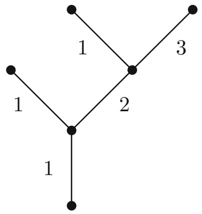


Fig. 8 The labelled planar rooted tree corresponding to the pairing of Fig. 6

From a labelled Dyck path of size $2r$, one constructs a pairing on $2r$ points as follows: one reads the diagram from left to right, opening a bond when the path is ascending, and closing the i-th opened bond available from right to left when the path is descending with label i. For instance, if one starts from the Dyck path of Fig. 7, one obtains the pairing of Fig. 6. This provides a first bijection between pairings ρ and labelled Dyck paths δ.

By considering a Dyck path as the code of the depth-first traversal of a rooted tree, one obtains a second bijection between pairings of size $2r$ and labelled planar rooted trees with r edges. Here, by labelled planar rooted tree, we mean a planar rooted tree with a label i on each edge e that is between 1 and the height $h(e)$ of the edge (with respect to the root). For instance, the following labelled tree T corresponds to the Dyck path of Fig. 7 and to the pairing of Fig. 6: see Fig. 8.

We shall denote $\mathfrak{T}_r$ the set of planar rooted trees with r edges (without label), and $\mathfrak{D}_{2r}$ the corresponding set of Dyck paths (again without label); they have cardinality

$$\operatorname{card} \mathfrak{T}_r = \operatorname{card} \mathfrak{D}_{2r} = C_r = \frac{1}{r+1}\binom{2r}{r}.$$

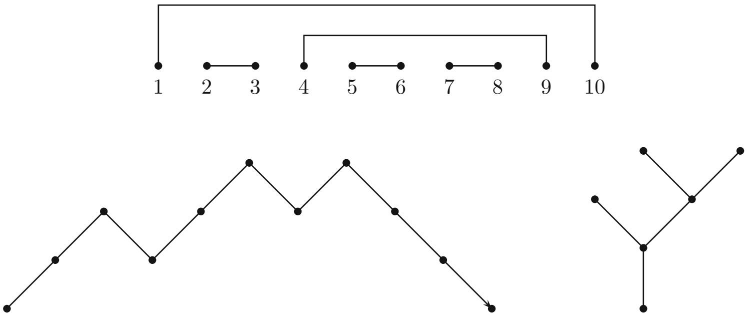


Fig. 9 Bijection between non-crossing pairings, Dyck paths and planar rooted trees

They correspond to the subset $\mathfrak{N}_{2r}$ of $\mathfrak{P}_{2r}$ that consists in non-crossing pair partitions of $[\![1, 2r]\!]$; a bijection is obtained by labelling each edge or descending step by 1, and by using the previous constructions. For instance, the non-crossing pairing, the Dyck path and the planar rooted tree of Fig. 9 do correspond.

In what follows, we shall always use the letters ν, δ and T respectively for non-crossing pairings, for Dyck paths and for planar rooted trees. We shall then use constantly the bijections described above, and denote for instance $\nu(T)$ for the non-crossing pairing associated to a tree T, or $\delta(\nu)$ for the Dyck path associated to a non-crossing pairing ν. We shall also use the exponent $+$ to indicate the following operations on these combinatorial objects:

- transforming a non-crossing pairing ν of size $2r-2$ in a non-crossing pairing ν^+ of size $2r$ by adding the bond $\{1, 2r\}$ "over" the bonds of ν.
- transforming a Dyck path δ of length $2r-2$ in a Dyck path δ^+ of length $2r$ by adding an ascending step before δ and a descending step after δ.
- transforming a rooted tree T with $r-1$ edges in a rooted tree T^+ with r edges by adding an edge "below" the root.

All these operations are compatible with the aforementioned bijections, so for instance $\nu(T^+) = (\nu(T))^+$ and $\delta(\nu^+) = (\delta(\nu))^+$.

5.1.2 Uncrossing Pairings and the Associated Poset

Let us now see how the combinatorics of pairings, Dyck paths and planar rooted trees intervene in Formula (12). We start by gathering the set partitions Π with the same associated pairing $\rho = p(\Pi)$. Thus, let us write

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r})) = \sum_{\rho \in \mathfrak{P}_{2r}} x^\rho \left(\sum_{\substack{\Pi \in \mathfrak{Q}_{2r,\mathrm{even}} \\ p(\Pi) = \rho}} \mu(\Pi) \right) = \sum_{\rho \in \mathfrak{P}_{2r}} x^\rho \, F(\rho),$$

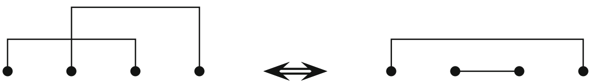


Fig. 10 The operation of uncrossing on a pairing

where $F(\rho)$ stands for the sum in parentheses. Notice that x^ρ is invariant if one replaces in a pairing two crossing pairs $\{a_1, a_3\}, \{a_2, a_4\}$ with $a_1 < a_2 < a_3 < a_4$ by two nested pairs (but non-crossing) $\{a_1, a_4\}, \{a_2, a_3\}$; indeed,

$$(a_3 - a_1) + (a_4 - a_2) = (a_4 - a_1) + (a_3 - a_1).$$

We call uncrossing the operation on pairings which consists in replacing two crossing pairs by two nested pairs as described above, and we denote $\rho_1 \succeq \rho_2$ if there is a sequence of uncrossings from the pairing ρ_1 to the pairing ρ_2; this is a partial order on the set $\mathfrak{P}_{2r}$ (Fig. 10).

Proposition 25 *The poset* $(\mathfrak{P}_{2r}, \preceq)$ *is a disjoint union of lattices, and each lattice contains a unique non-crossing set partition* ν*, which is the minimum of this connected component of the Hasse diagram of* $(\mathfrak{P}_{2r}, \preceq)$*. Moreover:*

1. *On the lattice* $L(\nu)$ *associated to* $\nu \in \mathfrak{N}_{2r}$*, the monomial* x^ρ *and the functional* $F(\rho)$ *are constant (equal to* x^ν *and* $F(\nu)$*).*
2. *The cardinality* $\operatorname{card} L(\nu) = N(\nu)$ *is given by:*

$$N(\nu) = \prod_{e \in E(T(\nu))} h(e, T(\nu)),$$

where $h(e,T)$ *is the height of the edge* e *in the (planar) rooted tree* T*, and* $E(T)$ *is the set of edges of a tree* T*.*

Proof First, notice that if $\rho_1 \preceq \rho_2$ in $\mathfrak{P}_{2r}$, then there is a sequence of pairings going from ρ_1 to ρ_2 such that every two consecutive terms μ and ρ of the sequence differ only by the replacement of a simple nesting by a simple crossing. By that we mean that we do not need to do replacements such as the one on Fig. 11, which creates 3 crossings at once.

Indeed, denoting (i,j) the crossing of the i-th bond with the j-th bond, bonds being numeroted from their starting point, one has $(1,3) = (1,2) \circ (2,3) \circ (1,2)$, which is a composition of simple operations of crossing; and the same idea works for nestings of higher depth. Thus, the Hasse diagram of the poset $(\mathfrak{P}_{2r}, \preceq)$ has edges that consist in replacements of simple nestings by simple crossings.

This being clarified, it suffices now to notice that *via* the bijection between pairings and labelled Dyck paths explained in Sect. 5.1.1, the replacing a simple nesting by a simple crossing corresponds to the raising of a label by 1: see Fig. 12.

In particular, if ρ_1 and ρ_2 are two comparable pairings in $(\mathfrak{P}_{2r}, \preceq)$, then the corresponding labelled Dyck paths have the same shape; and for a given shape

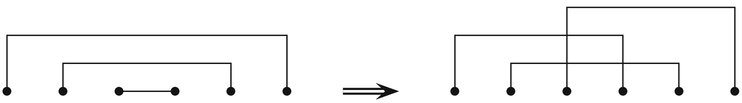


Fig. 11 The crossing of a nesting that is not simple

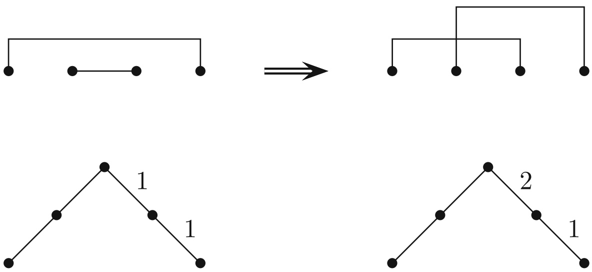


Fig. 12 The operation of uncrossing is a change of labels on Dyck paths

$\delta \in \mathfrak{D}_{2r}$, there is exactly one corresponding non-crossing pair partition $\nu = \nu(\delta)$, which is minimal in its connected component in the Hasse diagram of $(\mathfrak{P}_{2r}, \preceq)$. Endowed with $\preceq$, this connected component $L(\nu)$ is isomorphic as a poset to the product of intervals

$$\prod_{e \in T(\nu)} [\![1, h(e, T(\nu))]\!] \,.$$

Indeed, the order on the set of labelled trees of shape $T(\nu)$ induced by $(L(\nu), \preceq)$ and by the bijection between pairings and labelled trees is simply the product of the orders of the intervals of labels. This proves all of the Proposition but the invariance of $F(\cdot)$ on $L(\nu)$ (the invariance of $x^{(\cdot)}$ was shown at the beginning of this paragraph); we devote Sect. 5.1.3 to this last point and to the actual computation of the functional $F(\cdot)$. □

Assuming the invariance of $F(\cdot)$ on each lattice $L(\nu)$, we thus get:

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r})) = \sum_{\rho \in \mathfrak{P}_{2r}} x^{\rho}\, F(\rho) = \sum_{\nu \in \mathfrak{N}_{2r}} x^{\nu}\, N(\nu)\, F(\nu), \tag{13}$$

where $N(\nu)$ is explicit. Hence, it remains to compute the functional $F(\rho)$.

5.1.3 Computation of the Functional *F*

The main result of this paragraph is:

Proposition 26 *The functional $F(\cdot)$ is constant on $L(\nu)$, and if ν is a non-crossing pairing, then*

$$F(\nu) = (-1)^{r-1} \prod_{\substack{e \in T(\nu) \\ h(e, T(\nu)) \neq 1}} (h(e, T(\nu)) - 1)$$

if $T(\nu)$ has a single edge of height 1*, and* 0 *otherwise.*

Lemma 27 *The functional F vanishes on pairings associated to labelled rooted trees with more than one edge of height* 1.

Proof Suppose that Π is an even set partition with $p(\Pi) = \rho$; ρ being a pairing of size $2r$ associated to a labelled Dyck path that reaches 0 after $2a$ steps, with $2r = 2a + 2b$, $a > 0$ and $b > 0$ (this is equivalent to the statement "having more than one edge of height 1"). We denote ρ_1 and ρ_2 the pairings associated to the two parts of the Dyck path. There are several possibilities:

- either Π can be split as two even set partitions Π_1 and Π_2 of $[\![1, 2a]\!]$ and $[\![2a+1, 2r]\!]$, with respectively k and l parts, and with $p(\Pi_1) = \rho_1$ and $p(\Pi_2) = \rho_2$;
- or, Π is one of the $k \times l$ possible ways to unite two such even set partitions Π_1 and Π_2 by joining one part of Π_1 with one part of Π_2;
- or, Π is one of the $\binom{k}{2} \times \binom{l}{2} \times 2!$ possible ways to unite two such even set partitions Π_1 and Π_2 by joining two parts of Π_1 with two parts of Π_2;
- or, Π is one of the $\binom{k}{3} \times \binom{l}{3} \times 3!$ possible ways to unite two such even set partitions Π_1 and Π_2 by joining three parts of Π_1 with three parts of Π_2;
- *etc.*

So, $F(\rho)$ can be rewritten as

$$\sum_{\substack{p(\Pi_1)=\rho_1\\ p(\Pi_2)=\rho_2}} (-1)^{t-1} \left((t-1)! - kl\,(t-2)! + \binom{k}{2}\binom{l}{2} 2!\,(t-3)! \right.$$
$$\left. - \binom{k}{3}\binom{l}{3} 3!\,(t-4)! + \cdots \right),$$

where $t = k + l$. However, for every possible value of $k \geq 1$ and $l \geq 1$, the term in parentheses vanishes. Indeed, assuming for instance $k \leq l$, we look at

$$(k+l-1)! \sum_{x=0}^{k} (-1)^x \binom{k}{x}\binom{l}{x}\binom{k+l-1}{x}^{-1}$$
$$= k!\,(l-1)! \sum_{x=0}^{k} (-1)^x \binom{l}{x}\binom{k+l-1-x}{k-x}$$
$$= k!\,(l-1)! \binom{k-1}{k} = 0$$

by using Riordan's array rule for the second identity. □

Thus, F vanishes on pairings ρ associated to labelled trees with more than one edge of height 1. In other words, if $F(\rho) \neq 0$, then $\{1, 2r\}$ is a pair in ρ, and we can look at the restricted pairing $\tilde{\rho} = \rho_{|[\![2,2r-1]\!]}$, which is of size $2r-2$; and we can consider F as a functional on $\mathfrak{P}_{2r-2}$. To avoid any ambiguity, we denote this new functional

$$G(\rho \in \mathfrak{P}_{2r}) = \sum_{p(\Pi)=\rho} (-1)^{\ell(\Pi)} \, (\ell(\Pi))!$$

We then expect the formula $G(\rho) = (-1)^r \prod_{e \in E(T(\rho))} h(e)$. We proceed by induction on labelled rooted planar trees, and we look at the action of adding a leave of label 1 to the tree, and of increasing a label of an edge by 1. To fix the ideas, it is convenient to consider the following example of pairing ρ, and the associated set of set partitions Π with $p(\Pi) = \rho$. The pairing ρ of Fig. 13 is associated to the labelled planar rooted tree on Fig. 14, and it has functional $G(\rho) = (-1)^3\, 3! + 2 \times (-1)^2\, 2! = -2$. We denote $N(l, \rho)$ the number of set partitions such that $p(\Pi) = \rho$ and $\ell(\Pi) = l$. Hence,

$$G(\rho) = \sum_{l=1}^{r} N(l, \rho)\, (-1)^l \, l!$$

1. *Adding an edge*. Suppose that one adds an edge with label 1, to obtain for instance: see Fig. 15. Set ρ' for the new pairing; notice that it is obtained from ρ by inserting a simple bond •—•. The set partitions Π' with $p(\Pi') = \rho'$ are of two kinds:

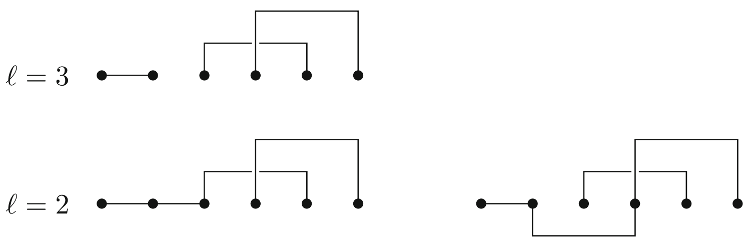


Fig. 13 A pairing of size $2r = 6$ (the *upper diagram*) and the associated set of set partitions, which contains 3 elements

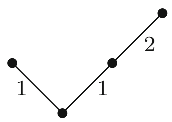


Fig. 14 The labelled planar rooted tree associated to the pairing of Fig. 13

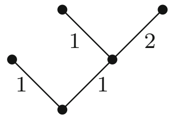


Fig. 15 Addition of an new edge of label 1 to the planar rooted tree

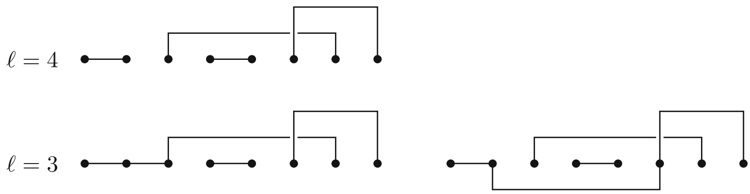


Fig. 16 Set partitions where the new bond is left alone

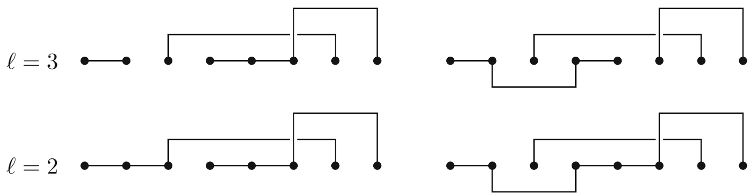


Fig. 17 Set partitions where the new bound is integrated in another part

a. those where the new bond is left alone. They all come from a set partition Π with $p(\Pi) = \rho$ by simply inserting the new bond: see Fig. 16.

These terms give the following contribution to $G(\rho')$:

$$G_{(a)}(\rho') = -\sum_{l=1}^{r} N(l, \rho)\,(-1)^l\,(l+1)!.$$

b. those where the new bond is linked to another part of a set partition Π with $p(\Pi) = \rho$. Starting from a set partition Π with $p(\Pi) = \rho$, the number of parts of Π that can actually receive the new bond is $\ell(\Pi) - (h(e) - 1)$, because the new bond cannot be linked to the $h(e) - 1$ parts that go above him. In our example: see Fig. 17.

These other terms give the following contribution to $G(\rho')$:

$$G_{(b)}(\rho') = \sum_{l=1}^{r} N(l, \rho)\,(-1)^l\,l!\,(l+1-h(e)).$$

We conclude that $G(\rho') = G_{(a)}(\rho') + G_{(b)}(\rho') = -h(e)\, G(\rho)$, so the formula for G stays true when one adds an edge of label 1.

2. *Raising a label*. As explained before, raising a label corresponds to adding a simple crossing to the pairing ρ, which is done by exchanging two ends b and d of two simply nested pairs $\{a < b\}$ and $\{c < d\}$ of ρ. This does not change the structure of the set of even set partitions Π with $p(\Pi) = \rho$; that is, $N(l, \rho) = N(l, \rho')$ for every l. So, the formula for G also stays true when one raises a label.

Since every labelled rooted tree is obtained inductively from the empty tree by adding edges and raising labels, the proof of Proposition 26 is done.

5.1.4 Expansion of the Joint Cumulants as Sums Over Dyck Paths

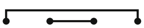
Recall that x^ν stands for $x^{(\mathbf{a}_2-\mathbf{a}_1)+\cdots+(\mathbf{a}_{2r}-\mathbf{a}_{2r-1})}$ if ν is the pairing $\{a_1 < a_2\}, \ldots, \{a_{2r-1} < a_{2r}\}$. We adopt the same notations with Dyck paths and planar rooted trees, so x^δ or x^T stands for x^ν if $\delta = \delta(\nu)$ or if $T = T(\nu)$. We also denote $\mathfrak{D}^*_{2r}$ the image of $\mathfrak{D}_{2r-2}$ in $\mathfrak{D}_{2r}$ by the operation $\delta \mapsto \delta^+$. Notice that if $\Delta = (\delta(T))^+$ with T tree with $r-1$ edges, then

$$\prod_{e\in E(T)} h(e)\,(h(e)+1) = \prod_{i=1}^{2r-1} \Delta_i,$$

Δ_i denoting the value of the Dyck path Δ after i steps. Starting from Eq. (13) and using the explicit formulas that we have obtained for $N(\nu)$ and $F(\nu)$, we therefore get:

Theorem 28 *For every indices* $\mathbf{1} \leq \cdots \leq \mathbf{2r}$,

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r})) = (-1)^{r-1} \sum_{\delta\in\mathfrak{D}^*_{2r}} \left(\prod_{i=1}^{2r-1} \delta_i\right) x^\delta.$$

Example 29 The two non-crossing pairings of size 4 are and , the associated powers of x are

$$x^{(\mathbf{6}-\mathbf{1})+(\mathbf{5}-\mathbf{4})+(\mathbf{3}-\mathbf{2})} \quad \text{and} \quad x^{(\mathbf{6}-\mathbf{1})+(\mathbf{5}-\mathbf{2})+(\mathbf{4}-\mathbf{3})},$$

and the associated quantities $G(\nu)$ are 4 and 12, so, with $r = 3$,

$$\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{6})) = 4\,x^{\mathbf{6}+\mathbf{5}+\mathbf{3}-\mathbf{4}-\mathbf{2}-\mathbf{1}} + 12\,x^{\mathbf{6}+\mathbf{5}+\mathbf{4}-\mathbf{3}-\mathbf{2}-\mathbf{1}}.$$

Theorem 28 has several easy corollaries. First of all, we see immediately from it that the sign of a joint cumulant of spins is prescribed, which was a priori non-obvious. On the other hand, applying Theorem 28 to the case $r = 1$ yields

$$\kappa(\sigma(i), \sigma(j)) = x^{|j-i|},$$

that is, the correlation between two spins decreases exponentially with the distance between the spins. More generally, one can use Theorem 28 to get a useful bound on cumulants. Notice that the minimal exponent of x that appears in the right-hand side of the formula is

$$x^{(\mathbf{2r})+((\mathbf{2r-1})-(\mathbf{2r-2}))+((\mathbf{2r-3})-(\mathbf{2r-4}))+\cdots+(\mathbf{3-2})-\mathbf{1}}.$$

Indeed, it is easily seen that the exponent of x in x^T increases when one makes a rotation of a leaf of T in the sense of Tamari (*cf.* [20]). Since all trees are generated by leaf rotations from the tree with all edges of height 1 (*cf.* [11]), the previous claim is shown. It follows that

$$|\kappa(\sigma(\mathbf{1}), \ldots, \sigma(\mathbf{2r}))| \leq \left(\sum_{\delta \in \mathfrak{D}^*_{2r}} \prod_{i=1}^{2r-1} \delta_i\right) x^{(\mathbf{2r})+((\mathbf{2r-1})-(\mathbf{2r-2}))+\cdots+(\mathbf{3-2})-\mathbf{1}}.$$

The quantity

$$Q(r) = \sum_{\delta \in \mathfrak{D}^*_{2r}} \left(\prod_{i=1}^{2r-1} \delta_i\right) = \sum_{T \in \mathfrak{T}_{r-1}} \left(\prod_{e \in E(T)} h(e)\,(h(e)+1)\right)$$

has for first values $1, 2, 16, 272, 7936, \ldots$, and a simple bound on $Q(r)$ is $(2r-2)!$, see Proposition 37 hereafter. Hence, a generalization of the exponential decay of covariances is given by:

Proposition 30 *For any positions of spins $i_1 \leq i_2 \leq \cdots \leq i_{2r}$,*

$$|\kappa(\sigma(i_1), \ldots, \sigma(i_{2r}))| \leq (2r-2)!\; x^{i_{2r}+(i_{2r-1}-i_{2r-2})+\cdots+(i_3-i_2)-i_1}.$$

5.2 Bounds on the Cumulants of the Magnetization

As explained in the introduction, we now have to gather the estimates given by Theorem 28 to get the asymptotics of the cumulants $\kappa^{(2r)}(M_n)$ of the magnetization.

5.2.1 Reordering of Indices and Compositions

Since the joint cumulants of spins have been computed for ordered spins $i_1 \leq i_2 \leq \ldots \leq i_{2r}$, in the right-hand side of the expansion

$$\kappa^{(2r)}(M_n) = \sum_{i_1,\ldots,i_{2r}=1}^{n} \kappa(\sigma(i_1),\ldots,\sigma(i_{2r})),$$

we need to reorder the indices $i_1,\ldots,i_{2r}$, and take care of the possible identities between these indices. We shall say that a sequence of indices $i_1,\ldots,i_r$ has type $c = (c_1,\ldots,c_l)$ with the c_i positive integers and $|c| = \sum_{i=1}^{l} c_i = r$ if, after reordering, the sequence of indices writes as

$$i'_1 = i'_2 = \ldots = i'_{c_1} < i'_{c_1+1} = i'_{c_1+2} = \cdots = i'_{c_1+c_2} < i'_{c_1+c_2+1} = \cdots .$$

Here, i'_k stands for the k-th element of the reordered sequence. For instance, the sequence of indices $(3,2,3,5,1,2)$ becomes after reordering $(1,2,2,3,3,5)$, so it has type $(1,2,2,1)$. The type of a sequence of indices of length r can be any composition of size r, and we denote $\mathfrak{C}_r$ the set of these compositions. Conversely, given a composition of size r and length l, in order to construct a sequence of indices $(i_1,\ldots,i_r)$ with type c and with values in $[\![1,n]\!]$, one needs:

- to choose which indices i will fall into each class $(i'_1,\ldots,i'_{c_1})$, $(i'_2,\ldots,i'_{c_1+c_2})$, *etc.*; there are

$$\binom{r}{c} = \frac{r!}{c_1!\,c_2!\cdots c_l!}$$

  possibilities there.
- and then to choose $1 \leq j_1 < j_2 < \cdots < j_l \leq n$ so that $j_1 = i'_1 = \cdots = i'_{c_1}$, $j_2 = i'_2 = \cdots = i'_{c_1+c_2}$, *etc.*

As a consequence,

$$\begin{aligned}\kappa^{(2r)}(M_n) &= \sum_{c\in\mathfrak{C}_{2r}} \sum_{1\leq j_1<j_2<\cdots<j_{\ell(c)}\leq n} \binom{2r}{c} \kappa\left(\sigma(j_1)^{c_1},\ldots,\sigma(j_{\ell(c)})^{c_{\ell(c)}}\right) \\ &= (-1)^{r-1} \sum_{c\in\mathfrak{C}_{2r}} \sum_{\delta\in\mathfrak{D}^*_{2r}} \binom{2r}{c} C(\delta)\,B(n,c,\delta)\end{aligned}$$

where $C(\delta) = \prod_{i=1}^{2r-1} \delta_i$ is the quantity computed in the previous paragraph, and

$$B(n, c, \delta) = \sum_{1 \leq j_1 < j_2 < \cdots < j_{\ell(c)} \leq n} x^{\sum_{\{a<b\} \in \nu(\delta)} (i_b - i_a)},$$

the indices i being computed from the indices j according to the rule previously explained, namely,

$$\begin{aligned} j_1 &= i_1 = \cdots = i_{c_1}; \\ j_2 &= i_{c_1+1} = \cdots = i_{c_1+c_2}; \\ &\vdots \qquad\qquad \vdots \\ j_{\ell(c)} &= i_{c_1+\cdots+c_{\ell(c)-1}+1} = \cdots = i_{2r}. \end{aligned}$$

Example 31 Suppose $r = 1$. There are two compositions of size 2, namely, (2) and (1, 1), and one trivial tree with 0 edge; therefore,

$$\begin{aligned} \kappa^{(2)}(M_n) &= B(n, (2), \bullet) + 2\, B(n, (1, 1), \bullet) \\ &= \sum_{j_1=1}^{n} 1 \ + \ 2 \sum_{1 \leq j_1 < j_2 \leq n} x^{j_2 - j_1}. \end{aligned}$$

The double geometric sum has the same asymptotics as $\sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{\infty} x^{j_2-j_1} = n\, \frac{x}{1-x}$, so

$$\kappa^{(2)}(M_n) \simeq n\, \frac{1+x}{1-x} = n\, \mathrm{e}^{2\beta}.$$

It is not hard to convince oneself that the approximation performed in the previous example can be done in any case, so that a correct estimate of $B(n, c, \delta)$ is $n\, B(c, \delta)$, with

$$B(c, \delta) = \sum_{0 = j_1 < j_2 < \cdots < j_{\ell(c)}} x^{\sum_{\{a<b\} \in \nu(\delta)} (i_b - i_a)}.$$

In this new expression, the indices j are unbounded (except the first one, fixed to 0), and what we mean by approximation is that

$$n\, B(c, \delta) = B(n, c, \delta) + O(1),$$

with a positive remainder corresponding to terms of the geometric series with indices larger than n. So:

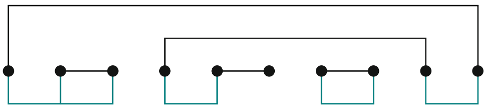


Fig. 18 Identifications of indices corresponding to the composition $c = (3, 2, 1, 2, 2)$

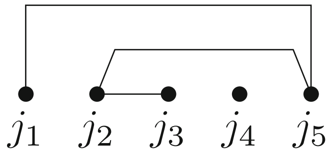


Fig. 19 Contraction of the diagram of a non-crossing partition along a composition

Proposition 32 *An upper bound, and in fact an estimate of* $|\kappa^{(2r)}(M_n)|$ *is*

$$|\kappa^{(2r)}(M_n)| \leq n \sum_{c \in \mathfrak{C}_{2r}} \sum_{\delta \in \mathfrak{D}^*_{2r}} \binom{2r}{c} B(c,\delta)\, C(\delta).$$

5.2.2 Computation of the Functional *B*

There is a simple algorithm that allows to compute $B(c,\delta)$ for any Dyck path δ and any composition c. Let us explain it with the path δ associated to the non-crossing pairing ν of Fig. 9 and with the composition $c = (3, 2, 1, 2, 2)$. This composition c corresponds to some identifications of indices, which we make appear on the diagram of the pairing ν as follows: see Fig. 18.

We now contract the green edges added above, obtaining thus: see Fig. 19.

This new diagram corresponds to the following simplification of the sum $B(c,\delta)$:

$$\begin{aligned} B(c,\delta) &= \sum_{0=i_1=i_2=i_3<i_4=i_5<i_6<i_7=i_8<i_9=i_{10}} x^{i_{10}+i_9+i_8-i_7+i_6-i_5-i_4+i_3-i_2-i_1} \\ &= \sum_{0=i_1<i_4<i_6<i_7<i_9} x^{2i_9+i_5-2i_4-i_1} \qquad \text{because of the identities of indices;} \\ &= \sum_{0=j_1<j_2<j_3<j_4<j_5} x^{(j_5-j_1)+(j_5-j_2)+(j_3-j_2)} \qquad \text{by relabeling the indices.} \end{aligned}$$

So, the new diagram, which we call the *contraction of* ν *along* c and denote $\nu \downarrow_c$, can be read similarly as the previous diagrams of pairings, that is to say that

$$B(c,\delta) = \sum_{0=j_1<j_2<j_3<j_4<j_5} x^{(\nu(\delta))\downarrow_c},$$

where $x^{\nu\downarrow_c}$ stands for the product of factors x^{b-a}, $\{a < b\}$ running over the bonds of the contracted diagram $\nu \downarrow_c$.

Given a contracted diagram $\rho = \nu \downarrow_c$ of length $\ell(c)$, denote $\delta_1(\rho)$ the number of bonds opened between j_1 and j_2; $\delta_2(\rho)$ the number of bonds opened between j_2 and j_3; $\delta_3(\rho)$ the number of bonds opened between j_3 and j_4; *etc.* up to $\delta_{\ell(c)-1}(\rho)$. For instance, in the previous example, there is one bond opened between j_1 and j_2 (the one starting from j_1); 3 bonds opened between j_2 and j_3 (the previous bond, which has not been closed, and the two bonds starting from j_2); and 2 bonds opened between j_3 and j_4 and between j_4 and j_5. So $(\delta_1, \delta_2, \delta_3, \delta_4) = (1, 3, 2, 2)$.

Proposition 33 *Set $\rho = (\nu(\delta)) \downarrow_c$. One has*

$$B(c, \delta) = \prod_{i=1}^{\ell(c)-1} \frac{x^{\delta_i(\rho)}}{1 - x^{\delta_i(\rho)}}.$$

Example 34 Consider the previous contracted diagram ρ_5, and the corresponding sum

$$B_5 = \sum_{0=j_1<j_2<j_3<j_4<j_5} x^{(j_5-j_1)+(j_5-j_2)+(j_3-j_2)}.$$

We reduce inductively the size of the contracted diagram as follows. We first write

$$\begin{aligned} B_5 &= \sum_{0=j_1<j_2<j_3<j_4<j_5} x^{2(j_5-j_4)+(j_4-j_1)+(j_4-j_2)+(j_3-j_2)} \\ &= \left(\sum_{0=j_1<j_2<j_3<j_4} x^{(j_4-j_1)+(j_4-j_2)+(j_3-j_2)} \right) \left(\sum_{j_5=j_4+1}^{\infty} x^{2(j_5-j_4)} \right) \\ &= \frac{x^2}{1-x^2} \left(\sum_{0=j_1<j_2<j_3<j_4} x^{(j_4-j_1)+(j_4-j_2)+(j_3-j_2)} \right) = \frac{x^{\delta_4}}{1-x^{\delta_4}} B_4, \end{aligned}$$

where B_4 is the sum corresponding to the diagram ρ_4 which is obtained from ρ_5 by identifying j_4 and j_5: see Fig. 20.

We can then do it again to go to size 3:

$$\begin{aligned} B_4 &= \sum_{0=j_1<j_2<j_3<j_4} x^{2(j_4-j_3)+(j_3-j_1)+(j_3-j_2)+(j_3-j_2)} \\ &= \left(\sum_{0=j_1<j_2<j_3} x^{(j_3-j_1)+2(j_3-j_2)} \right) \left(\sum_{j_4=j_3+1}^{\infty} x^{2(j_4-j_3)} \right) \\ &= \frac{x^2}{1-x^2} \left(\sum_{0=j_1<j_2<j_3} x^{(j_3-j_1)+2(j_3-j_2)} \right) = \frac{x^{\delta_3}}{1-x^{\delta_3}} B_3, \end{aligned}$$

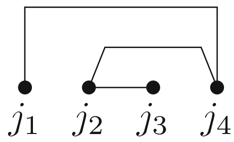


Fig. 20 Reduction of the diagram of Fig. 19

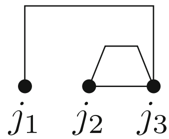


Fig. 21 Further reduction of the diagram of Fig. 19

where B_3 is the sum corresponding to the diagram ρ_3 which is obtained from ρ_4 by identifying j_3 and j_4: see Fig. 21. Two more operations yield similarly the factors $\frac{x^{\delta_2}}{1-x^{\delta_2}}$ and $\frac{x^{\delta_1}}{1-x^{\delta_1}}$.

Proof (of Proposition 33) The algorithm presented above on the example gives clearly a proof of the formula by induction on $\ell(c)$. Indeed, at each step of the induction, the term that is factorized is

$$\sum_{j_{\ell(c)}=j_{\ell(c)-1}+1}^{\infty} x^{\delta_{\ell(c)-1}(j_{\ell(c)}-j_{\ell(c)-1})} = \frac{x^{\delta_{\ell(c)-1}}}{1-x^{\delta_{\ell(c)-1}}},$$

because $\delta_{\ell(c)-1}$ is the number of bonds ending at $j_{\ell(c)}$. Then, as for the other factor, one obtains it by replacing $j_{\ell(c)}$ by $j_{\ell(c)-1}$ in the sum $B(c,\delta)$, and this amounts to do the identification between $j_{\ell(c)-1}$ and $j_{\ell(c)}$ in the contracted diagram. This identification and reduction to lower length does not change the values $\delta_1,\ldots,\delta_{\ell(c)-2}$, so the formula is proven. □

We recall that a descent of a composition $c=(c_1,\ldots,c_\ell)$ is one of the integers

$$c_1, c_1+c_2, c_1+c_2+c_3, \ldots, c_1+\cdots+c_{\ell-1}.$$

For instance, the descents of $c=(3,2,1,2,2)$ are 3, 5, 6 and 8. The set of descents $D(c)$ of a composition c of size r can be any subset of $[\![1,r-1]\!]$, so in particular, $\mathrm{card}\,\mathfrak{C}_r = 2^{r-1}$. The contraction of diagrams along compositions presented at the beginning of this paragraph satisfies the rule:

$$\{\delta_1(\rho),\ldots,\delta_{\ell(c)-1}(\rho)\} = \{\delta_d,\ d\in D(c)\} \quad \text{if } \rho=(\nu(\delta))\downarrow_c .$$

So, $B(c,\delta)=\prod_{d\in D(c)} \frac{x^{\delta_d}}{1-x^{\delta_d}}$, and Proposition 32 becomes:

Theorem 35 *An upper bound, and in fact an estimate of* $|\kappa^{(2r)}(M_n)|$ *is*

$$\frac{|\kappa^{(2r)}(M_n)|}{n} \leq \sum_{c \in \mathfrak{C}_{2r}} \sum_{\delta \in \mathfrak{D}^*_{2r}} A(c)\, B(c,\delta)\, C(\delta)$$

with $A(c) = \binom{2r}{c}$, $B(c,\delta) = \prod_{d \in D(c)} \frac{x^{\delta_d}}{1-x^{\delta_d}}$ *and* $C(\delta) = \prod_{i=1}^{2r-1} \delta_i$.

Example 36 Suppose $r = 2$. There is one Dyck path in $\mathfrak{D}^*_4$, with $C(\delta) = 2$ since $\delta_1 = \delta_3 = 1$ and $\delta_2 = 2$. The compositions of size 4 are (4), $(3,1)$, $(2,2)$, $(1,3)$, $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ and $(1,1,1,1)$; their contributions $A(c)\, B(c,\delta)$ are equal to

$$1,\ \frac{4x}{1-x},\ \frac{6x^2}{1-x^2},\ \frac{4x}{1-x},\ \frac{12x^3}{(1-x)(1-x^2)},\ \frac{12x^2}{(1-x)^2},\ \frac{12x^3}{(1-x)(1-x^2)},$$
$$\frac{24x^4}{(1-x)^2(1-x^2)}.$$

So,

$$|\kappa^{(4)}(M_n)| \simeq 2n \left(1 + \frac{8x}{1-x} + \frac{6x^2}{1-x^2} + \frac{12x^2}{(1-x)^2} + \frac{24x^3}{(1-x)(1-x^2)} + \frac{24x^4}{(1-x)^2(1-x^2)}\right)$$
$$\simeq 2n\, \frac{(1+x)(1+4x+x^2)}{(1-x)^3} = n\,(3\,\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta}).$$

5.2.3 Explicit Bound on Cumulants and the Mod-Gaussian Convergence

By examining the asymptotics of the first cumulants written as rational functions in x, one is lead to the following result. Set

$$P_r(x) = \left(\sum_{c \in \mathfrak{C}_{2r}} \sum_{\delta \in \mathfrak{D}^*_{2r}} A(c)\, B(c,\delta)\, C(\delta)\right) (1-x)^{2r-1}.$$

For instance, $P_1(x) = 1 + x$ and $P_2(x) = 2\,(1+x)(1+4x+x^2)$.

Proposition 37 *For every* $r \geq 1$ *and every* $x \in (0,1)$,

$$0 \leq P_r(x) \leq \frac{(2r)!}{r!}\, \frac{(2r-2)!}{(r-1)!}.$$

Proof For every composition c and every path δ, $B(c,\delta)\,(1-x)^{2r-1}$ is a non-negative and convex function of x on $[0,1]$. Therefore, $0 \leq P_r(x) \leq x\,P_r(0) + (1-x)\,P_r(1)$. When $x = 1$, all the rational functions $B(c,\delta)\,(1-x)^{2r-1}$ vanish, except when c has $2r-1$ descents, that is to say that $c = (1,1,\ldots,1)$. Then, $A(c) = (2r)!$, and

$$\lim_{x\to 1} B(c,\delta) = \prod_{i=1}^{2r-1} \frac{1}{\delta_i} = \frac{1}{C(\delta)}.$$

Therefore,

$$P_r(1) = (2r)!\,(\mathrm{card}\,\mathfrak{D}^*_{2r}) = \frac{(2r)!}{r!}\,\frac{(2r-2)!}{(r-1)!}.$$

On the other hand, when $x = 0$, all the rational functions $B(c,\delta)\,(1-x)^{2r-1}$ vanish, except when c has no descent, that is to say that $c = (2r)$. Then, $A(c) = 1$ and

$$P_r(0) = Q(r) = \sum_{\delta\in\mathfrak{D}^*_{2r}} A(\delta).$$

Among all Dyck paths in $\mathfrak{D}^*_{2r}$, the one with the maximal product of values $G(\delta)$ is $(0,1,2,\ldots,r-1,r,r-1,\ldots,2,1,0)$. So,

$$P_r(0) \leq r!\,(r-1)!\,(\mathrm{card}\,\mathfrak{D}^*_{2r}) = (2r-2)! \leq P_r(1).$$

It follows that $P_r(x) \leq x\,P_r(1) + (1-x)\,P_r(1) = P_r(1)$. □

Corollary 38 *For every r,*

$$|\kappa^{(2r)}(M_n)| \leq n\,(2r-1)!!\,(2r-3)!!\,(\mathrm{e}^{2\beta}+1)^{2r-1}.$$

Proof Indeed,

$$\begin{aligned}|\kappa^{(2r)}(M_n)| &\leq n\left(\sum_{c\in\mathfrak{C}_{2r}}\sum_{\delta\in\mathfrak{D}^*_{2r}} A(c)\,B(c,\delta)\,C(\delta)\right) = n\,\frac{P_r(x)}{(1-x)^{2r-1}}\\ &\leq n\,\frac{P_r(1)}{(1-x)^{2r-1}} = n\left(\frac{1}{1-x}\right)^{2r-1}\frac{(2r)!}{r!}\,\frac{(2r-2)!}{(r-1)!}.\end{aligned}$$

Replacing x by $\tanh\beta$ allows to conclude, and this gives another proof of Theorem 4. We rewrite the logarithm of the Laplace transform of $n^{-1/4}M_n$ as

$$\sum_{r=1}^{\infty}\frac{\kappa^{(2r)}(M_n)}{(2r)!}\,z^{2r}\,n^{-r/2} = \frac{\kappa^{(2)}(M_n)\,z^2}{2n^{1/2}} + \frac{\kappa^{(4)}(M_n)\,z^4}{24n} + \sum_{r=3}^{\infty}\frac{\kappa^{(2r)}(M_n)}{(2r)!}\,z^{2r}\,n^{-r/2}.$$

The series on the right-hand side is smaller than

$$\sum_{r=3}^{\infty} \frac{(2r-1)!!\,(2r-3)!!}{(2r)!}\,(\mathrm{e}^{\beta}+1)^{2r-1}\,z^{2r}\,n^{1-r/2} \leq n^{-1/2}\sum_{r=3}^{\infty}((\mathrm{e}^{2\beta}+1)z)^{2r}\,n^{-(r-3)/2}$$
$$\leq n^{-1/2}\,\frac{((\mathrm{e}^{2\beta}+1)z)^6}{1-((\mathrm{e}^{2\beta}+1)z)^2\,n^{-1/2}},$$

so it goes uniformly to zero on every compact set of the plane. On the other hand, we have seen that $\kappa^{(2)}(M_n) = n\,\mathrm{e}^{2\beta} - O(1)$ and $-\kappa^{(4)}(M_n) = n\,(3\,\mathrm{e}^{6\beta} - \mathrm{e}^{2\beta}) - O(1)$, so we conclude that

$$\mathbb{E}\left[\mathrm{e}^{z\,\frac{M_n}{n^{1/4}}}\right]\mathrm{e}^{-\frac{n^{1/2}\mathrm{e}^{2\beta}\,z^2}{2}} = \mathrm{e}^{-\frac{(3\,\mathrm{e}^{6\beta}-\mathrm{e}^{2\beta})\,z^4}{24}}\,\left(1+O(n^{-1/2})\right),$$

and this is indeed the content of Theorem 4. □

Remark 39 The method of cumulants that leads to Corollary 38, and eventually to Theorem 4, is developed in much more details in [8]. In particular, our approach to the computation of cumulants of sums of dependent random variables coming from complex systems can be made quite general. Thus, given a sum $S = \sum_{v\in V} X_v$, one can obtain powerful bounds on $\kappa^{(r)}(S)$ by:

1. first, computing explicitly the elementary joint cumulants $\kappa^{(r)}(X_{v_1}, X_{v_2}, \ldots, X_{v_r})$, as in Theorem 28;
2. then, find clever rules in order to sum these cumulants and keep correct bounds. In this second part, one needs in particular to identify which elementary cumulants $\kappa^{(r)}(X_{v_1}, X_{v_2}, \ldots, X_{v_r})$ give the main contribution to $\kappa^{(r)}(S) = \sum_{v_1,\ldots,v_r} \kappa^{(r)}(X_{v_1}, X_{v_2}, \ldots, X_{v_r})$.

An important open problem of combinatorial and geometric nature would be to adapt the arguments of this section to the two-dimensional Ising model, for which one cannot compute explicitly the generating function $\mathbb{E}[\mathrm{e}^{zM_n}]$. We expect the methods of cluster expansion (*cf.* [2, Chap. 5]) to be powerful tools in this setting.

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On Sharp Large Deviations for the Bridge of a General Diffusion

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Abstract We provide sharp Large Deviation estimates for the probability of exit from a domain for the bridge of a d-dimensional general diffusion process X, as the conditioning time tends to 0. This kind of results is motivated by applications to numerical simulation. In particular we investigate the influence of the drift b of X. It turns out that the sharp asymptotics for the exit probability are independent of the drift b, provided it satisfies a simple condition that is always satisfied in dimension 1. On the other hand we produce an example where this assumption is not satisfied and the drift is actually influential.

AMS 2000 subject classification: 60F10, 60J60

1 Introduction

Even if the subject of Large Deviations was not one of the most visited among the many objects of investigation in the large scientific production of Marc Yor, he was able to provide three original contributions in this field [14, 22, 23]. On the other hand bridges and conditioned processes have been at the heart of many of his most important contributions. In this short note we investigate some points concerning the asymptotics of conditioned processes when the conditioning time goes to 0.

The investigation of Large Deviation and sharp Large Deviation estimates in this context goes back to [4], where the case of the Brownian bridge and the asymptotics of the exit probability from a general domain D were investigated. This line of research was continued in the subsequent years [5, 7, 17].

These results were motivated by applications to simulation: actually when simulating the path of a stochastic process (with the Euler scheme e.g.) which is killed at the exit from some domain D it is important to be able to compute the

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probability for the conditioned diffusion with $X_{t_n} = x$, $X_{t_{n+1}} = y$ to exit from D in the time interval $[t_n, t_{n+1}]$, where t_n, t_{n+1} are consecutive times in the time grid and $X_{t_n}, X_{t_{n+1}}$ denote the corresponding simulated positions.

Established numerical evidence indicates that the naive approximation of the exit time by the smallest of the values t_i such that $X_{t_i} \notin D$ produces an error that decreases to 0 very slowly, so that this approach is in practice useless. The value of the exit probability for the conditioned process, or at least its asymptotics, allows to produce an improved algorithm that better detects the exit time, see [4, pp. 1645–1646], for a more complete explanation. See also [11, 17] for estimates concerning this improved simulation scheme.

When the simulation concerns a general diffusion, usually the exit probability of the conditioned diffusion is approximated by the corresponding quantity of the diffusion obtained by freezing its coefficients (at $x = X_{t_n}$ e.g.), thus taking advantage of the well known asymptotics of the Brownian bridge. The need of a more thorough investigation is now prompted by applications to finance, e.g. for the numerical computation of barrier options. This is of particular importance in the case of stochastic volatility models, the question being of producing a more accurate estimate or, possibly, to assess the accuracy of the freezing procedure.

Let us remark that problems concerning the simulation of conditioned processes have recently received much attention in connection to a rich context of applications, see [10, 12, 13] e.g.

In this note we investigate sharp Large Deviation estimates for the exit probability, highlighting a particular feature that has some interest from a theoretical point of view. It has been proved [3, 8] that the (non sharp) Large Deviation asymptotics for conditioned diffusions do not depend on the drift b of the non conditioned process X as in (1). It has been a general belief that this remains true also for the sharp Large Deviation asymptotics of the bridge of a diffusion. This was actually proved for a large family of one dimensional diffusion processes in [5]. We prove that in the multidimensional setting this property holds only if the drift satisfies a simple condition, always satisfied in dimension 1.

Our results stem from two main tools: the asymptotics of the exit probability from a domain provided by W.H. Fleming and M.R. James [15], and the asymptotics for small time of the transition density of a diffusion investigated by S.A. Molčanov [20] and G. Ben Arous [9].

Our goal here is mainly to put forward the main ideas and techniques, without trying to look for minimal regularity assumptions.

2 Conditioned Diffusions

Let X be a d-dimensional (possibly inhomogeneous) diffusion process with transition density p. The conditioned diffusion given $X_t = y$, $t > 0$, is associated to the transition density

$$\widehat{p}(u, v, x, z) = \frac{p(u, v, x, z)p(v, t, z, y)}{p(u, t, x, y)}\ , \quad 0 \le u \le v < t,\ x, z \in \mathbb{R}^d\ .$$

Remark that this is a time inhomogeneous transition density, even if X was time homogeneous. Let us assume moreover that X is the solution of the Stochastic Differential Equation (SDE)

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \tag{1}$$

(we consider therefore a process X that is time homogeneous) and let us denote by L its generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(z)\frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{d} b_i(z)\frac{\partial}{\partial z_i}\ ,$$

where, as usual, $a = \sigma\sigma^*$. By a straightforward computation (see also [19]), the generator $\widehat{L}_v$ of the conditioned diffusion is

$$\widehat{L}_v = L + \sum_{i=1}^{d}\widehat{b}_i^{y,t}(z,v)\frac{\partial}{\partial z_i}\ , \quad 0 \le v < t\ ,$$

where

$$\begin{aligned}\widehat{b}_i^{y,t}(z,v) &= \frac{1}{p(t-v,z,y)}\sum_{j=1}^{d} a_{ij}(z)\frac{\partial}{\partial z_j}p(t-v,z,y)\\ &= \sum_{j=1}^{d} a_{ij}(z)\frac{\partial}{\partial z_j}\log p(t-v,z,y)\ ,\end{aligned}$$

i.e.

$$\widehat{b}^{y,t}(z,v) = a(z)\nabla_z \log p(t-v,z,y)\ . \tag{2}$$

The conditioned diffusion has therefore the same distribution as the solution of the SDE

$$d\xi_v = \big(b(\xi_v) + \widehat{b}^{y,t}(\xi_v,v)\big)\,dv + \sigma(\xi_v)\,dB_v$$

for $v < t$. Let $\eta_v^t = \xi_{vt}$, the time changed conditioned diffusion so that it is defined on $[0,1[$; η^t is the solution of

$$d\eta_v^t = \big(b(\eta_v^t) + \widehat{b}^{y,t}(\eta_v^t, tv)\big)t\,dv + \sqrt{t}\sigma(\eta_v^t)\,dB_v \tag{3}$$

(with respect to a possibly different Brownian motion). We can therefore obtain estimates concerning the conditioned diffusion using the Freidlin-Wentzell Large Deviation estimates as soon as we are able to compute the limit

$$\widehat{b}^{y}(z,v) := \lim_{t\to 0} t\,\widehat{b}^{y,t}(z,tv) \tag{4}$$

uniformly on compact sets [1, 6, 21] and prove that the limit function $\widehat{b}^y$ is smooth enough. Recall that

$$\widehat{b}^{y,t}(z,tv) = a(z)\,\nabla_z \log p(t(1-v),z,y)\ . \tag{5}$$

Let us denote by $\widehat{\mathrm{P}}^{y,t}_{x,s}$ the law of η^t with the starting condition $\eta^t_s = x$. Let $D \subset \mathbb{R}^d$ be an open set with a smooth boundary and $\tau = \tau_{x,s}$ the exit time from D. Let us now assume that $a(x) = \sigma\sigma^*(x)$ is non degenerate for $x \in \overline{D}$. The Freidlin-Wentzell theory of Large Deviations states that

$$\lim_{t\to 0} t \log \widehat{\mathrm{P}}^{y,t}_{x,s}(\tau < 1) = -\inf_{\gamma(s)=x,\tau(\gamma)<1} I_s(\gamma) =: -\ell_{x,s}\ , \tag{6}$$

I_s denoting the rate Freidlin-Wentzell function

$$I_s(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_s^1 \langle a(\gamma_v)^{-1}(\dot\gamma_v - \widehat{b}^y(\gamma_v,v)), \dot\gamma_v - \widehat{b}^y(\gamma_v,v)\rangle\, dv & \text{if } \gamma \text{ is absolutely continuous} \\ +\infty & \text{otherwise}\ . \end{cases} \tag{7}$$

3 The Sharp Asymptotics

We want to prove the stronger result

$$q_t(x,s) := \widehat{\mathrm{P}}^{y,t}_{x,s}(\tau < 1) \sim c_{x,s}\,\mathrm{e}^{-\frac{1}{t}\ell_{x,s}}\ , \tag{8}$$

as $t \to 0$, for some constant $c_{x,s} > 0$. We stress the relevant fact that $c_{x,s}$ is a constant independent of t (see Remark 2 for more comments).

We shall investigate the situation where the positions x (the starting point of the process) and y (the conditioning position) are close to each other. This is justified by the application mentioned in Sect. 1, x and y being consecutive positions in a simulation scheme.

The computation of the asymptotics (8) was performed in [4] in the case where X is a multidimensional Brownian motion. The idea there was to take advantage of the results of W.H. Fleming and M.R. James [15]. Let us recall these estimates. Let X^ε be the solution of

$$\begin{cases} dX^\varepsilon_v = \tilde b_\varepsilon(X^\varepsilon_v,v)\,dv + \sqrt{\varepsilon}\,\sigma(X^\varepsilon_v)\,dB_v\ , & v > s \\ X^\varepsilon_s = x \in D\ . \end{cases} \tag{9}$$

Let $T > 0$ be fixed and let us assume that

$$\lim_{\varepsilon\to 0} \tilde b_\varepsilon(x,v) = \tilde b(x,v)$$

uniformly for (x, v) on the compact sets of $D \times [0, T]$. Let us define the function $u : D \times [0, T[\to \mathbb{R}$ by

$$u(x, s) = \inf_{\gamma(s)=x, \tau(\gamma)<T} \tilde{I}_s(\gamma) \tag{10}$$

where $\tilde{I}_s$ is, similarly as in (7),

$$\tilde{I}_s(\gamma) = \begin{cases} \frac{1}{2}\int_s^T \langle a(\gamma_v)^{-1}(\dot{\gamma}_v - \tilde{b}(\gamma_v, v)), \dot{\gamma}_v - \tilde{b}(\gamma_v, v)\rangle\, dv & \text{if } \gamma \text{ is absolutely continuous} \\ +\infty & \text{otherwise}\,. \end{cases} \tag{11}$$

It can be shown that u is the solution of the Hamilton-Jacobi problem

$$\begin{cases} \frac{\partial u}{\partial s} + \langle \tilde{b}, \nabla u\rangle - \frac{1}{2}\langle a\nabla u, \nabla u\rangle = 0 & \text{in } D\times]0, T[\\ u(x, s) = 0 & \text{on } \partial D \times [0, T] \\ u(x, s) \to +\infty & \text{as } s \nearrow T,\ x \in D \end{cases} \tag{12}$$

to be considered in the sense of viscosity solutions [16] and in the classical sense at each point of differentiability of u.

Now let $N \subset D \times [0, T'],\ T' < T$ and define

$$\beta(x, s) = \tilde{b}(x, s) - a(x)\nabla u(x, s), \qquad (x, s) \in \overline{N}\,. \tag{13}$$

Let $\gamma_{x,s}$ be the solution of

$$\begin{cases} \dot{\gamma}_{x,s}(v) = \beta(\gamma_{x,s}(v), v) \\ \gamma_{x,s}(s) = x \end{cases} \tag{14}$$

and set $t^*_{x,s} = \inf\{v > s, (\gamma_{x,s}(v), v) \notin N\}$, moreover define

$$\Gamma_1 = \{(\gamma_{x,s}(t^*_{x,s}), t^*_{x,s}), (x, s) \in N\}\,.$$

Assumption 1

a) N is an open set;
b) $u \in \mathscr{C}^\infty(\overline{N})$;
c) N is a Region of Strong Regularity (RSR) w.r.t. β, i.e. Γ_1 is a C^∞ manifold, relatively open in ∂N, $(\gamma_{x,s}(v), v)_{v\in[s,t^*_{x,s}]}$ crosses Γ_1 non tangentially and $\Gamma_1 \subset \partial D \times (0, T')$.

The following result is a particular case of Theorem 4.2 of [15] which provides an asymptotic expansion for the exit probability, at least for starting points (x, s) belonging to a set N as in Assumption 1. We shall discuss later the meaning of this assumption.

Theorem 1 *Let D be a bounded open set with a smooth boundary. Let* $N \subset D \times [0, T']$ *,* $T' < T$ *satisfy Assumption 1. For the SDE (9) assume that* σ *is infinitely many times differentiable and bounded, the drift* $\tilde{b}_\varepsilon$ *is Lipschitz continuous uniformly with respect to* ε *and enjoys the development*

$$\tilde{b}_\varepsilon = \tilde{b} + \varepsilon \tilde{b}_1 + o(\varepsilon) \;, \tag{15}$$

uniformly on compact sets of N where $\tilde{b}$*,* $\tilde{b}_1$ *are* C^∞ *functions. Then for* $(x, s) \in N$ *the following expansion holds*

$$\mathrm{P}^\varepsilon_{x,s}(\tau \le T) = \mathrm{e}^{-w(x,s)}\, \mathrm{e}^{-\frac{1}{\varepsilon}u(x,s)} \big(1 + o(\varepsilon)\big) \tag{16}$$

uniformly on compact subsets of N, where u is given in (10) or (12) and $w : N \to \mathbb{R}^+$ *is the solution of*

$$\begin{cases} \dfrac{\partial w}{\partial s} + \langle \tilde{b} - a\nabla u, \nabla w\rangle = -\dfrac{1}{2}\, \mathrm{tr}(a \cdot \mathrm{Hess}\, u) - \langle \tilde{b}_1, \nabla u\rangle & \text{in } N \\ w = 0 & \text{on } \partial D \times [0, T[\cap \overline{N} \;. \end{cases} \tag{17}$$

Remark 1 The original result in [15] deals with a more general situation in particular providing a full development of $\varepsilon \mapsto \mathrm{P}^\varepsilon_{x,s}(\tau \le T)$. Beware of some notation mismatch between Theorem 1 and the original Theorem 4.2 of [15]; in particular our $\tilde{b}_1$ corresponds to b_2 there. Remark also that Theorem 1 applies to a general diffusion, not necessarily obtained by conditioning.

The hypotheses in Theorem 1 ensure that for $(x, s) \in N$, there exists a unique minimizing path for the quantity in the right-hand side of (10), which moreover coincides with the solution $\gamma_{x,s}$ of the ordinary Eq. (14) for $v \in [s, t^*_{x,s}]$, $t^*_{x,s}$ turning out to be the first time at which $\gamma_{x,s}$ reaches ∂D. Furthermore, the differential systems (17) for w can be solved by characteristics: one has to solve the ordinary Eq. (14) and then

$$w(x, s) = \int_s^{t^*_{x,s}} \Big(\frac{1}{2}\, \mathrm{tr}(a \cdot \mathrm{Hess}\, u)(\gamma_{x,s}(v), v) + \langle \tilde{b}_1(\gamma_{x,s}(v), v), \nabla u(\gamma_{x,s}(v), v)\rangle\Big)\, dv \;. \tag{18}$$

Remark 2 It is useful to point out two features of Theorem 1. First, because of Assumption 1c), it holds for starting points (x, s) such that the characteristic $\gamma_{x,s}$ reaches the boundary ∂D at a time $t^*_{x,s} < T$.

Second, remark that Large Deviations estimates state that the asymptotics, as $\varepsilon \to 0$, of the quantity of interest $\mathrm{P}^\varepsilon_{x,s}(\tau \le T)$ are, in general, of the form $c(\varepsilon)\, \mathrm{e}^{-\ell/\varepsilon}$, where c is a subexponential function of ε, i.e. such that $\lim_{\varepsilon\to 0} \varepsilon \log c(\varepsilon) = 0$. Theorem 1 states that, under the assumptions considered, the term before the exponential, c, is a constant as a function of ε.

A typical situation where Theorem 1 does not apply, for instance, is when $dX^\varepsilon_v = \sqrt{\varepsilon}\, dB_v$ and $D =]-\infty, L[$ for some $L > 0$. In this case $\gamma_{x,s}(v) = x + \frac{v-s}{1-s}(L-x)$, so that for every $(x, s) \in D \times [0, T]$, $\gamma_{x,s}$ reaches the boundary $\partial D = \{L\}$ at time $T = 1$. Therefore there exists no subset $N \subset D \times [0, T]$ satisfying Assumption 1 and we are outside the range of Theorem 1. Actually by the reflection principle,

$$\begin{aligned} P^{\varepsilon}_{x,s}(\tau \le 1) &= P\Big(\sup_{s \le v \le 1} \{x + \sqrt{\varepsilon}(B_v - B_s)\} \ge L\Big) \\ &= 2P\Big(B_1 - B_s \ge \frac{L-x}{\sqrt{\varepsilon}}\Big) \sim \frac{2}{\sqrt{2\pi(1-s)}(L-x)} \sqrt{\varepsilon}\, e^{-\frac{(L-x)^2}{2(1-s)\varepsilon}} , \end{aligned}$$

so that here the term before the exponential is not a constant, whatever the starting point.

On the other hand the assumptions of Theorem 1 are satisfied in most cases where X^{ε} is the time changed bridge of a diffusion conditioned to be at some point $y \in D$ at time ε, in the sense that, up to a time-change, a “large” subset N of $D \times [0, 1[$ satisfies Assumption 1.

4 Applications and Remarks

In this section we see how to adapt Theorem 1 to the case of the asymptotics (8) for the exit probability of a conditioned diffusion.

A first problem arises from the fact that the drift of the time changed conditioned diffusion has a singularity at time $v = 1$ (think of the case of the Brownian bridge where $\tilde{b}(x, t) = -\frac{x}{1-t}$) so that Theorem 1 cannot be applied with $T = 1$. This difficulty is easily overcome, as remarked in [4], because it turns out that

$$\widehat{P}^{y,t}_{x,s}(\tau < 1) \sim \widehat{P}^{y,t}_{x,s}(\tau < 1 - \delta) \tag{19}$$

for some $\delta > 0$, in the sense that the difference between these two probabilities is exponentially negligible as $t \to 0$. In order to see this, recall Large Deviation estimates recently obtained for conditioned diffusions (see [3, 18] for the case of a compact manifold). These state that, as $t \to 0$, the time changed conditioned diffusion starting at x at time s satisfies a Large Deviation Principle with rate function given by

$$\widehat{J}_s(\gamma) = \begin{cases} J_s(\gamma) - J_s(\overline{\gamma}) & \text{if } \gamma_1 = y \\ +\infty & \text{otherwise} \end{cases} \tag{20}$$

where

$$J_s(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_s^1 \langle a^{-1}(\gamma_v)\dot{\gamma}_v, \dot{\gamma}_v \rangle \, dv & \text{if } \gamma \text{ is absolutely continuous} \\ +\infty & \text{otherwise} \end{cases}$$

and $\overline{\gamma}$ denotes a minimizing geodesic (see below) joining x to y. Therefore we have

$$t\log \widehat{\mathrm{P}}_{x,s}^{y,t}(\tau < 1) \sim -\inf_{\gamma_s = x, \tau(\gamma)<1} \widehat{J}_s(\gamma)\ .$$

Assume that there exists a unique $\hat{\gamma}$ minimizing the right-hand side above, then we can split

$$\widehat{\mathrm{P}}_{x,s}^{y,t}(\tau < 1) = \widehat{\mathrm{P}}_{x,s}^{y,t}(\tau < 1, U(\eta,\hat{\gamma})) + \widehat{\mathrm{P}}_{x,s}^{y,t}(\tau < 1, U(\eta,\hat{\gamma})^c)\ , \tag{21}$$

where $U(\eta,\hat{\gamma})$ denotes a neighborhood of radius η of the minimizer $\hat{\gamma}$. As the infimum of $\widehat{J}_s$ on the set of paths $\{\tau < 1, U(\eta,\hat{\gamma})^c\}$ is strictly larger than the infimum over $\{\tau < 1, U(\eta,\hat{\gamma})\}$, the rightmost term in (21) is exponentially negligible. Let us choose $\eta = \frac{1}{4}\,dist(y,\partial D)$ and let δ be such that $dist(\hat{\gamma}_v, y) \le \eta$ for every $v \ge 1-\delta$. Then for every $\gamma \in U(\eta,\hat{\gamma})$ and $v \ge 1-\delta$ we have $dist(\gamma_v, y) \le 2\eta = \frac{1}{2}\,dist(y,\partial D)$. Therefore if $\tau(\gamma) < 1$, then necessarily $\tau(\gamma) < 1-\delta$.

A second question in order to apply the Fleming-James Theorem 1 to our problem is to determine the development of the drift in Eq. (3), i.e. of finding vector fields $\tilde{b}$ and $\tilde{b}_1$ (of course depending on the conditioning point y) such that

$$\big(b(z) + \widehat{b}^{y,t}(z,tv)\big)t = \tilde{b}(z,v) + t\tilde{b}_1(z,v) + o(t), \quad \text{as } t \to 0$$

uniformly on compact sets and then to compute the corresponding quantities u and w of Theorem 1. As explained in Sect. 2 this requires the development of the quantity $\nabla_z \log p(t(1-v),z,y)$, appearing in the expression of $\widehat{b}^{y,t}$ given in (5).

The tool to this goal is provided by Molčanov results [20] (see also [2], Theorem 1.1, p. 56) together with those of Ben Arous [9]. Let us assume that $a = \sigma\sigma^*$ is elliptic. One can then consider on $\mathbb{R}^d$ the Riemannian metric associated to the matrix field a^{-1}: let us define the length of a smooth curve $\zeta : [0,1] \to \mathbb{R}^d$ by

$$l(\zeta) = \int_0^1 \sqrt{\langle a^{-1}(\zeta_v)\dot{\zeta}_v, \dot{\zeta}_v\rangle}\, dt$$

and the corresponding Riemannian distance by

$$d(x,y) = \inf_{\zeta,\zeta(0)=x,\zeta(1)=y} l(\zeta)\ . \tag{22}$$

Under an assumption of closeness of the points x, y, to be made precise below, we have [20] the development as $t \to 0$

$$\log p(t,x,y) \sim -\frac{d}{2}\log(2\pi t) + \log H(x,y) - \frac{1}{2t} d(x,y)^2 + A(x,y) \tag{23}$$

where d denotes the Riemannian distance (22) of the metric a^{-1}, $H(x,y) = (\det \exp_x'(\xi))^{-1/2}$, $\exp_x$ denoting the exponential map of the Riemann structure associated to the metric a^{-1} and ξ the tangent vector at $t = 0$ of the minimizing geodesic joining x to y, and

$$A(x,y) = \int_0^1 \langle a^{-1}(\overline{\gamma}_v) b(\overline{\gamma}_v), \dot{\overline{\gamma}}_v \rangle \, dt \tag{24}$$

$\overline{\gamma}$ denoting again the unique geodesic joining x to y. These results are obtained under some regularity assumptions on the coefficients b and σ, that should be 4 times differentiable.

As mentioned above this development holds under the hypothesis for the two points x, y to be close i.e. such that they are joined by a unique geodesic along which they are not conjugated. It is a well known fact in Riemannian geometry that for every y there exists a neighborhood $\mathcal{U}_y$ of points x such that this assumption is satisfied for every $x \in \mathcal{U}_y$.

Both H and d are quantities only depending on the metric a^{-1} and not on the drift b which appears only in the quantity (24).

Moreover Théorème 3.4 in [9] allows to state that the behavior as $t \to 0$ of $\nabla_x \log p(t,x,y)$ is obtained by taking formally the derivatives of the right-hand side in (23). We have therefore, as $t \to 0$,

$$t\,\nabla_x \log p(t(1-v),x,y) \sim -\frac{1}{2(1-v)} \nabla_x d(x,y)^2 + t(\nabla_x \log H(x,y) + \nabla_x A(x,y)) \,. \tag{25}$$

In addition the expansions (23) and (25) are uniform for x in compact sets of points that are connected to y by a unique geodesic along which they are not conjugated.

We plan, in a forthcoming paper, to use the development (25) in order to be able to obtain explicitly the values of the constants $c = \mathrm{e}^{-w(x,s)}$, $\ell_{x,s} = u(x,s)$ appearing in the asymptotics (8) for the most common models of stochastic volatility. In this note, as pointed out in the introduction, we just wish to investigate the question whether the drift b has an influence in the development (8). We already know that the answer is no for a large class of diffusions in dimension $d = 1$ [5] and also, in a multidimensional setting, if X is a Brownian motion with a constant drift: the bridge of a Brownian motion with a constant drift is exactly equal to a Brownian bridge, so here to the (constant) drift has no effect. We start first with an example.

Example 1 Let

$$dX_t = MX_t\, dt + dB_t \tag{26}$$

be a d-dimensional Ornstein-Uhlenbeck process where M is a $d \times d$-dimensional matrix. Let us start computing the development of the drift of this diffusion conditioned by $X_t = y$ as $t \to 0$.

The transition density $p(t, x, \cdot)$ is the density of a $N(e^{Mt}x, S_t)$-distributed r.v., where

$$S_t = \int_0^t e^{Mu} e^{M^* u}\, du\ .$$

Therefore

$$\begin{aligned}\log p(t(1-v), z, y) = &-\frac{d}{2}\, \log 2\pi - \frac{1}{2}\, \log\det(S_{t(1-v)})\\ &-\frac{1}{2}\left\langle S^{-1}_{t(1-v)}(y - e^{Mt(1-v)}z), (y - e^{Mt(1-v)}z)\right\rangle\end{aligned}$$

and

$$\begin{aligned}\nabla_z\, \log p(t(1-v), z, y) = &\ \frac{1}{2}\, e^{M^* t(1-v)} S^{-1}_{t(1-v)}\, (y - e^{Mt(1-v)}z)\\ &+ \frac{1}{2}\, (y - e^{Mt(1-v)}z)^* S^{-1}_{t(1-v)} e^{Mt(1-v)}\ .\end{aligned}$$

Writing down the developments as $t \to 0$ of the various terms appearing above we have

$$\begin{aligned}S_t &= tI + (M + M^*)\frac{t^2}{2} + o(t^2)\ , \quad S_t^{-1} = \frac{1}{t}\left(I - (M + M^*)\tfrac{t}{2} + o(t)\right)\\ e^{tM^*} &= I + tM^* + o(t), \quad e^{tM} = I + tM + o(t)\end{aligned}$$

so that

$$\begin{aligned}e^{tM^*} S_t^{-1} &= \frac{1}{t}\, I - \frac{1}{2}\, (M + M^*) + M^* + o(1) = \frac{1}{t}\, I + \frac{1}{2}\, (M^* - M) + o(1)\\ S_t^{-1} e^{tM} &= \frac{1}{t}\, I - \frac{1}{2}\, (M + M^*) + M + o(1) = \frac{1}{t}\, I + \frac{1}{2}\, (M - M^*) + o(1)\ .\end{aligned}$$

Also $y - e^{Mt}z = y - z + z - e^{Mt}z = y - z - Mtz + o(t)$, hence, uniformly for z in compact sets,

$$\begin{aligned}&\frac{1}{2}\, e^{M^* t(1-v)} S^{-1}_{t(1-v)}\, (y - e^{Mt(1-v)}z) = \frac{1}{2}\left(\frac{1}{t(1-v)}\, I + \frac{1}{2}\, (M - M^*) + o(1)\right)(y - z - Mt(1-v)z + o(t))\\ &= \frac{1}{2}\left(\frac{y - z}{t(1-v)} + \frac{1}{2}\, (M - M^*)(y - z) - Mz + o(1)\right)\ .\end{aligned}$$

Similarly

$$\frac{1}{2}\,(y-\mathrm{e}^{Mt(1-v)}z)^* S_{t(1-v)}^{-1}\mathrm{e}^{Mt(1-v)}$$
$$=\frac{1}{2}\Big(\frac{y-z}{t(1-v)}+\frac{1}{2}\,(M^*-M)(y-z)-M^*z+o(1)\Big)$$

and putting things together, after some straightforward computations, we find, uniformly for z in compact sets,

$$t\nabla_z\,\log p(t(1-v),z,y)=\frac{y-z}{1-v}-\frac{1}{2}\,t(M+M^*)z+o(t)\ .$$

Remark that the same result would have been obtained very quickly using (25), as here $H(x,y)\equiv 1$ and $d(x,y)=|x-y|$. Therefore the asymptotics for the drift of the bridge of X given $X_t=y$ is

$$\begin{aligned} t\big(b(z)+\widehat{b}^{y,t}(z,tv)\big) &= \frac{y-z}{1-v}-\frac{1}{2}\,t(M+M^*)z+tMz+o(t)\\ &= \frac{y-z}{1-v}+\frac{1}{2}\,t(M-M^*)z+o(t)\ . \end{aligned}$$

Hence we are as in (15) with

$$\tilde{b}(z,v)=\frac{y-z}{1-v},\quad \tilde{b}_1(z,v)=\frac{1}{2}\,(M-M^*)z\ .$$

Remark that $\tilde{b}_1\equiv 0$ if and only if the matrix M is symmetric. Therefore, in general, the quantity w which determines the value of the constant c in the expansion (8) depends on the drift $z\mapsto Mz$, unless M is symmetric.

To be more precise let us consider the case where D is the half-space $\{z,\langle \mathbf{v},z\rangle<k\}$ for some $\mathbf{v}\in\mathbb{R}^d$, $|\mathbf{v}|=1$ and $k>0$. Let $x,y\in D$ and let us evaluate the expansion (8) for the process X conditioned by $X_t=y$, where τ denotes the exit time from D.

Note first that the function u defined in (10) coincides with the one for the bridge of the Brownian motion, i.e.

$$u(x,s)=\frac{2}{1-s}\,(k-\langle x,\mathbf{v}\rangle)\,(k-\langle y,\mathbf{v}\rangle) \tag{27}$$

(it is also easy to check that such a function u satisfies (12)). Of course $\Delta_x u\equiv 0$ (u is a linear function of x) and

$$\nabla u(x,s)=-\frac{2}{1-s}\,(k-\langle y,\mathbf{v}\rangle)\,\mathbf{v}\ .$$

Therefore the sharp asymptotics as the conditioning time t tends to 0 of the exit probability $q_t(x,s)$ (8) for the diffusion starting at $X_s = x$ and conditioned by $X_t = y$, is

$$q_t(x,s) \sim \mathrm{e}^{-w(x,s)} \mathrm{e}^{-\frac{1}{t}\, u(x,s)}\ , \tag{28}$$

where, recalling (18),

$$w(x,s) = -\,(k - \langle y, \mathbf{v}\rangle) \int_s^\tau \frac{1}{1-t} \langle (M - M^*)\gamma_{x,s}(t), \mathbf{v}\rangle \, dt\ , \tag{29}$$

$\gamma_{x,s}$ being the solution of

$$\dot\gamma_{x,s}(v) = \tilde b(\gamma_{x,s}(v), v) - \nabla u(\gamma_{x,s}(v), v)\ , \qquad \gamma_{x,s}(s) = x \tag{30}$$

and $\tau = t^*_{x,s}$ the time at which $\gamma_{x,s}$ reaches the boundary ∂D. Straightforward computations lead to the solution

$$\gamma_{x,s}(v) = x + \frac{v-s}{\tau-s}(\eta - x) \qquad s \le v \le \tau\ , \tag{31}$$

where

$$\tau = s+(1-s)\frac{k-\langle x,\mathbf{v}\rangle}{2k-\langle x+y,\mathbf{v}\rangle} \quad \text{and} \quad \eta = x+\frac{k-\langle x,\mathbf{v}\rangle}{2k-\langle x+y,\mathbf{v}\rangle}(y-x+2(k-\langle y,\mathbf{v}\rangle)\mathbf{v})\ . \tag{32}$$

Remark that $\tau < 1$ and $\eta \in \partial D$ does not depend on s; $\gamma_{x,s}$ is the line segment connecting x to η. Going back to (29) we have

$$w(x,s) = -\,(k - \langle y, \mathbf{v}\rangle) \int_s^\tau \frac{1}{1-t} \big\langle \mathbf{v}, (M - M^*)(x + \tfrac{t-s}{\tau-s}(\eta - x))\big\rangle \, dt$$

which gives easily

$$w(x,s) = \big(k-\langle y,\mathbf{v}\rangle\big)\Big[\frac{k-\langle x,\mathbf{v}\rangle}{2k-\langle x+y,\mathbf{v}\rangle}\langle \mathbf{v}, (M-M^*)(y-x)\rangle + \log\Big(\frac{k-\langle y,\mathbf{v}\rangle}{2k-\langle x+y,\mathbf{v}\rangle}\Big)\,\langle \mathbf{v}, (M-M^*)y\rangle\Big] \tag{33}$$

($w(x,s)$ does not depend on s). Therefore for the quantity $q_t(x,s)$ in (8) we have

$$c_{x,s} = \Big(\frac{k-\langle y,\mathbf{v}\rangle}{2k-\langle x+y,\mathbf{v}\rangle}\Big)^{-(k-\langle y,\mathbf{v}\rangle)\langle \mathbf{v},(M-M^*)y\rangle}\ \mathrm{e}^{-\frac{(k-\langle y,\mathbf{v}\rangle)(k-\langle y,\mathbf{v}\rangle)}{2k-\langle x+y,v\rangle}\,\langle \mathbf{v},(M-M^*)(y-x)\rangle}$$

and

$$\ell_{x,s} = \frac{2}{1-s}\,(k - \langle x, \mathbf{v}\rangle)\,(k - \langle y, \mathbf{v}\rangle)\ .$$

We stress that the expansion (8) depends indeed on the matrix M unless it is symmetric: if M is any symmetric matrix, its value has no influence on (8), which is then exactly the same as if X was the Brownian motion, i.e. $w \equiv 0$.

We did not bother to check the assumptions of Theorem 1. It is not however difficult, given the computations above, for a given $x \in D$, to construct a RSR containing $(x, 0)$. Indeed remark that thanks to the expression of τ in (32), every characteristic $\gamma_{x,s}$ reaches ∂D at a time that is strictly smaller than 1. One can therefore construct a RSR of the form $N = \{(z, s); z \in D, t^*_{z,s} < T'\}$, for some T' such that $t^*_{x,s} < T' < 1 - \delta$ where δ is given in (19). The only remaining assumption to be checked is that D is assumed there to be bounded, which is not our case. This point is explained in the next remark.

Remark 3 It is easy to show that the asymptotics for the probability of exit from an open set D is, by a standard localization argument, the same as for the exit from a suitable bounded subset $\tilde D \subset D$. To be precise, a repetition of the argument leading to (19) yields

$$q_t(x, s) \sim \widehat{\mathrm{P}}^{y,t}_{x,s}(\tau < 1, U(\eta, \hat\gamma))\ ,$$

where $U(\eta, \hat\gamma)$ is a neighborhood of radius η of the minimizer $\hat\gamma$ for $\inf_{\gamma_0=x,\tau(\gamma)<1} \widehat{J}_0(\gamma)$, $\widehat{J}_s$ being given in (20). One can then set $\tilde D$ to be the intersection of D with a bounded neighborhood of the support of $\hat\gamma$, chosen in such a way as to preserve the smoothness of the boundary.

The previous example shows, among other things, that the sharp Large Deviation estimate of the exit probability of the bridge of a multidimensional Ornstein-Uhlenbeck process depends on the drift of the original process, unless the matrix M is symmetric. This is a phenomenon that is better investigated in the following statement.

Proposition 1 *Let X be the d-dimensional diffusion process that is the solution of the SDE*

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \tag{34}$$

and assume that $a = \sigma\sigma^*$ is elliptic and that b and σ are four times differentiable. Let us denote η^t the corresponding process conditioned by $X_t = y$, $t > 0$ and time changed (see (3)). Let x be close enough to y, in the sense that x and y are joined by a unique geodesic $\overline{\gamma}$ of the metric a^{-1} along which they are not conjugated. Then if there exists a potential $U : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla U = a^{-1}b$, the development for the drift of η^t up to the first order as $t \to 0$ does not depend on b.

Proof We must prove for the drift $(b(x)+\widehat{b}^{y,t}(x,tv))t$ of the time changed conditioned process η^t a development of the form $(b(x)+\widehat{b}^{y,t}(x,tv))t=\tilde{b}(x,v)+t\tilde{b}_1(x,v)+o(t)$ where neither $\tilde{b}$ nor $\tilde{b}_1$ depend on b and which is moreover uniform on a compact neighborhood of y. Recall the development (25): thanks to (24), if $\nabla U=a^{-1}b$ we have of course

$$A(x,y)=U(\overline{\gamma}_1)-U(\overline{\gamma}_0)=U(y)-U(x)$$

and $\nabla_x A(x,y)=-\nabla U(x)=-a^{-1}(x)b(x)$. Hence, by (25), as $t\to 0$, uniformly in a compact neighborhood of y

$$\begin{aligned}\widehat{b}^{y,t}(x,tv)&=a(x)\nabla_x\log p(t(1-v),x,y)\\&\sim a(x)\Big(\nabla_x\log H(x,y)-\frac{1}{2t(1-v)}\nabla_x d(x,y)^2-a^{-1}b(x)\Big)\end{aligned}$$

so that the drift of the time changed conditioned process is

$$\begin{aligned}\big(b(x)+\widehat{b}^{y,t}(x,tv)\big)t&\sim tb(x)\\&\quad+ta(x)\Big(\nabla_x\log H(x,y)-\frac{1}{2t(1-v)}\nabla_x d(x,y)^2-a^{-1}b(x)\Big)\\&=ta(x)\nabla_x\log H(x,y)-\frac{1}{2(1-v)}a(x)\nabla_x d(x,y)^2\ ,\end{aligned}$$

thus $\tilde{b}(x,v)=-\frac{1}{2(1-v)}a(x)\nabla_x d(x,y)^2$, whereas $\tilde{b}_1(x,v)=a(x)\nabla_x\log H(x,y)$ (remark that $\tilde{b}_1$ does not depend on v). Both $d(x,y)$ and $H(x,y)$ are quantities associated to the Riemann metric a^{-1} and neither of them depends on b. □

Remark 4 If the hypotheses in Theorem 1 are satisfied, then the drift of the unconditioned diffusion does not influence the sharp asymptotics for the probability of exit from the domain D, i.e. neither u nor w in (16) depends on b in (34). Recall that D could be unbounded, actually the argument in Example 1 holds in great generality.

Coming back to Example 1, of course if M is symmetric then the drift $z\mapsto Mz$ turns out to be the gradient field of the potential $U(z)=\frac{1}{2}\langle Mz,z\rangle$. We have therefore proved that, whenever the Fleming-James Theorem 1 can be applied, the asymptotics (8) do not depend on the drift b of the original diffusion as far as $a^{-1}b$ is a gradient field and also (Example 1) that the drift can be influential if this assumption is not satisfied.

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Large Deviations for Clocks of Self-similar Processes

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In memoriam, Marc Yor

Abstract The Lamperti correspondence gives a prominent role to two random time changes: the exponential functional of a Lévy process drifting to ∞ and its inverse, the clock of the corresponding positive self-similar process. We describe here asymptotical properties of these clocks in large time, extending the results of Yor and Zani (Bernoulli **7**, 351–362, 2001).

1 Introduction

This problem is an extension of a question raised by Marc Yor during the defense of the thesis of Marguerite, under the supervision of Alain, long time ago in 2000. The last part of this thesis was dedicated to the study of large deviations principles (LDP) for Maximum Likelihood Estimates of diffusion coefficients (for squared-radial Ornstein–Uhlenbeck processes, squared Bessel processes and Jacobi processes). The main tool there was a convenient Girsanov change of probability. This method allowed to convert the computation of Laplace transform of some additive functionals into the computation of Laplace transform of a single variable. This trick was used before in [24], p. 26, or [19], p. 30, where Marc Yor called it

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"reduction method". In those times, Marc was interested in exponential functionals of Brownian motion, Lamperti transform and Asian options and he guessed that this LDP could be applied to the Bessel clock and solved effectively the problem with Marguerite in [26] a couple of weeks later. Marc suggested to extend it to the Cauchy clock and gave a sketch of proof, but a technical difficulty stopped the project. Meanwhile, Marguerite and Nizar published a paper [12] on Jacobi diffusions where the reduction method is again crucial. Recently, the three of us felt the need to revisit the problem of Cauchy clock with the hope of new ideas.

We planned to discuss with Marc, and promised him we would keep him informed of our progress. We did not have the time…

In this paper, we extend the methods and results of [26] to a large class of clocks issued from positive self-similar Markov processes. In Sect. 2 we recall some basic results about the Lamperti correspondence between these processes and Lévy processes and we give the definition of the clocks. We also define a generalized Ornstein-Uhlenbeck process which will be useful in the sequel. In Sects. 3 and 4, we present the main results: Law of Large Numbers and Large Deviations for the clocks. In Sect. 5 we show some examples illustrating our main theorems.

2 Positive Self-similar Markov Processes and Lamperti Transformation

In [17] Lamperti defined (positive) semi-stable process which are nowadays called positive self-similar Markov process.

Definition 1 For $\alpha > 0$, a positive self-similar Markov process (pssMp) of index α, is a $[0,\infty)$-valued strong Markov process $(X, \mathbb{Q}_a)$, $a > 0$ with càdlàg paths, fulfilling the scaling property

$$(\{bX_{b^{-\alpha}t}, t \geq 0\}, \mathbb{Q}_a) \overset{(law)}{=} (\{X_t, t \geq 0\}, \mathbb{Q}_{ba}) \tag{1}$$

for every $a, b > 0$.

Lamperti [17] has shown that these processes can be connected to Lévy processes by a one-to-one correspondence, that we develop below. We refer to Kyprianou [15] especially Chap. 13 for properties of Lévy processes and pssMp. One can also see [4] for the Lamperti's correspondence. One can notice that there is a little confusion in the notion of index of these processes. In [17], [3] and [2] the index is $1/\alpha$, and in [4, 6, 15], the index is α. We take this latter convention. These processes have a natural application in the theory of self-similar fragmentations: see [1], references therein, and [5]. For other areas of application, such as diffusions in random environments, see Sect. 6 of [4].

2.1 From X to ξ

Any pssMp X which never reaches the boundary state 0 may be expressed as the exponential of a Lévy process not drifting to $-\infty$, time changed by the inverse of its exponential functional. More formally, if $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α which never reaches 0, set

$$T^{(X)}(t) = \int_0^t \frac{ds}{X_s^\alpha}\ ,\ (t \geq 0) \tag{2}$$

and let $A^{(X)}$ be its inverse, defined by

$$A^{(X)}(t) = \inf\{u \geq 0 : T^{(X)}(u) \geq t\}\,, \tag{3}$$

and let ξ be the process defined by

$$\xi_t = \log X_{A^{(X)}(t)} - \log X_0\ ,\ (t \geq 0)\,. \tag{4}$$

Then, for every $a > 0$, the distribution of $(\xi_t, t \geq 0)$ under $\mathbb{Q}_a$ does not depend on a and is the distribution of a Lévy process starting from 0.

Moreover, if we set

$$\mathscr{A}^{(\xi)}(t) = \int_0^t e^{\alpha \xi_s} ds \tag{5}$$

and $\tau^{(\xi)}$ its inverse defined by

$$\tau^{(\xi)}(t) = \inf\{u \geq 0 : \mathscr{A}^{(\xi)}(u) \geq t\} \tag{6}$$

we have

$$\tau^{(\xi)}(t) = T^{(X)}(tX_0^\alpha)\,. \tag{7}$$

Let us remark that the self-similarity property (1) leads to the following relation:

$$\text{the law of } \ T^{(X)}(.\, b^{-\alpha}) \ \text{ under } \ \mathbb{Q}_a \ \text{ is the law of } \ T^{(X)}(.) \ \text{ under } \ \mathbb{Q}_{ba}\,. \tag{8}$$

2.2 From ξ to X

Let (ξ_t) be a Lévy process starting from 0 and let $\mathbb{P}$ and $\mathbb{E}$ denote the underlying probability and expectation, respectively.

Fix $\alpha > 0$. Let $\mathscr{A}^{(\xi)}$ be its exponential functional defined by (5). When ξ drifts to $-\infty$, this functional is very popular in mathematical finance (see [25]), with important properties of the perpetuity $\mathscr{A}^{(\xi)}(\infty)$. Here, we rather assume that ξ does not drift to $-\infty$ i.e. satisfies, $\limsup_{t\uparrow\infty} \xi_t = \infty$. We define the inverse process $\tau^{(\xi)}$ of $\mathscr{A}^{(\xi)}$ by (6).

For every $a > 0$, let $\mathbb{Q}_a$ be the law under $\mathbb{P}$ of the time-changed process

$$X_t = a \exp \xi_{\tau^{(\xi)}(ta^{-\alpha})} \,, \; (t \geq 0) \,, \tag{9}$$

then $(X, (\mathbb{Q}_a)_{a>0})$ is a pssMp of index α which never reaches 0 and we have the fundamental relation (7).

2.3 Index and Starting Point

If $(X, (\mathbb{Q}_a)_{a>0})$ is an pssMa of index α, then the process $Y = (X^\alpha, (\mathbb{Q}_{a^\alpha})_{a>0})$, is a pssMp of index 1. Conversely if $(Y, (\mathbb{Q}_a)_{a>0})$ is a pssMp if index 1 then, for any $\alpha > 0$, the process $(X = Y^{1/\alpha}, (\mathbb{Q}_{a^{1/\alpha}})_{a>0})$ is a pssMp of index α. Hence *we may and will assume without loss of generality, that* $\alpha = 1$, except some examples in Sect. 5.

Let us stress also the fact that the distribution of $T^{(X)}$ is considered under $\mathbb{Q}_a$ for $a > 0$, and the distribution of $\tau^{(\xi)}$ is considered under $\mathbb{P}$.

2.4 Ornstein-Uhlenbeck Process

Having defined the above pair of associated processes (ξ, X) it is usual to consider a third process, very useful in the sequel. To introduce it, we need additional properties on the pssMp X. In Theorem 1 of [3], Bertoin and Yor have shown that whenever the support of ξ is not arithmetic and $\mathbb{E}\xi_1 > 0$, then as $a \downarrow 0$, the sequence of probability measures $\mathbb{Q}_a$ (defined in Sect. 2.2) converges to a probability measure denoted by $\mathbb{Q}_0$. This latter measure is an entrance law for the semigroup $Q_t f(x) = \mathbb{E}_x f(X_t)$ and satisfies

$$\mathbb{Q}_0(f(X_t)) = \frac{1}{\mathbb{E}\xi_1} \mathbb{E}[I_\infty^{-1} f(t/I_\infty)] \tag{10}$$

where

$$I_\infty = \int_0^\infty \mathrm{e}^{-\xi_s} \mathrm{d}s \,.$$

Now we define the associated generalized Ornstein-Uhlenbeck (OU) process by

$$U(t) := \mathrm{e}^{-t}X(\mathrm{e}^t)\,.$$

This process has been studied by Carmona, Petit, Yor in [8], and further on by Rivero [23] and Caballero and Rivero [5]. It is strictly stationary, Markovian and ergodic under $\mathbb{E}_0$, and its invariant measure is the law of X_1 under $\mathbb{Q}_0$, i.e.

$$\mu(f) = \frac{1}{\mathbb{E}\xi_1}\mathbb{E}[I_\infty^{-1}f(1/I_\infty)]\,.$$

The purpose of this paper is the study of the asymptotic behavior of the process $T^{(X)}$, called the clock of the pssMp X, as $t \to \infty$ under $\mathbb{Q}_a$ for any $a > 0$, or equivalently the asymptotic behavior of the process $\tau^{(\xi)}$, as $t \to \infty$ under $\mathbb{P}$. The two points of view are complementary: the clock as a functional of the pssMp or the clock as a functional of the Lévy process. We will restrict us to the case of $\mathbb{E}\xi_1 \in (0,\infty)$, which ensures a Law of Large Numbers (LLN). To get a Large Deviation Principle (LDP), we will make an assumption on the Laplace transform of ξ_1.

3 Law of Large Numbers

The first step of our study is the a.s. convergence of our functionals.

Theorem 1 *If* $\mathbb{E}\xi_1 \in (0,\infty)$*, then for every* $a > 0$*, we have as* $t \to \infty$

$$\frac{T^{(X)}(t)}{\log t} \to (\mathbb{E}\xi_1)^{-1}\,,\quad \mathbb{Q}_a - a.s. \tag{11}$$

or equivalently

$$\frac{\tau^{(\xi)}(t)}{\log t} \to (\mathbb{E}\xi_1)^{-1}\,,\quad \mathbb{P} - a.s. \tag{12}$$

Proof From (7) it is enough to prove (11). Let us now repeat verbatim the trick from [5], Sect. 2. By the ergodic theorem, we have that for $f \in L^1(\mu)$:

$$\frac{1}{t}\int_0^t f(U_s)\mathrm{d}s \to \mu(f)\,,\quad \mathbb{Q}_0 - \text{a.s.}$$

or, by scaling

$$\frac{1}{\log t}\int_1^t f(u^{-1}X_u)\frac{\mathrm{d}u}{u} \to \mu(f)\,,\quad \mathbb{Q}_0 - \text{a.s.} \tag{13}$$

For $f(x) = x^{-1}$ this yields

$$\frac{1}{\log(1+t)} \int_1^{1+t} \frac{\mathrm{d}u}{X_u} \to (\mathbb{E}\xi_1)^{-1} \ , \ \ \mathbb{Q}_0 - \text{a.s.}$$

If $\mathscr{H}$ denotes the set where the convergence holds, then the Markov property yields:

$$\mathbb{E}_0 \left[\mathbb{Q}_{X_1} \left(\frac{T^{(X)}(t)}{\log(1+t)} \to \ (\mathbb{E}\xi_1)^{-1} \right) \right] = \mathbb{P}_0(\mathscr{H}) = 1$$

If we remember that μ is the law of X_1 under $\mathbb{Q}_0$, we have

$$\mathbb{Q}_x \left(\frac{T^{(X)}(t)}{\log(1+t)} \to \ (\mathbb{E}\xi_1)^{-1} \right) = 1 \ , \ \ \mu - \text{a.e.} \, .$$

Fixing such an x and using the scaling property (8), we get (11). □

It is then natural to look for a Large Deviation Principle (LDP) to characterize the speed of this convergence. For definition and notation about large deviations, we follow [11].

4 LDP

We present a LDP for the clock $T^{(X)}(t)$ (Theorem 2), whose proof relies on the reduction method (Lemma 1). The remaining of the section is devoted to some illustrative remarks and the proof of a technical lemma. In view of (7), the following statements are valid when $T^{(X)}$ and $\mathbb{Q}_a$ are replaced by $\tau^{(\xi)}$ and $\mathbb{P}$, respectively.

As is usual in large deviations techniques, we need an assumption on the existence of some exponential moments. Let ψ be the Laplace exponent of (ξ_t); it is a convex function with values in $(-\infty, \infty]$ given by

$$\mathbb{E}(\exp m\xi_t) = \exp(t\psi(m)) \quad (m \in \mathbb{R}) \, , \tag{14}$$

and $\operatorname{dom} \psi := \{m : \psi(m) < \infty\}$ is an interval containing 0. In the sequel we will assume the following:

Assumption 4.1 *The interior of dom ψ contains 0, we denote it by (m_-, m_+). It is known that ψ is analytic on (m_-, m_+). We assume furthermore $\psi'(0) > 0$.*

Under this assumption, we set

$$m_0 = \inf\{\theta : \psi'(\theta) > 0\} \ \ , \tau_+ = \frac{1}{\psi'(m_+)} \ , \ \tau_0 = \frac{1}{\psi'(m_0)} \text{ and } \Delta = (\tau_+, \tau_0) \, ,$$

where $1/\psi'(\pm\infty) := \lim_{m\to\pm\infty} m/\psi(m)$. Let us notice also the important trivial relation

$$\mathbb{E}\xi_1 = \psi'(0)\,. \tag{15}$$

4.1 Main Result

Theorem 2 *Let $a > 0$.*

1. *For every $x \in \bar{\Delta}$ we have*

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{Q}_a\left(\frac{T^{(X)}(t)}{\log t} \in [x-\varepsilon, x+\varepsilon]\right) = -\mathscr{I}(x) \tag{16}$$

   where

$$\mathscr{I}(x) = \sup_{m\in(m_0,m_+)} \{m - x\psi(m)\}\,. \tag{17}$$

2. *Moreover*

$$\lim_{A\to\infty}\limsup_{t\to\infty} \frac{1}{\log t} \log \mathbb{Q}_a\left(\frac{T^{(X)}(t)}{\log t} > A\right) = -\infty\,. \tag{18}$$

3. *If either $\Delta = (0,\infty)$ or the complement of $\bar{\Delta}$ is exponentially negligible, i.e. if*

$$\forall \varepsilon > 0 \quad \limsup_{t\to\infty} \frac{1}{\log t} \log \mathbb{Q}_a\left(\inf_{x\in\bar{\Delta}}\left\{\left|x - \frac{T^{(X)}(t)}{\log t}\right|\right\} > \varepsilon\right) = -\infty \tag{19}$$

   then the family of distributions of $T^{(X)}(t)/\log t$ under $\mathbb{Q}_a$ satisfies the LDP on $[0,\infty)$ at scale $\log t$ with good rate function:

$$\tilde{\mathscr{I}}(x) = \begin{cases} \mathscr{I}(x) & \text{if } x \in \bar{\Delta}, \\ \infty & \text{otherwise.} \end{cases}$$

The following proposition describes some properties of $\mathscr{I}$ which are direct consequences of (17) and of the convexity of ψ. The proof is left to the reader.

Proposition 1

1. *If ψ^* denotes the Fenchel-Legendre dual of ψ defined by*

$$\psi^*(x) = \sup_{m\in\mathbb{R}}\{mx - \psi(m)\}\ \ ,\ \ (x \in \mathbb{R})\,,$$

then

$$\mathscr{I}(x) = x\psi^*(x^{-1}) \ \ , \ (x \in \Delta)\,. \tag{20}$$

2. *The function* $\mathscr{I}$ *is convex on* Δ *and has a minimum* 0 *at*

$$\tau_e = \frac{1}{\mathbb{E}\xi_1} \in (\tau_+, \tau_0)\,.$$

3. a. *If* m_0 *is a true minimum, i.e. if* $m_0 > -\infty$ *and* $\psi'(m_0) = 0$*, then* $\tau_0 = \infty$*, and as* $x \to \infty$*,* $\mathscr{I}$ *admits the asymptote* $y = -x\psi(m_0) + m_0$*.*
   b. *If* $m_0 = -\infty$ *and* $\psi'(m_0) \in (0, \infty)$ *, set* $\lim_{m\to m_0} \psi(m) - m\psi'(m_0) = -b \in [-\infty, 0)$*. Then* $\tau_0 < \infty$ *and* $\mathscr{I}(\tau_0) = b\tau_0$ *and* $\mathscr{I}'(\tau_0) = -\psi(m_0) = \infty$*.*
   c. *If* $m_0 = -\infty$ *and* $-\infty < \psi(m_0) < 0$*, hence* $\psi'(m_0) = 0$*, then* $\tau_0 = \infty$ *and* $\mathscr{I}(x)/x \to -\psi(m_0)$ *and* $\mathscr{I}(x) + \psi(m_0)x \to -\infty$*, as* $x \to \infty$*.*
4. a. *If* $m_+ < \infty$ *and* $\psi(m_+) = \infty$*, then* $\psi'(m_+) = \infty$*. We get* $\tau_+ = 0$ *and* $\mathscr{I}(0) = m_+$ *with* $\mathscr{I}'(0) = \infty$*.*
   b. *If* $m_+ = \infty$ *and* $\psi'(m_+) < \infty$*, set* $\lim_{m\to m_+} \psi(m) - m\psi'(m_+) = -b \in [-\infty, 0)$*. Then* $\tau_+ > 0$*,* $\mathscr{I}(\tau_+) = b\tau_+ \le \infty$ *and* $\mathscr{I}'(\tau_+) = -\psi(m_+) = -\infty$*.*
   c. *If* $m_+ = \infty$*, with* $\psi(m_+) = \infty$ *and* $\psi'(m_+) = \infty$ *then* $\tau_+ = 0$ *and* $\mathscr{I}(\tau_+) = m_+ = \infty$*.*

Proof of Theorem 2 1) A slight adaptation of the Gärtner-Ellis method ([11], Th. 2.3.6) allows to deduce (16) from the asymptotic behaviour of the normalized log-Laplace transform of $T^{(X)}(t)$, given by the following lemma. Recall that ψ is increasing on (m_0, m_+).

Lemma 1 *For* $\theta \in (-\psi(m_+), -\psi(m_0))$*, set* $L_t(\theta) = \log \mathbb{E}_a \exp(\theta T^{(X)}(t))$*. Then as* $t \to \infty$ *we have*

$$\frac{1}{\log t} L_t(\theta) \to L(\theta) \tag{21}$$

where

$$L(\theta) = -m \iff \theta = -\psi(m)\,. \tag{22}$$

The function L is differentiable on $(-\psi(m_+), -\psi(m_0))$ and satisfies $L'(\theta) = 1/\psi'(m)$, so that the range of L' is precisely Δ. Then the left-hand-side of (16) admits the limit $\mathscr{I}(x) = x\theta - L(\theta)$ where θ is the unique solution of $L'(\theta) = x$ i.e., thanks to (22), $\mathscr{I}(x) = -x\psi(m) + m$ which is exactly the right-hand-side of (17).

Proof of Lemma 1 To compute the Laplace transform of $T^{(X)}(t)$, we use a Girsanov type change of probability.

For $m \in (m_-, m_+)$ let

$$\psi_m(\theta) = \psi(m+\theta) - \psi(m)\,,$$

and let $\{(\xi_t, t \geq 0); \mathbb{P}^m\}$ be a Lévy process starting from 0 whose exponent is ψ_m (Esscher transform). Finally let $\{(X_t, t \geq 0); (\mathbb{Q}_a^m)_{a>0}\}$ be the associated pssMp.

Besides, from [10] the following relation between the infinitesimal generators L^ξ of ξ and L^X of X under $\mathbb{Q}_a$:

$$L^X f(x) = \frac{1}{x} L^\xi (f \circ \exp)(\log x)\,,$$

implies (see (2.7) in [10]),

$$L^X f_m = \psi(m) f_{m-1}$$

for $f_m(x) = x^m$ so that

$$\mathbb{Q}_a^m|_{\mathscr{F}_t} = \left(\frac{X_t}{a}\right)^m \exp\left(-\psi(m)\int_0^t \frac{\mathrm{d}s}{X_s}\right) . \mathbb{Q}_a|_{\mathscr{F}_t}\ ,\ (t \geq 0)\,. \tag{23}$$

We deduce that

$$\mathbb{E}_a \exp\left(-\psi(m)\int_0^t \frac{ds}{X_s}\right) = a^m \mathbb{E}_a^m[(X_t)^{-m}] \tag{24}$$

and, owing to the scaling property

$$\mathbb{E}_a \exp\left(-\psi(m)\int_0^t \frac{ds}{X_s}\right) = a^m t^{-m} \mathbb{E}_{a/t}^m[(X_1)^{-m}]\,. \tag{25}$$

Let us choose m such that $\psi'(m) > 0$.

We can now use the results on entrance boundary, for example from [3]. Since

$$\mathbb{E}^m \xi_1 = \psi_m'(0) = \psi'(m) > 0$$

then Theorem 1i) therein entails

$$\lim_{t\uparrow\infty} \mathbb{E}_{a/t}^m[(X_1)^{-m}] = \mathbb{E}_0^m[(X_1)^{-m}]\,. \tag{26}$$

Lemma 2 *With the notations of Theorem 2, the quantity* $F(m) := \mathbb{E}_0^m[(X_1)^{-m}]$ *is finite for* $m \in (m_0, m^+)$.

From (25)

$$L_t(\theta) = -m \log t + m \log a + \log \mathbb{E}^m_{a/t}[(X_1)^{-m}]$$

and then, as $t \to \infty$, thanks to Lemma 2:

$$\frac{L_t(\theta)}{\log t} \to L(\theta) = -m\,. \tag{27}$$

This ends the proof of Lemma 1 and consequently the proof of part 1) in Theorem 2, up to the result of Lemma 2 whose proof is postponed.

2) The statement (18) is a consequence of the Chernov inequality. Indeed for every $\theta \in (0, -\psi(m_0))$

$$\log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} > A \right) \le -\theta A \log t + L_t(\theta)$$

and it remains to divide by $\log t$, to let $t \to \infty$ (with θ fixed) and then let $A \to \infty$.

3) In both cases, the above assertion (18) gives the exponential tightness. It is then enough to prove a weak LDP.

Let us first assume that $\Delta = (0, \infty)$. Assertion (16) is equivalent to a weak LDP on $[0, \infty)$ with rate function $\mathscr{I}$, so that we get a full LDP on $[0, \infty)$.

Assume now that $\Delta \neq (0, \infty)$ and that the condition (19) is satisfied. For every x in the complement of $\bar{\Delta}$ we have

$$\lim_{\delta \to 0} \lim_{t \to \infty} \frac{1}{\log t} \log \mathbb{Q}_a \left(\frac{T^{(X)}(t)}{\log t} \in [x - \delta, x + \delta] \right) = -\infty\,,$$

so that a weak LDP is satisfied on $[0, \infty)$ with rate function $\tilde{\mathscr{I}}$, so that we get a full LDP on $[0, \infty)$. □

4.2 Remarks

Remark 1 (Reciprocal Pairs) The relation (20) between two rate functions is known to hold for pairs of inverse processes. This problem arose historically when people deduced a LDP for renewal processes whose interarrival times have sums satisfying a LDP. It was extended to more general processes, see [13] and the bibliography therein.

If (ξ_t) is a spectrally negative Lévy process with Laplace exponent ψ, then $t^{-1}\xi_t$ satisfies the LDP at scale t with rate function ψ^*. The subordinator

$$\hat{\tau}(u) = \inf\{t > 0 : \xi_t > u\} \tag{28}$$

has a Laplace exponent which is exactly L defined by the relation (22)

$$\psi(m) = -\theta\ ,\ L(\theta) = -m\,. \tag{29}$$

Then $t^{-1}\hat{\tau}(t)$ satisfies the LDP at scale t with rate function $x \to x\psi^*\,(1/x)$. This can be seen as an application of Theorem 1 of [13].

The relation (6) between $\mathscr{A}_u^{(\xi)}$ and $\tau^{(\xi)}(t)$:

$$\tau^{(\xi)}(t) = \inf\{u \geq 0 : \mathscr{A}_u^{(\xi)} = \int_0^u e^{\xi_s} ds \geq t\}\ \ ,\ (t \geq 0)$$

is more involved than (28). An alternative proof of our Theorem 2 about LDP for $(\log t)^{-1}\tau^{(\xi)}(t)$, in the spirit of Theorem 1 of [13], would require an LDP for $t^{-1}\log \mathscr{A}_t^{(\xi)}$ which is far from obvious. At first glance, the lower bound of large deviations seems accessible since we have for convenient x and δ,

$$(\xi_s/s \in [x-\delta, x+\delta]\ \forall s \in [0,u]) \Rightarrow \left(\mathscr{A}_u^{(\xi)} \in \left[\frac{\mathrm{e}^{(x-\delta)u}-1}{(x-\delta)}, \frac{\mathrm{e}^{(x+\delta)u}-1}{(x+\delta)}\right]\right)\,.$$

Conversely, we can apply Theorem 1 of [13] to get

Proposition 2 *The family of distributions of* $t^{-1}\log \mathscr{A}_t^{(\xi)}$ *(under* $\mathbb{P}$*) satisfies the LDP at scale t, with good rate function* ψ^**.*

To end this first remark, let us add two more comments:

- The statement of Lemma 1, viewed in terms of ξ can be rephrased as the striking relation

$$\lim_{t\to\infty} \frac{1}{\log t}\mathbb{E}\exp\theta\tau^{(\xi)}(t) = \log \mathbb{E}\exp\theta\hat{\tau}(1)\,.$$

- The relation (29) is also well known in the study of exponential families (inverse families or reciprocal pairs), see [14], Sect. 5.4 and [18], Sect. 5A.

Remark 2 (Another Approach to the LDP) We saw in a previous section that the core of the proof of the LLN is the convergence of the occupation measure

$$\frac{1}{t}\int_0^t \delta_{U_s}\mathrm{d}s \to \mu$$

followed by a scaling and by the choose of $f(x) = x^{-1}$ as a test function. It could be natural to look for a LDP according to the same line of reasoning. Such a proof would consist of three steps. First establish an LDP for the law of the occupation measure under $\mathbb{Q}_0$. The rate functional is expressed with the help of the infinitesimal generator of U. Then apply the contraction by the (non-continuous) mapping

$\nu \mapsto \nu(f)$, hence solve a variational problem. This would give an LDP for the law of

$$\frac{1}{t}\int_0^t \frac{\mathrm{d}s}{U_s} = \frac{1}{t}\int_1^{e^t} \frac{\mathrm{d}s}{X_s},$$

under $\mathbb{Q}_0$. It would remain to convert it into an LDP under $\mathbb{Q}_a$. In the Appendix of [26], the authors give the first rate functional and solved the variational problem, but with the lack of justification for the contraction principle. The result fits with the rate function of Theorem 2.

Remark 3 (Functional LDP) In the Bessel clock, a functional LDP was stated ([26], Th. 4.1). Here it is possible to consider the same problem, i.e. the study of the LDP for the sequence of processes

$$u \in [0, 1] \mapsto \frac{1}{n} T^{(X)}(\mathrm{e}^{nu})$$

under $\mathbb{Q}_a$. The rate function involves an action functional built on $\mathscr{I}$ and a recession function which will be $x \mapsto rx$ where $r = -\psi(m_0)$ (when it is finite).

Remark 4 (Central Limit Theorem) It is known for the Bessel clock (Theorem 1.1 in [26]) but seems unknown otherwise. We can conjecture that

$$\sqrt{\log t}\left(\frac{T^{(X)}(t)}{\log t} - \frac{1}{\mathbb{E}\xi_1}\right) \Longrightarrow \mathscr{N}(0; \psi''(0)/\psi'(0)^3) \tag{30}$$

as soon as $\mathbb{E}(\xi_1)^2 < \infty$ (or $\psi''(0) < \infty$).

4.3 Proof of Lemma 2

From (10) applied to the Esscher transform we have for $m > m_0$

$$F(m) = \frac{1}{\psi'(m)} \mathbb{E}^m(I_\infty^{m-1}).$$

The finiteness of $F(m)$ is a consequence of the following lemma summarizing properties of the moments of the exponential functionals of Lévy processes which we detail for completeness.

Lemma 3 *Let ζ be a Lévy process of Laplace exponent φ given by*

$$\mathbb{E}\exp(\lambda\zeta_t) = \exp(t\varphi(\lambda))$$

such that $\varphi'(0) \in (0,\infty)$ *and let*

$$I_\infty = \int_0^\infty e^{-\zeta_s} ds\,.$$

Then

1)

$$\mathbb{E}(I_\infty^s) < \infty \text{ for all } s \in [-1,0] \text{ and all } s > 0 : \varphi(-s) < 0\,. \tag{31}$$

2) For $r > 0$*, if* $\varphi(r) < \infty$ *and* $\mathbb{E}(I_\infty^{-r}) < \infty$*, then*

$$\mathbb{E}(I_\infty^{-r-1}) = \frac{\varphi(r)}{r}\,\mathbb{E}(I_\infty^{-r})\,. \tag{32}$$

Proof of Lemma 3 The statement 1) comes from a rephrased part of a lemma in Maulik and Zwart on the existence of moments of exponential functionals (see [20], Lemma 2.1).

The statement 2) comes from a recursive argument due to Bertoin and Yor (see [4], Theorems 2(i) and 3), which we detail here for the sake of completeness. Set, for $t \geq 0$

$$J_t = \int_t^\infty \mathrm{e}^{-\zeta_s}\mathrm{d}s \;\; (t \geq 0) \;\; ; \; J_0 = I_\infty = \int_0^\infty \mathrm{e}^{-\zeta_s}\mathrm{d}s$$

For all $r > 0$ we have

$$J_t^{-r} - J_0^{-r} = r\int_0^t \mathrm{e}^{-\zeta_s} J_s^{-r-1}\mathrm{d}s \tag{33}$$

Besides, from the properties of Lévy processes, we deduce

$$J_s = \mathrm{e}^{-\zeta_s}\hat{I}(s)$$

and

$$\hat{I}(s) = \int_0^\infty \mathrm{e}^{-(\zeta_{s+u}-\zeta_s)}\mathrm{d}u \overset{(law)}{=} I_\infty$$

with $\hat{I}(s)$ independent of J_s. Plugging into (33) we get

$$\mathrm{e}^{r\zeta_t}\hat{I}(t)^{-r} - I_\infty^{-r} = r\int_0^t \mathrm{e}^{r\zeta_s}\hat{I}(s)^{-r-1}\mathrm{d}s \tag{34}$$

Assume that $\varphi(r) < \infty$ and $\mathbb{E}I^{-r-1} = \infty$, taking expectations on both sides of the inequality

$$e^{r\zeta_t}\hat{I}(t)^{-r} > r\int_0^t e^{r\zeta_s}\hat{I}(s)^{-r-1}\mathrm{d}s$$

we would get $\mathbb{E}(I^{-r}) = \infty$. Consequently, if $\mathbb{E}(I^{-r}) < \infty$, we have $\mathbb{E}(I^{-r-1}) < \infty$. Moreover, taking expectations on both sides of (34) we get (32). □

Now, to end the proof of Lemma 2, we apply Lemma 3 with $\varphi = \psi_m$.

For $m > 0$, we choose $s = m-1$ hence $\varphi(-s) = \psi(1) - \psi(m)$ which is negative if $m > 1$. So from 1) of Lemma 3, $F(m)$ is finite for $m > 0$.

Now, if $m \in [m_0, 0]$, set $k = \lfloor -m \rfloor$. Then $m + k \in [-1, 0]$, and from 1) of Lemma 3, $\mathbb{E}^m(I_\infty^{m+k})$ is finite. Besides, we have $\psi(m) < \psi(-j)$ for all integers j such that $0 \leq j \leq k$, hence $\varphi(-m-j) = \psi(-j) - \psi(m) > 0$ and applying (32) recursively, we have

$$F(m) = \frac{\varphi(-m)\varphi(-m-1)\cdots\varphi(-m-k)}{(-m)(-m-1)\cdots(-m-k)}\mathbb{E}^m(I_\infty^{m+k}) \tag{35}$$

and this quantity is finite. □

5 Examples

For all the following examples, we give the Laplace exponent ψ and the parameters τ_+, τ_e. In Sects. 5.1 and 5.2 we obtain an explicit expression for τ_0 and $\mathscr{I}$. In Sect. 5.3 we do not have an explicit expression for $\mathscr{I}$ (and sometimes for τ_0). For all examples (but one) we identify the behaviour at the boundary, according to the classification of Proposition 1.

5.1 The Brownian Process with Drift (Bessel Clock [26])

For $\nu > 0$, we consider the Lévy process

$$\xi_t = 2B_t + 2\nu t\,.$$

The pssMp (X_t) is the squared Bessel process of dimension $d = 2(1+\nu)$. Its index is $\alpha = 1$ and it is the only continuous pssMp (of index 1). The clock is related to the scale function of a diffusion in random environment (see [25], Chap. 10).

We have

$$\psi(m) = 2m(m+\nu),\ m_\pm = \pm\infty,\ \psi(m_\pm) = +\infty\,.$$

Here $m_0 = -\nu/2$ and $\psi(m_0) = -\nu^2/2$, so that

$$\tau_+ = 0,\quad \tau_0 = \infty,\quad \tau_e = \frac{1}{2\nu},\quad \Delta = (0,\infty)\,.$$

The rate function is

$$\mathscr{I}(x) = \frac{(1-2\nu x)^2}{8x}\ ,\ (x>0)\,.$$

Notice that here the function L may be obtained by explicit inversion. Recall that it is defined in Lemma 1, where we proved, via the change of probability (23), the equivalence (22) i.e.

$$L(\theta) = -m \Longleftrightarrow \theta = -\psi(m)\,.$$

for $\theta \in (-\psi(m_+), -\psi(m_0))$ and $m \in (m_0, m_+)$.

Here, for $\theta < \nu^2/2$, m is the unique solution of

$$2m(m+\nu) + \theta = 0$$

in $(-\nu/2, \infty)$, so that

$$m = \frac{-\nu + \sqrt{\nu^2 - 2\theta}}{2}$$

and therefore

$$L(\theta) = \frac{\nu - \sqrt{\nu^2 - 2\theta}}{2}\ ,\ (\theta < \nu^2/2)\,.$$

The boundary $\tau_+ = 0$ is in the situation 4c of Proposition 1 and the boundary τ_0 is in situation 3a, with asymptote $y = \frac{\nu^2}{2}x - \frac{\nu}{2}$. The minimum of $\mathscr{I}$, reached in τ_e, corresponds to the LLN (Theorem 1) which is in [22], Exercise (4.23), Chap. IV and in [26], Theorem 1.1 (Fig. 1).

Let us now comment the change of probability. Under $\mathbb{P}^m$, the processes (ξ_t) and (X_t) remain in the same family as under $\mathbb{P}$: they are Brownian motion with drift and squared Bessel, respectively, but with ν replaced by $\nu + 2m$. Moreover, it is known from [25], Chap. 8 (or equivalently [7]), that under $\mathbb{P}$,

$$I_\infty \overset{(law)}{=} (2Z_\nu)^{-1}$$

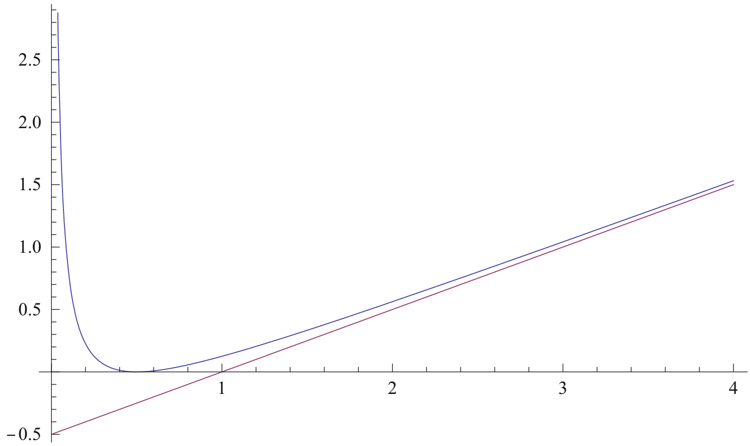

Fig. 1 Example 5.1: the rate function $\mathscr{I}$ for $\nu = 1$

where Z_ν is a gamma variable of parameter ν. This allows to see directly that the assertion of Lemma 2 holds.

5.2 *Lévy Processes of the Form* $\xi_t = \mathrm{d}t \pm \mathit{Pois}(\beta, \gamma)_t$

Here $\mathrm{Pois}(a, b)_t$ is the compound Poisson process of parameter a whose jumps are exponential r.v. of parameter b. They are studied in Sect. 8.4.3 of [25] (or equivalently [7]), with very informative Tables. In particular, these families are invariant by changes of probability and the law of the exponential functional I_∞ is known, which allows again to see directly that the assertion of Lemma 2 holds. Notice that, as in Sect. 5.1, the function L could be obtained explicitly by inversion

5.2.1 $\xi_t = \mathrm{d}t + \mathrm{Pois}(\beta, \gamma)_t$ with $\mathrm{d} \geq 0$

(The particular case of the compound Poisson process corresponds to the case $\mathrm{d} = 0$). We have

$$\psi(m) = m\left(\mathrm{d} + \frac{\beta}{\gamma - m}\right),\ m_- = -\infty\ ,\ m_+ = \gamma\ ,\ \psi(m_+) = \infty\,,$$

Here $m_0 = -\infty$ and $\psi(m_0) = -\infty$ so that

$$\tau_+ = 0\,,\ \tau_0 = \mathrm{d}^{-1}\,,\ \tau_e = \frac{\gamma}{\gamma\mathrm{d} + \beta}\,,\ \Delta = (0, \mathrm{d}^{-1})\,.$$

The rate function is

$$\mathscr{I}(x) = \left(\sqrt{\gamma(1-\mathrm{d}x)} - \sqrt{\beta x}\right)^2 \quad , \quad (0 \le x \le \mathrm{d}^{-1}) \, .$$

To check the assumption (19) we go back to the process ξ. Notice that $\xi_s \ge \mathrm{d}s$ for every $s \ge 0$, so that $\tau^{(\xi)}(t) \le \mathrm{d}^{-1} \log(1 + \mathrm{d}t)$, whence, for every $\varepsilon > 0$

$$\mathbb{P}\left(\frac{\tau^{(\xi)}(t)}{\log t} > \tau_0 + \varepsilon\right) = 0$$

for t large enough, which is equivalent to (19) thanks to (7).

The boundary τ_+ is in situation 4a and τ_0 in situation 3b, i.e. vertical tangents (Fig. 2).

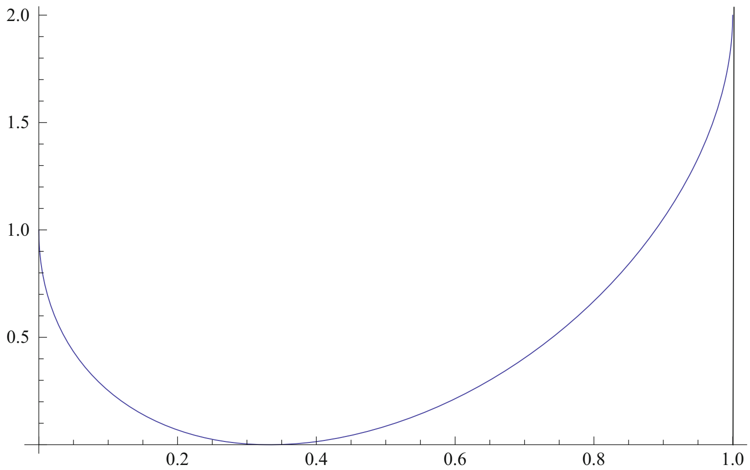

Fig. 2 Example 5.2.1: the rate function $\mathscr{I}$ for $\beta = 2, \gamma = 1, \mathrm{d} = 1$

In the following examples, when $d < 0$ with no loss of generality we may and will assume that $\mathrm{d} = -1$, since (with obvious notations)

$$\tau^{(\xi;\mathrm{d},\beta)}(t) \overset{(law)}{=} -\mathrm{d}^{-1} \tau^{(\xi;-1,\beta/-\mathrm{d})}(-\mathrm{d}\, t) \, .$$

5.2.2 $\xi_t = -t + \mathrm{Pois}(\beta, \gamma)_t$ with $0 < \gamma < \beta$

We have

$$\psi(m) = m\left(-1 + \frac{\beta}{\gamma - m}\right), m_- = -\infty \quad , \ m_+ = \gamma \ , \ \psi(m_+) = \infty \, .$$

Here

$$m_0 = \gamma - \sqrt{\beta\gamma}\ ,\ \psi(m_0) = -(\sqrt{\gamma} - \sqrt{\beta})^2\ ,$$

so that

$$\tau_+ = 0\ ,\ \tau_0 = \infty\ ,\ \tau_e = \frac{\gamma}{\beta - \gamma}\ ,\ \Delta = (0, \infty)\ .$$

The rate function is

$$\mathscr{I}(x) = \left(\sqrt{\gamma(1+x)} - \sqrt{\beta x}\right)^2 \quad (x > 0)\ .$$

The boundary τ_+ is in situation 4a and τ_0 in situation 3a with asymptote having equation $y = (\sqrt{\beta} - \sqrt{\gamma})^2 x + \gamma - \sqrt{\beta\gamma}$ (Fig. 3).

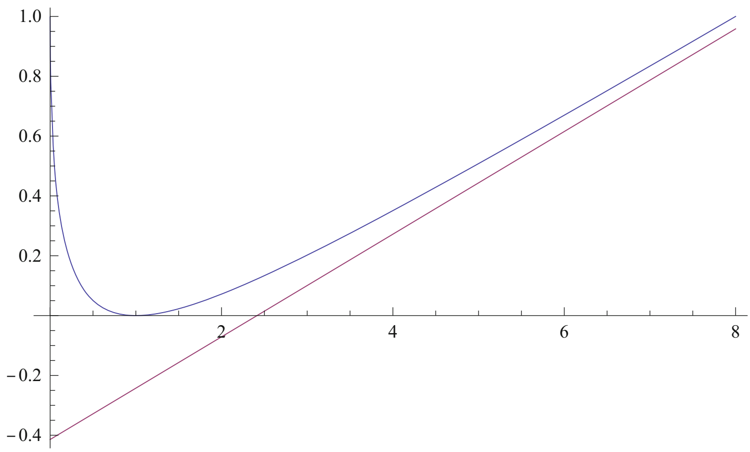

Fig. 3 Example 5.2.2: the rate function $\mathscr{I}$ for $\beta = 2, \gamma = 1$

5.2.3 The Saw-Tooth Process

It is a particular case of the Cramer-Lundberg risk process ([15], Sect. 1.3.1). $0 < \beta < \gamma$. The Lévy process ξ is defined by

$$\xi_t = t - \mathrm{Pois}(\beta, \gamma)_t\ . \tag{36}$$

The self-similar process X_t is described precisely in ([9], p. 327).

The Laplace exponent of (ξ_t) is given by:

$$\psi(m) := m\frac{\gamma-\beta+m}{\gamma+m}\ ,\ m_- = -\gamma\ ,\ m_+ = \infty\ ,\ \psi(m_+) = \infty\ ,\ \psi'(m_+) = 1\,. \tag{37}$$

We have easily

$$m_0 = -\gamma + \sqrt{\beta\gamma}\ ,\ \psi(m_0) = -\left(\sqrt{\gamma} - \sqrt{\beta}\right)^2$$

so that

$$\tau_+ = 1\ ,\ \tau_0 = \infty\ ,\ \tau_e = \frac{\gamma}{\gamma-\beta}\ ,\ \varDelta = (1,\infty)\,.$$

The rate function is

$$\mathscr{I}(x) = \left(\sqrt{\gamma(x-1)} - \sqrt{\beta x}\right)^2\ ,\ (x \geq 1)\,.$$

The condition (19) is fulfilled since by definition $\xi_t \leq t$ for all $t > 0$, so we have $\tau^{(\xi)}(t) \geq \log t$ for all $t > 0$. The minimum of the rate function $\mathscr{I}$ is reached at $x = \frac{\gamma}{\gamma-\beta}$ in accordance with the law of large numbers ([9], Theorem 4.7ii)).

The boundary $\tau_+ = 1$ is in situation 4b and the boundary τ_0 in situation 3a, with an asymptote of equation $y = (\sqrt{\gamma} - \sqrt{\beta})^2 x + \sqrt{\beta\gamma} - \gamma$ (Fig. 4).

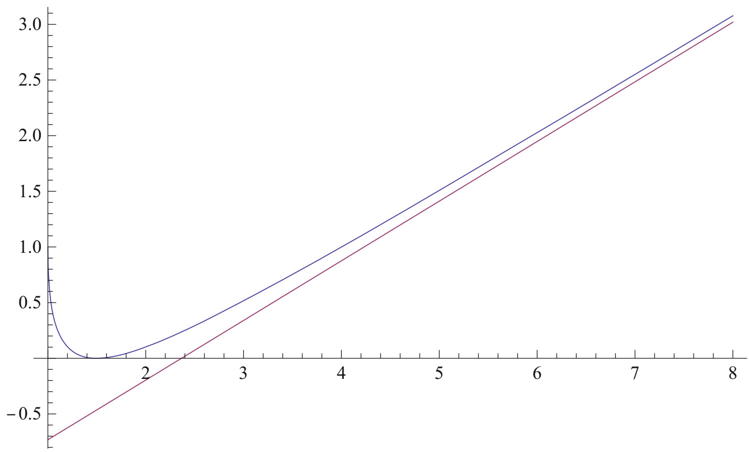

Fig. 4 Example 5.2.3: the rate function $\mathscr{I}$ for $\beta = 1, \gamma = 3$

5.3 Three More Examples

The first and the second may be seen for instance in [21]. The third one is a general class (see [16]).

5.3.1 First

For $\alpha \in (1,2)$, let $X^{\uparrow}$ be the spectrally negative regular α-stable Lévy process conditioned to stay positive. It is a pssMp of index α and its associated Lévy process $\xi^{\uparrow}$ has Laplace exponent

$$\psi(m) = c\frac{\Gamma(m+\alpha)}{\Gamma(m)}$$

and

$$m_- = -\alpha \ , \ m_+ = \infty \ , \ \psi(m_+) = \infty \ , \ \psi'(m_+) = \infty$$

where c is a positive constant. We have

$$\frac{\psi'(m)}{\psi(m)} = \Psi(m+\alpha) - \Psi(m)$$

where Ψ is the Digamma function. Then $m_0 = \gamma_\alpha$ where γ_α is the (unique) solution in $(-1,0)$ of the equation

$$\Psi(\gamma+\alpha) = \Psi(\gamma)\,,$$

and then $\psi(\gamma_\alpha) = c\Gamma(\gamma_\alpha+\alpha)/\Gamma(\gamma_\alpha) < 0$. We have

$$\tau_+ = 0\ ,\ \tau_0 = \infty\ , \tau_e = \frac{1}{c\Gamma(\alpha)}\ ,\ \Delta = (0,\infty)\,.$$

The boundary τ_+ is in situation 4c and τ_0 in situation 3a with asymptote.

Notice that when $\alpha \uparrow 2$ we end up with the Laplace exponent of a Brownian motion with drift $1/2$.

5.3.2 Second

Let $\kappa \in (0,1]$ and $\delta > \kappa/(1+\kappa)$. Let X be the continuous state branching process with immigration whose branching mechanism (see [21], Lemma 4.8) is

$$\varphi(u) = -\frac{c}{\kappa}u^{\kappa+1}$$

and immigration mechanism is

$$\chi(u) = c\frac{\kappa+1}{\kappa}u^{\kappa}\,.$$

It is a pssMp of index κ and the associated Lévy process has Laplace exponent

$$\psi(m) = c(\kappa - (\kappa+1)\delta - m)\frac{\Gamma(-m+\kappa)}{\Gamma(-m)}\ ,\ m_- = -\infty\ ,\ m_+ = \kappa\ , \psi(m_+) = \infty\,.$$

First, we see that $\psi'(0) = \lim \psi(m)/m = c((\kappa+1)\delta - \kappa)\Gamma(\kappa) > 0$ and $\tau_e = 1/\psi'(0)$. Since as $m \to -\infty$ we have $\psi(m) \sim c(-m)^{\kappa+1} \to \infty$, we deduce the existence of $m_0 > -\infty$ such that $\psi'(m_0) = 0$ (and $\psi(m_0) < 0$). Besides, when $m \uparrow \kappa$ or $h = \kappa - m \downarrow 0$ then

$$\psi'(m) \sim \frac{(\kappa+1)\delta\Gamma'(h)}{\Gamma(-\kappa)} \uparrow \infty\,.$$

We have then

$$\tau_+ = 0\ ,\ \tau_0 = \infty\ ,\ \Delta = (0,\infty)\,.$$

The boundary $\tau_+ = 0$ is in situation 4a and τ_0 is in situation 3a with asymptote.

5.3.3 Hypergeometric-Stable Process

The modulus of a Cauchy process in $\mathbb{R}^d$ for $d > 1$ is a pssMp of index 1 with infinite lifetime. Actually the associated Lévy process is a particular case of hypergeometric-stable process of index α as defined in [6], with $\alpha < d$.

The characteristic exponent is given therein by Theorem 7, hence the Laplace exponent is:

$$\psi(m) = -2^{\alpha}\frac{\Gamma((-m+\alpha)/2)}{\Gamma(-m/2)}\frac{\Gamma((m+d)/2)}{\Gamma((m+d-\alpha)/2)}$$

and

$$m_- = -d\ ,\ m_+ = \alpha\ ,\ \psi(m_+) = \infty\,.$$

Since $m\Gamma(-m/2) = -2\Gamma((2-m)/2)$ we have

$$\psi'(0) = \lim_{m\to 0}\psi(m)/m = 2^{\alpha-1}\frac{\Gamma(\alpha/2)\Gamma(d/2)}{\Gamma((d-\alpha)/2)} > 0\,.$$

Moreover, for $m \neq 0$

$$2\frac{\psi'(m)}{\psi(m)} = -\Psi((-m+\alpha)/2)) + \Psi((m+d)/2) + \Psi(-m/2) - \Psi(m+d-\alpha)/2)\,,$$

where Ψ is the digamma function.

It is then easy to see that ψ' vanishes at $m = m_0 := (\alpha - d)/2$ and

$$\psi(m_0) = -2^\alpha \left(\frac{\Gamma((d+\alpha)/4)}{\Gamma((d-\alpha)/4)}\right)^2 .$$

We have then

$$\tau_+ = 0\,,\ \tau_0 = \infty\,,\ \Delta = (0,\infty)\,.$$

In the particular case $\alpha = 1$ and $d = 3$, applying the identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

we have

$$\psi(m) = (m+1)\tan\frac{\pi m}{2} \tag{38}$$

with $m_- = -3,\ \ m_+ = 1$ and $\psi(-1) = -2/\pi$ by continuity. A direct computation gives

$$\psi'(m) = \frac{\sin \pi m + \pi m + \pi}{2\cos^2[(\pi m)/2]}$$

so that

$$\psi'(0) = \pi/2\,,\quad m_0 = -1\,,\quad \theta_0 = -\psi(m_0) = 2/\pi \text{ and } \Delta = (0,\infty)$$

We have no close expression for the rate function. It is the situation 3a and the asymptote as $x \to \infty$ is the line $y = \frac{2x}{\pi} - 1$ (Fig. 5).

Notice that in [10], the authors announced a study of the modulus of a multidimensional Cauchy process. They found the expression (38) but never published it (F. Petit, personal communication).

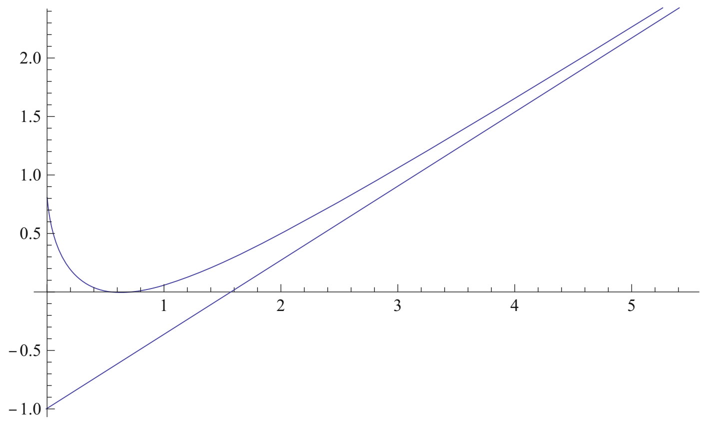

Fig. 5 Example 5.3.3: the rate function $\mathscr{I}(x)$ for $\alpha = 1$, $\mathrm{d} = 3$

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Stochastic Bäcklund Transformations

Neil O'Connell

Dedicated to the memory of Marc Yor

Abstract How does one introduce randomness into a classical dynamical system in order to produce something which is related to the 'corresponding' quantum system? We consider this question from a probabilistic point of view, in the context of some integrable Hamiltonian systems.

1 Introduction

Let $\mu \geq 1/2$ and consider the evolution $\dot{x} = \mu/x$ on the positive half-line. Then $\ddot{x} = -\mu^2/x^3$, which is the equation of motion for the rational Calogero-Moser system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{\mu^2}{2x^2}.$$

If we add noise, that is, if we consider the stochastic differential equation

$$dX = dB + \frac{\mu}{X}dt,$$

where B is a standard one-dimensional Brownian motion, then X is a diffusion process on the positive half-line with infinitesimal generator

$$L = \frac{1}{2}\partial_x^2 + \frac{\mu}{x}\partial_x.$$

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The assumption $\mu \geq 1/2$ ensures that X never hits zero. The operator L is related to the quantum Calogero-Moser Hamiltonian

$$H = \frac{1}{2}\partial_x^2 - \frac{\mu(\mu-1)}{2x^2},$$

via the ground state transform

$$L = \psi(x)^{-1} H \psi(x),$$

where $\psi(x) = x^\mu$. Ignoring for the moment the discrepancy between the coupling constants $\mu^2/2$ and $\mu(\mu-1)/2$, this provides a very simple example of a classical Hamiltonian system which has the property that if we add noise in a suitable way we obtain a diffusion process whose infinitesimal generator is simply related to the corresponding quantum system.

Appealing as it is, this example is quite unique and, in fact, somewhat misleading. The only constant of motion is the Hamiltonian itself and, as $\dot{x} = p$, the evolution $\dot{x} = \mu/x$ necessarily has $p^2/2 - \mu^2/2x^2 = 0$. It is not clear how to extend this construction—*together with its stochastic counterpart*—to allow for other values. In fact, there is a kind of 'explanation' for this limitation which will come later.

In the papers [21, 22] a certain probabilistic relation between the classical and quantum Toda lattice was observed. This relation can be loosely described as follows: starting with a particular construction of the classical flow on a given sub-Lagrangian manifold, *adding white noise to the constants of motion* yields a diffusion process whose infinitesimal generator is simply related to the corresponding quantum system.

As we shall see, this relation extends naturally to some other integrable many-body systems, specifically rational and hyperbolic Calogero-Moser systems. The basic construction can be formulated in terms of *kernel functions* and *Bäcklund transformations*. For more background on the (interrelated) role of kernel functions and Bäcklund transformations in integrable systems see, for example, [9, 16, 24, 27] and references therein. In the present paper, to illustrate the main ideas, we will focus on rank-one (two particle) systems although most of the constructions extend naturally to higher rank systems.

The examples we consider are of course very special, having the property that there are kernel functions which unite the classical and quantum systems through a kind of exact stationary phase property. Nevertheless, they should provide a useful benchmark for exploring similar relations for other Hamiltonian systems.

The outline of the paper is as follows. In the next section, we illustrate the basic construction of [21, 22] in the context of the rank one Toda lattice. In this setting it is closely related to earlier results of Matsumoto and Yor [17] and Baudoin [1]. In Sects. 3–5, we give analogous constructions for the rational and hyperbolic Calogero-Moser systems. As we shall see, the above example should in fact be seen as a particular degeneration of a more general construction for the hyperbolic Calogero-Moser system, based on the kernel functions of Hallnäs and

Ruijsenaars [9, 10]. In Sect. 6, we conclude with some remarks on how the solution to the Kardar-Parisi-Zhang equation can also be interpreted from this point of view.

The following notation will be used throughout. If E is a topological space, we denote by $B(E)$ the set of Borel measurable functions on E, by $C_b(E)$ the set of bounded continuous functions on E and by $\mathscr{P}(E)$ the set of Borel probability measures on E. If E is an open subset of $\mathbb{R}^n$, we denote by $C_c^2(E)$ the set of continuously twice differentiable, compactly supported, functions on E.

2 The Toda Lattice

For the rank-one Toda lattice we consider the kernel function

$$K(x,u) = \exp\left(-e^{-x}\cosh u\right),$$

and note that K satisfies

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = e^{-2x}, \tag{1}$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = 0. \tag{2}$$

The corresponding Bäcklund transformation

$$\dot{u} = -\partial_u \ln K = e^{-x}\sinh u, \qquad \dot{x} = \partial_x \ln K = e^{-x}\cosh u \tag{3}$$

has the property that, if (3) holds, then x satisfies the equations of motion of the Toda system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{1}{2}e^{-2x},$$

and $\dot{u} = \lambda$ is a conserved quantity for the coupled system. Indeed, differentiating (1) with respect to x yields

$$\ddot{x} = \partial_x^2 \ln K\, \partial_x \ln K - \partial_u\partial_x \ln K\, \partial_u \ln K = -e^{-2x}, \tag{4}$$

and differentiating (1) with respect to u gives

$$\ddot{u} = \partial_u^2 \ln K\, \partial_u \ln K - \partial_x\partial_u \ln K\, \partial_x \ln K = 0. \tag{5}$$

It also follows from (1) and (3) that λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & e^{-x} \\ -e^{-x} & -p \end{pmatrix}.$$

Now the equation $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely

$$\sinh u = \lambda e^x, \tag{6}$$

we can rewrite the evolution equation (3) as

$$\dot{u} = \lambda, \qquad \dot{x} = \lambda + e^{-u-x} = (\partial_x + \partial_u) \ln K_\lambda. \tag{7}$$

We note that (6) has a unique solution $u_\lambda(x) = \sinh^{-1}(\lambda e^x)$ for any $\lambda, x \in \mathbb{R}$. The relation (6) is stable under the new evolution equations (7), and is now required to be in force in order to guarantee that (x, p) evolves according to the Toda flow on the iso-spectral manifold corresponding to λ. Given any $\lambda \in \mathbb{R}$, the evolution equations (7) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (6)) in the sense that they admit a unique semi-global solution. For any $\lambda \in \mathbb{R}$ and initial condition $x(0) = x_0$, the solution is given explicitly for all $t \geq 0$ by

$$u(t) = u_\lambda(x_0) + \lambda t, \qquad x(t) = \begin{cases} \ln\left(\frac{1}{\lambda} \sinh u(t)\right) & \lambda \neq 0 \\ \ln\left(e^{x_0} + t\right) & \lambda = 0. \end{cases}$$

The evolution equations (7) provide the correct framework into which we can introduce noise with the desired outcome.

Let $H = (\partial_x^2 - e^{-2x})/2$, and write $H_\lambda = H - \lambda^2/2$. Combining (1) and (2) gives the intertwining relation

$$H_\lambda K_\lambda = \left(\frac{1}{2}\partial_u^2 - \lambda \partial_u\right) K_\lambda. \tag{8}$$

It follows, using the Leibnitz rule, that

$$\psi_\lambda(x) = \int_{-\infty}^{\infty} K_\lambda(x, u) du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$. We note that $\psi_\lambda(x) = 2K_\lambda(e^{-x})$, where $K_\nu(z)$ is the modified Bessel function of the second kind, also known as Macdonald's function.

Remark 1 The intertwining relation (8) and associated integral formula for the eigenfunctions can be seen as a special case of those obtained by Gerasimov, Kharchev, Lebedev and Oblezin [8] for the n-particle open Toda chain. The above Bäcklund transformation is a special case of the one given by Wojciechowski [29] which, as remarked in that paper, is closely related to a construction of Kac and Van Moerbeke [11] for the periodic Toda chain. It can also be seen as a particular degeneration of the Bäcklund transformation for the infinite particle system given in Toda's monograph [28].

Consider the integral operator defined, for suitable $f : \mathbb{R}^2 \to \mathbb{R}$, by

$$\tilde{K}_\lambda f(x) = \int_{-\infty}^{\infty} K_\lambda(x,u) f(x,u)\, du,$$

and the differential operator, defined on $\mathscr{D}(A_\lambda) = C_c^2(\mathbb{R}^2)$, by

$$A_\lambda = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_u^2 + \partial_x \partial_u + \lambda \partial_u + (\lambda + e^{-u-x})\, \partial_x.$$

Proposition 1 *For $f \in \mathscr{D}(A_\lambda)$,*

$$H_\lambda \tilde{K}_\lambda f = \tilde{K}_\lambda A_\lambda f. \tag{9}$$

Proof This follows from the intertwining relation (8). Recall that

$$(\partial_x + \partial_u) \ln K_\lambda = \lambda + e^{-u-x}.$$

By Leibnitz' rule and integration by parts,

$$\begin{aligned}
H_\lambda \tilde{K}_\lambda f(x) &= H_\lambda \int_{-\infty}^{\infty} K_\lambda f du \\
&= \int_{-\infty}^{\infty} \Big[(H_\lambda K_\lambda) f + (\partial_x K_\lambda) \partial_x f + K_\lambda \tfrac{1}{2} \partial_x^2 f \Big] du \\
&= \int_{-\infty}^{\infty} \Big[(\tfrac{1}{2}\partial_u^2 K_\lambda - \lambda \partial_u K_\lambda) f + K_\lambda (\partial_x \ln K_\lambda) \partial_x f + K_\lambda \tfrac{1}{2}\partial_x^2 f \Big] du \\
&= \int_{-\infty}^{\infty} \Big[K_\lambda (\tfrac{1}{2}\partial_u^2 f + \lambda \partial_u f) + K_\lambda ((\partial_x + \partial_u) \ln K_\lambda) \partial_x f - (\partial_u K_\lambda) \partial_x f + K_\lambda \tfrac{1}{2}\partial_x^2 f \Big] du \\
&= \int_{-\infty}^{\infty} \Big[K_\lambda (\tfrac{1}{2}\partial_u^2 f + \lambda \partial_u f) + K_\lambda ((\partial_x + \partial_u) \ln K_\lambda) \partial_x f + K_\lambda \partial_u \partial_x f + K_\lambda \tfrac{1}{2}\partial_x^2 f \Big] du \\
&= K_\lambda A_\lambda f,
\end{aligned}$$

as required. □

Now, if $\lambda \in \mathbb{R}$, the intertwining relation (9) has a *probabilistic* meaning, which we will soon make precise. It implies that there is a two-dimensional diffusion process, characterized by the differential operator A_λ, which has the property that, with particular initial condition specified by the kernel K_λ, its projection onto the x-coordinate is a diffusion process in $\mathbb{R}$ which is characterised by a renormalisation of the operator H_λ. Moreover, the two-dimensional diffusion process characterized by A_λ is precisely the Bäcklund transformation, in the form of (7), with white noise added to the constant of motion λ. We will now make this statement precise.

Suppose $\lambda \in \mathbb{R}$, let B be a standard one-dimensional Brownian motion and consider the coupled stochastic differential equations obtained by adding white noise to λ in (7), that is

$$dU = dB + \lambda dt, \qquad dX = dU + e^{-U-X}dt. \tag{10}$$

This can be solved explicitly: for any initial condition (X_0, U_0),

$$U_t = U_0 + B_t + \lambda t, \qquad X_t = U_t + \ln\left(e^{X_0-U_0} + \int_0^t e^{-2U_s}ds\right). \tag{11}$$

As the function $(x, u) \mapsto (\lambda + e^{-u-x},\ \lambda)$ is locally Lipschitz, it follows that, for any initial condition, (11) is the unique solution to (10). Moreover, it is a diffusion process in $\mathbb{R}^2$ with infinitesimal generator A_λ and the martingale problem for (A_λ, ν) is well-posed for any $\nu \in \mathscr{P}(\mathbb{R}^2)$. For more background on the relation between stochastic differential equations and martingale problems see, for example, [6, 15].

Next we consider the diffusion process on $\mathbb{R}$ with infinitesimal generator

$$L_\lambda = \psi_\lambda(x)^{-1}H_\lambda\psi_\lambda(x) = \frac{1}{2}\partial_x^2 + \partial_x \ln\psi_\lambda(x)\cdot\partial_x.$$

This process was introduced by Matsumoto and Yor [17]. Observe that the drift

$$\partial_x \ln\psi_\lambda(x) = e^{-x}\frac{K_{\lambda+1}(e^{-x})}{K_\lambda(e^{-x})},$$

is locally Lipschitz, behaves like e^{-x} at $-\infty$ and vanishes at $+\infty$. It follows that $-\infty$ is an entrance boundary, $+\infty$ is a natural boundary and, for any $\rho \in \mathscr{P}(\mathbb{R})$, the martingale problem for (L_λ, ρ), with $\mathscr{D}(L_\lambda) = C_c^2(\mathbb{R})$, is well-posed.

Using the theory of Markov functions (see Appendix), the intertwining relation (9) yields the following result of Matsumoto and Yor [17] and Baudoin [1].

Theorem 1 *Let $\rho \in \mathscr{P}(\mathbb{R})$ and $\nu = \rho(dx)\nu_x(du) \in \mathscr{P}(\mathbb{R}^2)$, where*

$$\nu_x(du) = \psi_\lambda(x)^{-1}K_\lambda(x,u)du.$$

Let (X, U) be a diffusion process in $\mathbb{R}^2$ with initial condition ν and infinitesimal generator A_λ. Then X is a diffusion process in $\mathbb{R}$ with infinitesimal generator L_λ. Moreover, for each $t \ge 0$ and $g \in B(\mathbb{R})$,

$$E[g(U_t)|\, X_s,\ 0 \le s \le t] = \int_{-\infty}^{\infty} g(u)\nu_{X_t}(du),$$

almost surely.

Proof This follows from the intertwining relation (9), using Theorem 4. The map $\gamma : \mathbb{R}^2 \to \mathbb{R}$ defined by $\gamma(x, u) = x$ is continuous and the Markov transition kernel Λ from $\mathbb{R}$ to $\mathbb{R}^2$ defined by

$$\Lambda f(x) = \int_{-\infty}^{\infty} \nu_x(du) f(x, u), \qquad f \in B(\mathbb{R})$$

satisfies $\Lambda(g \circ \gamma) = g$ for $g \in B(\mathbb{R})$. Moreover, by (9),

$$L_\lambda \Lambda f = \Lambda A_\lambda f, \qquad f \in \mathscr{D}(A_\lambda). \tag{12}$$

Now, $\mathscr{D}(A_\lambda) = C_c^2(\mathbb{R}^2)$ is closed under multiplication, separates points and is convergence determining. Finally, by Itô's lemma and the intertwining relation (12), the martingale problem for (L_λ, ρ), now taking $\mathscr{D}(L_\lambda) = \Lambda(\mathscr{D}(A_\lambda)) \cup C_c^2(\mathbb{R})$, is also well-posed, so we are done. □

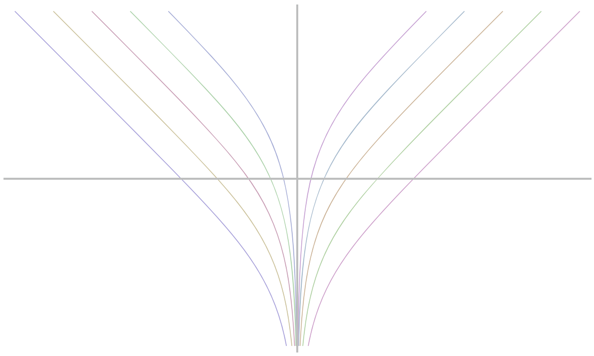

Fig. 1 The Toda flow in u (horizontal) and x (vertical) coordinates

To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$, the classical flow in $\mathbb{R}^2$ is along the curve $\sinh u = \lambda e^x$ (see Fig. 1), according to the evolution equations

$$\dot{u} = \lambda, \qquad \dot{x} = \dot{u} + e^{-u-x}, \tag{13}$$

and the x-coordinate satisfies the equation of motion $\ddot{x} = -e^{-2x}$. If we add noise to the constant of motion λ, then the evolution is described by the stochastic differential equations

$$dU = dB + \lambda dt, \qquad dX = dU + e^{-U-X} dt \tag{14}$$

and, for appropriate (random) initial conditions, the u-coordinate evolves as a Brownian motion with drift λ and the x-coordinate evolves as a diffusion process in $\mathbb{R}$ with infinitesimal generator L_λ. As (13) is essentially a rewriting of the Bäcklund transformation (3), and in view of Theorem 1, it seems natural to refer to (14) as a *stochastic Bäcklund transformation*, hence the title of this paper.

To relate this to the semi-classical limit, consider the Hamiltonian

$$H^{(\varepsilon)} = \frac{\varepsilon}{2}\partial_x^2 - \frac{1}{\varepsilon}e^{-2x}.$$

Now the eigenfunctions are given by

$$\psi_\lambda^{(\varepsilon)}(x) = \int_{-\infty}^{\infty} K_\lambda(x,u)^{1/\varepsilon}du,$$

and Theorem 1 can be restated as follows. Let $X_0 = x$ and choose U_0 at random according to the probability distribution

$$\nu_x^{(\varepsilon)} = \psi_\lambda^{(\varepsilon)}(x)^{-1}K_\lambda(x,u)^{1/\varepsilon}du.$$

Let (X,U) be the unique solution to the SDE

$$dU = \sqrt{\varepsilon}dB + \lambda dt, \qquad dX = dU + e^{-U-X}dt,$$

with this initial condition. Then X is a diffusion process in $\mathbb{R}$ with infinitesimal generator given by

$$\frac{\varepsilon}{2}\partial_x^2 + \varepsilon\,\partial_x \ln\psi_\lambda^{(\varepsilon)}(x)\cdot\partial_x.$$

As $\varepsilon \to 0$, the evolution of (X,U) reduces to the evolution equations (7) and the initial distribution of U_0 concentrates on the unique solution $u_\lambda(x)$ to the critical point equation $\partial_u \ln K_\lambda = 0$. On the other hand, one might expect

$$\varepsilon\,\partial_x \ln\psi_\lambda^{(\varepsilon)}(x) \to \partial_x\big[\ln K_\lambda(x,u_\lambda(x))\big],$$

as is indeed the case, and the evolution of X reduces to the gradient flow

$$\dot{x} = \partial_x\big[\ln K_\lambda(x,u_\lambda(x))\big],$$

which is equivalent to (7) thanks to the remarkable identity

$$\partial_x\big[\ln K_\lambda(x,u_\lambda(x))\big] = \big[\partial_x \ln K_\lambda\big](x,u_\lambda(x)).$$

3 Rational Calogero-Moser System

In this section, we formulate an analogous construction for the one-dimensional rational Calogero-Moser system. Consider the kernel function

$$K(x,u) = \frac{x^2-u^2}{x}, \qquad |u| \le x,$$

and note that K satisfies

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = 1/x^2 \tag{15}$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = 1/x^2. \tag{16}$$

The corresponding Bäcklund transformation

$$\dot{u} = \frac{1}{x-u} - \frac{1}{x+u} = -\partial_u \ln K, \qquad \dot{x} = \frac{1}{x-u} + \frac{1}{x+u} - \frac{1}{x} = \partial_x \ln K \tag{17}$$

has the property that, if (17) holds, then x satisfies the equations of motion of the rational Calogero-Moser system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{1}{2x^2},$$

and $\dot{u} = \lambda$ is a conserved quantity for the coupled system. Indeed, as in the Toda case, differentiating (15) with respect to x and u yields, respectively,

$$\ddot{x} = \partial_x^2 \ln K \; \partial_x \ln K - \partial_u \partial_x \ln K \; \partial_u \ln K = -1/x^3, \tag{18}$$

and

$$\ddot{u} = \partial_u^2 \ln K \; \partial_u \ln K - \partial_x \partial_u \ln K \; \partial_x \ln K = 0. \tag{19}$$

It also follows from (15) and (17) that λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & 1/x \\ -1/x & -p \end{pmatrix}.$$

As before, $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely

$$2u = \lambda(x^2 - u^2), \tag{20}$$

we can rewrite the evolution equations as

$$\dot{u} = \lambda, \qquad \dot{x} = \lambda + \frac{2}{x+u} - \frac{1}{x} = (\partial_x + \partial_u) \ln K_\lambda. \tag{21}$$

The critical point equation (20) has a unique solution $u_\lambda(x) \in (-x, x)$ for any $\lambda \in \mathbb{R}$ and $x > 0$. The relation (20) is stable under the new evolution equation (21), and is now required to be in force in order to guarantee that (x, p) evolves according to the rational Calogero-Moser flow on the iso-spectral manifold corresponding to λ. Given any $\lambda \in \mathbb{R}$, the evolution equations (21) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (20)) in the sense that they admit a unique semi-global solution. For any $\lambda \in \mathbb{R}$ and initial condition $x(0) = x_0 > 0$, the solution is given explicitly for all $t \geq 0$ by

$$u(t) = u_\lambda(x_0) + \lambda t, \qquad x(t) = \begin{cases} \sqrt{u(t)^2 + 2u(t)/\lambda} & \lambda \neq 0 \\ \sqrt{x_0^2 + 2t} & \lambda = 0. \end{cases}$$

As in the Toda case, the evolution equation (21) provide the correct framework into which we can introduce noise with the desired outcome.

Let

$$H = \frac{1}{2}\partial_x^2 - \frac{1}{x^2},$$

and write $H_\lambda = H - \lambda^2/2$. Combining (15) and (16) gives the intertwining relation

$$H_\lambda K_\lambda = \big(\frac{1}{2}\partial_u^2 - \lambda \partial_u\big) K_\lambda. \tag{22}$$

It follows that

$$\psi_\lambda(x) = \int_{-x}^{x} K_\lambda(x, u) du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$. To see this, first note that

$$\partial_x K_\lambda = e^{\lambda u}\left(1 + \frac{u^2}{x^2}\right), \qquad \partial_u K_\lambda = -\frac{2u}{x} e^{\lambda u} + \lambda K_\lambda,$$

and

$$K_\lambda(x, x) = K_\lambda(x, -x) = 0.$$

By the Leibnitz rule,

$$\partial_x \psi_\lambda = \int_{-x}^{x} \partial_x K_\lambda du + K_\lambda(x, x) + K_\lambda(x, -x) = \int_{-x}^{x} \partial_x K_\lambda du,$$

and so

$$\begin{aligned} \partial_x^2 \psi_\lambda &= \int_{-x}^{x} \partial_x^2 K_\lambda du + \partial_x K_\lambda(x, x) + \partial_x K_\lambda(x, -x) \\ &= \int_{-x}^{x} \partial_x^2 K_\lambda du + 2(e^{\lambda x} + e^{-\lambda x}). \end{aligned}$$

It follows, using (22), that

$$H_\lambda \psi_\lambda = \int_{-x}^{x} H_\lambda K_\lambda du + (e^{\lambda x} + e^{-\lambda x})$$
$$= \int_{-x}^{x} (\frac{1}{2}\partial_u^2 - \lambda \partial_u) K_\lambda du + (e^{\lambda x} + e^{-\lambda x})$$
$$= (\frac{1}{2}\partial_u - \lambda) K_\lambda \Big|_{u=-x}^{u=x} + (e^{\lambda x} + e^{-\lambda x}) = 0,$$

as required.

Remark 2 The above integral representation is a special case of the *Dixon-Anderson formula* [7]. The corresponding Bäcklund transformation is a special case of the one introduced in [3], see also [2, 29].

We note that $\psi_0(x) = 2x^2/3$, $\psi_{-\lambda}(x) = \psi_\lambda(x)$ and, for $\lambda > 0$,

$$\psi_\lambda(x) = \lambda^{-3/2} \sqrt{2\pi x}\, I_{3/2}(\lambda x),$$

where $I_\nu(z)$ is the modified Bessel function of the first kind.

Let

$$D = \{(x, u) \in \mathbb{R}^2 : \ |u| < x\}.$$

Consider the integral operator defined, for suitable $f : D \to \mathbb{R}$, by

$$\tilde{K}_\lambda f(x) = \int_{-x}^{x} K_\lambda(x, u) f(x, u) du,$$

and the differential operator, defined on $\mathscr{D}(A_\lambda) = C_c^2(D)$, by

$$A_\lambda = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_u^2 + \partial_x \partial_u + \lambda \partial_u + \left(\lambda + \frac{2}{x+u} - \frac{1}{x}\right) \partial_x.$$

Proposition 2 *For $f \in \mathscr{D}(A_\lambda)$,*

$$H_\lambda \tilde{K}_\lambda f = \tilde{K}_\lambda A_\lambda f. \tag{23}$$

Proof This follows from (22), as in the proof of Proposition 1. □

Now suppose $\lambda \in \mathbb{R}$. Let B be a standard one-dimensional Brownian motion and consider the coupled stochastic differential equations obtained by adding white noise to λ in (21), that is

$$dU = dB + \lambda dt, \qquad dX = dU + \left(\frac{2}{X+U} - \frac{1}{X}\right) dt. \tag{24}$$

Lemma 1 *For any initial condition $\nu \in \mathscr{P}(D)$, the stochastic differential equation* (24) *has a unique strong solution with continuous sample paths in D. It is a diffusion process in D with infinitesimal generator A_λ and the martingale problem for (A_λ, ν) is well-posed.*

Proof The function

$$(x,u) \mapsto \left(\lambda + \frac{2}{x+u} - \frac{1}{x},\ \lambda\right)$$

is uniformly Lipschitz and bounded on

$$D_\epsilon = \{(x,u) \in D :\ x+u > \epsilon,\ x-u > \epsilon\}$$

for any $\epsilon > 0$, so by standard arguments, for any fixed initial condition $(x,u) \in D$, the SDE (24) has a unique strong solution with continuous sample paths up until the first exit time τ from the domain D. We are therefore required to show that $\tau = +\infty$ almost surely. As $X_t - U_t$ is non-decreasing, this is equivalent to showing that $Y_t = X_t + U_t$ almost surely never vanishes. We show this by a simple comparison argument. Set

$$b(x,u) = \frac{2}{x+u} - \frac{1}{x} = \frac{x-u}{x+u}\,\frac{1}{x},$$

and note that for $(x,u) \in D$ with $x - u \geq \delta$, where $\delta > 0$,

$$b(x,u) > \frac{2}{x+u} - \frac{2}{\delta}.$$

Indeed, if $x \leq \delta/2$ then

$$b(x,u) = \frac{x-u}{x+u}\,\frac{1}{x} \geq \frac{\delta}{x+u}\,\frac{2}{\delta} > \frac{2}{x+u} - \frac{2}{\delta};$$

on the other hand, if $x > \delta/2$, then

$$b(x,u) = \frac{2}{x+u} - \frac{1}{x} > \frac{2}{x+u} - \frac{2}{\delta}.$$

Now,

$$dY = 2dU + b(X,U)dt,$$

and it is straightforward to see that the one-dimensional SDE

$$dR = 2dU + \left(\frac{2}{R} - \frac{2}{\delta}\right) dt$$

has a unique strong solution with continuous sample paths in $(0,\infty)$ for any $R_0 = r > 0$; by the usual boundary classification 0 is an entrance boundary for this diffusion. Thus, if $(X_0, U_0) = (x, u)$ and we set $\delta = x - u$ and $r = x + u$, then $Y_t \geq R_t > 0$ almost surely for all $t \geq 0$, proving the first claim. The second claim follows. □

Combining this with the intertwining relation (23), we obtain:

Theorem 2 *Let* $\rho \in \mathscr{P}((0,\infty))$ *and* $\nu = \rho(dx)\nu_x(du) \in \mathscr{P}(D)$, *where*

$$\nu_x(du) = \psi_\lambda(x)^{-1} K_\lambda(x,u)du.$$

Let (X, U) *be a diffusion process in* D *with initial condition* ν *and infinitesimal generator* A_λ. *Then* X *is a diffusion process in* $(0,\infty)$ *with infinitesimal generator*

$$L_\lambda = \psi_\lambda(x)^{-1} H_\lambda \psi_\lambda(x) = \frac{1}{2}\partial_x^2 + \partial_x \ln \psi_\lambda(x) \cdot \partial_x.$$

Moreover, for each $t \geq 0$ *and* $g \in B(\mathbb{R})$,

$$E[g(U_t)|\, X_s,\ 0 \leq s \leq t] = \int_{-X_t}^{X_t} g(u)\nu_{X_t}(du),$$

almost surely.

Proof This follows from the intertwining relation (23) using Theorem 4. First note that we can identify D with $\mathbb{R}^2$ via the one-to-one mapping $(x,u) \mapsto (\ln(x + u), \ln(x - u))$ and thus regard D, equipped with the metric induced from the Euclidean metric on $\mathbb{R}^2$, as a complete, separable, locally compact metric space. Similarly, we identify $(0,\infty)$ with $\mathbb{R}$ via the one-to-one mapping $x \mapsto \ln x$ and regard $(0,\infty)$, equipped with the metric induced from the Euclidean metric on $\mathbb{R}$, as a complete, separable metric space. Note that this does not alter the topologies on D and $(0,\infty)$, or the definitions of $B(D)$, $C_b(D)$, $\mathscr{P}(D)$, $C_c^2(D)$, $B((0,\infty))$, $C_b((0,\infty))$, $\mathscr{P}((0,\infty))$, $C_c^2((0,\infty))$, and so on: it is just a smooth change of variables.

The map $\gamma : D \to (0,\infty)$ defined by $\gamma(x,u) = x$ is continuous and the Markov transition kernel Λ from $(0,\infty)$ to D defined by

$$\Lambda f(x) = \int_{-x}^{x} \nu_x(du) f(x,u), \qquad f \in B(D)$$

satisfies $\Lambda(g \circ \gamma) = g$ for $g \in B((0,\infty))$. Moreover, by (23),

$$L_\lambda \Lambda f = \Lambda A_\lambda f, \qquad f \in \mathscr{D}(A_\lambda). \tag{25}$$

Now, $\mathscr{D}(A_\lambda) = C_c^2(D)$ is closed under multiplication, separates points and is convergence determining. Thus, all that remains to be shown is that the martingale problem for (L_λ, ρ), for some $\mathscr{D}(L_\lambda) \supset \Lambda(\mathscr{D}(A_\lambda))$, is well-posed.

As $\psi_\lambda(x) = \psi_{-\lambda}(x)$, we can assume $\lambda \geq 0$. The drift $b_\lambda(x) = \partial_x \ln \psi_\lambda(x)$ is given by $2/x$ if $\lambda = 0$ and, for $\lambda > 0$,

$$b_\lambda(x) = \frac{1}{2x} + \lambda \frac{I'_{3/2}(\lambda x)}{I_{3/2}(\lambda x)} = \frac{1}{2x} + \lambda \frac{I_{1/2}(\lambda x) + I_{5/2}(\lambda x)}{2I_{3/2}(\lambda x)}.$$

This is bounded below by $1/2x$ and converges to λ as $x \to +\infty$. In fact, $H_\lambda \psi_\lambda = 0$ implies

$$\partial_x^2 \ln \psi(x) = 2/x^2 - \lambda^2 - b_\lambda(x)^2,$$

hence $b_\lambda(x)$ is uniformly Lipschitz and bounded on (a, ∞) for any $a > 0$. It follows that 0 is an entrance boundary and $+\infty$ is a natural boundary for this one-dimensional diffusion process and the martingale problem for (L_λ, ρ) with $\mathscr{D}(L_\lambda) = C_c^2((0,\infty))$ is well-posed. By Itô's lemma and the intertwining relation (25), we conclude that the martingale problem for (L_λ, ρ) with $\mathscr{D}(L_\lambda) = \Lambda(\mathscr{D}(A_\lambda)) \cup C_c^2((0,\infty))$ is also well-posed, as required. □

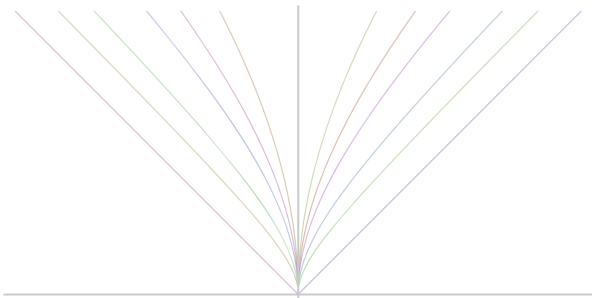

Fig. 2 The rational Calogero-Moser flow in D, shown here with u as the horizontal and x as the vertical coordinate

To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$, the classical flow in D is along the curve $2u = \lambda(x^2 - u^2)$ (see Fig. 2), according to the evolution equations

$$\dot{u} = \lambda, \qquad \dot{x} = \dot{u} + \frac{2}{x+u} - \frac{1}{x},$$

and the x-coordinate satisfies the equation of motion $\ddot{x} = -1/x^3$. Adding noise to the constant of motion λ gives the *stochastic Bäcklund transformation*

$$dU = dB + \lambda dt, \qquad dX = dU + \left(\frac{2}{X+U} - \frac{1}{X} \right) dt;$$

according to Theorem 2, for appropriate (random) initial conditions, U evolves as a Brownian motion with drift λ and the X evolves as a diffusion process in $(0, \infty)$ with infinitesimal generator L_λ.

When $\lambda = 0$, as $u_0(x) = 0$, the Bäcklund transformation reduces to $\dot{x} = 1/x$, as in the example discussed in the introduction. Note however that in this setting

$$L_0 = \frac{1}{2}\partial_x^2 + \frac{2}{x}\partial_x,$$

and the stochastic differential equations (24) do *not* reduce to the one discussed in the introduction which, for example, gives the simpler construction of the diffusion process with generator L_0 as the solution to the stochastic differential equation

$$dX = dB + \frac{2}{X}dt.$$

To see how the above construction relates to the semi-classical limit, let us introduce a parameter $\mu \geq 1$ and consider

$$H = \frac{1}{2\mu}\partial_x^2 - \frac{1+\mu}{2x^2}.$$

Then all of the above carries over with K_λ replaced by $(K_\lambda)^\mu$ and

$$\psi_\lambda^{(\mu)}(x) = \int_{-x}^{x} K_\lambda(x,u)^\mu du.$$

In this setting, Theorem 2 can be restated as follows. Let B be a Brownian motion and (X, U) the unique strong solution in D to

$$dU = \mu^{-1/2}dB + \lambda dt, \qquad dX = dU + \left(\frac{2}{X+U} - \frac{1}{X}\right)dt \tag{26}$$

with $X_0 = x > 0$ and U_0 chosen at random in $(-x, x)$ according to

$$\nu_x^{(\mu)}(du) = \psi_\lambda^{(\mu)}(x)^{-1} K_\lambda(x,u)^\mu du.$$

Then X evolves as a diffusion process in $(0, \infty)$ with infinitesimal generator

$$\frac{1}{2\mu}\partial_x^2 + \frac{1}{\mu}\partial_x \ln \psi_\lambda^{(\mu)}(x)\partial_x.$$

When $\mu \to \infty$, the SDE (26) reduces to the deterministic evolution (21) and the initial distribution $\delta_x \times \nu_x^{(\mu)}$ concentrates on $\delta_x \times \delta_{u_\lambda(x)}$ where $u_\lambda(x)$ is the unique solution in $(-x, x)$ to the critical point equation $\partial_u \ln K_\lambda = 0$ or, equivalently

$2u = \lambda(x^2 - u^2)$. On the other hand, one might expect

$$\frac{1}{\mu}\partial_x \ln \psi_\lambda^{(\mu)}(x) \to \partial_x \left[\ln K_\lambda(x, u_\lambda(x))\right],$$

(which is indeed the case) and so in the limit as $\mu \to \infty$, the evolution of X is according to the gradient flow

$$\dot{x} = \partial_x \left[\ln K_\lambda(x, u_\lambda(x))\right].$$

Comparing this with (21) gives, as in the Toda case,

$$\partial_x \left[\ln K_\lambda(x, u_\lambda(x))\right] = \left[\partial_x \ln K_\lambda\right](x, u_\lambda(x)),$$

which can be verified directly.

If $-1/2 \le \mu \le 1$ and we consider

$$H = \frac{1}{2}\partial_x^2 - \frac{\mu(\mu+1)}{2x^2},$$

then things are more complicated, because now the evolution

$$dU = dB + \lambda dt, \qquad dX = dU + \mu\left(\frac{2}{X+U} - \frac{1}{X}\right)dt$$

can reach the boundary of D and one needs to introduce reflecting boundary conditions on the boundary $x+u=0$ in the x direction to ensure that the appropriate intertwining relation holds; even then, proving the analogue of Theorem 2 is considerably more technical. One can also consider the case $-3/2 \le \mu < -1/2$, but then the diffusion with infinitesimal generator L_λ will also require either reflecting (for $\mu > -3/2$) or absorbing (for $\mu = -3/2$) boundary conditions at zero.

Formally it can be seen that the analogue of Theorem 2, in the case $\mu = 0$, corresponds to Pitman's '$2M-X$' theorem, for general drift and initial condition [25, 26], which can be stated as follows. Let $x \ge 0$ and U be a Brownian motion with drift λ and U_0 chosen at random in $[-x, x]$ with probability density proportional to $e^{\lambda u}$. Set

$$X_t = U_t - \min\{2 \inf_{s \le t} U_s, U_0 - x\}, \qquad t \ge 0.$$

Then (X, U) is a reflected Brownian motion (with singular covariance) in the closure of D and X is a diffusion process in $[0, \infty)$ started at x with infinitesimal generator

$$\frac{1}{2}\partial_x^2 + \lambda \coth(\lambda x)\partial_x.$$

4 Hyperbolic Calogero-Moser System I

The above example extends to the hyperbolic case, taking

$$K(x,u) = \left[\frac{\sinh\left(\epsilon\frac{x+u}{2}\right)\sinh\left(\epsilon\frac{x-u}{2}\right)}{\sinh\epsilon x}\right]^{\mu}, \qquad |u| < x.$$

We will assume for convenience that $\mu \geq 1$. Now,

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = \frac{\epsilon^2\mu^2}{\sinh^2 \epsilon x} \tag{27}$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = \frac{\epsilon^2\mu}{\sinh^2 \epsilon x}. \tag{28}$$

The corresponding Bäcklund transformation

$$\dot{u} = -\partial_u \ln K, \qquad \dot{x} = \partial_x \ln K \tag{29}$$

agrees with the one given in [29] and has the property that, if (29) holds, then x satisfies the equations of motion of the hyperbolic Calogero-Moser system with Hamiltonian

$$\frac{1}{2}p^2 - \frac{\epsilon^2\mu^2}{2\sinh^2 \epsilon x}, \tag{30}$$

and $\dot{u} = \lambda$ is a conserved quantity for the coupled system. Indeed, as before, differentiating (27) with respect to x and u yields, respectively,

$$\ddot{x} = \partial_x^2 \ln K\ \partial_x \ln K - \partial_u\partial_x \ln K\ \partial_u \ln K = \partial_x \frac{\epsilon^2\mu^2}{2\sinh^2 \epsilon x}, \tag{31}$$

and

$$\ddot{u} = \partial_u^2 \ln K\ \partial_u \ln K - \partial_x\partial_u \ln K\ \partial_x \ln K = 0. \tag{32}$$

It also follows from (27) and (29) that λ is an eigenvalue of the Lax matrix

$$\begin{pmatrix} p & \epsilon\mu/\sinh\epsilon x \\ -\epsilon\mu/\sinh\epsilon x & -p \end{pmatrix}. \tag{33}$$

The equation $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$. Using this equation, namely

$$\coth\big(\epsilon\frac{x-u}{2}\big) - \coth\big(\epsilon\frac{x+u}{2}\big) = \frac{2\lambda}{\epsilon\mu}, \tag{34}$$

we can rewrite the evolution equation (29) as

$$\dot{u} = \lambda, \qquad \dot{x} = \lambda + b(x,u) = (\partial_x + \partial_u)\ln K_\lambda, \tag{35}$$

where

$$b(x,u) = (\partial_x + \partial_u)\ln K = \mu\epsilon\left[\coth\left(\epsilon\,\frac{x+u}{2}\right) - \coth\epsilon x\right].$$

Equation (34) has a unique solution $u_\lambda(x) \in \mathbb{R}$ for each $x > 0$ and $\lambda \in \mathbb{R}$. The relation (34) is stable under the new evolution equations (35), and is now required to be in force in order to guarantee that (x,p) evolves according to the hyperbolic Calogero-Moser flow on the iso-spectral manifold corresponding to λ.

Note that $u_0(x) = 0$ for all $x > 0$, so when $\lambda = 0$, we must have $u(t) = 0$ for all $t \geq 0$ and the equation for x simplifies to $\dot{x} = \epsilon\mu/\sinh\epsilon x$, which admits a unique semi-global solution for any initial condition $x(0) = x_0 > 0$, defined for all $t \geq 0$ by

$$x(t) = \frac{1}{\epsilon}\cosh^{-1}\big(\cosh\epsilon x_0 + \epsilon^2\mu t\big). \tag{36}$$

For $\lambda \in \mathbb{R}\backslash\{0\}$, the function u_λ is a bijection from $(0,\infty)$ to $\mathbb{R}$, with inverse given by

$$u_\lambda^{-1}(u) = \frac{2}{\epsilon}\cosh^{-1}\sqrt{\frac{\epsilon\mu}{2\lambda}\sinh\epsilon u + \cosh^2\frac{\epsilon u}{2}}.$$

It follows that, for any given $\lambda \in \mathbb{R}$, the evolution equations (35) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (34)) in the sense that they admit a unique semi-global solution. For $\lambda \in \mathbb{R}\backslash\{0\}$ and initial condition $x(0) = x_0 > 0$, the solution is given explicitly for all $t \geq 0$ by $u(t) = u_\lambda(x_0) + \lambda t$ and $x(t) = u_\lambda^{-1}(u(t))$. As before, the equations (35) provide the correct framework into which we can introduce noise with the desired outcome.

Combining (27) and (28) gives the intertwining relation

$$H_\lambda K_\lambda = \big(\frac{1}{2}\partial_u^2 - \lambda\partial_u\big)K_\lambda \tag{37}$$

where $K_\lambda = e^{\lambda u}K$, $H_\lambda = H - \lambda^2/2$, and

$$H = \frac{1}{2}\partial_x^2 - \frac{\epsilon^2\mu(\mu+1)}{2\sinh^2 \epsilon x}. \tag{38}$$

As before, it follows, using (37) and the Leibnitz rule, that

$$\psi_\lambda(x) = \int_{-x}^{x} K_\lambda(x,u)du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$. Indeed, if $\mu > 1$, then K_λ, $\partial_x K_\lambda$ and $\partial_u K_\lambda$ vanish for $u = \pm x$ and the claim is immediate. If $\mu = 1$, then

$$\partial_x K_\lambda(x,x) = \frac{\epsilon}{2}e^{\lambda x}, \quad \partial_x K_\lambda(x,-x) = \frac{\epsilon}{2}e^{-\lambda x},$$
$$\partial_u K_\lambda(x,x) = -\frac{\epsilon}{2}e^{\lambda x}, \qquad \partial_u K_\lambda(x,-x) = \frac{\epsilon}{2}e^{-\lambda x}$$

and

$$K_\lambda(x,x) = K_\lambda(x,-x) = 0.$$

By the Leibnitz rule,

$$\partial_x\psi_\lambda = \int_{-x}^{x}\partial_x K_\lambda du + K_\lambda(x,x) + K_\lambda(x,-x) = \int_{-x}^{x}\partial_x K_\lambda du,$$

and so

$$\begin{aligned}\partial_x^2\psi_\lambda &= \int_{-x}^{x}\partial_x^2 K_\lambda du + \partial_x K_\lambda(x,x) + \partial_x K_\lambda(x,-x)\\ &= \int_{-x}^{x}\partial_x^2 K_\lambda du + \frac{\epsilon}{2}(e^{\lambda x} + e^{-\lambda x}).\end{aligned}$$

It follows, using (22), that

$$\begin{aligned}H_\lambda\psi_\lambda &= \int_{-x}^{x} H_\lambda K_\lambda du + \frac{\epsilon}{4}(e^{\lambda x} + e^{-\lambda x})\\ &= \int_{-x}^{x}\left(\frac{1}{2}\partial_u^2 - \lambda\partial_u\right)K_\lambda du + \frac{\epsilon}{4}(e^{\lambda x} + e^{-\lambda x})\\ &= \left(\frac{1}{2}\partial_u - \lambda\right)K_\lambda\Big|_{u=-x}^{u=x} + \frac{\epsilon}{4}(e^{\lambda x} + e^{-\lambda x}) = 0,\end{aligned}$$

as required.

For example, when $\mu = 1$,

$$\psi_\lambda(x) = \frac{\epsilon}{\epsilon^2 - \lambda^2}\left[\frac{\epsilon}{\lambda}\coth\epsilon x \sinh\lambda x - \cosh\lambda x\right].$$

In particular, $\psi_0(x) = x\coth\epsilon x - 1/\epsilon$.

Continuing as before, this kernel function leads to a hyperbolic version of Theorem 2, valid for any $\lambda \in \mathbb{R}$.

5 Hyperbolic Calogero-Moser System II

There is another choice of kernel function which leads to a very different 'version' of Theorem 2, valid only for a restricted range of λ. It is based on the kernel functions considered in [9, 10] and, in the rational case, reduces to the example discussed in the introduction.

Let $D = (0, \infty) \times \mathbb{R}$ and consider the kernel function

$$K(x,u) = \left[\tanh\left(\epsilon\,\frac{x+u}{2}\right) + \tanh\left(\epsilon\,\frac{x-u}{2}\right)\right]^\mu, \qquad (x,u) \in D.$$

Note that we can also write

$$K(x,u) = \left[\frac{\sinh\epsilon x}{\cosh(\epsilon(x+u)/2)\cosh(\epsilon(x-u)/2)}\right]^\mu.$$

Now,

$$(\partial_x \ln K)^2 - (\partial_u \ln K)^2 = \frac{\epsilon^2\mu^2}{\sinh^2\epsilon x} \tag{39}$$

and

$$\partial_x^2 \ln K - \partial_u^2 \ln K = -\frac{\epsilon^2\mu}{\sinh^2\epsilon x}. \tag{40}$$

The corresponding Bäcklund transformation

$$\dot{u} = -\partial_u \ln K, \qquad \dot{x} = \partial_x \ln K \tag{41}$$

has the property that, if (41) holds, then x satisfies the equations of motion of the hyperbolic Calogero-Moser system with Hamiltonian (30) and $\dot{u} = \lambda$ is a conserved quantity for the coupled system, as can be seen by differentiating (39) with respect to x and u, respectively. It also follows from (39) and (41) that λ is an eigenvalue of the Lax matrix (33).

Now the equation $\dot{u} = \lambda$ is equivalent to the critical point equation $\partial_u \ln K_\lambda = 0$, where $K_\lambda = e^{\lambda u} K$. Using this equation, namely

$$\tanh\left(\epsilon\,\frac{x+u}{2}\right) - \tanh\left(\epsilon\,\frac{x-u}{2}\right) = \frac{2\lambda}{\epsilon\mu}, \tag{42}$$

we can rewrite the evolution equation (29) as

$$\dot{u} = \lambda, \qquad \dot{x} = \lambda + b(x,u) = (\partial_x + \partial_u)\ln K_\lambda, \tag{43}$$

where

$$b(x,u) = (\partial_x + \partial_u)\ln K = \mu\epsilon\left[\coth\epsilon x - \tanh\left(\epsilon\,\frac{x+u}{2}\right)\right].$$

In this setting, the critical point equation (42) only has a solution $u_\lambda(x) \in \mathbb{R}$ if $|\lambda| < \mu\epsilon$, in which case it is unique. We note that $u_0(x) = 0$ for all $x > 0$ and $u_\lambda(x) \to \pm\infty$ when $\lambda \to \pm\mu\epsilon$. The relation (42) is stable under the new evolution equations (43), and is now required to be in force in order to guarantee that (x, p) evolves according to the hyperbolic Calogero-Moser flow on the iso-spectral manifold corresponding to λ.

When $\lambda = 0$, we must have $u(t) = 0$ for all $t \geq 0$ and the equation for x simplifies to $\dot{x} = \epsilon\mu/\sinh\epsilon x$, as in the previous example, which admits a unique solution for any initial condition $x(0) = x_0 > 0$, defined for all $t \geq 0$ by (36).

For $\lambda > 0$, the function u_λ is a bijection from $(0,\infty)$ to $(0,\infty)$, with inverse

$$u_\lambda^{-1}(u) = \frac{2}{\epsilon}\cosh^{-1}\sqrt{\frac{\epsilon\mu}{2\lambda}\sinh\epsilon u - \sinh^2\frac{\epsilon u}{2}}.$$

Note that the constraint $\lambda < \epsilon\mu$ ensures that the quantity in the square root is positive. For $\lambda < 0$, u_λ is a bijection from $(0,\infty)$ to $(-\infty, 0)$, with inverse given by the same formula. It follows that, given any $\lambda \in \mathbb{R}$, the evolution equations (43) are well-posed on the corresponding iso-spectral manifold (defined in these coordinates by the relation (42)) in the sense that they admit a unique semi-global solution. For $\lambda \in \mathbb{R}\backslash\{0\}$ and initial condition $x(0) = x_0 > 0$, the solution is given for all $t \geq 0$ by $u(t) = u_\lambda(x_0) + \lambda t$ and $x(t) = u_\lambda^{-1}(u(t))$. As before, the evolution equations (43) provide the correct framework into which we can introduce noise with the desired outcome.

Now let

$$H = \frac{1}{2}\partial_x^2 - \frac{\epsilon^2\mu(\mu-1)}{2\sinh^2\epsilon x},$$

and write $H_\lambda = H - \lambda^2/2$. Note that this Hamiltonian has a different coupling constant to the one in (38), reflecting the difference between (40) and (28).

Combining (39) and (40) gives the intertwining relation

$$H_\lambda K_\lambda = \big(\frac{1}{2}\partial_u^2 - \lambda\partial_u\big)K_\lambda \tag{44}$$

and it follows, using the Leibnitz rule, that for $|\mathrm{Re}\,\lambda| < \epsilon\mu$,

$$\psi_\lambda(x) = \int_{-\infty}^{\infty} K_\lambda(x,u)du$$

is an eigenfunction of H with eigenvalue $\lambda^2/2$.

As noted in [10, Eq. (4.16)], the eigenfunction ψ_λ is related to the associated Legendre function of the first kind by

$$\psi_\lambda(x) = \frac{2^{2\mu+3/2}}{\sqrt{\pi}\epsilon}(\sinh\epsilon x)^{1/2}\frac{\Gamma(\mu+\lambda/\epsilon)\Gamma(\mu-\lambda/\epsilon)}{\Gamma(\mu)}P^{\frac{1}{2}-\mu}_{\frac{\lambda}{\epsilon}-\frac{1}{2}}(\cosh\epsilon x). \tag{45}$$

We note also that $\psi_\lambda(x) = \psi_{-\lambda}(x)$, as can be seen, for example, from the functional equation $P^a_{-b}(z) = P^a_{b-1}(z)$, and

$$\psi_0(x) = \frac{2\sqrt{\pi}\Gamma(\mu)}{\epsilon\Gamma(\mu+1/2)}(\sinh\epsilon x)^\mu.$$

These are *not* the same eigenfunctions which were obtained in the previous section, even taking account of the different coupling constants. For example, taking $\mu = 2$ here gives $\psi_0(x) = 8(\sinh\epsilon x)^2/3\epsilon$, which is different from the eigenfunction $\psi_0(x) = x\coth\epsilon x - 1/\epsilon$ of the previous section with $\mu = 1$; both are positive on $(0,\infty)$, vanish at zero, and satisfy

$$\psi_0'' - \frac{\epsilon^2}{\sinh^2\epsilon x}\psi_0 = 0,$$

but they are not equal. On the other hand, they agree (up to a constant factor) in the limit as $\epsilon \to 0$, which corresponds to the rational case.

Consider the integral operator defined, for suitable $f : D \to \mathbb{R}$, by

$$\tilde{K}_\lambda f(x) = \int_{-\infty}^{\infty} K_\lambda(x,u)f(x,u)du,$$

and the differential operator, defined on $\mathscr{D}(A_\lambda) = C_c^2(D)$, by

$$A_\lambda = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_u^2 + \partial_x\partial_u + \lambda\partial_u + (\lambda + b(x,u))\,\partial_x.$$

Proposition 3 *For* $|\mathrm{Re}\,\lambda| < \epsilon\mu$ *and* $f \in \mathscr{D}(A_\lambda)$,

$$H_\lambda \tilde{K}_\lambda f = \tilde{K}_\lambda A_\lambda f. \tag{46}$$

Proof This follows from (44), as in the proof of Proposition 1. □

Now, let B be a standard one-dimensional Brownian motion and consider the coupled stochastic differential equations obtained by adding white noise to λ in (7), that is

$$dU = dB + \lambda dt, \qquad dX = dU + b(X, U)dt. \tag{47}$$

Lemma 2 *Suppose* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \epsilon\mu$ *and* $\mu \geq 1/2$. *For any initial condition* $\nu \in \mathscr{P}(D)$, *the stochastic differential equation* (47) *has a unique strong solution with continuous sample paths in D. It is a diffusion process in D with infinitesimal generator* A_λ *and the martingale problem for* (A_λ, ν) *is well-posed.*

Proof The function $(x, u) \mapsto (\lambda + b(x, u),\ \lambda)$ is uniformly Lipschitz and bounded on $D_\delta = \{(x, u) \in D : \ x > \delta\}$ for any $\delta > 0$, so by standard arguments, for any fixed initial condition $(x, u) \in D$, the SDE (47) has a unique strong solution with continuous sample paths up until the first exit time τ from the domain D. We are therefore required to show that $\tau = +\infty$ almost surely or equivalently, that X_t almost surely never vanishes. We show this by a comparison argument, using the fact that on D we have

$$b(x, u) > \mu\epsilon(\coth \epsilon x - 1).$$

Now,

$$dX = dU + b(X, U)dt,$$

and it is straightforward to see that the one-dimensional SDE

$$dR = dU + \mu\epsilon(\coth(\epsilon R) - 1)dt$$

has a unique strong solution with continuous sample paths in $(0, \infty)$ for any $R_0 = r > 0$; since $\mu \geq 1/2$, by the usual boundary classification 0 is an entrance boundary for this diffusion. Thus, if $(X_0, U_0) = (x, u) \in D$ and $R_0 = x - u$, then $X_t \geq R_t > 0$ almost surely for all $t \geq 0$, as required, proving the first claim. The second claim follows. □

Combining this with the intertwining relation (46), we obtain:

Theorem 3 *Suppose* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \epsilon\mu$ *and* $\mu > 1/2$. *Let* $\rho \in \mathscr{P}((0, \infty))$ *and* $\nu = \rho(dx)\nu_x(du) \in \mathscr{P}(D)$, *where* $\nu_x(du) = \psi_\lambda(x)^{-1} K_\lambda(x, u)du$. *Let* (X, U) *be a diffusion process in D with initial condition ν and infinitesimal generator* A_λ. *Then*

X is a diffusion process in $(0,\infty)$ *with infinitesimal generator*

$$L_\lambda = \psi_\lambda(x)^{-1} H_\lambda \psi_\lambda(x) = \frac{1}{2}\partial_x^2 + \partial_x \ln \psi_\lambda(x) \cdot \partial_x.$$

Moreover, for each $t \geq 0$ *and* $g \in B(\mathbb{R})$,

$$E[g(U_t)|\, X_s,\ 0 \leq s \leq t] = \int_{-\infty}^{\infty} g(u)\nu_{X_t}(du),$$

almost surely.

Proof This follows from the intertwining relation (44) using Theorem 4. As before, we identify D with $\mathbb{R}^2$ via the one-to-one mapping $(x,u) \mapsto (\ln x, u)$ and thus regard D, equipped with the metric induced from the Euclidean metric on $\mathbb{R}^2$, as a complete, separable, locally compact metric space. Similarly, we identify $(0,\infty)$ with $\mathbb{R}$ via the one-to-one mapping $x \mapsto \ln x$ and regard $(0,\infty)$, equipped with the metric induced from the Euclidean metric on $\mathbb{R}$, as a complete, separable metric space.

The map $\gamma : D \to (0,\infty)$ defined by $\gamma(x,u) = x$ is continuous and the Markov transition kernel Λ from $(0,\infty)$ to D defined by

$$\Lambda f(x) = \int_{-\infty}^{\infty} \nu_x(du) f(x,u), \qquad f \in B(D)$$

satisfies $\Lambda(g \circ \gamma) = g$ for $g \in B((0,\infty))$. Moreover, by (46),

$$L_\lambda \Lambda f = \Lambda A_\lambda f, \qquad f \in \mathscr{D}(A_\lambda). \tag{48}$$

Now, $\mathscr{D}(A_\lambda) = C_c^2(D)$ is closed under multiplication, separates points and is convergence determining. Thus, all that remains to be shown is that the martingale problem for (L_λ, ρ), for some $\mathscr{D}(L_\lambda) \supset \Lambda(\mathscr{D}(A_\lambda))$, is well-posed.

By (45) and the relation

$$(z^2-1)\frac{d}{dz}P_b^a(z) = bzP_b^a(z) - (a+b)P_{b-1}^a(z),$$

the drift $b_\lambda(x) = \partial_x \ln \psi_\lambda(x)$ is given by

$$b_\lambda(x) = \lambda \coth \epsilon x + \frac{\epsilon\mu - \lambda}{\sinh \epsilon x}\left[P^{\frac{1}{2}-\mu}_{\frac{\lambda}{\epsilon}-\frac{3}{2}}(\cosh \epsilon x) \Big/ P^{\frac{1}{2}-\mu}_{\frac{\lambda}{\epsilon}-\frac{1}{2}}(\cosh \epsilon x)\right].$$

As $x \to 0^+$,

$$P^{\frac{1}{2}-\mu}_{\frac{\lambda}{\epsilon}-\frac{3}{2}}(\cosh \epsilon x) \Big/ P^{\frac{1}{2}-\mu}_{\frac{\lambda}{\epsilon}-\frac{1}{2}}(\cosh \epsilon x) \to 1.$$

Now $\mu > 1/2$, so this implies that $b_\lambda(x) > 1/2x$ for x sufficiently small, which classifies 0 as an entrance boundary. On the other hand, as $x \to +\infty$, the second term vanishes and $b_\lambda(x) \to \lambda$, which shows that $+\infty$ is a natural boundary. The relevant asymptotics can be found, for example, in [19, §14.8.7, §14.8(iii)]. Thus, as b_λ is locally Lipschitz, the martingale problem for (L_λ, ρ) with $\mathscr{D}(L_\lambda) = C_c^2((0,\infty))$ is well-posed. By Itô's lemma and the intertwining relation (25), it follows that the martingale problem for (L_λ, ρ) with $\mathscr{D}(L_\lambda) = \Lambda(\mathscr{D}(A_\lambda)) \cup C_c^2((0,\infty))$ is also well-posed, as required. □

To summarise, for any given value of the constant of motion $\lambda = \dot{u} \in \mathbb{R}$ with $|\lambda| \le \mu\epsilon$, the classical flow in D evolves according to the evolution equations

$$\dot{u} = \lambda, \qquad \dot{x} = \dot{u} + b(x,u).$$

If we add noise to the constant of motion λ, then the evolution is described by the stochastic Bäcklund transformation

$$dU = dB + \lambda dt, \qquad dX = dU + b(X,U)dt$$

and, for appropriate (random) initial conditions, U evolves as a Brownian motion with drift λ and X evolves as a diffusion process in $(0,\infty)$ with infinitesimal generator L_λ.

As in the previous examples, we can let $\mu \to \infty$ to study the semi-classical limit and the result is analogous. As before, if $\mu = 1$ and $|\lambda| < \epsilon$, and $u_\lambda(x)$ denotes the unique solution to the critical point equation $\partial_u \ln K_\lambda = 0$, then

$$\partial_x \left[\ln K_\lambda(x, u_\lambda(x))\right] = \left[\partial_x \ln K_\lambda\right](x, u_\lambda(x)).$$

It is natural to ask what happens to the statement of Theorem 3 when $\lambda \to \mu\epsilon$. In this limit, $b_\lambda(x) \to \mu\epsilon \coth \epsilon x$ and

$$\Gamma(\mu - \lambda/\epsilon)^{-1}\psi_\lambda(x) \to \frac{2^{\mu+2}\Gamma(2\mu)}{\sqrt{\pi}\epsilon\Gamma(\mu)\Gamma(\mu+1/2)}(\sinh \epsilon x)^\mu =: \tilde{\psi}_{\mu\epsilon}(x).$$

Furthermore, since $u_\lambda(x) \to +\infty$, it is easy to see that the measure ν_x concentrates at $+\infty$. Now, when $\lambda \to \mu\epsilon$ and $u \to +\infty$, $\lambda + b(x,u) \to \mu\epsilon \coth \epsilon x$. The Bäcklund transformation simplifies: if

$$\dot{x} = \mu\epsilon \coth \epsilon x$$

then x evolves according to the hyperbolic Calogero-Moser flow with the constant to motion $\lambda = \mu\epsilon$. The statement of Theorem 3 carries over trivially: if X evolves according to the SDE

$$dX = dB + \mu\epsilon \coth(\epsilon X)dt$$

then X is a diffusion process on $(0, \infty)$ with infinitesimal generator

$$L_{\mu\epsilon} = \tilde{\psi}_{\mu\epsilon}(x)^{-1} H_{\mu\epsilon} \tilde{\psi}_{\mu\epsilon}(x) = \frac{1}{2}\partial_x^2 + \mu\epsilon \coth \epsilon x \cdot \partial_x.$$

Similar remarks apply when $\lambda \to -\mu\epsilon$, and in fact the limiting statements are the same. When $\epsilon \to 0$, this reduces to the example discussed in the introduction, and now we can see from the fundamental restriction $|\lambda| < \epsilon\mu$ that in fact we can only hope for the above structure to remain intact in this limit when $\lambda = 0$, as indeed it does with the evolution of the x-coordinate in (43) becoming autonomous and reducing to $\dot{x} = \mu/x$, and the analogue of Theorem 3 carrying over trivially.

6 The KPZ Equation and Semi-infinite Toda Chain

As remarked in the introduction, most of the above constructions extend naturally to higher rank systems. For the n-particle Toda chain, this has been developed in the papers [21, 22]. The construction given in [21] is related to the geometric RSK correspondence. In [23] it was extended to a semi-infinite setting and related to the Kardar-Parisi-Zhang (or stochastic heat) equation. In this context it can be represented formally as a semi-infinite system of coupled stochastic partial differential equations, the first of which is the stochastic heat equation. In the language of the present paper, the construction given in [23] is a stochastic Bäcklund transformation and should be related (in a way that has yet to be fully understood) to a semi-infinite version of the quantum Toda chain. See also [4, 18] for further related work in this direction.

With this picture in mind, it is natural to expect the construction given in [23], without noise, to be related to the semi-infinite classical Toda chain. This is indeed the case, as we will now explain directly. The conclusion is that the fixed-time solution to the KPZ equation, with 'narrow wedge' initial condition, can be viewed as the trajectory of the first particle in a stochastic perturbation of a particular solution to the semi-infinite Toda chain.

The stochastic heat equation can be written formally as

$$u_t = \frac{1}{2}u_{xx} + \xi u$$

where $\xi(t, x)$ is space-time white noise. It is related to the KPZ equation

$$h_t = \frac{1}{2}h_{xx} + \frac{1}{2}(h_x)^2 + \xi$$

via the Cole-Hopf transformation $h = \log u$. The extension given in [23] starts with a solution $u(t, x, y)$ to the stochastic heat equation with delta initial condition $u(0, x, y) = \delta(x - y)$ and defines a sequence of 'τ-functions' τ_n which can be expressed formally as the bi-Wronskians

$$\tau_n = \det[\partial_x^{i-1}\partial_y^{j-1}u]_{i,j=1,\dots,n}.$$

Their evolution can be described, again formally, by the coupled equations

$$\partial_t a_n = \frac{1}{2}\partial_x^2 a_n + \partial_x[a_n\partial_x h_n]$$

where $a_n = \tau_{n-1}\tau_{n+1}/\tau_n^2$ and $h_n = \log(\tau_n/\tau_{n-1})$ with the convention $\tau_0 = 1$. Moreover, formally it can be seen that the τ_n are τ-functions for the 2d Toda chain, that is, $(\ln\tau_n)_{xy} = a_n$.

If we switch off the noise by setting $\xi = 0$, then u is given by the heat kernel

$$u(t,x,y) = \frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}$$

and

$$\tau_n = t^{-n(n-1)/2}\left(\prod_{j=1}^{n-1} j!\right)u^n.$$

Note that $a_n = \tau_{n-1}\tau_{n+1}/\tau_n^2 = n/t$ and

$$h_{n+1} = -(x-y)^2/2t - \ln\left[\sqrt{2\pi t}\,\frac{t^n}{n!}\right].$$

These τ_n satisfy the 2d Toda equations $(\ln\tau_n)_{xy} = a_n$ as before, but now it also holds that $(\ln\tau_n)_{xx} = -a_n$ or, equivalently,

$$(h_n)_{xx} = e^{h_n - h_{n-1}} - e^{h_{n+1}-h_n}$$

(with $h_0 \equiv +\infty$) which are the equations of motion of the semi-infinite Toda chain.

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Appendix

The theory of Markov functions is concerned with the question: when does a function of a Markov process inherit the Markov property? The simplest case is when there is symmetry in the problem, for example, the norm of Brownian motion in $\mathbb{R}^n$ has the Markov property, for any initial condition, because the heat kernel in $\mathbb{R}^n$ is invariant under rotations. A more general formulation of this idea is the well-known *Dynkin criterion* [5]. There is another, more subtle, criterion which has been proved at various levels of generality by, for example, Kemeny and Snell [13], Rogers and Pitman [26] and Kurtz [14]. It can be interpreted as a time-reversal of Dynkin's criterion [12] and provides sufficient conditions for a function of a Markov process to have the Markov property, but only for very particular initial conditions. For our purposes, the martingale problem formulation of Kurtz [14] is best suited, as it is quite flexible and formulated in terms of infinitesimal generators.

Let E be a complete, separable metric space. Let $A : \mathscr{D}(A) \subset B(E) \to B(E)$ and $\nu \in \mathscr{P}(E)$. A progressively measurable E-valued process $X = (X_t,\ t \geq 0)$ is a solution to the *martingale problem* for (A, ν) if X_0 is distributed according to ν and there exists a filtration $\mathscr{F}_t$ such that

$$f(X_t) - \int_0^t Af(X_s)ds$$

is a $\mathscr{F}_t$-martingale, for all $f \in \mathscr{D}(A)$. The martingale problem for (A, ν) is *well-posed* is there exists a solution X which is unique in the sense that any two solutions have the same finite-dimensional distributions.

The following is a special case of Corollary 3.5 (see also Theorems 2.6, 2.9 and the remark at the top of page 5) in the paper [14].

Theorem 4 (Kurtz [14]) *Assume that E is locally compact, that $A : \mathscr{D}(A) \subset C_b(E) \to C_b(E)$, and that $\mathscr{D}(A)$ is closed under multiplication, separates points and is convergence determining. Let F be another complete, separable metric space, $\gamma : E \to F$ continuous and $\Lambda(y, dx)$ a Markov transition kernel from F to E such that $\Lambda(g \circ \gamma) = g$ for all $g \in B(F)$, where $\Lambda f(x) = \int_E f(x)\Lambda(y, dx)$ for $f \in B(E)$. Let $B : \mathscr{D}(B) \subset B(F) \to B(F)$, where $\Lambda(\mathscr{D}(A)) \subset \mathscr{D}(B)$, and suppose*

$$B\Lambda f = \Lambda A f, \qquad f \in \mathscr{D}(A).$$

Let $\mu \in \mathscr{P}(F)$ and set $\nu = \int_F \mu(dy)\Lambda(y, dx) \in \mathscr{P}(E)$. Suppose that the martingale problems for (A, ν) and (B, μ) are well-posed, and that X is a solution to the martingale problem for (A, ν). Then $Y = \gamma \circ X$ is a Markov process and a solution to the martingale problem for (B, μ). Furthermore, for each $t \geq 0$ and $g \in B(F)$ we have, almost surely,

$$E[g(X_t)|\ Y_s,\ 0 \leq s \leq t] = \int_E g(x)\Lambda(Y_t, dx).$$

We remark that, under the hypotheses of the above theorem, X is a Markov process and the forward equation

$$\nu_t f = \nu f + \int_0^t \nu_s A f ds, \qquad f \in \mathscr{D}(A)$$

has a unique continuous solution in $\mathscr{P}(D)$; also the assumption of uniqueness for the martingale problem for (B, μ) is not necessary, as it is implied by the other hypotheses; we refer the reader to [14] for more details.

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The Kolmogorov Operator and Classical Mechanics

Nobuyuki Ikeda and Hiroyuki Matsumoto

Abstract The Kolmogorov operator is a quadratic differential operator which gives a typical example of a degenerate and hypoelliptic operator. The purpose of this note is to remark that the explicit expression for the transition probability density of the diffusion process generated by the Kolmogorov operator may be regarded as the Van Vleck formula. In fact, we show that it is given by the critical value of the action integral in some adequate path space.

1 Introduction

A quadratic functional of Brownian motion is one of the fundamental objects in probability theory, but new results are being published every year for many reasons as Marc Yor has mentioned in [10]. He himself was always enthusiastic to find new formulae, good understanding and so on.

It may be worth mentioning that the distributions of the quadratic functionals are infinitely divisible and that, in the typical cases like Lévy's stochastic area, the Mellin transforms of the corresponding Lévy measures are given by the Riemann zeta function [8]. Marc Yor was also very much interested in the zeta function and discussed it in the context of both Brownian motions and Bessel processes. See for example [1, 9, 11].

In some special cases, the quadratic functionals come from the Schrödinger operators which are quadratic in the sense that the Hamiltonians of the corresponding classical mechanics are given by the quadratic polynomials. Lévy's stochastic area appears in a study of the Schrödinger operator with a constant magnetic field.

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The integral in time of the squared norm of Brownian motion appears in that of quantum harmonic oscillators, as well as in the Cramér-von Mises test. In such cases the classical mechanics plays an important role. For example, the propagators (the fundamental solution) for the Schrödinger operators are expressed by using the action integrals of the classical paths. This is the so-called Van Vleck formula. An analogue for the heat kernels is also known [5] and we will use it. It is known that the Van Vleck formula holds for the non-degenerate quadratic operators.

The purpose of this note is to remark that, also for the Kolmogorov operator which is a typical example of a degenerate and hypoelliptic differential operator, the transition probability density for the corresponding diffusion process (the heat kernel) may be represented by the minimum action integral in an adequate subspace of the usual path space. The subspace given by (5) below is guessed from the explicit expression (1) for the corresponding diffusion process and we show that its transition density may be represented by means of the action integral in the same way as the usual Van Vleck formula. Moreover we show that these results may be obtained by considering a non-degenerate perturbation and taking a limit.

As mentioned above, the Kolmogorov operator gives an example of a degenerate hypoelliptic operator. In fact, it is mentioned on the first page of the celebrated paper by Hörmander [4]. Needless to say, one of the applications of Malliavin calculus [7] at the beginning was to the hypoellipticity problem. Moreover, we should mention that the Kolmogorov operator has been used to construct a motivating example in the recent works by Bismut (see, e.g., [2, 3]), where a family of hypoelliptic operators which interpolate between the Laplacian and the generator of the geodesic flow on a manifold are constructed.

This note is organized as follows. In Sects. 2 and 3, we review the Kolmogorov operator and consider a perturbation, respectively. The corresponding classical mechanics and their relation to the transition probability densities given in the preceding sections are discussed in Sect. 4.

2 The Kolmogorov Operator

The second order differential operator A on $\mathbf{R}^2$ defined by

$$A = \frac{1}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y} = V_1^2 + V_2$$

is called the Kolmogorov operator, where the vector fields V_1 and V_2 are given by

$$V_1 = \frac{\partial}{\partial x} \qquad \text{and} \qquad V_2 = x\frac{\partial}{\partial y},$$

respectively. Since

$$[V_1, V_2] \equiv V_1 V_2 - V_2 V_1 = \frac{\partial}{\partial y},$$

the operator A is hypoelliptic by virtue of Hörmander's theorem [4] and the diffusion process generated by A admits a smooth transition probability density.

In fact, as was computed by Kolmogorov himself [6], the corresponding diffusion process $Z = \{(X(t), Y(t))\}_{t \geqq 0}$ starting from (x, y) is given explicitly as a Wiener functional by

$$X(t) = x + B(t), \qquad Y(t) = y + \int_0^t X(s)\mathrm{d}s, \tag{1}$$

where $B = \{B(t)\}_{t \geqq 0}$ is a one-dimensional standard Brownian motion with $B(0) = 0$, and the transition probability density $p(t, (x, y), (x', y'))$ with respect to the Lebesgue measure is also explicitly given by

$$p(t, (x, y), (x', y')) = \frac{\sqrt{3}}{\pi t^2} \exp\Big(-\frac{(x' - x)^2}{2t} - \frac{6}{t^3}\big(y' - y - \frac{x' + x}{2} t\big)^2 \Big). \tag{2}$$

Kolmogorov introduced the diffusion process Z as a model of motion with random acceleration.

Identity (2) is obtained as follows. Z is a Gaussian process and, for fixed $t > 0$, the distribution of $(X(t), Y(t))$ is the Gaussian distribution with mean $(x, y + xt)$ and covariance matrix

$$V = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix}.$$

Hence, for every bounded and continuous function φ on $\mathbf{R}^2$, we have

$$\begin{aligned} &E[\varphi(X(t), Y(t))] \\ &\quad = \iint_{\mathbf{R}^2} \varphi(x + \xi, y + xt + \eta) \frac{1}{2\pi \sqrt{\det V}} \exp\Big(-\frac{1}{2} \langle \begin{pmatrix} \xi \\ \eta \end{pmatrix}, V^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rangle \Big) \mathrm{d}\xi \mathrm{d}\eta \\ &\quad = \iint_{\mathbf{R}^2} \varphi(x + \xi, y + xt + \eta) \frac{\sqrt{12}}{2\pi t^2} \exp\Big(-\frac{2}{t}\xi^2 + \frac{6}{t^2}\xi\eta - \frac{6}{t^3}\eta^2 \Big) \mathrm{d}\xi \mathrm{d}\eta \end{aligned}$$

and, by changing the variables from (ξ, η) into (x', y') by $x' = x + \xi$ and $y' = y + xt + \eta$, we get

$$\begin{aligned} &E[\varphi(X(t), Y(t))] \\ &= \iint_{\mathbf{R}^2} \varphi(x', y') \frac{\sqrt{3}}{\pi t^2} \exp\Big(-\frac{2}{t}(x'-x)^2 + \frac{6}{t^2}(x'-x)(y'-y-xt) \\ &\qquad\qquad - \frac{6}{t^3}(y'-y-xt)^2 \Big) \mathrm{d}x' \mathrm{d}y' \\ &= \iint_{\mathbf{R}^2} \varphi(x', y') \frac{\sqrt{3}}{\pi t^2} \exp\Big(-\frac{(x'-x)^2}{2t} - \frac{6}{t^3}\big(y'-y-\frac{x'+x}{2}t\big)^2 \Big) \mathrm{d}x' \mathrm{d}y', \end{aligned}$$

which shows (2).

3 Perturbation

In this section, we consider a perturbation of the Kolmogorov operator, which is non-degenerate and is also quadratic.

For $\varepsilon > 0$ we consider the following second order differential operator on $\mathbf{R}^2$

$$A_\varepsilon = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\varepsilon^2 \frac{\partial^2}{\partial y^2} + x\frac{\partial}{\partial y}.$$

The corresponding diffusion process $Z^\varepsilon = \{(X^\varepsilon(t), Y^\varepsilon(t))\}_{t \geqq 0}$ starting from (x, y) is also explicitly given as

$$X^\varepsilon(t) = x + B_1(t), \qquad Y^\varepsilon(t) = y + \varepsilon B_2(t) + \int_0^t X^\varepsilon(s)\mathrm{d}s,$$

where $\{(B_1(t), B_2(t))\}_{t \geqq 0}$ is a two-dimensional standard Brownian motion starting from $(0, 0)$.

Proposition 1 *The transition probability density $p^\varepsilon(t, (x, y), (x', y'))$ of the diffusion process $\{(X^\varepsilon(t), Y^\varepsilon(t))\}$ with respect to the Lebesgue measure is given by*

$$\begin{aligned} &p^\varepsilon(t, (x, y), (x', y')) \\ &= \frac{\sqrt{3}}{\pi t\sqrt{t^2 + 12\varepsilon^2}} \exp\Big(-\frac{(x'-x)^2}{2t} - \frac{6}{t(t^2+12\varepsilon^2)}\big(y'-y-\frac{x'+x}{2}t\big)^2 \Big). \end{aligned} \tag{3}$$

Proof By a similar way to the proof of Formula (2), we have

$$E[\varphi(X^\varepsilon(t), Y^\varepsilon(t))] = \iiint_{\mathbf{R}^3} \varphi(x+\xi, y+xt+\varepsilon\zeta+\eta) \\ \times \frac{\sqrt{3}}{\pi t^2} \exp\Bigl(-\frac{2}{t}\xi^2 + \frac{6}{t^2}\xi\eta - \frac{6}{t^3}\eta^2\Bigr) \frac{1}{\sqrt{2\pi t}} e^{-\zeta^2/2t} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

for every bounded and continuous function φ. Changing the variables from (ξ, η, ζ) into (x', y', u) by

$$x' = x + \xi, \qquad y' = y + xt + \varepsilon\zeta + \eta \qquad \text{and} \qquad u = \varepsilon\zeta,$$

we get

$$E[\varphi(X^\varepsilon(t), Y^\varepsilon(t))] = \iiint_{\mathbf{R}^3} \varphi(x', y') \\ \times \frac{\sqrt{3}}{\pi t^2} \exp\Bigl(-\frac{(x'-x)^2}{2t} - \frac{6}{t^3}\Bigl(y' - y - \frac{x'+x}{2}t - u\Bigr)^2\Bigr) \\ \times \frac{1}{\sqrt{2\pi t \varepsilon}} e^{-u^2/(2\varepsilon^2 t)} \mathrm{d}x' \mathrm{d}y' \mathrm{d}u$$

and, carrying out the integration with respect to u,

$$E[\varphi(X^\varepsilon(t), Y^\varepsilon(t))] = \iint_{\mathbf{R}^2} \varphi(x', y') \frac{\sqrt{3}}{\pi t \sqrt{t^2 + 12\varepsilon^2}} \\ \times \exp\Bigl(-\frac{(x'-x)^2}{2t} - \frac{6}{t(t^2+12\varepsilon^2)}\Bigl(y' - y - \frac{x'+x}{2}t\Bigr)^2\Bigr) \mathrm{d}x' \mathrm{d}y'.$$

4 Classical Mechanics

We first show that the expression (3) for the heat kernel of the perturbed operator A_ε may be obtained by the Van Vleck formula, and the corresponding classical path converges as $\varepsilon \to 0$. It is natural to regard the limiting path as the classical path which corresponds to the diffusion process generated by the degenerate operator A. In fact, we show that the identity (2) may be represented by using the action integral of this classical path.

The Van Vleck formula for quadratic Hamiltonians was studied in Ikeda et al. [5] and the corresponding heat kernel can be obtained exactly in the same manner as the formula in quantum mechanics if we make some formal modification.

Let us consider the classical Hamiltonian $H_\varepsilon(p,q)$ on $\mathbf{R}^2$ given by

$$H_\varepsilon(p,q) = \frac{1}{2}p_1^2 + \frac{1}{2}\varepsilon^2 p_2^2 + q_1 p_2, \qquad p=(p_1,p_2), q=(q_1,q_2).$$

The Hamilton equation is

$$\dot{q}_1 = \frac{\partial H_\varepsilon}{\partial p_1} = p_1, \qquad \dot{q}_2 = \frac{\partial H_\varepsilon}{\partial p_2} = \varepsilon^2 p_2 + q_1,$$
$$\dot{p}_1 = -\frac{\partial H_\varepsilon}{\partial q_1} = -p_2, \qquad \dot{p}_2 = -\frac{\partial H_\varepsilon}{\partial q_2} = 0.$$

Hence, setting $p_2(s) = \text{const} = \alpha$ and $\dot{q}_1(0) = p_1(0) = \beta$, we see that the classical path starting from $z=(x,y)$ is given as

$$q_1(s) = x + \beta s - \frac{\alpha}{2}s^2,$$
$$q_2(s) = y + \varepsilon^2 \alpha s + \big(xs + \frac{1}{2}\beta s^2 - \frac{\alpha}{6}s^3\big).$$

The Lagrangian is

$$L_\varepsilon(q,\dot{q}) = \dot{q}_1 p_1 + \dot{q}_2 p_2 - H_\varepsilon(p,q) = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2\varepsilon^2}(\dot{q}_2 - q_1)^2,$$

where $p = \partial L_\varepsilon / \partial \dot{q}$.

Now fix $t>0, z'=(x',y')$ and determine the constants α and β so that $q_1(t) = x', q_2(t) = y'$. We easily see that the classical path with this boundary condition is given by

$$\overline{q}_1^\varepsilon(s) = x + (x'-x)\frac{s}{t} + \frac{6}{t^2+12\varepsilon^2}\big(y'-y-\frac{x'+x}{2}t\big)\big(1-\frac{s}{t}\big)s,$$
$$\overline{q}_2^\varepsilon(s) = y + \frac{12\varepsilon^2}{t^2+12\varepsilon^2}\big(y'-y-\frac{x'+x}{2}t\big)\frac{s}{t} + xs + \frac{x'-x}{2t}s^2$$
$$+ \frac{3}{t^2+12\varepsilon^2}\big(y'-y-\frac{x'+x}{2}t\big)s^2 - \frac{2}{t(t^2+12\varepsilon^2)}\big(y'-y-\frac{x'+x}{2}t\big)s^3.$$

Moreover it is also easy to compute the action integral for this path, which is given by

$$S_{cl}^\varepsilon(t,z,z') := \int_0^t L_\varepsilon(\overline{q}^\varepsilon(s), \dot{\overline{q}}^\varepsilon(s))\mathrm{d}s = \frac{(x'-x)^2}{2t} + \frac{6}{t(t^2+12\varepsilon^2)}\big(y'-y-\frac{x'+x}{2}t\big)^2.$$

The Van Vleck-Morette determinant is

$$\det\left(\frac{\partial^2 S^{\varepsilon}_{cl}(t,z,z')}{\partial z \partial z'}\right) = \frac{12}{t^2(t^2+12\varepsilon^2)}$$

and we see from (3) that the Van Vleck formula holds for the heat kernel for A_ε, that is, we have

$$p^{\varepsilon}(t,z,z') = \frac{1}{2\pi}\left\{\det\left(\frac{\partial^2 S^{\varepsilon}_{cl}(t,z,z')}{\partial z \partial z'}\right)\right\}^{1/2} e^{-S^{\varepsilon}_{cl}(t,z',z)}.$$

Next we consider the limit as $\varepsilon \downarrow 0$. We then have

$$\overline{q}^{\varepsilon}_1 \to \overline{q}_1(s) := x + (x'-x)\frac{s}{t} + \frac{6}{t^2}\big(y'-y-\frac{x'+x}{2}t\big)\big(1-\frac{s}{t}\big)s, \qquad (4)$$

$$\overline{q}^{\varepsilon}_2 \to \overline{q}_2(s) := y + \int_0^s \overline{q}_1(u)\mathrm{d}u,$$

$$S^{\varepsilon}_{cl}(t,z,z') \to S_{cl}(t,z,z') := \frac{(x'-x)^2}{2t} + \frac{6}{t^3}\big(y'-y-\frac{x'+x}{2}t\big)^2.$$

Hence we can rewrite the identity (2) as

$$p(t,z,z') = \frac{1}{2\pi}\left\{\det\left(\frac{\partial^2 S_{cl}(t,z,z')}{\partial z \partial z'}\right)\right\}^{1/2} e^{-S_{cl}(t,z',z)}.$$

Finally we give a meaning of the path $\overline{q}$ or its first component $\overline{q}_1$. For this purpose, letting H_x be the set of all absolutely continuous functions ϕ on $[0,t]$ with $\phi(0) = x$ which has square integrable derivative, we introduce the path space $\mathscr{D}_{z,z'}$ by

$$\mathscr{D}_{z,z'} = \{\phi \in H_x \; ; \; \phi(t) = x', \int_0^t \phi(s)ds = y'-y\}. \qquad (5)$$

Note that $\overline{q}_1 \in \mathscr{D}_{z,z'}$.

We put, for $\phi, \psi \in \mathscr{D}_{z,z'}$,

$$\langle \phi, \psi \rangle = \int_0^t \dot{\phi}(s)\dot{\psi}(s)\mathrm{d}s \quad \text{and} \quad S[\phi] = \frac{1}{2}\langle \phi, \phi \rangle.$$

Then, we obtain the following characterization for $\overline{q}_1$.

Theorem 1

(1) $S[\overline{q}_1] = S_{cl}(t,z,z')$ *holds for any* $t > 0, z, z' \in \mathbf{R}^2$.

(2) *The limiting path* $\{\overline{q}_1(s)\}_{0 \leqq s \leqq t}$ *is the unique critical point of the action integral* S *in the path space* $\mathscr{D}_{z,z'}$.

Proof The assertion of (1) is easily checked and we only show (2). Let $\overline{q}_1$ be the path defined by (4) and take any path $\phi \in \mathscr{D}_{z,z'}$. Then, setting $\psi = \overline{q}_1 - \phi$, we have

$$\int_0^t \frac{d}{ds}(\dot{\overline{q}}_1\psi)(s)\mathrm{d}s = \dot{\overline{q}}_1(t)\psi(t) - \dot{\overline{q}}_1(0)\psi(0) = 0$$

since $\psi(0) = \psi(t) = 0$, and

$$\int_0^t \frac{d}{ds}(\dot{\overline{q}}_1\psi)(s)\mathrm{d}s = \int_0^t \ddot{\overline{q}}_1(s)\psi(s)\mathrm{d}s + \int_0^t \dot{\overline{q}}_1(s)\dot{\psi}(s)\mathrm{d}s = \int_0^t \dot{\overline{q}}_1(s)\dot{\psi}(s)\mathrm{d}s$$

since $\ddot{\overline{q}}_1(s)$ is constant and $\int_0^t \psi(s)\mathrm{d}s = 0$. Hence we get $\langle \overline{q_1}, \psi\rangle = 0$ and

$$S(\phi) = S(\overline{q}_1) + S(\psi) \geqq S(\overline{q}_1).$$

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Explicit Formulae in Probability and in Statistical Physics

Alain Comtet and Yves Tourigny

Abstract We consider two aspects of Marc Yor's work that have had an impact in statistical physics: firstly, his results on the windings of planar Brownian motion and their implications for the study of polymers; secondly, his theory of exponential functionals of Lévy processes and its connections with disordered systems. Particular emphasis is placed on techniques leading to explicit calculations.

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1 Introduction

In this article, dedicated to Marc Yor, we would like to highlight some aspects of his work which have had a direct impact in statistical physics taken in its broader sense. Although Marc did not draw his inspiration from physics, he firmly believed in the unity of science and in the fruitfulness of approaching a problem from many different angles. The seminar "Physique et Probabilités" at the Institut Henri Poincaré, which he promoted, and in which he participated actively, attests his desire to share his work beyond a specialist circle. We were all struck by his fascination for explicit formulae. More than mere curiosities, explicit formulae can sometimes reveal deep connections between different probabilistic objects; in his hands, they generated further formulae and their inspection would often be illuminating. This attitude was in stark contrast with that adopted by certain mathematical physicists—"the austere guardians of the temple" who often view their rôle as one of providing

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rigorous existence proofs. Without sacrificing rigour, Marc taught us to value the special case—not only as a possible clue for the general case, but also for its own intrinsic beauty. In the following, we will discuss two among the many themes to which Marc made significant contributions during his career: (1) the winding properties of planar Brownian motion and (2) exponential functionals of Lévy processes. Our aim is to provide instances of application of his work to physical systems and to emphasize its relevance and innovative character.

2 The Winding Number of Planar Brownian Motion

The subject takes off in 1958 with the pioneering work of Spitzer [47], in which he proves

$$\frac{2\theta_t}{\log t} \xrightarrow[t\to\infty]{\text{law}} C_1$$

where $Z_t = X_t + \mathrm{i}Y_t$ is a planar Brownian motion, $\theta_t = \arg Z_t$ a continuous determination of the angle swept up to time t and C_1 a Cauchy random variable of parameter 1. Note that the same logarithmic scale is already present in the work of Kallianpur and Robbins on occupation times [29].

This problem turns out to be closely related to certain questions that arise in polymer physics. Polymers are extended objects which may assume different conformations that affect the medium. In particular, topologically non-trivial configurations in which the polymer has knots or wounds around an obstacle will affect the elastic properties of the medium. The standard approach is to model the ideal polymer by a random walk on a lattice which, in the continuum limit, converges to a Brownian motion. For planar ideal polymers wound around a point, the problem then reduces to computing the winding number distribution of a Brownian path with fixed ends; Prager and Frisch [43] and Edwards [22] solved it simultaneously and independently by different methods in 1967. Edwards' approach, which enables the determination of the elastic properties of ideal polymers in the presence of more general topological constraints, is based on a formal correspondence with the problem of a quantum mechanical particle in a Bohm–Aharonov magnetic field, and uses path integrals.

A few years later, several authors, Rudnik and Hu [45] among them, realized that the angular distribution has a non trivial behaviour in the continuum limit: all the moments of θ_t are divergent. The origin of this divergence and, more importantly, the correct way of handling this problem, are in fact discussed in a seminal paper of Messulam and Yor [36]. In order to probe deeper into the winding process, the winding angle is written as a sum of two terms $\theta_t = \theta_t^+ + \theta_t^-$. Although the terminology had not yet settled at the time, these two terms represent the "large" and "small" windings respectively. The idea is to describe the winding process as

Fig. 1 The angles θ_+ et θ_- for a typical realisation of planar Brownian motion

consisting of a series of phases: those in which the particle is far from the origin contribute to θ^+, whilst those in which the particle makes a lot of turns near the origin bring an important contribution to θ^-; see Fig. 1. This decomposition turned out to be one of the key concepts enabling the general derivation of the asymptotic laws around several points [42]; the works that followed, both in the physics [13, 27, 46] and in the mathematics [3] literatures, brought further confirmation of its usefulness. In particular, Bélisle showed that the asymptotic law for the windings of random walks involves the large-winding component

$$\frac{2\theta_t^+}{\log t}.$$

This random variable, unlike $2\theta_t^-/\log t$, has moments of all orders. We emphasise the practical implications of these facts, not only for polymer physics but also in the context of flux lines in type-two superconductors [19, 37], as well as in the more exotic context of magnetic lines in the solar corona [4].

The study of winding properties in relation to experimental and theoretical work on DNA elasticity has seen a revival in recent years. Although the description of the elastic properties of a single supercoiled DNA molecule requires more complicated models, the short-distance behaviour is similar to the familiar Brownian case [9]. Marc Yor, with D. Holcman and S. Vakeroudis, returned to this theme recently and sought to apply these ideas to a polymer model inspired by biophysics [48]. In that work, the polymer is modelled as a collection of n planar rods attached to a point—each rod making a Brownian angle $\theta_k(t)$, $1 \leq k \leq n$ with a fixed direction. Although the model does not take into account exclusion constraints, it is sufficiently rich and non trivial to be worth investigating. The object is to compute the mean rotation time—which is the characteristic time for a polymer to wind around a point:

$$\tau_n = \inf\{t > 0 : \varphi_n(t) = |2\pi|\}$$

where $\varphi_n(t) = \arg Z_n(t)$ is the angle swept by the last rod. This reduces to the study of a sum of independent, identically-distributed variables on the complex unit circle with one-dimensional Brownian motions as arguments. An asymptotic formula for the mean rotation time $\mathbb{E}(\tau_n)$ in terms of the initial angles of the chain $\theta_k(0)$ is obtained. The physical interest of this work is that it provides an estimate of the mean rotation time as a function of the number of monomers and of the diffusion constant.

3 Brownian Windings and the Bohm–Aharonov Effect

To conclude this discussion we find it particularly instructive to revisit—albeit briefly—the link between the windings of Brownian motion and the Bohm-Aharonov effect.

In this context, the central mathematical object is the index of a curve; its physical counterpart is the notion of magnetic flux. Recall that the index n_z of a closed planar curve γ with respect to a point z is an integer giving the number of turns of the curve around that point (Fig. 2). Let A_n be the area of the set of points of a given index n. Then the algebraic area of the curve may be decomposed into its winding sectors:

$$A(\gamma) = \int n_z \, \mathrm{d}z = \sum_{n \in Z} n \, A_n \, .$$

It should be pointed out that the A_n are non trivial random variables which depend on the whole history of the curve. Nevertheless, when the curve is a two-dimensional Brownian bridge, their expectation may, for $n \neq 0$, be found explicitly [14]:

$$\mathbb{E}(A_n) = \frac{t}{2\pi n^2} \, , \quad n \neq 0 \, .$$

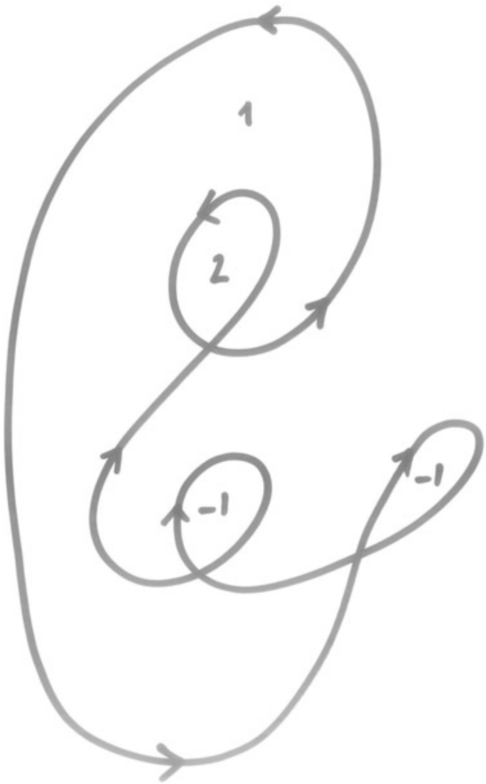
The idea is to relate these quantities to the quantum partition function of a charged particle coupled to a Bohm-Aharonov flux carrying a magnetic flux $\phi = 2\pi\alpha$. They are in fact the Fourier coefficients of the difference of two partition functions:

$$Z_\alpha(t) - Z_0(t) = \frac{1}{2\pi t} \sum_{n \in \mathbb{Z}} (\mathrm{e}^{-2i\pi\alpha n} - 1) \mathbb{E}(A_n)$$

where Z_0 is the “free” partition function, and Z_α, $\alpha \neq 0$, is the “interacting” one. It is, however, not so easy to give a precise meaning to these objects. Indeed when γ is not a smooth curve, n_z is not necessarily locally integrable owing to the small turns that the particle makes around the point z. W. Werner [49] has shown that there exists a regularized version n_z^ϵ of the index such that

$$A(\gamma) = \lim_{\epsilon \to 0} \int n_z^\epsilon \mathrm{d}z$$

Fig. 2 The index n_z of a planar curve with respect to different points z



provides a rigorous decomposition of the Lévy stochastic area into its different components $n \in \mathbb{Z}\backslash\{0\}$. Note that the stochastic area A does not involve the $n = 0$ sector, in contrast with the arithmetic area $\mathscr{A}$, which includes every sector inside the Brownian bridge:

$$\mathscr{A}(\gamma) = \sum_{n\in\mathbb{Z}} A_n \,.$$

A few years ago, Garban and Trujillo-Ferreras [24] managed to compute exactly the expected value of the arithmetic area by using LSE techniques:

$$\mathbb{E}(\mathscr{A}) = \frac{\pi t}{5} \,.$$

Then, by using Yor's result of 1980 [50], they also obtained rigorously the expected values of the area of the $n \neq 0$ sectors, and thence deduced

$$\mathbb{E}(A_0) = \frac{\pi t}{5} - \sum_{n\in\mathbb{Z}} \frac{t}{2\pi n^2} = \frac{\pi t}{30} \,.$$

The exploitation of these ideas in other physical problems, such as that of determining the density of states of a quantum particle in a random magnetic field, has led to several more papers [18, 25]—thus providing yet another illustration of Wigner's famous observation concerning "the unreasonable effectiveness of mathematics in the natural sciences".

4 Exponentials of Lévy Processes

We now turn to our second theme; Marc's interest in this topic had its origin in mathematical finance [51]. The following identity, discovered by Dufresne [20] in that context, provides a good example of the kind of result which was sure to draw his attention: for $\mu > 0$, and B_t a standard Brownian motion,

$$\int_0^\infty \mathrm{e}^{-\mu t - B_t}\,\mathrm{d}t \stackrel{\text{law}}{=} \frac{2}{\Gamma_{2\mu}} \tag{1}$$

where Γ_μ is a gamma-distributed random variable with parameter μ. This motivates the general study of integrals of the form

$$I_t := \int_0^t \mathrm{e}^{-W(s)}\,\mathrm{d}s \tag{2}$$

where W is a Lévy process, i.e. a process started at zero, with right-continuous, left-limited paths, and stationary increments [5].

We recall here for the reader's convenience that a Lévy process W is completely characterised by its Lévy exponent Λ, defined via

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta W(t)}\right] = \mathrm{e}^{t\Lambda(\theta)}\,,$$

and that Λ is necessarily of the form

$$\Lambda(\theta) = \mathrm{i}a\theta - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}_*} \left(\mathrm{e}^{\mathrm{i}\theta y} - 1 - \frac{\mathrm{i}\theta y}{1+y^2}\right) \Pi(\mathrm{d}y) \tag{3}$$

for some real numbers a and σ, and some measure Π on $\mathbb{R}_* := \mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}_*} \left(1 \wedge y^2\right) \Pi(\mathrm{d}y) < \infty\,.$$

The case $\Pi \equiv 0$ corresponds to Brownian motion with drift, i.e. $W(t) = at + \sigma B_t$. The case $\sigma = 0$ and Π a finite measure yields a compound Poisson process with drift; the drift coefficient is given by

$$\mu := a - \int_{\mathbb{R}_*} \frac{y}{1+y^2} \Pi(\mathrm{d}y)\,, \tag{4}$$

the intensity of the process is $\Pi(\mathbb{R}_*)$ and the probability distribution of the jumps is

$$(\Pi(\mathbb{R}_*))^{-1}\,\Pi(\mathrm{d}y)\,.$$

An important subset of the Lévy processes for which the theory of the integral I_t takes on a particularly elegant form consists of the *subordinators*. By definition, a subordinator is a non-decreasing Lévy process; hence $\sigma = 0$, the support of Π is contained in $(0,\infty)$,

$$\int_{(0,\infty)} (1 \wedge y)\,\Pi(\mathrm{d}y) < \infty\,,$$

and μ, as defined in Eq. (4), is non-negative. The importance of this class stems from the fact that the inverse local time of a Feller diffusion is a subordinator.

Yor's work in this area displayed a characteristic fondness for illustrating the theory with concrete calculations. Our purpose in what follows is to review some of the techniques and ideas, and to show how they can be adapted to the study of some disordered systems to yield new results in that field.

5 The Disordered System

Mathematically speaking, by "disordered system" we simply mean an operator-valued random variable. The motivations for considering such systems are many, but for physicists the introduction of randomness is, roughly speaking, a means of modelling a very complex phenomenon. Their study was initiated by Dyson [21], and gained prominence when P.W. Anderson used a linear difference equation with random coefficients to explain why the presence of impurities in metals has a dramatic effect on their conduction properties [1]. This is one instance of a general phenomenon, now known as *Anderson localisation*, whereby linear waves propagating in a medium with randomness tend to have a spatial support that is localised rather than extended. This phenomenon has since been intensively studied by mathematicians and physicists, particularly in the context of linear second-order differential or difference operators, where the effect of randomness on the spectrum has been the focus of attention [40]. The main challenges in this respect are to determine conditions under which localisation takes place and, when it does, to quantify the typical localisation length. The latter problem—which in many cases is tantamount to computing the growth rate of a certain subadditive process—is notoriously difficult [11, 16, 30]; in what follows, we shall see how Marc's ideas suggested new cases where calculations proved possible.

The connection between exponential functionals and disordered systems is through their respective applications to diffusion in a random environment. Indeed, Bouchaud et al. studied the return probability of a particle diffusing in such an environment by considering a certain quantum mechanical disordered system [8], while Carmona, Petit and Yor showed how the moments of I_t yield information on the hitting time of the diffusing particle [12].

Let us describe informally the particular disordered system which will occupy us in the remainder. Consider a diffusion $X(t)$, started from zero, with infinitesimal generator

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left[e^{-W(x)}\frac{d}{dx}\right].$$

The Laplace transform, say $u(x,\lambda)$, of the distribution of the hitting time

$$T_x := \inf\{t : X(t) = x\}$$

solves the equation

$$\frac{1}{2}e^{W(x)}\frac{d}{dx}\left[e^{-W(x)}\frac{du}{dx}\right] = \lambda u\,.$$

If we now set $\psi := e^{-W/2}u$ and $E = -2\lambda$, then ψ solves (formally) the Schrödinger equation

$$-\psi'' + V(x)\,\psi = E\psi \tag{5}$$

where

$$V(x) := \frac{w^2(x)}{4} - \frac{w'(x)}{2}\,, \quad w(x) := W'(x) \tag{6}$$

and the prime symbol indicates differentiation with respect to x.

Of particular interest is the *complex Lyapunov exponent* [33]

$$\Omega(E) := \lim_{x\to\infty}\frac{\ln\psi(x,E)}{x}$$

where $\psi(\cdot,E)$ is the particular solution of Eq. (5) satisfying $\psi(0,E) = 0$ and $\psi'(0,E) = 1$. When W is a Lévy process, the limit on the right-hand side is a self-averaging (non-random) quantity, i.e. its value is the same for almost every realisation of W. Furthermore, we have

$$\Omega(E) = \gamma(E) - i\pi N(E) \tag{7}$$

where the real numbers $\gamma(E)$ and $N(E)$ are, respectively, the reciprocal of the localisation length of the disordered system and the integrated density of states per unit length.

The recipe for computing the complex Lyapunov exponent is as follows [15, 23, 31]: rewrite the Schrödinger equation with the supersymmetric potential (6) as the first-order system

$$-\psi' - \frac{w}{2}\psi = \sqrt{E}\,\phi \tag{8}$$

$$\phi' - \frac{w}{2}\phi = \sqrt{E}\,\psi \tag{9}$$

and introduce the Riccati variable

$$Z := \frac{-1}{\sqrt{E}}\frac{\psi}{\phi}\,. \tag{10}$$

Then

$$Z' = 1 + EZ^2 - wZ \tag{11}$$

and, for $E < 0$,

$$\Omega(E) = \frac{c(0)}{2} - E\,\mathbb{E}\left(Z_\infty\right)\,. \tag{12}$$

In this expression, Z_∞ is the unique positive stationary solution of the Riccati equation (11), the coefficient $c(0)$ is the limit as $s \to 0$ of

$$c(s) := -\frac{\Lambda(\mathrm{i}s)}{s} \tag{13}$$

and the expectation is over the realisations of the Lévy process.

Remark 1 For a Lévy process W, the meaning of the stochastic differential equation (11) needs to be spelled out. When W is a Brownian motion, the equation should be understood in the sense of Stratonovich [39, 44]. When W is a compound Poisson process, the equation should be understood as

$$Z'(x) = 1 + EZ^2(x) \quad \text{for } x \neq x_j$$

and

$$Z(x_j+) = \exp\left\{-\left[W(x_j+) - W(x_j-)\right]\right\} Z(x_j-)$$

where the x_j are the "times" when W jumps; see [15, 17]. Hence the equation makes sense when W is an interlacing process, i.e. the sum of a Brownian motion and of a compound Poisson process. More general Lévy processes may be viewed as limits of interlacing processes.

Let us now elaborate the relationship between the Riccati variable Z and the exponential I_t, defined by Eq. (2). For $E = 0$, the Riccati equation (11) becomes linear and, by using an integrating factor, we obtain

$$\begin{aligned} Z(x) &= Z(0)\, \mathrm{e}^{-W(x)} + \int_0^x \mathrm{e}^{-[W(x)-W(y)]}\, \mathrm{d}y \\ &\overset{\text{law}}{=} Z(0)\, \mathrm{e}^{-W(x)} + \int_0^x \mathrm{e}^{-W(x-y)}\, \mathrm{d}y \\ &= Z(0)\, \mathrm{e}^{-W(x)} + \int_0^x \mathrm{e}^{-W(s)}\, \mathrm{d}s\,. \end{aligned}$$

In particular, if we suppose that $Z(0) = 0$, then we see that the zero-energy Riccati variable has the same law as I_x. So the problem of computing the complex Lyapunov exponent of the disordered system may be viewed as a generalisation of the problem of computing the first moment of I_∞.

6 Explicit Formulae for the Distribution

One approach which Carmona et al. used to determine the distribution of I_∞ consists of expressing it as the stationary distribution of a certain generalised Ornstein–Uhlenbeck process. This process has an infinitesimal generator, and the stationary density therefore solves a forward Kolmogorov equation involving the adjoint of this generator [12, 41].

In the context of our disordered system, the counterpart of the generalised Ornstein–Uhlenbeck process is of course the Riccati process, and the forward Kolmogorov equation for the probability density $f(x,z)$ of the random variable $Z(x)$ is

$$\begin{aligned} \frac{\partial f}{\partial x}(x,z) = \frac{\partial}{\partial z} \Bigg\{ & \left(az - 1 - Ez^2\right) f(x,z) + \frac{\sigma^2}{2} z \frac{\partial}{\partial z}\left[z f(x,z)\right] \\ & + \int_{\mathbb{R}_*} \left[\int_z^{z\mathrm{e}^y} f(x,t)\, \mathrm{d}t - \frac{yz}{1+y^2} f(x,z) \right] \Pi(\mathrm{d}y) \Bigg\}\,. \end{aligned} \tag{14}$$

In particular, the stationary density $f_\infty(z)$ of Z_∞ solves an equation that generalises to the case $E \neq 0$ Equation (2.2) of [12]:

$$\begin{aligned} & \left(az - 1 - Ez^2\right) f_\infty(z) + \frac{\sigma^2}{2} z \frac{\partial}{\partial z}\left[z f_\infty(z)\right] \\ & + \int_{\mathbb{R}_*} \left[\int_z^{z\mathrm{e}^y} f_\infty(t)\, \mathrm{d}t - \frac{yz}{1+y^2} f_\infty(z) \right] \Pi(\mathrm{d}y) = \text{const}\,. \end{aligned} \tag{15}$$

It may be shown that, for $E \leq 0$, the density is supported on $(0, \infty)$ and that the constant on the right-hand side of this last equation is in fact zero. The calculation therefore simplifies if we set, in the first instance, $E = -k^2$, k real. The complex Lyapunov exponent $\Omega(E)$ is analytic in E except for a branch cut along the positive real axis. Hence its value elsewhere in the complex plane may be obtained by analytic continuation.

Example 1 The simplest case arises when

$$\Pi \equiv 0 .$$

Then

$$f_\infty(z) = C(a, \sigma^2, k)\, z^{-2a/\sigma^2 - 1} \exp\left[-\frac{2}{\sigma^2}(k^2 z + 1/z)\right].$$

For $k = 0$, this is Dufresne's result. For $k > 0$, one deduces from Formula (12) the result of Bouchaud et al. which expresses $\Omega(-k^2)$ in terms of the MacDonald functions [8].

When the Lévy measure Π is non-trivial, there is no systematic method for solving the integro-differential equation (15). Nevertheless, in the particular case where the density of Π satisfies a differential equation with constant coefficients, it is possible to eliminate the integral term in (15) and so reduce it to a purely differential form [26].

Example 2 Let $\sigma = 0$ and

$$\Pi(\mathrm{d}y) = p\, q\, \mathrm{e}^{-qy}\, \mathbf{1}_{(0,\infty)}(y)\, \mathrm{d}y .$$

Then one may deduce from Eq. (15) that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(\mu z - 1 + k^2 z^2) f_\infty(z)\right] - p f_\infty(z) - \frac{q}{z}\left[(\mu z - 1 + k^2 z^2) f_\infty(z)\right] = 0$$

where μ is defined in Eq. (4). The solution is given by

$$C(\mu, p, q, k)\, z^q\, (z - z_-)^{-\nu - 1} (z_+ - z)^{\nu - 1} \quad \text{for } 0 < z < z_+ , \tag{16}$$

where

$$\nu := \frac{p}{\sqrt{4k^2 + \mu^2}} \quad \text{and} \quad z_\pm := \frac{1}{2k^2}\left[-\mu \pm \sqrt{4k^2 + \mu^2}\right].$$

The normalisation constant is

$$1/C(\mu, p, q, k) = z_+^{q+\nu} |z_-|^{-\nu - 1} \mathrm{B}(\nu, q + 1)\, {}_2F_1\,(\nu + 1, q + 1; q + \nu + 1; z_+/z_-)$$

and it follows easily that the expectation in Formula (12) for the complex Lyapunov exponent is a ratio of hypergeometric functions. This generalises to an arbitrary negative energy Example B of Carmona, Petit and Yor [12], and to an arbitrary drift the example first discussed in [17]. Yet more examples may be found in [16].

7 Explicit Calculation of the Moments

Another approach to finding the distribution of the exponential functional uses the fact that, at least for subordinators, the positive moments can be computed exactly; see [7], Theorem 2. In order to generalise this result to our disordered system, we work with the Mellin transform of the density $f(x, z)$ of the Riccati variable:

$$\hat{f}(x, s) := \mathbb{E}\left[Z^s(x)\right] = \int_0^\infty z^s f(x, z)\, \mathrm{d}z\,. \tag{17}$$

To find an equation for these moments, we multiply the forward Kolmogorov equation (14) by z^s and integrate over z. For $E \leq 0$ and $s \geq 0$, the result is

$$\frac{\partial \hat{f}(x, s)}{\partial x} = s\left\{E\hat{f}(x, s+1) - c(s)\hat{f}(x, s) + \hat{f}(x, s-1)\right\} \tag{18}$$

where $c(s)$ was defined in Eq. (13), and it is assumed that $f(x, z)$ decays sufficiently quickly at infinity; see [15]. To proceed, we introduce the Laplace transform

$$\hat{F}(\lambda, s) := \int_0^\infty \mathrm{e}^{-\lambda x} \hat{f}(x, s)\, \mathrm{d}x\,. \tag{19}$$

Then

$$\lambda \hat{F}(\lambda, s) - \hat{f}(0, s) = s\left\{E\hat{F}(\lambda, s+1) - c(s)\,\hat{F}(\lambda, s) + \hat{F}(\lambda, s-1)\right\}\,.$$

In particular, if $E = 0$ and the Riccati variable starts at zero, this reduces to the first-order recurrence relation

$$\hat{F}(\lambda, s) = \frac{s}{\lambda + s\, c(s)} \hat{F}(\lambda, s-1)\,,$$

which is precisely the conclusion of the aforementioned theorem.

The stationary version of Eq. (18) is of course simpler:

$$E\hat{f}_\infty(s+1) - c(s)\hat{f}_\infty(s) + \hat{f}_\infty(s-1) = 0\,. \tag{20}$$

It has particularly interesting consequences in the subordinator case, where it may be shown that the complex Lyapunov exponent has the continued fraction expansion

$$\Omega(E) = \frac{c(0)}{2} + \cfrac{-E}{c(1) + \cfrac{-E}{c(2) + \cdots}}.$$

Comtet et al. [15] used this in order to determine the low-energy behaviour of the density of states.

Let us end by making it clear that Marc's work on exponential functionals was of course much more than a collection of striking formulae. With his prodigious knowledge of probability, he was able to use exponential functionals as a tool to re-derive or extend results obtained by other methods in other parts of the theory of stochastic processes. In the particular case where the Lévy process is a Brownian motion with drift, his work with H. Matsumoto led among other things to extensions of Bougerol's identity [10] and of Pitman's $2M - X$ theorem [2, 6, 34, 35, 38]. For more general Lévy processes, a dominant theme was the correspondence between exponential functionals and the semi-stable processes introduced by J. Lamperti [32]. There, the main applications are, on the one hand, to the description of the entrance law of the semi-stable process and, on the other hand, to the factorisations of the exponential variable [7, 28].

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Matsumoto–Yor Process and Infinite Dimensional Hyperbolic Space

Philippe Bougerol

A la mémoire de Marc Yor, avec admiration

Abstract The Matsumoto–Yor process is $\int_0^t \exp(2B_s - B_t)\,ds, t \geq 0$, where (B_t) is a Brownian motion. It is shown that it is the limit of the radial part of the Brownian motion at the bottom of the spectrum on the hyperbolic space of dimension q, when q tends to infinity. Analogous processes on infinite series of non compact symmetric spaces and on regular trees are described.

1 Introduction

The aim of this paper is mainly to see in the first part that the Matsumoto–Yor process [26]

$$\eta_t = \int_0^t e^{2B_s - B_t}\,ds, t \geq 0,$$

where (B_t) is a standard Brownian motion, appears naturally as the radial part of the Brownian motion on the hyperbolic space $\mathbb{H}_q$ at the bottom of the spectrum when the dimension $q \to +\infty$. Marc Yor (1999, private communication) asked in 1999 whether there is a geometric interpretation of this process which provides a direct proof of its Markovianity. Notice that

$$\int_0^t e^{\mu B_s - B_t}\,ds, t \geq 0,$$

is a Markov process only for $\mu = 1$ and $\mu = 2$ (this follows from [25] by a scaling and limit argument). The case $\mu = 1$ is easy since it follows from Ito's formula that

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this process is solution of a stochastic differential equation. Things are more subtle for $\mu = 2$ because the process is not Markov for the filtration $\sigma(B_s, 0 \le s \le t)$.

In a second part, we describe the limit of the radial part of the Brownian motion for the three infinite series of symmetric spaces of higher rank, namely $SO(p,q)$, $SU(p,q)$, $Sp(p,q)$ when p is fixed and $q \to +\infty$. For $SU(p,q)$ we give the generator of the limit by studying the asymptotic behaviour of spherical functions (such an analysis is not available yet for the other cases).

In the higher rank case, these processes are different from the Whittaker processes obtained by O'Connell [29] and Chhaibi [10] who also generalize the Matsumoto–Yor process for real split semi simple groups, but in a different (and more interesting) direction linked with representation theory and geometric crystals.

In the last part we show that in the q-adic case of rank one, or more generally on regular trees $\mathbb{T}_q$, the radial part of the simple random walk at the bottom of the spectrum converges when $q \to \infty$ to the discrete Pitman process

$$2 \max_{0 \le k \le n} \Sigma_k - \Sigma_n, n \in \mathbb{N},$$

where (Σ_n) is the simple random walk on $\mathbb{Z}$. Contrary to the real case we use here only elementary arguments and the treatment is self-contained.

In an appendix we describe how to modify Shimeno [38] in order to obtain the needed asymptotics for spherical functions in rank one.

2 The Matsumoto–Yor Process

Let us first recall a now classical theorem of Pitman [35]. Let $B_t, t \ge 0$, be a standard real Brownian motion starting at 0.

Theorem 1 (Pitman [35]) *The process*

$$2 \max_{0 \le s \le t} B_s - B_t, t \ge 0,$$

is a Markov process on $\mathbb{R}^+$. *It is the Bessel(3) process with generator*

$$\frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}.$$

In 1999, Matsumoto and Yor [26–28], have found the following exponential generalization of Pitman's theorem.

Theorem 2 (Matsumoto and Yor [26]) *The process*

$$\eta_t = \int_0^t e^{2B_s - B_t}\, ds, t \ge 0,$$

is a Markov process and the generator of $\log \eta_t$ *is*

$$\frac{1}{2}\frac{d^2}{dr^2} + (\frac{d}{dr}\log K_0(e^{-r}))\frac{d}{dr},$$

where K_0 *is the Macdonald function.*

Recall that, for $\lambda \in \mathbb{R}$, the classical Macdonald function K_λ is

$$K_\lambda(x) = \frac{1}{2}(\frac{x}{2})^\lambda \int_0^\infty \frac{e^{-t}e^{-x^2/4t}}{t^{1+\lambda}}\, dt.$$

A geometric intuition of the Markovianity of this process does not follow clearly from the rather intricate known proofs (either the original ones [26, 27] or Baudoin [2], see also [3]). Pitman's theorem can be recovered by Brownian scaling and Laplace's method.

3 Brownian Motion on Hyperbolic Spaces

3.1 Hyperboloid Model

There are many realizations of the hyperbolic spaces (see for instance Cannon et al. [7]). We will consider two of them: the hyperboloid and the upper half-space models. A very convenient reference for us is Franchi and Le Jan's book [15], which we will follow. A more Lie theoretic approach will be applied in Sect. 4 for the higher rank case (and of course could also be used here).

For $\xi, \xi' \in \mathbb{R}^{q+1}$ let

$$\langle \xi, \xi' \rangle = \xi_0 \xi_0' - \sum_{k=1}^{q} \xi_k \xi_k'.$$

The hyperbolic space $\mathbb{H}_q$ of dimension q is

$$\mathbb{H}_q = \{\xi \in \mathbb{R}^{q+1}; \langle \xi, \xi \rangle = 1, \xi_0 > 0\},$$

which is the upper sheet of an hyperboloid, with the Riemannian metric d defined by

$$\cosh d(\xi, \xi') = \langle \xi, \xi' \rangle.$$

The group $SO_0(1, q)$ is the connected component of the identity in

$$\{g \in Gl(q+1, \mathbb{R}); \langle g\xi, g\xi' \rangle = \langle \xi, \xi' \rangle, \xi, \xi' \in \mathbb{R}^{q+1}\}.$$

It acts by isometry on $\mathbb{H}_q$ by matrix multiplication on $\mathbb{R}^{q+1}$. Let $\{e_0, \cdots, e_q\}$ be the canonical basis of $\mathbb{R}^{q+1}$. We consider e_0 as an origin in $\mathbb{H}_q$ and we let $o = e_0$. The subgroup

$$K = \{g \in SO_o(1,q); ge_0 = e_0\}$$

is isomorphic to $SO(q)$, it is a maximal compact subgroup of $SO_o(1,q)$ and

$$\mathbb{H}_q = SO_o(1,q)/K.$$

Each $\xi \in \mathbb{H}_q$ can be written uniquely as

$$\xi = (\cosh r)e_0 + (\sinh r)\varphi,$$

where $r \geq 0, \varphi \in S^{q-1} = \{\sum_{k=1}^q \varphi_k e_k; \sum_{k=1}^q \varphi_k^2 = 1\}$. One has $d(o,\xi) = r$ and (r,φ) are called the polar coordinates of ξ. The Laplace Beltrami operator Δ on $\mathbb{H}_q$ is given in these coordinates by (see [15, Prop. 3.5.4])

$$\Delta = \frac{\partial^2}{\partial r^2} + (q-1)\coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta^{\varphi}_{S^{q-1}}, \tag{1}$$

where $\Delta^{\varphi}_{S^{q-1}}$ is the Laplace operator on the sphere S^{q-1} acting on the φ-variable.

We denote (ξ_t) the Riemannian Brownian motion on $\mathbb{H}_q$, defined as the diffusion process with generator $\Delta/2$, starting from the origin $o = e_0$. Since the group K acts transitively on the spheres $\{\xi \in \mathbb{H}_q; d(o,\xi) = r\}$, $d(o,\xi_t)$ is a Markov process, called the radial part of ξ_t. It follows from (1) that its generator is $\Delta_R/2$ where

$$\Delta_R = \frac{d^2}{dr^2} + (q-1)\coth r \frac{d}{dr}. \tag{2}$$

3.2 Upper Half Space Model

We introduce the upper half space model following Franchi and Le Jan [15]: we consider the square matrices $\tilde{E}_j$, $1 \leq j \leq 1+q$, of order $q+1$ given by the expression

$$t\tilde{E}_1 + \sum_{j=1}^{q-1} x_j \tilde{E}_{j+1} = \begin{pmatrix} 0 & t & x_1 & x_2 & .. & x_{q-1} \\ t & 0 & x_1 & x_2 & .. & x_{q-1} \\ x_1 & -x_1 & 0 & 0 & .. & 0 \\ x_2 & -x_2 & 0 & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ x_{q-1} & -x_{q-1} & 0 & 0 & .. & 0 \end{pmatrix},$$

when $t, x_1, \cdots, x_{q-1} \in \mathbb{R}$. Then

$$\mathfrak{S} = \{t\tilde{E}_1 + \sum_{j=1}^{q-1} x_j \tilde{E}_{j+1}, t \in \mathbb{R}, x \in \mathbb{R}^{q-1}\}$$

is a solvable subalgebra of the Lie algebra of $SO_0(1, q)$. For $x \in \mathbb{R}^{q-1}$ and $y > 0$ let

$$T_{x,y} = \exp(\sum_{j=1}^{q-1} x_j \tilde{E}_{j+1}) \exp((\log y)\tilde{E}_1).$$

Then

$$S = \{T_{x,y}; (x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*\}$$

is the Lie subgroup of $SO_o(1, q)$ with Lie algebra $\mathfrak{S}$. Moreover, $T_{x,y}T_{x',y'} = T_{x+yx',yy'}$ (see [15, Proposition 1.4.3]). Therefore the map $(x, y) \mapsto T_{x,y}$ is an isomorphism between the affine group of $\mathbb{R}^{q-1}$, namely the semi-direct product $\mathbb{R}^{q-1} \rtimes \mathbb{R}_+^*$, and the group S. One has ([15, Proposition 2.1.3, Corollary 3.5.3]),

Proposition 1 *The map $\tilde{T} : \mathbb{R}^{q-1} \times \mathbb{R}_+^* \to \mathbb{H}_q$ given by $\tilde{T}(x, y) = T_{x,y}e_0$ is a diffeomorphism. In these so-called horocyclic or Poincaré coordinates $(x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$, the hyperbolic distance is given by*

$$\cosh d(\tilde{T}(x, y), \tilde{T}(x', y')) = \frac{\|x - x'\|^2 + y^2 + y'^2}{2yy'}.$$

The pull back of the Laplace Beltrami operator Δ on $\mathbb{H}_q$ is

$$y^2 \frac{\partial^2}{\partial y^2} + (2 - q)y \frac{\partial}{\partial y} + \Delta_{q-1}^x,$$

where Δ_{q-1}^x is the Euclidean Laplacian of $\mathbb{R}^{q-1}$ acting on the coordinate x.

In horocyclic coordinates on $\mathbb{H}_q$, the hyperbolic Brownian motion ξ_t has a nice probabilistic representation which comes from the fact that it can be seen as a Brownian motion on the group S where

Definition 1 On a Lie group a process is called a Brownian motion if it is a continuous process with independent stationary multiplicative increments.

Indeed, let $(X_t, Y_t) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$ be the horocyclic coordinates of ξ_t, and let

$$\varrho = \frac{q - 1}{2},$$

then (see [5], or [15, Theorem 7.6.5.1]):

Proposition 2 *One can write*

$$X_t = \int_0^t e^{B_s^{-\rho}}\, dW_s^{(q-1)},\ Y_t = e^{B_t^{-\rho}},$$

where $B_t^{-\rho} = B_t - t\varrho$ *is a real Brownian motion on* $\mathbb{R}$ *with drift* $-\varrho$ *and* $W_s^{(q-1)}$ *is a* $q-1$*-dimensional standard Brownian motion, independent of* $B^{-\varrho}$.

Proof This follows immediately from Ito's formula and the expression of the generator given in Proposition 1.

3.3 Ground State Processes

3.3.1 Ground State Processes on a Manifold

We will need the notion of ground state process. In order to introduce it rapidly we use the set up presented in Pinsky [34] (see also Pinchover [33]). On a manifold M we consider an elliptic operator which can be written locally as

$$D = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i} + V(x),$$

where the coefficients are C^∞ and the matrix a is symmetric positive definite (i.e. hypothesis H_{loc} in [34, p. 124]).

Let

$$C_D = \{u \in C^\infty(M); Du(x) = 0, u(x) > 0,\ \text{for all } x \in M\}.$$

There exists $\lambda_0 \in (-\infty, +\infty]$ such that, for any $\lambda < \lambda_0$, $C_{D-\lambda}$ is empty, and for $\lambda > \lambda_0$, $C_{D-\lambda}$ is not empty ([34, Theorem 4.3.2]). For a self adjoint operator $-\lambda_0$ coincide with the bottom of the spectrum on L^2 under general conditions ([34, Proposition 4.10.1]), for instance for the Brownian motion on a Riemannian manifold.

Definition 2 When $\lambda_0 < +\infty$, λ_0 is the generalized principal eigenvalue of D.

A positive function $h \in C^\infty(M)$ such that $Dh = \lambda_0 h$ is called a ground state (it does not always exist and is in general not unique). The Doob h-transform of D is the operator D^h defined by

$$D^h f = \frac{1}{h} D(hf) - \lambda_0 f.$$

The associated Markov process is called the h-ground state process. When a is the identity matrix, then

$$D^h = D + 2\sum_i \frac{\partial \log(h)}{\partial x_i}\frac{\partial}{\partial x_i}, \tag{3}$$

(see [34, Sect. 4.1]). For the Brownian motion on a Riemannian manifold, when $V = 0$ and once an origin is fixed, there is sometimes a canonical choice of a ground state for which the ground state process has a probabilistic interpretation as an infinite Brownian loop. This infinite Brownian loop is, loosely speaking, the limit of the first half of the Brownian bridge around the origin when its length goes to infinity (see [1] for details).

3.3.2 A Radial Ground State Process on $\mathbb{H}_q$

In Proposition 2 the component Y_t of the Brownian motion on $\mathbb{H}_q$ depends on the dimension q only through the drift $\varrho = (q-1)/2$. In order to see what happens when $q \to +\infty$, it is natural to first kill this drift. This is why we will consider the Brownian motion at the bottom of its spectrum.

The generalized principal eigenvalue of the Laplace Beltrami operator Δ is $-\varrho^2$ and there exists a unique radial function φ_0 on $\mathbb{H}_q$, called the basic Harish Chandra function, such that

$$\Delta\varphi_0 = -\varrho^2\varphi_0$$

and $\varphi_0(o) = 1$ (see, e.g., Davies [13, 5.7.1], Gangolli and Varadarajan [16], Helgason [18]).

Definition 3 The infinite Brownian loop on $\mathbb{H}_q$ is the φ_0-ground state process (ξ_t^0) with generator

$$\frac{1}{2}\Delta^{\varphi_0}f = \frac{1}{2\varphi_0}\Delta(\varphi_0 f) + \frac{\varrho^2}{2}f.$$

The function φ_0 is radial, which means that $\varphi_0(\xi)$ is a function of $r = d(o,\xi)$; we define $\tilde{\varphi}_0 : \mathbb{R} \to \mathbb{R}$ by,

$$\tilde{\varphi}_0(r) = \varphi_0(\xi).$$

It follows from (2) and (3) that,

Proposition 3 *The process $\{d(o,\xi_t^0), t \geq 0\}$ is a Markov process on $\mathbb{R}^+$ with generator $\Delta_R^{\tilde{\varphi}_0}/2$ where*

$$\Delta_R^{\tilde{\varphi}_0} = \frac{d^2}{dr^2} + ((q-1)\coth r + 2\frac{\tilde{\varphi}_0'(r)}{\tilde{\varphi}_0(r)})\frac{d}{dr}.$$

3.3.3 A Non Radial Ground State Process on $\mathbb{H}_q$

Although our main interest is in ξ_t^0 we will actually need another ground state process for which computations are easier. It is not invariant under rotations but its radial part is the same as the one of ξ_t^0 (see Proposition 4 below). Let $\Psi_0 : \mathbb{H}_q \to \mathbb{R}$ be the function defined by, if $(x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$ are the horocyclic coordinates of $\xi \in \mathbb{H}_q$,

$$\Psi_0(\xi) = y^{(q-1)/2} = e^{\varrho \log y}. \tag{4}$$

Notice that $\Psi_0(o) = 1$ and

$$\Delta\Psi_0(\xi) = y^2 \frac{\partial^2}{\partial y^2} y^\varrho - (q-2) y \frac{\partial}{\partial y} y^\varrho = -\varrho^2 \Psi_0(\xi).$$

Thus Ψ_0 is, like φ_0, a positive ground state of Δ. Let $\{S_t, t \geq 0\}$ be the Ψ_0-ground state process of the Brownian motion (ξ_t) on $\mathbb{H}_q$. By definition, for all $T > 0$, when $f : C([0, T], \mathbb{H}_q) \to \mathbb{R}^+$ is measurable,

$$\mathbb{E}(f(S_t, 0 \leq t \leq T)) = e^{\varrho^2 T/2} \mathbb{E}(f(\xi_t, 0 \leq t \leq T) \frac{\Psi_0(\xi_T)}{\Psi_0(\xi_0)}) \tag{5}$$

where $\Psi_0(\xi_0) = 1$ since $\xi_0 = o$. Then, it is easy to see that:

Lemma 1 ([5]) *In horospherical coordinates,*

$$S_t = (\int_0^t e^{B_s} \, dW_s^{(q-1)}, e^{B_t}),$$

where B_t is a standard (i.e. driftless) one dimensional Brownian motion, and $W^{(q-1)}$ is a standard $q-1$ dimensional Brownian motion, independent of B.

Let dk be the Haar measure on K, normalized as a probability measure. The function

$$\int_K \Psi_0(k \cdot \xi) \, dk$$

is a positive radial eigenvector of Δ with eigenvalue $-\varrho^2$. Therefore, by uniqueness, we have the well known formula of Harish Chandra (see Helgason [18])

$$\varphi_0(\xi) = \int_K \Psi_0(k \cdot \xi) \, dk.$$

The processes S_t and ξ_t^0 do not have the same law, and S_t is not rotation invariant. However,

Proposition 4 *The two processes $\{d(o, S_t), t \geq 0\}$ and $\{d(o, \xi_t^0), t \geq 0\}$ have the same law.*

Proof By invariance under rotation of the Brownian motion on $\mathbb{H}_q$, for any $k \in K$, the processes (ξ_t) and $(k \cdot \xi_t)$ have the same law. Hence, after integration over K, for $T > 0$ and any measurable function $f : C([0,T], \mathbb{R}^+) \to \mathbb{R}^+$,

$$\begin{aligned}
\mathbb{E}(f(d(o, S_t), 0 \le t \le T)) &= e^{\varrho^2 T/2}\mathbb{E}(f(d(o,\xi_t), 0 \le t \le T)\Psi_0(\xi_T)) \\
&= e^{\varrho^2 T/2}\mathbb{E}(f(d(o,k\cdot\xi_t), 0 \le t \le T)\Psi_0(k\cdot\xi_T)) \\
&= e^{\varrho^2 T/2}\mathbb{E}(f(d(o,\xi_t), 0 \le t \le T)\Psi_0(k\cdot\xi_T)) \\
&= e^{\varrho^2 T/2}\mathbb{E}(f(d(o,\xi_t), 0 \le t \le T)\int_K \Psi_0(k\cdot\xi_T)\,dk) \\
&= e^{\varrho^2 T/2}\mathbb{E}(f(d(o,\xi_t), 0 \le t \le T)\varphi_0(\xi_T)) \\
&= \mathbb{E}(f(d(o,\xi_t^0), 0 \le t \le T)).
\end{aligned}$$

Remark 1 In horocyclic coordinates in (4) the function $(x, y) \mapsto \log y$ is the Busemann function on $\mathbb{H}_q$ associated with the point at infinity $y = +\infty$ and Ψ_0 is a minimal eigenfunction. The process (S_t) can be interpreted as the Brownian motion on $\mathbb{H}_q$ conditioned to have 0 speed (i.e. $d(o, S_t)/t \to 0$ as $t \to +\infty$) and to exit at $y = +\infty$ (see [17]).

3.4 Matsumoto–Yor Process as a Limit

Our main result is the following.

Theorem 3 *As $q \to +\infty$, the process*

$$d(0, \xi_t^0) - \log q, t > 0,$$

converges in distribution to $\log \eta_t, t > 0$*, where*

$$\eta_t = \int_0^t e^{2B_s - B_t}\,ds$$

is the Matsumoto–Yor process, which is therefore a Markov process.

Proof By Proposition 4, it is enough to show that, almost surely,

$$\lim_{q\to\infty} d(o, S_t) - \log q = \log \int_0^t e^{2B_s - B_t}\,ds. \tag{6}$$

The origin o in $\mathbb{H}_q$ is $e_0 = \tilde{T}(0,1)$. By Proposition 1,

$$\cosh d(o, \tilde{T}(x,y)) = \frac{\|x\|^2 + y^2 + 1}{2y},$$

and

$$\cosh d(0, S_t) = \frac{\| \int_0^t e^{B_s}\, dW_s^{(q-1)} \|^2 + e^{2B_t} + 1}{2e^{B_t}},$$

thus

$$\frac{2\cosh d(0,S_t)}{q} = \frac{e^{B_t} + e^{-B_t}}{q} + e^{-B_t}\frac{1}{q}\sum_{k=1}^{q-1}(\int_0^t e^{B_s}\, d\beta_s^{(k)})^2,$$

where $\beta_s^{(k)}, k \geq 1$, are independent standard Brownian motions. Conditionally on the σ-algebra $\sigma(B_r, r \geq 0)$, the random variables $\int_0^t e^{B_s}\, d\beta_s^{(k)}, k \geq 1$, are independent with the same distribution and

$$\mathbb{E}((\int_0^t e^{B_s}\, d\beta_s^{(k)})^2 | \sigma(B_r, r \geq 0)) = \int_0^t e^{2B_s}\, ds.$$

Therefore, by the law of large numbers, a.s.

$$\lim_{q\to+\infty} \frac{1}{q} e^{d(0,S_t)} = \lim_{q\to+\infty} \frac{2}{q}\cosh d(0,S_t) = e^{-B_t}\int_0^t e^{2B_s}\, ds.$$

The limit is Markov as a limit of Markov processes.

Let us recover the generator of the Matsumoto–Yor process. Let

$$\delta(r) = \sinh^{q-1} r,$$

then, by Proposition 3,

$$\Delta_R^{\tilde{\varphi}_0} = \frac{d^2}{dr^2} + 2\frac{d}{dr}\log(\delta^{1/2}\tilde{\varphi}_0)(r)\frac{d}{dr},$$

hence the generator of $d(o, \xi_t^0) - \log q$ is

$$\frac{1}{2}\frac{d^2}{dr^2} + \frac{d}{dr}\log(\delta^{1/2}\tilde{\varphi}_0)(r + \log q)\frac{d}{dr}. \tag{7}$$

Therefore the next proposition follows from Corollary 4 (since there, $m_\alpha = q - 1$ for $SO(1,q)$).

Proposition 5 ([26]) *The generator of the Matsumoto–Yor process is*

$$\frac{1}{2}\frac{d^2}{dr^2} + (\frac{d}{dr}\log K_0(e^{-r}))\frac{d}{dr}.$$

Remark 2 We will clarify in Sect. 4 the occurrence of the ground state K_0 of the Toda operator $\frac{d^2}{dr^2} - e^{-2r}$.

3.5 A Conditional Law

The intertwining property which occurs in the proof of Matsumoto–Yor theorem can also be establish by our approach.

Proposition 6 ([26]) *When* $\eta_t = e^{-B_t}\int_0^t e^{2B_s}\,ds$, *and* $\lambda \in \mathbb{R}$,

$$\mathbb{E}(e^{\lambda B_t}|\sigma(\eta_s, 0 \le s \le t)) = \frac{K_\lambda(1/\eta_t)}{K_0(1/\eta_t)}.$$

Proof For $\xi \in \mathbb{H}_q$ with horocyclic coordinates $(x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$ and $\lambda \in \mathbb{R}$, let $\lambda(\xi) = \lambda \log y$. We consider the Harish Chandra function φ_λ (see [18, Theorem IV.4.3]) given by

$$\varphi_\lambda(\xi) = \int_K e^{(\lambda+\rho)(k\cdot\xi)}dk,$$

(it of course depends on q) and we write $\tilde{\varphi}_\lambda(r) = \varphi_\lambda(\xi)$ when $r = d(o, \xi)$. Since $\lambda(S_t) = \lambda B_t$ and $e^{\varrho(\xi)} = \Psi_0(\xi)$, it follows from (5) that, when $f : C([0, t]) \to \mathbb{R}^+$ is measurable,

$$\begin{aligned}
&e^{-\varrho^2 t/2}\mathbb{E}(f(d(o, S_s), 0 \le s \le t))e^{\lambda B_t}) \\
&= e^{-\varrho^2 t/2}\mathbb{E}(f(d(o, S_s), 0 \le s \le t))e^{\lambda(S_t)}) \\
&= \mathbb{E}(f(d(o, \xi_s), 0 \le s \le t)e^{(\lambda+\rho)(\xi_t)}) \\
&= \mathbb{E}(f(d(o, \xi_s), 0 \le s \le t)\int_K e^{(\lambda+\rho)(k\cdot\xi_t)}dk) \\
&= \mathbb{E}(f(d(o, \xi_s), 0 \le s \le t)\varphi_\lambda(\xi_t)) \\
&= \mathbb{E}(f(d(o, \xi_s), 0 \le s \le t)\frac{\varphi_\lambda(\xi_t)}{\varphi_0(\xi_t)}\varphi_0(\xi_t))
\end{aligned}$$

$$= \mathbb{E}(f(d(o,\xi_s), 0 \le s \le t)\frac{\varphi_\lambda(\xi_t)}{\varphi_0(\xi_t)}e^{\rho(\xi_t)})$$

$$= e^{-\varrho^2 t/2}\mathbb{E}(f(d(o,S_s), 0 \le s \le t)\frac{\tilde{\varphi}_\lambda(d(o,S_t))}{\tilde{\varphi}_0(d(o,S_t))}).$$

We have again used the fact that for any $k \in K$, $\{\xi_t, t \ge 0\}$ has the same law as $\{k \cdot \xi_t, t \ge 0\}$. For each q,

$$\mathbb{E}(f(d(0,S_s) - \log q, 0 \le s \le t))e^{\lambda B_t})$$

$$= \mathbb{E}(f(d(0,S_s) - \log q, 0 \le s \le t)\frac{\tilde{\varphi}_\lambda((d(o,S_t) - \log q) + \log q)}{\tilde{\varphi}_0((d(o,S_t) - \log q) + \log q)}).$$

We choose for f a continuous bounded function with bounded support. Letting $q \to +\infty$, it then follows from Theorem 3 and Corollary 4 that

$$\mathbb{E}(f(\log \eta_s, 0 \le s \le t)e^{\lambda B_t}) = \mathbb{E}(f(\log \eta_s, 0 \le s \le t)\frac{K_\lambda(1/\eta_t)}{K_0(1/\eta_t)}).$$

Remark 3 It is straightforward to deduce from this proposition, Theorem 3 and Cameron and Martin's theorem that, as proved by Matsumoto–Yor [26], when $B_t^{(\lambda)} = B_t + \lambda t$ is a Brownian motion with drift λ, then $\log \int_0^t e^{2B_s^{(\lambda)} - B_t^{(\lambda)}}\, ds, t \ge 0$, is a Markov process on $\mathbb{R}$ with generator

$$\frac{1}{2}\frac{d^2}{dr^2} + (\frac{d}{dr}\log K_\lambda(e^{-r}))\frac{d}{dr}.$$

4 Infinite Series of Symmetric Spaces

We will now consider the same problem as above for the infinite series of symmetric spaces of non positive curvature, when the rank is fixed and the dimension goes to infinity. We will see that in the rank one case one finds the same result as for hyperbolic spaces, but that new phenomenons occur in higher rank.

There are only three infinite series of (irreducible) Riemannian symmetric spaces of non positive (non zero) curvature namely the spaces G/K where

$$G = SO(p,q), SU(p,q) \text{ and } Sp(p,q),$$

and K is a maximal compact subgroup (see [18]). We will suppose that $p \le q$, then p is the rank of the symmetric space.

4.1 $SU(p,q)$

We first consider the symmetric spaces associated with the series $SU(p,q)$ with the rank p fixed, when $q \to +\infty$. As in the case of hyperbolic spaces, it is convenient to use two descriptions of $SU(p,q)$. The usual description is the following (see, e.g., [19, 23]): $SU(p,q)$ is the set of $(p+q)\times(p+q)$ matrices with entries in $\mathbb{C}$ and with determinant 1, which conserve the quadratic form

$$\sum_{i=1}^{p} z_i\bar{z}_i - \sum_{j=1}^{q} z_{p+j}\bar{z}_{p+j}.$$

Let I_p and I_q be the identity matrix of order p and q, and let $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ then

$$SU(p,q) = \{M \in Sl(p+q,\mathbb{C}); M^*JM = J\}.$$

If we write M by block as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where the size of the matrix M_1 is $p\times p$, M_2 is $p\times q$, M_3 is $q\times p$ and M_4 is $q\times q$, then $M \in SU(p,q)$ when $M_1^*M_1 - M_2^*M_2 = I_p$, $M_4^*M_4 - M_3^*M_3 = I_q$, $M_1^*M_3 = M_2^*M_4$, $\det(M) = 1$.

4.2 Cartan Decomposition and Radial Part

Let $\bar{K}$ be the following maximal compact subgroup of $SU(p,q)$ (we put the notation K aside for a later use)

$$\bar{K} = \{\begin{pmatrix} K_1 & 0 \\ 0 & K_4 \end{pmatrix} \in Sl(p+q,\mathbb{C}), K_1 \in U(p), K_4 \in U(q)\}.$$

When $r = (r_1, \cdots, r_p) \in \mathbb{R}^p$, let

$$D_p(r) = \begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix},$$

where

$$\cosh r = \begin{pmatrix} \cosh r_1 & 0 & \cdots & 0 \\ 0 & \cosh r_2 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cosh r_p \end{pmatrix},$$

and $\sinh r$ is the same matrix with cosh replaced by sinh. We consider the closed Weyl chamber

$$\mathfrak{A}^+ = \{(r_1, \cdots, r_p) \in \mathbb{R}^p, r_1 \geq r_2 \geq \cdots \geq r_p \geq 0\}.$$

The Cartan decomposition says that any M in $SU(p,q)$ can be written as $M = k_1 D_p(r) k_2$ with $k_1, k_2 \in \bar{K}$ and $r \in \mathfrak{A}^+$. Such a r is unique and is called the radial part of M. We let $r = \text{Rad}(M)$. Recall that,

Definition 4 Let N be a $p \times p$ square matrix, the vector of singular values of N is

$$\text{SingVal}(N) = (\sigma_1, \cdots, \sigma_p) \in \mathfrak{A}^+,$$

where the σ_i's are the square roots of the eigenvalues of NN^* written in decreasing order.

Lemma 2 *If* $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in SU(p,q)$*, then*

$$\cosh \text{Rad}(M) = \text{SingVal}(M_1).$$

Proof By the Cartan decomposition, there exists $K_1, \tilde{K}_1 \in U(p)$ and $K_4, \tilde{K}_4 \in U(q)$ such that, if r is the radial part of M then

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} K_1 & 0 \\ 0 & K_4 \end{pmatrix} \begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix} \begin{pmatrix} \tilde{K}_1 & 0 \\ 0 & \tilde{K}_4 \end{pmatrix}.$$

Therefore $M_1 = K_1(\cosh r)\tilde{K}_1$ and $M_1 M_1^* = K_1(\cosh^2 r) K_1^*$ which proves the lemma.

4.3 Iwasawa Decomposition and Horocyclic Coordinates

It will be convenient to write matrices in $SU(p,q)$ in another basis to make more tractable the solvable part of the Iwasawa decomposition (see, e.g., Lu [24], Iozzi and Morris [22] or Sawyer [37]). We still suppose that $q \geq p$. Let

$$P = \begin{pmatrix} I_p/\sqrt{2} & 0 & I_p/\sqrt{2} \\ I_p/\sqrt{2} & 0 & -I_p/\sqrt{2} \\ 0 & I_{q-p} & 0 \end{pmatrix}.$$

Then $P^* = P^{-1}$ and $P^{-1}JP = \bar{J}$ where

$$\bar{J} = \begin{pmatrix} 0 & 0 & -I_p \\ 0 & I_{q-p} & 0 \\ -I_p & 0 & 0 \end{pmatrix}.$$

We introduce

$$G = \{P^{-1}MP, M \in SU(p,q)\},$$

which is the set of matrices M in $Sl(p+q,\mathbb{C})$ such that $M^*\bar{J}M = \bar{J}$. The group G is obviously isomorphic to $SU(p,q)$. So, we will work with G instead of $SU(p,q)$.

Let l, b, c be $p \times p, p \times (q-p)$ and $p \times p$ complex matrices, respectively, where l is **lower** triangular and invertible, and define

$$S(l,b,c) = \begin{pmatrix} l & b & cl^{*-1} \\ 0 & I_{q-p} & b^*l^{*-1} \\ 0 & 0 & l^{*-1} \end{pmatrix}. \tag{8}$$

Let

$$S = \{S(l,b,c); l \text{ is lower triangular with positive diagonal}, c + c^* = bb^*\},$$

$$A = \{S(l,0,0) \in S; l \text{ is diagonal with positive diagonal}\},$$

$$N = \{S(l,b,c) \in S; \text{ the diagonal elements of } l \text{ are equal to } 1\},$$

$$K = P^{-1}\bar{K}P.$$

Notice that in general l is lower triangular and hence neither N nor S is made of upper triangular matrices. If $D = \{D_p(r), r \in \mathbb{R}^p\}$ then $A = P^{-1}DP$. The Iwasawa decomposition is $G = NAK$, N is the nilpotent component and $S = NA$. Notice that $SU(p,q)/\bar{K}$ is isomorphic with G/K. For $M \in G$, PMP^{-1} is in $SU(p,q)$ and we let $\mathrm{Rad}(M) = \mathrm{Rad}(PMP^{-1})$. In G/K, we choose as origin $o = K$, then $\mathrm{Rad}(M)$ plays the role of a generalized distance between the cosets $o = K$ and $\xi = MK$ in G/K.

Lemma 3 ([24]) *For* $S(l,b,c) \in S$,

$$\cosh \mathrm{Rad}(S(l,b,c)) = \frac{1}{2}\mathrm{SingVal}(l + l^{*-1} + cl^{*-1}).$$

Proof This follows immediately from Lemma 2 since, for $S(l,b,c) \in G$, the corresponding element in $SU(p,q)$ can be written as

$$PS(l,b,c)P^{-1} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where $M_1 = (l + (I+c)l^{*-1})/2$.

By the Iwasawa decomposition each element $\xi \in G/K$ can be written uniquely as $\xi = S(l,b,c)K$ with $S(l,b,c) \in S$. We see that we can and will identify S and G/K. We call $S(l,b,c)$ the horocyclic coordinates of ξ. They generalize the horocyclic–Poincaré coordinates in $\mathbb{H}_q$.

4.4 Brownian Motion on G/K and Infinite Brownian Loop

The Lie algebra $\mathfrak{S}$ of S is

$$\mathfrak{S} = \{\begin{pmatrix} l & b & c \\ 0 & 0 & b^* \\ 0 & 0 & -l^* \end{pmatrix}, \text{where } c \text{ is skew} - \text{Hermitian}, l \text{ is lower triangular with real diagonal}\}.$$

The Killing form on the Lie algebra of G allows to define a scalar product on $\mathfrak{S}$ by

$$< X, Y > = \frac{1}{2}\mathrm{Trace}(XY^*).$$

Then $\mathfrak{S}$ is a real Euclidean space. Let $\mathfrak{A}$ and $\mathfrak{N}$ be the Lie algebras of A and N, then $\mathfrak{S} = \mathfrak{A} \oplus \mathfrak{N}$. Let $X_1, \cdots, X_p$ be an orthonormal basis of $\mathfrak{A}$ and $N_1, \cdots, N_s$ be an orthonormal (real) basis of $\mathfrak{N}$ adapted to the root space decomposition. In horospherical coordinates the Laplace Beltrami operator on $G/K = S$ is (e.g. [5, proof of Proposition 2.2], [11], [17, p. 105])

$$\Delta = \sum_{i=1}^{p} X_i^2 + 2\sum_{j=1}^{s} N_j^2 - 2\sum_{i=1}^{p} \varrho(X_i)X_i,$$

where ϱ is given below by (11) and X_i and N_j are considered as left invariant vector fields. We consider the Riemannian Brownian motion $\{\xi_t, t \geq 0\}$ on $G/K = S$. It is the process with generator $\Delta/2$ starting from the origin o.

As in the hyperbolic case one can consider the ground state process $\{\xi_t^0, t \geq 0\}$ of this Brownian motion associated with the basic Harish Chandra spherical function φ_0. By [18, Theorem IV.4.3],

$$\varphi_0(g) = \int_K e^{\varrho(H(kg))}\, dk, \tag{9}$$

where for $g \in G$, we write $H(g)$ for the element of the Lie algebra $\mathfrak{A}$ of A such that $g \in Ne^{H(g)}K$ in the Iwasawa decomposition $G = NAK$. The generator of ξ^0 is $\Delta^{\varphi_0}/2$ and it corresponds to the infinite Brownian loop on G/K (see [1]). By invariance of Δ under K, the radial part of $\mathrm{Rad}(\xi_t)$ and $\mathrm{Rad}(\xi_t^0)$ are Markov processes with values in the closed Weyl chamber $\mathfrak{A}^+$.

4.5 Distinguished Brownian Motion on S

We define $\Psi_0 : G \to \mathbb{R}^+$ by

$$\Psi_0(g) = e^{\varrho(H(g))}.$$

Since $H(gk) = H(g)$ for any $k \in K$, Ψ_0 is well defined on G/K. We consider the Ψ_0-ground state process $\{S_t, t \geq 0\}$ of the Brownian motion ξ on G/K. Using the identification $G/K = S$, we see it as a process on S (it has the same interpretation as the one given in Remark 1). The following definition is used in harmonic analysis (see, e.g., [5, Proposition 1.2], [11, 12]),

Definition 5 The distinguished Brownian motion on S is the process $S_t, t \geq 0$.

The generator of (S_t) is

$$\frac{1}{2}\Big(\sum_{i=1}^{p} X_i^2 + 2\sum_{j=1}^{s} N_j^2\Big).$$

One shows as in the hyperbolic case that (see also [1, Proof of Theorem 6.1]),

Lemma 4 *The two processes* $\{\mathrm{Rad}(S_t), t \geq 0\}$ *and* $\{\mathrm{Rad}(\xi_t^{(0)}), t \geq 0\}$ *have the same law.*

The process (S_t) is a solution of a stochastic differential equation. Indeed, consider the Brownian motion (V_t) on the Lie algebra $\mathfrak{S}$, considered as an Euclidean space,

$$V_t = \begin{pmatrix} \lambda_t & \beta_t & \kappa_t \\ 0 & 0 & \beta_t^* \\ 0 & 0 & -\lambda_t^* \end{pmatrix} \in \mathfrak{S},$$

where the coefficients $\lambda_t^{r,r}, i\kappa_t^{r,r}/2, 1 \leq r \leq p$, and the real and imaginary parts of $\lambda_t^{r,s}/\sqrt{2}, r > s$, $\beta_t^{k,l}/\sqrt{2}, 1 \leq k \leq p, 1 \leq l \leq q-p$ and $\kappa_t^{r,s}/\sqrt{2}, 1 \leq r < s \leq q$ are standard real independent Brownian motions, $\lambda_t^{r,s} = 0$ if $1 \leq r < s \leq p$ and $\kappa_t^{r,s} = -\bar{\kappa}_t^{s,r}$, when $q \geq r > s \geq 1$ (we use $m^{i,j}$ to denote the (i,j) coefficient of a matrix m).

When (X_t) is a continuous semimartingale, we use δX_t for its Stratonovich differential and dX_t for its Ito one (see, e.g., [21]).

Proposition 7 *The distinguished Brownian motion* (S_t) *is the solution of the following Stratonovich stochastic differential equation in the set of* $(p+q)\times(p+q)$ *complex matrices,*

$$\delta S_t = S_t\, \delta V_t, S_0 = I_{p+q}.$$

In order to compute the radial component we use a decomposition of S which is slightly different from the factorization $S = NA$ coming from the Iwasawa factorization. We write, using notation (8)

$$S_t = S(l_t, b_t, c_t),$$

and

$$S_t = M_t L_t, \tag{10}$$

where

$$M_t = \begin{pmatrix} I_p & b_t & c_t \\ 0 & I_{q-p} & b_t^* \\ 0 & 0 & I_p \end{pmatrix}, \quad L_t = \begin{pmatrix} l_t & 0 & 0 \\ 0 & I_{q-p} & 0 \\ 0 & 0 & l_t^{*-1} \end{pmatrix}.$$

Recall that the matrix l_t is lower triangular. Its diagonal is the A component of S_t but it also has a part of the N component. We solve the equation satisfied by (S_t). By Stratonovich calculus,

$$\delta S_t = M_t\, \delta L_t + (\delta M_t) L_t,$$

therefore,

$$M_tL_t\,\delta V_t = M_t\,\delta L_t + (\delta M_t)L_t,$$

which implies that

$$\delta V_t = L_t^{-1}\,\delta L_t + L_t^{-1}M_t^{-1}(\delta M_t)L_t,$$

hence

$$L_t^{-1}\,\delta L_t = \begin{pmatrix} \delta\lambda_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta\lambda_t^* \end{pmatrix},$$

and

$$L_t^{-1}M_t^{-1}(\delta M_t)L_t = \begin{pmatrix} 0 & \delta\beta_t & \delta\kappa_t \\ 0 & 0 & \delta\beta_t^* \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain that $\delta l_t = l_t\,\delta\lambda_t$ and

$$\delta M_t = M_tL_t \begin{pmatrix} 0 & \delta\beta_t & \delta\kappa_t \\ 0 & 0 & \delta\beta_t^* \\ 0 & 0 & 0 \end{pmatrix} L_t^{-1},$$

which gives

$$\begin{pmatrix} 0 & \delta b_t & \delta c_t \\ 0 & 0 & \delta b_t^* \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_p & b_t & c_t \\ 0 & I_{q-p} & b_t^* \\ 0 & 0 & I_p \end{pmatrix} \begin{pmatrix} 0 & l_t\delta\beta_t & l_t\delta\kappa_t l_t^* \\ 0 & 0 & \delta\beta_t^* l_t^* \\ 0 & 0 & 0 \end{pmatrix}.$$

This show that:

Proposition 8

$$\delta l_t = l_t\,\delta\lambda_t, b_t = \int_0^t l_s\,\delta\beta_s, c_t = \int_0^t l_s(\delta\kappa_s)l_s^* + \int_0^t b_s(\delta\beta_s^*)l_s^*.$$

In particular,

Corollary 1 *The process (l_t) is a Brownian motion on the subgroup of $Gl(p,\mathbb{C})$ consisting of lower triangular matrices with positive diagonal.*

The linear equation $\delta l_t = l_t\,\delta\lambda_t$ is therefore easy to solve explicitly by induction on p. For instance, when $p = 2$, if

$$\lambda_t = \begin{pmatrix} \lambda_t^{(1)} & 0 \\ \lambda_t^{(3)} & \lambda_t^{(2)} \end{pmatrix},$$

we obtain that

$$l_t = \begin{pmatrix} e^{\lambda_t^{(1)}} & 0 \\ e^{\lambda_t^{(2)}} \int_0^t e^{\lambda_s^{(1)}-\lambda_s^{(2)}}\,\delta\lambda_s^{(3)} & e^{\lambda_t^{(2)}} \end{pmatrix}.$$

4.6 Limit as $q \to +\infty$

We study the asymptotic behaviour of the radial part $\mathrm{Rad}(S_t)$ of the distinguished Brownian motion on S, using the decomposition (10). We first consider Ito's integral.

Lemma 5 *Almost surely,*

$$\lim_{q\to+\infty} \frac{1}{q} \int_0^t [\int_0^s l_u d\beta_u](d\beta_s^*)l_s^* = 0.$$

Proof For $1 \le i,j \le p$,

$$[\int_0^t (\int_0^s l_u d\beta_u)(d\beta_s^*)l_s^*]^{i,j} = \sum_{r=1}^{q-p}\sum_{n=1}^{p}\sum_{m=1}^{p} \int_0^t (\int_0^s l_u^{i,n} d\beta_u^{n,r})\bar{l}_s^{j,m} d\bar{\beta}_s^{m,r}.$$

We fix i,j,n,m. Conditionally on the sigma-algebra $\sigma(l_s, s \ge 0)$, the random variables

$$\int_0^t (\int_0^s l_u^{i,n} d\beta_u^{n,r})\bar{l}_s^{j,m} d\bar{\beta}_s^{m,r}$$

for $r = 1, 2, \cdots$, are independent with the same law, and with expectation equal to 0 since they are martingales. Therefore, by the law of large numbers,

$$\frac{1}{q-p} \sum_{r=1}^{q-p} \int_0^t (\int_0^s l_u^{i,n} d\beta_u^{n,r})\bar{l}_s^{j,m} d\bar{\beta}_s^{m,r}$$

converges a.s. to 0 when $q \to +\infty$, which proves the lemma.

Proposition 9 *Let $S_t, t \geq 0$, be the distinguished Brownian motion. Then, a.s.,*

$$\lim_{q\to+\infty} \frac{1}{q} \cosh \mathrm{Rad}(S_t) = \mathrm{SingVal}(l_t^{-1} \int_0^t l_s l_s^* \, ds).$$

Proof By Proposition 8,

$$c_t = \int_0^t l_s(\delta\kappa_s) l_s^* + \int_0^t (\int_0^s l_u \, \delta\beta_u)(\delta\beta_s^*) l_s^*.$$

Since the processes (l_t) and (κ_t) do not depend on q,

$$\lim_{q\to+\infty} \frac{1}{q} c_t = \lim_{q\to+\infty} \frac{1}{q} \int_0^t (\int_0^s l_u \, \delta\beta_u)(\delta\beta_s^*) l_s^*.$$

Now, recall the link between Stratonovich and Ito integral: if X and Y are continuous semimartingales,

$$\int_0^t Y\delta X = \int_0^t Y dX + \frac{1}{2}\langle X, Y\rangle_t,$$

where, if X and Y are matrices, $\langle X, Y\rangle_t$ is the matrix with (i, j) entries

$$\langle X, Y\rangle_t^{i,j} = \sum_k \langle X^{i,k}, Y^{k,j}\rangle_t.$$

Since (β_t) is independent of (l_t), $\int_0^s l_u \delta\beta_u = \int_0^s l_u \, d\beta_u$, and

$$\int_0^t [\int_0^s l_u \delta\beta_u](\delta\beta_s^*) l_s^* = \int_0^t [\int_0^s l_u d\beta_u](d\beta_s^*) l_s^* + \frac{1}{2}\langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \rangle_t,$$

so it follows from the preceding lemma that

$$\lim_{q\to+\infty} \frac{1}{q} c_t = \frac{1}{2} \lim_{q\to+\infty} \langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \rangle_t.$$

On the other hand, for $1 \leq i, j \leq p$,

$$\begin{aligned} &\langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \rangle_t^{i,j} \\ &= \sum_{r=1}^{q-p} \langle (\int_0^t l_s d\beta_s)^{i,r}, (\int_0^t d\beta_s^* l_s^*)^{r,j} \rangle_t \end{aligned}$$

$$= \sum_{r=1}^{q-p} \sum_{k=1}^{p} \sum_{l=1}^{p} \langle \int_0^t l_s^{i,k} d\beta_s^{k,r}, \int_0^t \bar{l}_s^{j,l} d\bar{\beta}_s^{l,r} \rangle_t$$

$$= \sum_{r=1}^{q-p} \sum_{k=1}^{p} \langle \int_0^t l_s^{i,k} d\beta_s^{k,r}, \int_0^t \bar{l}_s^{j,k} d\bar{\beta}_s^{k,r} \rangle_t$$

$$= \sum_{r=1}^{q-p} \sum_{k=1}^{p} \int_0^t l_s^{i,k} \bar{l}_s^{j,k} d\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s.$$

Recall that $\beta_t^{k,r}/\sqrt{2}$ is a complex Brownian motions, hence

$$\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s = 4s,$$

therefore

$$\sum_{r=1}^{q-p} \sum_{k=1}^{p} \int_0^t l_s^{i,k} \bar{l}_s^{j,k} d\langle \beta^{k,r}, \bar{beta}^{k,r} \rangle_s = 4(q-p) \sum_{k=1}^{p} \int_0^t l_s^{i,k} \bar{l}_s^{j,k} \, ds,$$

and

$$\langle \int_0^t l_s \, d\beta_s, \int_0^t d\beta_s^* l_s^* \rangle_t = 4(q-p) \int_0^t l_s l_s^* \, ds.$$

Consequently, a.s.,

$$\lim_{q \to +\infty} \frac{1}{q} c_t = \lim_{q \to +\infty} \frac{2(q-p)}{q} \int_0^t l_s l_s^* \, ds = 2 \int_0^t l_s l_s^* \, ds,$$

and the proposition follows from Lemma 3 and from the equality $\mathrm{SingVal}(\int_0^t l_s l_s^* l_t^{*-1} \, ds) = \mathrm{SingVal}(l_t^{-1} \int_0^t l_s l_s^* \, ds)$.

The following is a direct consequence of the previous proposition and Lemma 4. Let $\mathbf{1} \in \mathbb{R}^p$ denote the vector $\mathbf{1} = (1, 1, \cdots, 1)$.

Theorem 4 *For $SU(p,q)$, the process $\eta_t, t \geq 0$, with values in $\{(r_1, \cdots, r_p) \in \mathbb{R}^p, r_1 \geq r_2 \geq \cdots \geq r_p\}$ given by*

$$\eta_t = \mathrm{SingVal}(l_t^{-1} \int_0^t l_s l_s^* \, ds)$$

is a Markov process. The process

$$\mathrm{Rad}(\xi_t^0) - \log(2q)\mathbf{1}, t \geq 0,$$

converges in distribution to $\log \eta_t, t \geq 0$.

We will compute the generator of (η_t) in Theorem 6.

4.7 $SO(p,q)$ and $Sp(p,q)$

For $SO(p,q)$ the only difference with $SU(p,q)$ is that all the entries are real. Hence, in the preceding computation,

$$\langle \beta^{k,r}, \bar{\beta}^{k,r}\rangle_s = \langle \beta^{k,r}, \beta^{k,r}\rangle_s = 2s.$$

Similarly, for $Sp(p,q)$ the entries are quaternionic, therefore, in that case,

$$\langle \beta^{k,r}, \bar{\beta}^{k,r}\rangle_s = 8s.$$

So we obtain,

Theorem 5 *For $SO(p,q)$, resp. $Sp(p,q)$, in distribution,*

$$\lim_{q\to+\infty} \mathrm{Rad}(\xi_t^0) - \log(\theta q)\mathbf{1} = \log \mathrm{SingVal}(l_t^{-1}\int_0^t l_s l_s^*\, ds),$$

with $\theta = 1$ for $SO(p,q)$, resp. $\theta = 4$ for $Sp(p,q)$, and where l is the Brownian motion on the group of $p\times p$ lower triangular matrices with real, resp. quaternionic, entries and with positive diagonal, solution of $\delta l_t = l_t\delta\lambda_t$.

Corollary 2 *In rank one, i.e. $p = 1$,*

$$\lim_{q\to+\infty} \mathrm{Rad}(\xi_t^0) - \log(\theta q)\mathbf{1} = \log \int_0^t e^{2B_s - B_t}\, ds,$$

where B is a standard Brownian motion.

5 Generator of (η_t)

5.1 Inozemtsev Limit to Quantum Toda Hamiltonian

In order to compute the generator L of the process (η_t) process in Theorem 4 it is convenient to relate $\mathrm{Rad}(\xi_t^0)$ to a Calogero Moser Sutherland model.

We consider the general case of a symmetric space G/K associated with one of the groups $G = SO(p,q), SU(p,q)$ and $Sp(p,q)$, where K is a maximal compact subgroup. The root system is of type BC_p. Recall that, if $\{e_1,\dots,e_p\}$ is an orthonormal basis of a real Euclidean space $\mathfrak{a}$ of dimension p, with dual basis (e_i^*) then the positive roots Σ^+ of the root system BC_p are given by

$$\Sigma^+ = \{e_k^*, 2e_k^*, (1\le k\le p), e_i^* + e_j^*, e_i^* - e_j^*, (1\le i<j\le p)\}.$$

We associate to each considered group a triplet $m = (m_1, m_2, m_3)$ given by, when $p \geq 2$, for $SO(p,q), m = (q-p, 0, 1)$, for $SU(p,q), m = (2(q-p), 1, 2)$ and for $Sp(p,q), m = (4(q-p), 3, 4)$. When $p = 1$, the only difference is that $m_3 = 0$. The quantum Calogero-Moser-Sutherland trigonometric Hamiltonian of type BC_p is (see [31, 32])

$$H_{CMS} = H_{CMS}(p,q;r) = \sum_{k=1}^{p} (\frac{\partial^2}{\partial r_k^2} - \frac{m_1(m_1 + 2m_2 - 2)}{4\sinh^2 r_k} - \frac{m_2(m_2-2)}{\sinh^2 2r_k})$$
$$- \sum_{1 \leq i < j \leq p} m_3(m_3 - 2)(\frac{1}{2\sinh^2(r_i - r_j)} + \frac{1}{2\sinh^2(r_i + r_j)}).$$

For $\alpha \in \Sigma^+$ we let $m_\alpha = m_1$ if $\alpha = e_k^*$, $m_\alpha = m_2$ if $\alpha = 2e_k^*$ and $m_\alpha = m_3$ if $\alpha = e_i^* + e_j^*$ or $\alpha = e_i^* - e_j^*$. Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \, \alpha, \;\; \delta = \prod_{\alpha \in \Sigma_+} (e^\alpha - e^{-\alpha})^{m_\alpha}. \tag{11}$$

The radial part Δ_R of the Laplace Beltrami operator on G/K is equal (see [18, p. 268]) to

$$\Delta_R = \sum_{k=1}^{p} \frac{\partial^2}{\partial r_k^2} + \sum_{k=1}^{p} \frac{\partial}{\partial r_k} (\log \delta) \frac{\partial}{\partial r_k},$$

which can be written as

$$\Delta_R = \delta^{-1/2} \circ \{H_{CMS} - (\rho, \rho)\} \circ \delta^{1/2}$$

(see [30–32]). On the other hand, we define $\tilde{\varphi}_0 : \mathbb{R}^p \to \mathbb{R}$ by $\tilde{\varphi}_0(r) = \varphi_0(D_p(r))$ where φ_0 is the basic Harish Chandra function. Then $\tilde{\varphi}_0(0) = 1$ and

$$\Delta_R \tilde{\varphi}_0 = -(\rho, \rho) \tilde{\varphi}_0.$$

Therefore

$$H_{CMS}(\delta^{1/2} \tilde{\varphi}_0) = 0.$$

The generator of the radial part $\mathrm{Rad}(\xi_t^0)$ of the ground state process (ξ_t^0) associated with φ_0 is $\Delta_R^{\tilde{\varphi}_0}/2$ where

$$\Delta_R^{\tilde{\varphi}_0} = \sum_{k=1}^{p} \frac{\partial^2}{\partial r_k^2} + 2 \sum_{k=1}^{p} \frac{\partial}{\partial r_k} (\log \delta^{1/2} \tilde{\varphi}_0) \frac{\partial}{\partial r_k}$$
$$= (\delta^{1/2} \tilde{\varphi}_0)^{-1} \circ H_{CMS} \circ (\delta^{1/2} \tilde{\varphi}_0).$$

For $SU(p,q)$, since $m_3 = 2$, we have the following, called the Inozemtsev limit [20],

$$\lim_{q\to+\infty} H_{CMS}(p,q;r+\log(2q)\mathbf{1}) = \sum_{k=1}^{p} \frac{\partial^2}{\partial r_k^2} - e^{-2r_k}.$$

Similarly, for $SO(p,q)$,

$$\lim_{q\to+\infty} H_{CMS}(p,q;r+\log q\,\mathbf{1}) = \sum_{k=1}^{p}(\frac{\partial^2}{\partial r_k^2} - e^{-2r_k}) + \sum_{1\le i<j\le p} \frac{1}{2\sinh^2(r_i-r_j)},$$

and for $Sp(p,q)$,

$$\lim_{q\to+\infty} H_{CMS}(p,q;r+\log(4q)\mathbf{1}) = \sum_{k=1}^{p}(\frac{\partial^2}{\partial r_k^2} - e^{-2r_k}) - \sum_{1\le i<j\le p} \frac{4}{\sinh^2(r_i-r_j)}.$$

We denote by H_T the respective limit. Since $\delta^{1/2}\tilde{\varphi}_0$ is a ground state of H_{CMS}, it is reasonable to infer that there is a ground state Ψ of H_T such that the generator L of (η_t) in Theorem 4 is given by

$$L = \frac{1}{2}\Psi^{-1} \circ H_T \circ \Psi.$$

Oshima and Shimeno [32] goes into that direction but is not precise enough to obtain this conclusion. We will see that this hold true for $SU(p,q)$, but that the result is quite subtle. The reason is that on the one hand, there are many ground states for H_T and on the other hand that L is the generator of a process with values in a proper subcone of $\mathbb{R}^p$ when $p>1$. We don't know if this is true for $S0(p,q)$ or $Sp(p,q)$.

5.2 Asymptotics for $SU(p,q)$

We compute the generator of the process η_t for $SU(p,q)$, using that in this case the spherical functions are explicitly known for all (p,q). In $SO(p,q)$ and $Sp(p,q)$, such an expression is not known up to now.

For $\lambda \in \mathbb{R}^p$, the Harish Chandra spherical function $\varphi_\lambda^{(p,q)}$ of $SU(p,q)$, is defined by (see [18, Theorem IV.4.3]), for $g \in SU(p,q)$,

$$\varphi_\lambda^{(p,q)}(g) = \int_K e^{(\lambda+\varrho)(H(kg))}\,dk.$$

Let

$$A(p,q) = (-1)^{\frac{1}{2}(p(p-1)}2^{2p(p-1)}\prod_{j=1}^{p-1}\{(q-p+j)^{p-j}j!\},$$

$$c(p,q) = A(p,q)(-1)^{\frac{1}{2}p(p-1)}(2!4!\cdots(2(p-1))!)^{-1}.$$

By Hoogenboom [19, Theorem 3], if for all i, $\lambda_i \notin \mathbb{Z}$, then when $r \in \mathbb{R}^p$,

$$\varphi_\lambda^{(p,q)}(D_p(r)) = \frac{A(p,q)\det(\varphi_{\lambda_i}^{(1,q-p+1)}(D_1(r_j))}{\prod_{1\le i<j\le p}(\cosh 2r_i - \cosh 2r_j)(\lambda_i^2-\lambda_j^2)}. \tag{12}$$

Lemma 6 *For $r \in \mathbb{R}^p$,*

$$\varphi_0^{(p,q)}(D_p(r)) = \frac{c(p,q)}{\prod_{1\le i<j\le p}(\cosh 2r_i - \cosh 2r_j)}\det(M_{(p,q)}(r)),$$

where $M_{(p,q)}(r)$ is the $p\times p$ matrix with (i,j) coefficient given by

$$M_{(p,q)}(r)^{i,j} = \frac{d^{2(j-1)}}{d\lambda^{2(j-1)}}\varphi_\lambda^{(1,q-p+1)}(D_1(r_i))_{\{\lambda=0\}}.$$

Proof Since; for $\lambda \in \mathbb{R}$, $\varphi_\lambda^{(1,q-p+1)} = \varphi_{-\lambda}^{(1,q-p+1)}$, the lemma follows from Hua's lemma [19, Lemma 4.1].

For $SU(p,q)$, $m_\alpha = 2(q-p)$ when $\alpha = e_k^*$, $m_\alpha = 1$ when $\alpha = 2e_k^*$ and $m_\alpha = 2$ when $\alpha = e_i^* + e_j^*$ or $\alpha = e_i^* - e_j^*$. Thus, by (11),

$$\delta(D_p(r)) = \prod_k (2\sinh r_k)^{2(q-p)}(2\sinh 2r_k)\prod_{i<j}4(\cosh 2r_i - \cosh 2r_j)^2.$$

On the other hand, for $r \in \mathbb{R}$, let

$$\delta_{q-p+1}(r) = 2^{2(q-p)+1}\sinh^{2(q-p)} r \sinh 2r.$$

(this is (14) adapted to the complex case). Then there is $C(p,q) > 0$ such that, for $r \in \mathbb{R}^p$,

$$(\delta^{1/2}\varphi_0^{(p,q)})(D_p(r)) = C(p,q)\det M_{(p,q)}(r)\prod_{k=1}^{p}\delta_{q-p+1}(r_k) = C(p,q)\det N_{(p,q)}(r),$$

where

$$(N_{(p,q)}(r))^{i,j} = \frac{d^{2(j-1)}}{d\lambda^{2(j-1)}}(\delta^{1/2}_{q-p+1}\tilde{\varphi}^{q-p+1}_{\lambda})(r_i)_{\{\lambda=0\}},$$

and where $\tilde{\varphi}^{q-p+1}_{\lambda}(s) = \varphi^{(1,q-p+1)}_{\lambda}(D_1(s))$ when $s \in \mathbb{R}$. Let (see (18) and (19))

$$a(q-p+1) = \frac{\Gamma(q-p)}{\Gamma(2(q-p))2^{5/2}},$$

and

$$g_{q-p+1}(\lambda, r) = a(q-p+1)(\delta^{1/2}_{q-p+1}\tilde{\varphi}^{q-p+1}_{\lambda})(r + \log 2(q-p)) - K_{\lambda}(e^{-r}).$$

By Corollary 4, $g_{q-p+1}(\lambda, r)$ and all its derivatives at $\lambda = 0$ converge to 0 as $q \to +\infty$. The generator of $\mathrm{Rad}(\xi^0_t)$ is

$$\frac{1}{2}\sum_{i=1}^{p}\frac{\partial^2}{\partial r_i^2} + \sum_{i=1}^{p}\frac{\partial}{\partial r_i}\log(\delta^{1/2}\varphi_0^{(p,q)})(D_p(r))\frac{\partial}{\partial r_i}.$$

Recall that

$$\eta_t = \mathrm{SingVal}(l_t^{-1}\int_0^t l_s l_s^* \, ds).$$

We deduce from Corollary 4 that:

Theorem 6 *The generator of the process* $\log \eta_t, t \geq 0$, *in Theorem 4 is*

$$\frac{1}{2}\sum_{i=1}^{p}\frac{\partial^2}{\partial r_i^2} + \sum_{i=1}^{p}\frac{\partial}{\partial r_i}\log \tilde{K}^{(p)}(e^{-r})\frac{\partial}{\partial r_i},$$

where $\tilde{K}^{(p)}(e^{-r})$ *is the determinant of the* $p \times p$ *matrix with* (i, j)*-coefficient*

$$\frac{d^{2(j-1)}}{d\lambda^{2(j-1)}}K_{\lambda}(e^{-r_i})_{\{\lambda=0\}}.$$

Remark 4 The function $r \mapsto \tilde{K}^{(p)}(e^{-r})$ is a ground state of the Toda Hamiltonian

$$\sum_{k=1}^{p}\frac{\partial^2}{\partial r_k^2} - e^{-2r_k}$$

equal to 0 on the walls $\{r_i = r_j\}$. Notice that $\prod_{k=1}^{p} K_0(e^{-r_k})$ is another ground state.

Remark 5 Exactly as in Proposition 6, but using (12), one has that, for $r = \log \eta_t$,

$$\mathbb{E}(e^{\sum_{k=1}^{p} \lambda_k l_t^{(k,k)}} | \sigma(\eta_s, 0 \le s \le t)) = \frac{c_p \tilde{K}_\lambda(e^{-r})}{\tilde{K}^{(p)}(e^{-r}) \prod_{i<j}(\lambda_i^2 - \lambda_j^2)},$$

where $c_p = (-1)^{\frac{1}{2}p(p-1)}(2!4!\cdots(2(p-1))!$ and $\tilde{K}_\lambda(e^{-r})$ is the determinant of the matrix $(K_{\lambda_i}(e^{-r_j}))$. As in Remark 3, this implies that when the diagonal part of (l_t) has a drift λ, then $\log \eta_t$ is a Markov process with generator given by

$$\frac{1}{2}\sum_{i=1}^{p} \frac{\partial^2}{\partial r_i^2} + \sum_{i=1}^{p} \frac{\partial}{\partial r_i} \log \tilde{K}_\lambda(e^{-r}) \frac{\partial}{\partial r_i}.$$

Remark 6 After the submission of this paper, Rider and Valko [36] have posted a paper where they consider a Brownian motion M_t on $Gl(p, \mathbb{R})$, solution of the stochastic differential equation $dM_t = M_t dB_t + (\frac{1}{2} + \mu) M_t\, dt$ where B_t is the $p \times p$ matrix made of p^2 independent standard real Brownian motions. They show in particular, using a similar approach as Matsumoto and Yor, that $Z_t = M_t^{-1} \int_0^t M_s M_s^*\, ds, t \ge 0$, is a Markov process if $|\mu| > (p-1)/2$ and describe its generator. For $Gl(p, \mathbb{C})$, when $\mu = 0$, $\mathrm{SingVal}(Z_t) = \mathrm{SingVal}(\eta_t)$, where η_t is given in the preceding remark with $\lambda = \varrho$.

6 Series of Homogeneous Trees

We now consider the same question for q-adic symmetric spaces of rank one. The most important series is given by the symmetric spaces associated with $Gl(2, \mathbb{Q}_q)$ where q is the sequence of prime numbers which are the homogeneous trees $\mathbb{T}_q$. The analogue of the Brownian motion is the simple random walk. Let us consider more generally the tree $\mathbb{T}_q$, where q is an arbitrary integer. We will deal to this case by an elementary treatment. By definition, $\mathbb{T}_q$ is the connected graph without cycle whose vertices have exactly $q+1$ edges. We choose an origin o in this tree. The simple random walk $W_n, n \ge 0$, starts at o and at each step goes to one of its $q+1$ neighbours with uniform probability.

Let us recall some (well known) elementary facts about (W_n) [14, 39]. For the convenience of the reader we give the simple proofs. Let us write $x \sim y$ when x and y are neighbours. The probability transition of W_n is

$$P(x, y) = \frac{1}{q+1}, \text{iff } x \sim y.$$

Let d be the standard distance on the tree. The radial part of W_n is the process $X_n = d(o, W_n)$, it is a Markov chain on $\mathbb{N}$ with transition probability R given by $R(0,1) = 1$, and if $n > 0$,

$$R(n, n-1) = \frac{1}{q+1}, R(n, n+1) = \frac{q}{q+1}.$$

A function $f : \mathbb{T}_q \to \mathbb{R}$ is called radial when $f(x)$ depends only on $d(o,x)$. In this case one defines $\tilde{f} : \mathbb{N} \to \mathbb{R}$ by

$$\tilde{f}(n) = f(x), \text{ when } d(o,x) = n.$$

Let us introduce the average operator A which associates to a function $f : \mathbb{T}_q \to \mathbb{R}$ the radial function $Af : \mathbb{T}_q \to \mathbb{R}$ defined by

$$Af(x) = \frac{1}{S(o,x)} \sum_{y \in S(o,x)} f(y),$$

where $S(o,x)$ is the sphere $S(o,x) = \{y \in \mathbb{T}_q; d(o,y) = d(o,x)\}$. It is easy to see that $PA = AP$, which implies that,

Lemma 7 *When a function $f : \mathbb{T}_q \to \mathbb{R}$ is a λ-eigenfunction of P (i.e. $Pf = \lambda f$), then Af is a radial λ-eigenfunction of P and $\widetilde{Af}$ is a λ-eigenfunction of R. Conversely, if $\tilde{g} : \mathbb{N} \to \mathbb{R}$ is a λ-eigenfunction of R, then $g(x) = \tilde{g}(d(o,x))$ is a radial λ-eigenfunction of P.*

Let $\lambda > 0$, $\tilde{g} : \mathbb{N} \to \mathbb{R}^+$ is a λ-eigenfunction of R when $\tilde{g}(1) = \lambda \tilde{g}(0)$ and

$$\tilde{g}(n-1) + q\tilde{g}(n+1) = \lambda(q+1)\tilde{g}(n), n \geq 1.$$

One sees easily that there exists a positive solution of this equation if and only if $\lambda \geq \frac{2\sqrt{q}}{q+1}$. Hence, the principal generalized eigenvalue ϱ of R is

$$\varrho = \frac{2\sqrt{q}}{q+1},$$

the associated eigenfunction $\tilde{\varphi}_0$ is

$$\tilde{\varphi}_0(n) = (1 + n\frac{q-1}{q+1})\frac{1}{q^{n/2}}.$$

It is the only one if we suppose that $\tilde{\varphi}_0(0) = 1$. It follows from Lemma 7 that ϱ is also the principal generalized eigenvalue of P. The function $\varphi_0 : \mathbb{T}_q \to \mathbb{R}$ given by $\varphi_0(x) = \tilde{\varphi}_0(d(o,x))$ for $x \in \mathbb{T}_q$, is a radial ϱ-eigenfunction of P. By uniqueness of $\tilde{\varphi}_0$, φ_0 is the unique radial ground state of the random walk $W_n, n \geq 0$, equal to 1 at o. We consider the φ_0-ground state process $W_n^{(0)}, n \geq 0$, on $\mathbb{T}_q$, defined as the Markov chain with transition probability $P^{(0)}$ given by, for $x, y \in \mathbb{T}_q$,

$$P^{(0)}(x, y) = \frac{1}{\varrho\phi_0(x)} P(x, y)\phi_0(y).$$

Its radial part is a Markov chain on $\mathbb{N}$ with transition probability, for $m, n \in \mathbb{N}$,

$$R^{(0)}(n, m) = \frac{1}{\varrho\tilde{\varphi}_0(n)} R(n, m)\tilde{\varphi}_0(m).$$

It is clear that

$$\lim_{q\to+\infty} R^{(0)}(m, n) = B(m, n),$$

where B is transition probability of the so-called discrete Bessel(3) Markov chain on $\mathbb{N}$, namely,

$$B(0, 1) = 1, B(n, n+1) = \frac{n+2}{2(n+1)}, B(n, n-1) = \frac{n}{2(n+1)}.$$

Therefore,

Proposition 10 *When $q \to +\infty$, the radial part of the φ_0-ground state process $W_n^{(0)}$ of the simple random walk on the tree $\mathbb{T}_q$ converges in distribution to the discrete Bessel(3) chain.*

Our aim is to obtain a path description of this limit process in terms of the simple random walk on $\mathbb{Z}$. We use the analogue of the distinguished Brownian motion. A geodesic ray ω is an infinite sequence $x_n, n \geq 0$, of distinct vertices in $\mathbb{T}_q$ such that $d(x_n, x_{n+1}) = 1$ for any $n \geq 0$. We fix such a geodesic ray $\omega = \{x_n, n \geq 0\}$ starting from $x_0 = o$. For any x in $\mathbb{T}_q$ let $\pi_\omega(x)$ be the projection of x on ω, defined by $d(x, \pi_\omega(x)) = \min\{d(x, y), y \in \omega\}$ and $\pi_\omega(x) \in \omega$. The height $h(x)$ of $x \in \mathbb{T}_q$ with respect to ω is

$$h(x) = d(x, \pi_\omega(x)) - d(o, \pi_\omega(x)),$$

(h is a Busemann function). For $n \in \mathbb{Z}$ the horocycle H_n is

$$H_n = \{x \in \mathbb{T}_q, h(x) = n\}.$$

As described by Cartier [8] (see also [9]), $\mathbb{T}_q$ can be viewed as an infinite genealogical tree with ω as the unique "mythical ancestor" (Fig. 1). Then H_n is the n-th generation. Any vertex (individual) x in $\mathbb{T}_q$ has q neighbours of height $h(x) + 1$ (his children) and one neighbour of height $h(x) - 1$ (his parent). This implies that

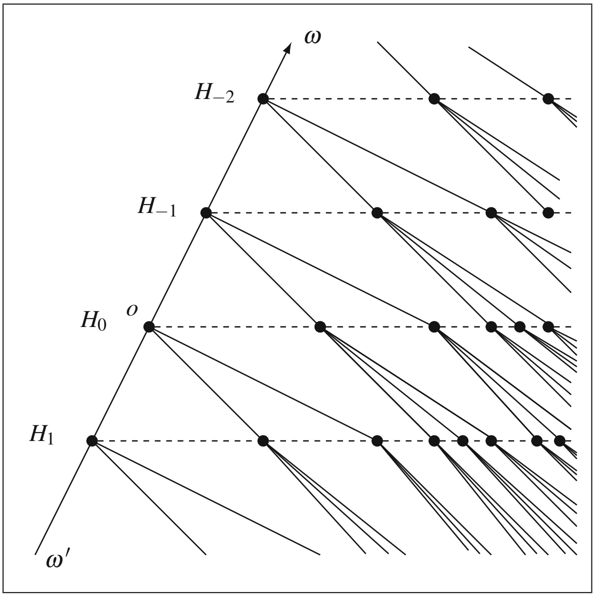

Fig. 1 Example $\mathbb{T}_3$

if W_n is the simple random walk on $\mathbb{T}_q$, then $h(W_n)$ is the random walk on $\mathbb{Z}$ with transition probability H given by, for any $n \in \mathbb{Z}$,

$$H(n, n-1) = \frac{1}{q+1}, H(n, n+1) = \frac{q}{q+1}.$$

A positive eigenfunction f of this kernel is a solution of

$$f(n-1) + qf(n+1) = \lambda(q+1)f(n), n \in \mathbb{Z}, f(0) = 1,$$

and one sees easily that it exists if and only if

$$\lambda \geq \frac{2\sqrt{q}}{q+1} = \varrho.$$

As a result ϱ is also the principal generalized eigenvalue (or spectral radius) of H. Moreover the associated eigenfunction is $f(n) = q^{-n/2}, n \in \mathbb{N}$. This implies that the function

$$\varphi_\omega(x) = q^{-h(x)/2},\ x \in \mathbb{T}_q,$$

is, like φ_0, a ground state of the Markov chain W_n. By Lemma 7, $A\varphi_\omega$ is a radial ground state of P. By uniqueness, this implies that

$$\varphi_0 = A\varphi_\omega. \tag{13}$$

Definition 6 Let $S_n, n \geq 0$, be the φ_ω-ground state process associated to (W_n) on $\mathbb{T}_q$ starting from o. We denote by Q_ω its probability transition.

We have,

$$Q_\omega(x,y) = P(x,y)\frac{\varphi_\omega(y)}{\varrho\varphi_\omega(x)} = P(x,y)\frac{q^{-h(y)/2}}{\varrho q^{-h(x)/2}} = P(x,y)\frac{(q+1)q^{h(x)-h(y)/2}}{2\sqrt{q}}.$$

Therefore, when $x \sim y$, since $P(x,y) = 1/(q+1)$,

$$Q_\omega(x,y) = \begin{cases} 1/2q, \text{ when } h(y) = h(x)+1, \\ 1/2, \text{ when } h(y) = h(x)-1. \end{cases}$$

As in Proposition 4,

Lemma 8 *The process* $d(o, S_n), n \geq 0$, *has the same law as* $d(o, W_n^{(0)})$, $n \geq 0$.

Proof Let $N \in \mathbb{N}$ and $F : \mathbb{Z}^{N+1} \to \mathbb{R}^+$. For any isometry k of the tree $\mathbb{T}_q$ which fixes o, $W_n, n \geq 0$, has the same law as $k.W_n, n \geq 0$. Hence, using also (13),

$$\begin{aligned}
\mathbb{E}(F(d(o,S_n), n \leq N)) &= \varrho^{-N}\mathbb{E}(F(d(o,W_n), n \leq N)\varphi_\omega(W_N)) \\
&= \varrho^{-N}\mathbb{E}(F(d(o,k.W_n), n \leq N)\varphi_\omega(k.W_N)) \\
&= \varrho^{-N}\mathbb{E}(F(d(o,W_n), n \leq N)A\varphi_\omega(W_N)) \\
&= \varrho^{-N}\mathbb{E}(F(d(o,W_n), n \leq N)\varphi_0(W_N)) \\
&= \mathbb{E}(F(d(o,W_n^{(0)}), n \leq N)).
\end{aligned}$$

We choose in the tree $\mathbb{T}_q$ another geodesic ray $\omega' = \{x_{-n}, n \geq 0\}$ such that $\omega \cap \omega' = \{o\}$. Then $\omega\omega' = \{x_n, n \in \mathbb{Z}\}$ is a two-sided geodesic. We consider the following (unoriented) graph G embedded in $\mathbb{Z}^2$ (see Fig. 2): the edges of G are the points with coordinates

$$(k+2n, k), k \in \mathbb{Z}, n \in \mathbb{N},$$

and the vertices are the first diagonal segments joining (k,k) and $(k+1,k+1)$ and the segments joining $(k+2n,k)$ and $(k+2n+1,k-1)$ for $n \geq 0$. It is obtained

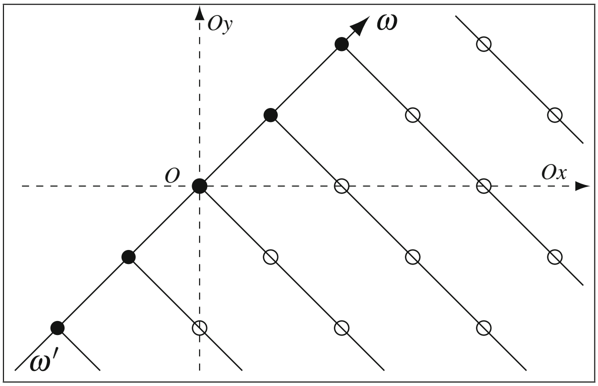

Fig. 2 The graph G

from the tree by gathering siblings (with one exception on each vertex of $\omega\omega'$). Let $\Psi : \mathbb{T}_q \to G$ defined as follows: for $x \in \mathbb{T}_q$,

$$\Psi(x) = (-h(x) + 2d(x, \omega'\omega), -h(x)).$$

Then $\Psi(S_n), n \geq 0$, is the nearest neighbour Markov chain on the graph G with probability transition P_G given by

$$P_G((k,k),(k-1,k-1)) = 1/2q,$$
$$P_G((k,k),(k+1,k+1)) = 1/2,$$
$$P_G((k,k),(k+1,k-1)) = (q-1)/2q,$$

and, for $n > 0$ and $\varepsilon = 1$ or -1,

$$P_G((k+2n,k),(k+2n-\varepsilon,k+\varepsilon)) = 1/2.$$

When $q \to +\infty$, the Markov chain $\Psi(S_n)$ converges to the Markov chain $\tau_n, n \in \mathbb{N}$, on the graph G with transition probability Q given by

$$Q((k,k),(k-1,k-1)) = 0,$$
$$Q((k,k),(k+1,k+1)) = 1/2,$$
$$Q((k,k),(k+1,k-1)) = 1/2,$$

and for $n > 0$ and $\varepsilon = 1$ or -1,

$$Q((k+2n,k),(k+2n-\varepsilon,k+\varepsilon)) = 1/2.$$

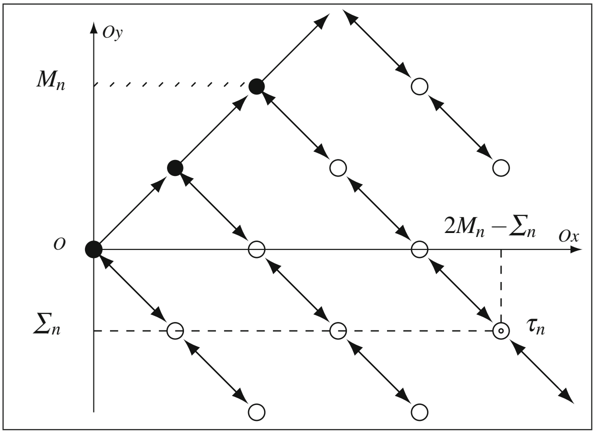

Fig. 3 Pitman's walk on the graph $\tilde{G}$

When the chain τ_n starts from $(0, 0)$ it cannot reach $(-1, -1)$ therefore it lives on the subgraph $\tilde{G}$ of G described in Fig. 3. It is the one which appears in the discrete time Pitman theorem (see Fig. 8 in Biane [4]). It has the following simple description in terms of the simple symmetric random walk $\Sigma_n, n \geq 0$, on $\mathbb{Z}$. Let $\Sigma_n = \varepsilon_1 + \cdots + \varepsilon_n$ where the (ε_n) are i.i.d. random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$ and

$$M_n = \max\{\Sigma_k, k \leq n\}.$$

The Markov chain τ_n on $\tilde{G}$ starting from $(0, 0)$ is, in xOy coordinates,

$$\tau_n = (2M_n - \Sigma_n, \Sigma_n),$$

(see Fig. 3). The radial component of S_n is $d(o, S_n) = \delta((0, 0), \Psi(S_n))$ for the graph distance δ on G. Since $\delta((0, 0), (a, b)) = a$, we obtain using Lemma 8 that,

Theorem 7 *The limit as $q \to +\infty$ of the processes $d(o, S_n), n \geq 0$, and $d(o, W_n^{(0)}), n \geq 0$, have the same law as*

$$2 \max_{0 \leq k \leq n} \Sigma_k - \Sigma_n, n \geq 0,$$

where Σ_n is the simple symmetric random walk on $\mathbb{Z}$.

By Proposition 10, we recover the following theorem of Pitman.

Corollary 3 ([35]) *The process $2\max_{0 \leq k \leq n} \Sigma_k - \Sigma_n, n \geq 0$, is the discrete Bessel(3) Markov chain.*

Appendix: Asymptotics of Spherical Functions in Rank One

We describe the needed asymptotic behaviour of spherical functions on $SO(1,q)$, $SU(1,q)$ and $Sp(1,q)$ when $q \to +\infty$. We adapt Shimeno [38] to this setting (which only considers the real split case) and Oshima and Shimeno [32]. The advantage of this approach is that it is adapted to higher rank cases. The details of the computations are quite long but straightforward. Therefore we only indicate the main points of the proof where they differ from [38].

We adapt the notions of Sect. 5.1 to the rank one case. In this case there are at most two roots, α and 2α and we may suppose that $\alpha(r) = r$ when $r \in \mathfrak{a} = \mathbb{R}$. Their multiplicity are $(m_\alpha, m_{2\alpha}) = (q-1, 0)$ for $SO(1,q)$, $(2(q-1), 1)$ for $SU(1,q)$ and $(4(q-1), 3)$ for $Sp(1,q)$. Let

$$\varrho_q = \frac{1}{2}(m_\alpha + 2m_{2\alpha}),$$

and

$$\delta_q(r) = (e^r - e^{-r})^{m_\alpha}(e^{2r} - e^{-2r})^{m_{2\alpha}}. \tag{14}$$

Then (see (5.1))

$$\begin{aligned} H_{CMS} &= \delta_q^{1/2} \circ \{\Delta_R + \rho_q^2\} \circ \delta_q^{-1/2} \\ &= \frac{d^2}{dr^2} - \frac{m_\alpha(m_\alpha + 2m_{2\alpha} - 2)}{\sinh^2 r} - \frac{m_{2\alpha}(m_{2\alpha} - 2)}{\sinh^2 2r}. \end{aligned}$$

For $\lambda \in \mathbb{C}$, the spherical function φ_λ satisfies

$$\Delta_R \tilde{\varphi}_\lambda = (\lambda^2 - \varrho_q^2)\tilde{\varphi}_\lambda,$$

where $\tilde{\varphi}_\lambda(r) = \varphi_\lambda(D_1(r))$, $r \in \mathbb{R}$, therefore,

$$H_{CMS}(\delta_q^{1/2}\tilde{\varphi}_\lambda) = \lambda^2 \delta_q^{1/2}\tilde{\varphi}_\lambda.$$

There exists a unique function $\Psi_{CMS}(\lambda, q, r)$, $r \in \mathbb{R}$, of the form

$$\Psi_{CMS}(\lambda, q, r) = \sum_{n \in \mathbb{N}} b_n(\lambda, q) e^{(\lambda - n)r}, \quad b_0(\lambda, q) = 1, \tag{15}$$

such that

$$H_{CMS}\Psi_{CMS} = \lambda^2 \, \Psi_{CMS}, \tag{16}$$

(see [38, (17)]). When $q \to +\infty$,

$$\lim H_{CMS}(r + \log m_\alpha) = H_T,$$

where H_T is the Toda type Hamiltonian

$$H_T = \frac{d^2}{dr^2} - e^{-2r}.$$

There is also a unique function $\Psi_T(\lambda, r)$, $r \in \mathbb{R}$, of the form

$$\Psi_T(\lambda, r) = \sum_{n \in \mathbb{N}} b_n(\lambda) e^{(\lambda - n)r}, \quad b_0(\lambda) = 1, \tag{17}$$

such that

$$H_T \Psi_T = \lambda^2 \Psi_T,$$

([38], notice that this is the function denoted $\Psi_T(-\lambda, -r)$ in [38]).

Lemma 9 *If* $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ *and* $2\lambda \notin \mathbb{Z}^*$, *then*

$$\lim_{q \to +\infty} \frac{1}{m_\alpha^\lambda} \Psi_{CMS}(\lambda, q, r + \log m_\alpha) = \Psi_T(\lambda, r)$$

Proof The proof is similar to the one of Proposition 1 in [38].

Let

$$a(q) = \frac{\Gamma(m_\alpha/2)^2}{\Gamma(m_\alpha) 2^{1+3m_{2\alpha}/2}}, \tag{18}$$

and

$$g_q(\lambda, r) = a(q)(\delta_q^{1/2} \tilde{\varphi}_\lambda)(\log m_\alpha + r) - K_\lambda(e^{-r}). \tag{19}$$

Proposition 11 *Uniformly on* λ *in a small neighborhood of* 0 *in* $\mathbb{C}$, $\lambda g_q(\lambda, r)$ *and its derivatives in* λ *converge to* 0 *as* $q \to +\infty$.

Proof Using the Harish Chandra spherical function expansion of φ_λ (see [18, Theorems IV.5.5 and IV.6.4] or [16]), when λ is for instance in the ball $\{\lambda \in \mathbb{C}; |\lambda| \leq 1/4\}$ and $\lambda \neq 0$, one has

$$\delta_q^{1/2}(r)\tilde{\varphi}_\lambda(r) = c(\lambda)\Psi_{CMS}(\lambda, q, r) + c(-\lambda)\Psi_{CMS}(-\lambda, q, r),$$

where

$$c(\lambda) = \frac{2^{\frac{1}{2}m_\alpha + m_{2\alpha} - \lambda}\Gamma(\frac{1}{2}(m_\alpha + m_{2\alpha} + 1))}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \lambda))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \lambda))}.$$

It follows from Lemma 9 that

$$\lim_{q\to+\infty} a(q)c(\lambda)\frac{1}{m_\alpha^\lambda}\Psi_{CMS}(\lambda, q, r + \log m_\alpha) = \Gamma(\lambda)2^{\lambda-1}\Psi_T(\lambda, r).$$

Hence

$$\lim_{q\to+\infty} a(q)(\delta_q^{1/2}\tilde{\varphi}_\lambda)(r + \log m_\alpha) = \Gamma(\lambda)2^{\lambda-1}\Psi_T(\lambda, r) + \Gamma(-\lambda)2^{-\lambda-1}\Psi_T(-\lambda, r).$$

The limit can be expressed in terms of the Whittaker function for $Sl(2,\mathbb{R})$ as in [38, Theorem 3]. Due to the relation between Whittaker and Macdonald functions (see, e.g., Bump [6]), one finds that this limit is $K_\lambda(e^{-r})$, thus $g_q(\lambda, r)$ tends to 0 when $q \to +\infty$.

Now we remark that, for r fixed, the functions $\lambda g_q(\lambda, r)$ are analytic functions in λ in the ball $\{\lambda \in \mathbb{C}; |\lambda| \le 1/4\}$ which are uniformly bounded in q by the computations of [38, Proposition 1]. Notice that we have to multiply by λ to avoid the singularity at 0 of the Harish Chandra c function. The uniform convergence of $\lambda g_q(\lambda, r)$ and its derivatives in λ thus follows from Montel's theorem.

Corollary 4 *As $q \to +\infty$, $g_q(\lambda, r)$ and all its derivatives at $\lambda = 0$ converge to* 0.

Proof This follows from the proposition and the fact that

$$\frac{d^n}{d\lambda^n}g_q(\lambda, r)_{\{\lambda=0\}} = \frac{1}{n+1}\frac{d^{n+1}}{d\lambda^{n+1}}(\lambda g_q(\lambda, r))_{\{\lambda=0\}}.$$

Remark 7 A small mistake in the constant in [38, Example 1] is corrected in [32].

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Breadth First Search Coding of Multitype Forests with Application to Lamperti Representation

Loïc Chaumont

Abstract We obtain a bijection between some set of multidimensional sequences and the set of d-type plane forests which is based on the breadth first search algorithm. This coding sequence is related to the sequence of population sizes indexed by the generations, through a Lamperti type transformation. The same transformation in then obtained in continuous time for multitype branching processes with discrete values. We show that any such process can be obtained from a d^2-dimensional compound Poisson process time changed by some integral functional. Our proof bears on the discretisation of branching forests with edge lengths.

1 Introduction

A famous result from Lamperti [10] asserts that any continuous state branching process can be represented as a spectrally positive Lévy process, time changed by the inverse of some integral functional. This transformation is invertible and defines a bijection between the set of spectrally positive Lévy processes and this of continuous state branching processes. The same type of transformation holds between continuous time, integer valued branching processes and downward skip free compound Poisson processes. Lamperti's result is the source of an extensive mathematical literature in which it is mainly used as a tool in branching theory. However, recently Lamperti representation itself has been the focus of some research papers. In [2] several proofs of this result are proposed and in [4] an extension of the transformation to the case of branching processes with immigration is proved. The case of affine processes, which includes continuous state multitype branching processes with immigration, was investigated in [8]. In the latter paper a Lamperti type representation in law of affine processes was obtained. Then right after our

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paper was submitted, the two prepublications [3, 6], appeared where pathwise construction of affine processes is obtained through Lamperti representation.

In the present work we show through a combinatorial method, an extension of Lamperti representation to continuous time, integer valued, multitype branching processes. More specifically, let $Z = (Z^{(1)}, \dots, Z^{(d)})$ be such a process issued from $x \in \mathbb{Z}_+^d$, then we shall prove that Z can be represented as

$$(Z_t^{(1)}, \dots, Z_t^{(d)}) = x + \left(\sum_{i=1}^d X^{i,1}_{\int_0^t Z_s^{(i)}\,ds}, \dots, \sum_{i=1}^d X^{i,d}_{\int_0^t Z_s^{(i)}\,ds} \right), \quad t \geq 0\,,$$

where $X^{(i)} = (X^{i,1}, \dots, X^{i,d})$, $i = 1, \dots, d$, are d independent $\mathbb{Z}_+^d$-valued compound Poisson processes. Conversely, given $X^{(i)}$, $i = 1, \dots, d$, the above equation admits a unique solution Z, and Z is a multitype branching process. Since 0 is an absorbing state for Z, it is plain that the whole path of $X = (X^{(1)}, \dots, X^{(d)})$ is not always needed in the above transformation. For instance, when $d = 1$, we see that the process X can be stopped at its first passage time T_x at level $-x$. In the multitype case, we shall prove that the processes $X^{(1)}, \dots, X^{(d)}$ can be stopped at times $T_x^{(1)}, \dots, T_x^{(d)}$, respectively, where $T_x = (T_x^{(1)}, \dots, T_x^{(d)})$ is defined as the 'first passage time' at level $-x$ by the multidimensional random field,

$$\mathrm{X} = (t_1, \dots, t_d) \mapsto \left(\sum_{i=1}^d X^{i,j}_{t_i},\ j \in [d] \right) = X^{(1)}_{t_1} + \dots + X^{(d)}_{t_d}\,.$$

The law of the multivariate random time T_x will be given in Theorem 1.

The multitype Lamperti representation is not invertible as in the one dimensional case. However, by considering the whole branching structure behind the multitype branching process, we actually obtain in Theorems 2 and 3 a one-to-one pathwise transformation between $(X^{(1)}, \dots, X^{(d)})$ and a particular family of processes $(Z^{i,j} : i,j = 1, \dots d)$ satisfying the decomposition

$$Z_t^{(i)} = \sum_{i=1}^d Z_t^{i,j} \quad \text{and} \quad Z_t^{i,j} = x_i\, \mathbf{1}_{i=j} + X^{i,j}_{\int_0^t Z_s^{(i)}\,ds}\,, \quad t \geq 0,$$

where $Z_t^{i,j}$ is the total number of individuals of type j whose parent has type i and who were born before time t.

The proofs of our results are achieved by using a special coding of multitype plane forests based on the breadth first search algorithm. This deterministic one-to-one correspondence between some set of multivariate sequences and multitype forests is stated in Theorem 4 and leads to a Lamperti type representation of discrete time, multitype branching processes in Theorem 5. Results in discrete time are displayed and proved in Sect. 3, whereas the next section is devoted to the statements of our results in continuous time. The latter will be proved in Sect. 4.

2 Main Results in Continuous Time

In all this work, we use the notation $\mathbb{R}_+ = [0,\infty)$, $\mathbb{Z}_+ = \{0,1,2,\dots\}$ and for any positive integer d, we set $[d] = \{1,\dots,d\}$. We also define the following partial order on $\mathbb{R}^d$ by setting $x = (x_1,\dots,x_d) \geq y = (y_1,\dots,y_d)$, if $x_i \geq y_i$, for all $i \in [d]$. Moreover, we write $x > y$ if $x \geq y$ and if there exists $i \in [d]$ such that $x_i > y_i$. We will denote by e_i the i-th unit vector of $\mathbb{Z}_+^d$.

Fix $d \geq 2$ and let $\nu = (\nu_1,\dots,\nu_d)$, where ν_i is any probability measure on $\mathbb{Z}_+^d$ such that $\nu_i(e_i) < 1$. Let $Z = (Z^{(1)},\dots,Z^{(d)})$ be a d-type continuous time and $\mathbb{Z}_+^d$-valued branching process with progeny distribution $\nu = (\nu_1,\dots,\nu_d)$ and such that type $i \in [d]$ has reproduction rate $\lambda_i > 0$. For $i,j \in [d]$, the quantity

$$m_{ij} = \sum_{x\in\mathbb{Z}_+^d} x_j \nu_i(x)\,,$$

corresponds to the mean number of children of type j, given by an individual of type i. Let $M := (m_{ij})_{i,j\in[d]}$ be the mean matrix of Z. We say that the progeny distribution ν is irreducible if the matrix $M = (m_{ij})$ satisfies $m_{ij}^{(n)} > 0$, for some n and for all $i,j \in [d]$, where $m_{ij}^{(n)}$ is the (i,j)-th element of the matrix M^n. Recall that if ν is irreducible, then according to Perron-Frobenius Theorem, it admits a unique eigenvalue ρ which is simple, positive and with maximal modulus. If moreover, ν is non degenerate, that is if individuals have exactly one offspring with probability strictly less than 1, then extinction holds if and only if $\rho \leq 1$, see [7, 12] and Chap. V of [1]. If $\rho = 1$, we say that Z is critical and if $\rho < 1$, we say that Z is subcritical.

We now define the underlying compound Poisson process in the Lamperti representation of Z that will be presented in Theorems 2 and 3. Let $X = (X^{(1)},\dots,X^{(d)})$, where $X^{(i)}$, $i \in [d]$ are d independent $\mathbb{Z}^d$-valued compound Poisson processes. We assume that $X_0^{(i)} = 0$ and that $X^{(i)}$ has rate λ_i and jump distribution

$$\mu_i(\mathrm{k}) = \frac{\tilde{\nu}_i(\mathrm{k})}{1-\tilde{\nu}_i(0)}\,, \quad \text{if } \mathrm{k} \neq 0 \text{ and } \quad \mu_i(0) = 0\,, \tag{1}$$

where

$$\tilde{\nu}_i(k_1,\dots,k_d) = \nu_i(k_1,\dots,k_{i-1},k_i+1,k_{i+1},\dots,k_d)\,. \tag{2}$$

In particular, with the notation $X^{(i)} = (X^{i,1},\dots,X^{i,d})$, for all $i = 1,\dots,d$, the process $X^{i,i}$ is a $\mathbb{Z}$-valued, downward skip free, compound Poisson process, i.e. $\Delta X_t^{i,i} = X_t^{i,i} - X_{t-}^{i,i} \geq -1$, $t \geq 0$, with $X_{0-} = 0$ and for all $i \neq j$, the process $X^{i,j}$ is an increasing compound Poisson process. We emphasize that in this definition, some of the processes $X^{i,j}$, $i,j \in [d]$ can be identically equal to 0.

We first present a result on passage times of the multidimensional random field

$$\mathrm{X} : (t_1, \dots, t_d) \mapsto \left(\sum_{i=1}^{d} X_{t_i}^{i,j}, \, j \in [d] \right) = X_{t_1}^{(1)} + \dots + X_{t_d}^{(d)} ,$$

which is a particular case of additive Lévy process, see [9] and the references therein. Henceforth, a process such as X will be called an *additive* (*downward skip free*) *compound Poisson process*.

Theorem 1 *Let* $x = (x_1, \dots, x_d) \in \mathbb{Z}_+^d$. *Then there exists a* (*unique*) *random time* $T_x = (T_x^{(1)} \dots, T_x^{(d)}) \in \overline{\mathbb{R}}_+^d$ *such that almost surely,*

$$x_j + \sum_{i, T_x^{(i)} < \infty} X^{i,j}(T_x^{(i)}) = 0 \, , \quad \textit{for all } j \textit{ such that } T_x^{(j)} < \infty \, , \tag{3}$$

and if T'_x *is any random time satisfying (3), then* $T'_x \geq T_x$. *The time* T_x *will be called the first passage time of the additive compound Poisson process X at level* $-x$.

The process $(T_x, x \in \mathbb{Z}_+^d)$ *is increasing and additive, that is, for all* $x, y \in \mathbb{Z}_+^d$,

$$T_{x+y} \stackrel{(d)}{=} T_x + \tilde{T}_y \, , \tag{4}$$

where $\tilde{T}_y$ *is an independent copy of* T_y. *The law of* T_x *on* $\mathbb{R}_+^d$ *is given by*

$$\begin{aligned} &\mathbb{P}(T_x \in dt, \, X_{t_i}^{i,j} = x_{i,j}, \, 1 \leq i,j \leq d) \\ &= \frac{det(-x_{i,j})}{t_1 t_2 \dots t_d} \prod_{i=1}^{d} \mathbb{P}(X_{t_i}^{i,j} = x_{i,j}, \, 1 \leq j \leq d) dt_1 dt_2 \dots dt_d \, , \end{aligned}$$

where the support of this measure is included in $\left\{ x_{ij} \in \mathbb{Z} : x_{ij} \geq 0, \, x_{ii} \leq 0, \, \sum_{i=1}^{d} x_{i,j} = -x_i \right\}$.

Note that from the additivity property (4) of $(T_x, \, x \in \mathbb{Z}_+^d)$, we derive that the law of this process is characterised by the law of the variables T_{e_i} for $i \in [d]$.

As the above statement suggests, some coordinates of the time T_x may be infinite. More specifically, we have:

Proposition 1 *Assume that* ν *is irreducible and non degenerate.*

1. *If* ν *is* (*sub*)*critical, then almost surely, for all* $x \in \mathbb{Z}_+^d$ *and for all* $i \in [d]$, $T_x^{(i)} < \infty$.
2. *If* ν *is super critical, then for all* $x \in \mathbb{Z}_+^d$, *with some probability* $p > 0$, $T_x^{(i)} = \infty$, *for all* $i \in [d]$ *and with probability* $1 - p$, $T_x^{(i)} < \infty$, *for all* $i \in [d]$.

There are instances of reducible distributions ν such that for some $x \in \mathbb{Z}_+^d$, with positive probability, $T_x^{(i)} < \infty$, for all $i \in A$ and $T_x^{(i)} = \infty$, for all $i \in B$, (A, B) being a non trivial partition of $[d]$. It is the case for instance when $d = 2$, for $x = (1, 1)$, $X^{1,2} = X^{2,1} \equiv 0$, $0 < m_{11} < 1$ and $m_{22} > 1$.

Then we define d-type branching forests with edge lengths as finite sets of independent branching trees with edge lengths. We say that such a forest is issued from $x = (x_1, \dots, x_d) \in \mathbb{Z}_+^d$ (at time $t = 0$), if it contains x_i trees whose root is of type i. The discrete skeleton of a branching forest with edge lengths is a discrete branching forest with progeny distribution ν. The edges issued from vertices of type i are exponential random variables with parameter λ_i. These random variables are mutually independent and are independent of the discrete skeleton. A realisation of such a forest is represented in Fig. 1. Then to each d-type forest with edge lengths, F, is associated the branching process $Z = (Z^{(1)}, \dots, Z^{(d)})$, where $Z^{(i)}$ is the number of individuals in F, alive at time t.

Definition 1 For $i \neq j$, we denote by $Z_t^{i,j}$ the total number of individuals of type j whose parent has type i and who were born before time t. The definition of $Z_t^{i,i}$ for $i \in [d]$ is the same, except that we add the number of roots of type i and we subtract the number of individuals of type i who died before time t.

More formal definitions of branching forests with edge lengths and processes $Z^{i,j}$ will be given in Sects. 4.2 and 4.3, see in particular (24).

Then we readily check that the branching process $Z = (Z_1, \dots, Z_d)$ which is associated to this forest can be expressed in terms of the processes $Z^{i,j}$, as follows:

$$Z_t^{(j)} = \sum_{i=1}^{d} Z_t^{i,j}, \quad j \in [d].$$

The next theorem asserts that from a given d-type forest with edge lengths F, issued from $x = (x_1, \dots, x_d)$, we can construct a d-dimensional additive compound

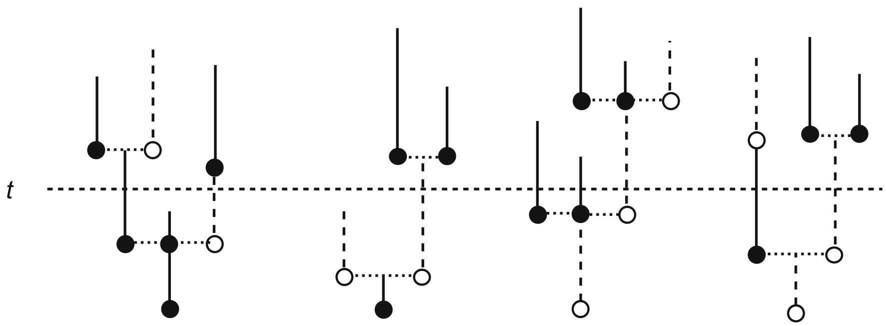

Fig. 1 A two type forest with edge lengths issued from $x = (2, 2)$. Vertices of type 1 (resp. 2) are represented in *black* (resp. *white*). At time t, $Z_t^{1,1} = 1$, $Z_t^{1,2} = 2$, $Z_t^{2,1} = 3$ and $Z_t^{2,2} = 2$

Poisson process $\mathrm{X} = (\sum_{i=1}^d X^{i,j}_{t_i}, j \in [d], (t_1, \dots, t_d) \in \mathbb{R}_+^d)$ stopped at its first passage time of $-x$, such that the branching process Z associated to F can be represented as a time change of X. This extends the Lamperti representation to multitype branching processes.

Theorem 2 *Let $x = (x_1, \dots, x_d) \in \mathbb{Z}_+^d$ and let F be a d-type branching forest with edge lengths, issued from x, with progeny distribution ν and reproduction rates λ_i. Then the processes $Z^{i,j}$, $i,j \in [d]$ introduced in Definition 1 admit the following representation:*

$$Z^{i,j}_t = x_i \mathbf{1}_{i=j} + X^{i,j}_{\int_0^t Z^{(i)}_s\, ds}, \quad t \geq 0, \tag{5}$$

where

$$X^{(i)} = (X^{i,1}, X^{i,2}, \dots, X^{i,d}), \quad i = 1, \dots, d,$$

are independent $\mathbb{Z}_+^d$-valued compound Poisson processes with respective rates λ_i and jump distributions μ_i defined in (1), stopped at the first hitting time T_x of $-x$ by the additive compound Poisson process, $\mathrm{X} = \left(\sum_{i=1}^d X^{i,j}_{t_i}, j \in [d], (t_1, \dots, t_d) \in \mathbb{R}_+^d\right)$*, that is,*

$$X^{(i)}_t \mathbf{1}_{\{t < T^{(i)}_x\}} + (X^{i,1}_{T^{(i)}_x}, \dots, X^{i,d}_{T^{(i)}_x}) \mathbf{1}_{\{t \geq T^{(i)}_x\}}, \quad t \geq 0.$$

In particular the multitype branching process Z, issued from $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$ admits the following representation,

$$(Z^{(1)}_t, \dots, Z^{(d)}_t) = x + \left(\sum_{i=1}^d X^{i,1}_{\int_0^t Z^{(i)}_s\, ds}, \dots, \sum_{i=1}^d X^{i,d}_{\int_0^t Z^{(i)}_s\, ds} \right), \quad t \geq 0. \tag{6}$$

Moreover, the transformation (5) is invertible, so that the processes $Z^{i,j}$, $i,j \in [d]$ can be recovered from the processes $X^{(i)}$, $i \in [d]$.

Note that in Theorem 2, $T^{(i)}_x$ actually represents the total length of the branches issued from vertices of type i in the forest F.

Conversely, the following theorem asserts that an additive compound Poisson process X being given, we can construct a multitype branching forest whose branching process Z is the unique solution to Eq. (6).

Theorem 3 *Let $x = (x_1, \dots, x_d) \in \mathbb{Z}_+^d$ and*

$$X^{(i)} = (X^{i,1}, X^{i,2}, \dots, X^{i,d}), \quad i = 1, \dots, d,$$

be independent $\mathbb{Z}_+^d$ valued compound Poisson processes with respective rates $\lambda_i > 0$ and jump distributions μ_i, stopped at the first hitting time T_x of $-x$ by the additive *compound Poisson process* $(t_1,\dots,t_d)\mapsto\left(\sum_{i=1}^d X^{i,j}_{t_i},\, j\in[d]\right)$. *Then there is a branching forest with edge lengths, with progeny distribution* ν *and reproduction rates* $\lambda_i>0$ *such that the processes* $Z^{i,j}$ *of Definition 1 satisfy relation (5). Moreover, the branching process* $Z=(Z^{(1)},\dots,Z^{(d)})$ *associated to this forest is the unique solution of the equation,*

$$(Z^{(1)}_t,\dots,Z^{(d)}_t)=x+\left(\sum_{i=1}^d X^{i,1}_{\int_0^t Z^{(i)}_s\,ds},\dots,\sum_{i=1}^d X^{i,d}_{\int_0^t Z^{(i)}_s\,ds}\right),\quad t\ge 0\,.$$

We emphasize that Theorems 2 and 3 do not define a bijection between the set of branching forests with edge lengths and this of additive compound Poisson processes, as in the discrete time case, see Sect. 3. Indeed, in the continuous time case, when constructing the processes $Z^{i,j}$ as in Definition 1, at each birth time, we lose the information of the specific individual who gives birth. In particular, the forest which is constructed in Theorem 3 is not unique. This lost information is preserved in discrete time and the breadth first search coding that is defined in Sect. 3.2 allows us to define a bijection between the set of discrete forests and this of coding sequences.

3 Discrete Multitype Forests

3.1 The Space of Multitype Forests

We will denote by $\mathscr{F}$ the set of plane forests. More specifically, an element $\mathbf{f}\in\mathscr{F}$ is a directed planar graph with no loops on a possibly infinite and non empty set of vertices $\mathbf{v}=\mathbf{v}(\mathbf{f})$, with a *finite* number of connected components, such that each vertex has a finite inner degree and an outer degree equals to 0 or 1. The elements of $\mathscr{F}$ will simply be called forests.The connected components of a forest are called the *trees*. A forest consisting of a single connected component is also called a tree. In a tree $\mathbf{t}$, the only vertex with outer degree equal to 0 is called the *root* of $\mathbf{t}$. It will be denoted by $r(\mathbf{t})$. The roots of the connected components of a forest $\mathbf{f}$ are called the roots of $\mathbf{f}$. For two vertices u and v of a forest $\mathbf{f}$, if (u,v) is a directed edge of $\mathbf{f}$, then we say that u is a *child* of v, or that v is the *parent* of u. The set $\mathbf{v}(\mathbf{f})$ of vertices of each forest $\mathbf{f}$ will be enumerated according to the usual breadth first search order, see Fig. 2. We emphasize that we begin by enumerating the roots of the forest from the left to the right. In particular, our enumeration is not performed tree by tree. If a forest $\mathbf{f}$ contains at least n vertices, then the n-th vertex of $\mathbf{f}$ is denoted by $u_n(\mathbf{f})$. When no confusion is possible, we will simply denote the n-th vertex by u_n.

A d-type forest is a couple $(\mathbf{f},c_{\mathbf{f}})$, where $\mathbf{f}\in\mathscr{F}$ and $c_{\mathbf{f}}$ is an application $c_{\mathbf{f}}:\mathbf{v}(\mathbf{f})\to[d]$. For $v\in\mathbf{v}(\mathbf{f})$, the integer $c_{\mathbf{f}}(v)$ is called the *type* (or the *color*) of v. The set of finite d-type forests will be denoted by $\mathscr{F}_d$. An element $(\mathbf{f},c_{\mathbf{f}})\in\mathscr{F}_d$ will

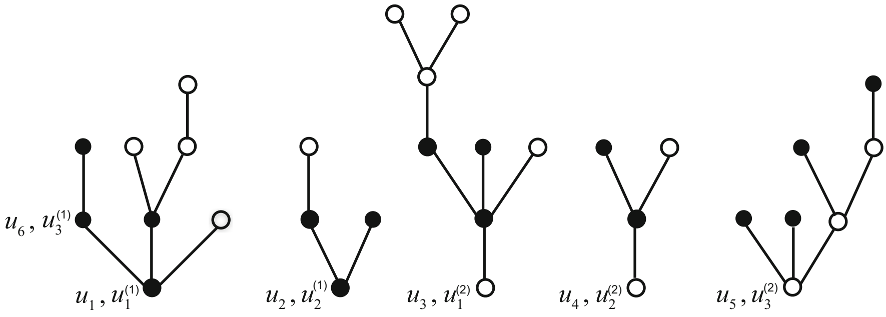

Fig. 2 A two type forest labeled according to the breadth first search order. Vertices of type 1 (resp. 2) are represented in *white* (resp. *black*)

often simply be denoted by $\mathbf{f}$. We assume that for any $\mathbf{f} \in \mathscr{F}_d$, if $u_i, u_{i+1}, \dots, u_{i+j} \in \mathbf{v}(\mathbf{f})$ have the same parent, then $c_{\mathbf{f}}(u_i) \le c_{\mathbf{f}}(u_{i+1}) \le \cdots \le c_{\mathbf{f}}(u_{i+j})$. Moreover, if $u_1, \dots, u_k$ are the roots of $\mathbf{f}$, then $c_{\mathbf{f}}(u_1) \le \cdots \le c_{\mathbf{f}}(u_k)$. For each $i \in [d]$ we will denote by $u_n^{(i)} = u_n^{(i)}(\mathbf{f})$ the n-th vertex of type i of the forest $\mathbf{f} \in \mathscr{F}_d$, see Fig. 2.

3.2 Coding Multitype Forests

The aim of this subsection is to obtain a bijection between the set of multitype forests and some particular set of integer valued sequences which has been introduced in [5]. This bijection, which will be called a *coding*, depends on the breadth first search ordering defined in the previous subsection. We emphasize that this coding is quite different from the one which is defined in [5].

Definition 2 Let S_d be the set of $\left[\mathbb{Z}^d\right]^d$-valued sequences, $x = (x^{(1)}, x^{(2)}, \dots, x^{(d)})$, such that for all $i \in [d]$, $x^{(i)} = (x^{i,1}, \dots, x^{i,d})$ is a $\mathbb{Z}^d$-valued sequence defined on some interval of integers, $\{0, 1, 2, \dots, n_i\}$, $0 \le n_i \le \infty$, which satisfies $x_0^{(i)} = 0$ and if $n_i \ge 1$, then

(i) for $i \ne j$, the sequence $(x_n^{i,j})_{0 \le n \le n_i}$ is nondecreasing,
(ii) for all i, $x_{n+1}^{i,i} - x_n^{i,i} \ge -1$, $0 \le n \le n_i - 1$.

A sequence $x \in S_d$ will sometimes be denoted by $x = (x_k^{i,j},\ 0 \le k \le n_i,\ i,j \in [d])$ and for more convenience, we will sometimes denote $x_k^{i,j}$ by $x^{i,j}(k)$. The vector $\mathrm{n} = (n_1, \dots, n_d) \in \overline{\mathbb{Z}}_+^d$, where $\overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{+\infty\}$ will be called the length of x.

Recall the definition of the order on $\mathbb{R}^d$, from the beginning of Sect. 2 and let us set $U_{\mathrm{s}} = \{i \in [d] : s_i < \infty\}$, for any $\mathrm{s} \in \overline{\mathbb{Z}}_+^d$. Then the next lemma extends Lemma 2.2 in [5] to the case where the smallest solution of a system such as (r, x) in (7) may have infinite coordinates.

Lemma 1 *Let $x \in S_d$ whose length* $\mathrm{n} = (n_1, \ldots, n_d)$ *is such that $n_i = \infty$ for all i (i.e. $U_{\mathrm{n}} = \emptyset$) and let* $\mathrm{r} = (r_1, \ldots, r_d) \in \mathbb{Z}_+^d$*. Then there exists* $\mathrm{s} = (s_1, \ldots, s_d) \in \overline{\mathbb{Z}}_+^d$ *such that*

$$(\mathrm{r}, x) \qquad r_j + \sum_{i=1}^{d} x^{i,j}(s_i) = 0\,, \quad j \in U_{\mathrm{s}}\,, \tag{7}$$

(*we will say that* s *is a solution of the system* (r, x)) *and such that any other solution* q *of* (r, x) *satisfies* $\mathrm{q} \geq \mathrm{s}$. *Moreover we have $s_i = \min\{q : x_q^{i,i} = \min_{0 \leq k \leq s_i} x_k^{i,i}\}$, for all $i \in U_{\mathrm{s}}$.*

The solution s *will be called* the smallest solution *of the system* (r, x). *We emphasize that in (7), we may have $U_{\mathrm{s}} = \emptyset$, so that according to this definition, the smallest solution of the system* (r, x) *may be infinite, that is $s_i = \infty$ for all $i \in [d]$. Note also that in (7) it is implicit that $\sum_{i \in [d] \setminus U_{\mathrm{s}}} x^{i,j}(s_i) < \infty$, for all $j \in U_{\mathrm{s}}$.*

Proof This proof is based on the simple observation that for fixed j, when at least one of the indices k_i's for $i \neq j$ increases, the term $\sum_{i \neq j} x^{i,j}(k_i)$ may only increase and when k_j increases, the term $x^{j,j}(k_j)$ may decrease only by jumps of amplitude -1.

First recall the notation U_{s}, for $\mathrm{s} \in \overline{\mathbb{Z}}_+^d$ introduced before Lemma 1 and set $v_j^{(1)} = r_j$ and for $n \geq 1$,

$$k_j^{(n)} = \inf\{k : x_k^{j,j} = -v_j^{(n)}\} \;\text{ and }\; v_j^{(n+1)} = r_j + \sum_{i \neq j} x^{i,j}(k_i^{(n)})\,,$$

where $\inf \emptyset = \infty$. Set also $\mathrm{k}^{(0)} = 0$ and $U_{\mathrm{k}^{(0)}} = [d]$. Then note that since for $i \neq j$, the $x^{i,j}$'s are positive and increasing, we have

$$\mathrm{k}^{(n)} \leq \mathrm{k}^{(n+1)} \;\text{ and }\; U_{\mathrm{k}^{(n+1)}} \subseteq U_{\mathrm{k}^{(n)}}\,, \quad n \geq 0\,.$$

Moreover, for each $n \geq 0$,

$$r_j + \sum_{i \neq j} x^{i,j}(k_i^{(n)}) + x^{j,j}(k_j^{(n)}) \geq 0\,, \quad j \in U_{\mathrm{k}^{(n)}}\,. \tag{8}$$

Define

$$n_0 = \inf\left\{n \geq 1 : r_j + \sum_{i \neq j} x^{i,j}(k_i^{(n)}) + x^{j,j}(k_j^{(n)}) = 0\,,\, j \in U_{\mathrm{k}^{(n)}}\right\},$$

where in this definition, we consider that $r_j + \sum_{i \neq j} x^{i,j}(k_i^{(n)}) + x^{j,j}(k_j^{(n)}) = 0$ is satisfied for all $j \in U_{\mathrm{k}^{(n)}}$ if $U_{\mathrm{k}^{(n)}} = \emptyset$. Note that $\mathrm{k}^{(n)} = \mathrm{k}^{(n_0)}$ and $U_{\mathrm{k}^{(n)}} = U_{\mathrm{k}^{(n_0)}}$, for all $n \geq n_0$. The index n_0 can be infinite and in general, we have $\mathrm{k}^{(n_0)} = \lim_{n \to \infty} \mathrm{k}^{(n)}$.

Then the smallest solution of the system (r, x) in the sense which is defined in Lemma 1 is $\mathrm{k}^{(n_0)}$.

Indeed, (7) is clearly satisfied for $\mathrm{s} = \mathrm{k}^{(n_0)}$. Then let $\mathrm{q} \in \overline{\mathbb{Z}}_+^d$ satisfying (7), that is

$$r_j + \sum_{i \neq j} x^{i,j}(q_i) + x^{j,j}(q_j) = 0\,,\; j \in U_{\mathrm{q}}\,. \tag{9}$$

We can prove by induction that $\mathrm{q} \geq \mathrm{k}^{(n)}$, for all $n \geq 1$. Firstly for (9) to be satisfied, we should have $q_j \geq \inf\{k : x^{j,j}(k) = -r_j\}$, for all $j \in U_{\mathrm{q}}$, hence $\mathrm{q} \geq \mathrm{k}^{(1)}$. Now assume that $\mathrm{q} \geq \mathrm{k}^{(n)}$. Then $U_{\mathrm{q}} \subseteq U_{\mathrm{k}^{(n)}}$ and from (8) and (9) for each $j \in U_{\mathrm{q}}$, $q_j \geq \inf\{k : x^{j,j}(k) = -(r_j + \sum_{i \neq j} x^{i,j}(k_i^{(n)}))\}$, hence $\mathrm{q} \geq \mathrm{k}^{(n+1)}$.

Finally the fact that $s_i = \min\{q : x_q^{i,i} = \min_{0 \leq k \leq s_i} x_k^{i,i}\}$, for all $i \in U_{\mathrm{s}}$ readily follows from the above construction of s_i. □

Let $(\mathbf{f}, c_{\mathbf{f}}) \in \mathscr{F}_d$, $u \in \mathbf{v}(\mathbf{f})$ and denote by $p_i(u)$ the number of children of type i of u. For each $i \in [d]$, let $n_i \geq 0$ be the number of vertices of type i in $\mathbf{v}(\mathbf{f})$. Then we define the application Ψ from $\mathscr{F}_d$ to S_d by

$$\Psi(\mathbf{f}, c_{\mathbf{f}}) = x\,, \tag{10}$$

where $x = (x^{(1)}, \ldots, x^{(d)})$ and for all $i \in [d]$, $x^{(i)}$ is the d-dimensional chain $x^{(i)} = (x^{i,1}, \ldots, x^{i,d})$, with length n_i, whose values belong to the set $\mathbb{Z}^d$, such that $x_0^{(i)} = 0$ and if $n_i \geq 1$, then

$$x_{n+1}^{i,j} - x_n^{i,j} = p_j(u_{n+1}^{(i)})\,, \quad \text{if } i \neq j \;\text{ and }\; x_{n+1}^{i,i} - x_n^{i,i} = p_i(u_{n+1}^{(i)}) - 1\,, \quad 0 \leq n \leq n_i - 1\,. \tag{11}$$

We recall that $u_n^{(i)}$ is the n-th vertex of type i in the breadth first search order of $\mathbf{f}$. We will see in Theorem 4 that $(n_1, \ldots, n_d)$ is actually the smallest solution of the system (r, x), where r_i is the number of roots of type i of the forest $\mathbf{f}$. This leads us to the following definition.

Definition 3 Fix $\mathrm{r} = (r_1, \ldots, r_d) \in \mathbb{Z}_+^d$, such that $\mathrm{r} > 0$.

(i) We denote by Σ_d^{r} the subset of S_d of sequences x with length $\mathrm{n} = (n_1, \ldots, n_d) \in \overline{\mathbb{Z}}_+^d$ such that n is the smallest solution of the system $(\mathrm{r}, \bar{x})$, where for all $i \in [d]$, $\bar{x}_k^{(i)} = x_k^{(i)}$, if $k \leq n_i$ and $\bar{x}_k^{(i)} = x_{k_i}^{(i)}$, if $k \geq n_i$. We will also say that n is the smallest solution of the system (r, x).

(ii) We denote by $\mathscr{F}_d^{\mathrm{r}}$, the subset of $\mathscr{F}_d$ of d-type forests containing exactly r_i roots of type i, for all $i \in [d]$.

The following theorem gives a one to one correspondence between the sets $\mathscr{F}_d^{\mathrm{r}}$ and Σ_d^{r}.

Theorem 4 *Let* $\mathrm{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d$*, be such that* $\mathrm{r} > 0$*. Then for all* $(f, c_f) \in \mathscr{F}_d^{\mathrm{r}}$*, the chain* $x = \Psi(f, c_f)$ *belongs to the set* Σ_d^{r}*. Moreover, the mapping*

$$\begin{aligned}\Psi : \mathscr{F}_d^{\mathrm{r}} &\to \Sigma_d^{\mathrm{r}} \\ (f, c_f) &\mapsto \Psi(f, c_f)\end{aligned}$$

is a bijection.

Proof In this proof, in order to simplify the notation, we will identify the sequence x with its extension $\bar{x}$ introduced in Definition 3.

Let $(\mathbf{f}, c_{\mathbf{f}})$ be a forest of $\mathscr{F}_d^{\mathrm{r}}$ and let $\mathrm{s} = (s_1, \dots, s_d) \in \overline{\mathbb{Z}}_+^d$, where s_i is the number of vertices of type i in $\mathbf{f}$. We define a subtree of type $i \in [d]$ of $(\mathbf{f}, c_{\mathbf{f}})$ as a maximal connected subgraph of $(\mathbf{f}, c_{\mathbf{f}})$ whose all vertices are of type i. Formally, $\mathbf{t}$ is a subtree of type i of $(\mathbf{f}, c_{\mathbf{f}})$, if it is a connected subgraph whose all vertices are of type i and such that either $r(\mathbf{t})$ has no parent or the type of its parent is different from i. Moreover, if the parent of a vertex $v \in \mathbf{v}(\mathbf{t})^c$ belongs to $\mathbf{v}(\mathbf{t})$, then $c_{\mathbf{f}}(v) \neq i$.

Let $i \in [d]$ and assume first that $s_i < \infty$ (i.e. $i \in U_{\mathrm{s}}$) and let $k_i \leq s_i$ be the number of subtrees of type i in $\mathbf{f}$. Then we can check that the length s_i of the sequence $x^{i,i}$ corresponds to its first hitting time of level $-k_i$, i.e.

$$s_i = \inf\{n : x_n^{i,i} = -k_i\}\,. \tag{12}$$

Indeed, let us rank the subtrees of type i in $\mathbf{f}$ according to the breadth first search order of their roots, so that we obtain the subforest of type i: $\mathbf{t}_1, \dots, \mathbf{t}_{k_i}$ and let x' be its Lukasiewicz-Harris coding path, that is $x'_0 = 0$ and

$$x'_{n+1} - x'_n = p(u_{n+1}) - 1\,, \quad 0 \leq n \leq s_i - 1\,,$$

where u_n is the n-th vertex in the breadth first search of this subforest and $p(u_n)$ is its number of children. We refer to [11] for the coding of forests through their Lukasiewicz-Harris coding path. Then we readily check that both sequences have the same length and end up at the same level, i.e.

$$\inf\{n : x'_n = -k_i\} = \inf\{n : x_n^{i,i} = -k_i\}\,. \tag{13}$$

If $s_i = \infty$ then either $k_i = \infty$ and (12) holds, or $k_i < \infty$ and at least one of the subtrees of type i in the forest is infinite. In this case, we can still compare $x^{i,i}$ to the Lukasiewicz-Harris coding path x' in order to obtain (13), so that (12) also holds.

Now let us check that s is a solution of the system (r, x), that is

$$r_j + \sum_{i=1}^{d} x^{i,j}(s_i) = 0\,, \quad j \in U_{\mathrm{s}}\,. \tag{14}$$

Let $j \in U_{\rm s}$, then $r_j + \sum_{i\neq j} x^{i,j}(s_i)$ clearly represents the total number of vertices of type j in $\mathbf{v}(\mathbf{f})$ which are either a root of type j or whose parent is of a type different from j, i.e. it represents the total number of subtrees of type j in $\mathbf{f}$. On the other hand, from (12), $-x^{j,j}(s_j) \geq 0$ is the number of these subtrees. We conclude that Eq. (14) is satisfied.

It remains to check that s is the smallest solution of the system $({\rm r}, x)$. As in Lemma 1, set ${\rm k}^{(0)} = 0$ and for all $j \in [d]$,

$$k_j^{(n)} = \inf\big\{k : x_k^{j,j} = -(r_j + \sum_{i\neq j} x^{i,j}(k_i^{(n-1)}))\big\}\,,\quad n \geq 1\,. \tag{15}$$

Then from the proof of Lemma 1, we have to prove that ${\rm s} = \lim_{\to\infty} {\rm k}^{(n)}$. Recall the coding which is defined in (11). For all $j \in [d]$, once we have visited the r_j first vertices of type j which are actually roots of the forest, we have to visit the whole corresponding subtrees, so that, if the total number of vertices of type j in $(\mathbf{f}, c_\mathbf{f})$ is finite, i.e. $j \in U_{\rm s}$, then the chain $x^{j,j}$ first hits $-r_j$ at time $k_j^{(1)} < \infty$. Then at time $k_j^{(1)}$, an amount of $\sum_{i\neq j} x^{i,j}(k_i^{(1)})$ more subtrees of type j have to be visited. So the chain $x^{j,j}$ has to hit $-(r_j + \sum_{i\neq j} x^{i,j}(k_i^{(1)}))$ at time $k^{(2)} < \infty$. This procedure is iterated until the last vertex of type j is visited, that is until time $s_j = \lim_{n\to\infty} k_j^{(n)} < \infty$ (note that the sequence $k_j^{(n)}$ is constant after some finite index). Besides, from (15), we have

$$s_j = \inf\{k : x_k^{j,j} = -(r_j + \sum_{i\neq j} x^{i,j}(s_i))\}\,,\quad n \geq 1\,,\quad j \in U_{\rm s}\,.$$

On the other hand, if the total number of vertices of type j in $(\mathbf{f}, c_\mathbf{f})$ is infinite, then $k_j^{(n)}$ tends to ∞ (it can be infinite by some rank). So that we also have $s_j = \lim_{n\to\infty} k_j^{(n)}$ in this case. Therefore, s is the smallest solution of $({\rm r}, x)$.

Now let $x \in \Sigma_d^{\rm r}$ with length s, then we construct a forest $(\mathbf{f}, c_\mathbf{f}) \in \mathscr{F}_d^{\rm r}$ such that $\Psi(\mathbf{f}, c_\mathbf{f}) = x$, generation by generation, using the definition of Ψ in (10) as follows. At generation $n = 1$, for each $i \in [d]$, we take r_i vertices of type i. We rank these $r_1 + \cdots + r_d$ vertices as it is defined in Sect. 3.1. Then to the k-th vertex of type i, we give $\Delta x_k^{i,j} := x_k^{i,j} - x_{k-1}^{i,j}$ children of type $j \in [d]$ if $j \neq i$ and $\Delta x_k^{i,i} + 1$ children of type i. We rank vertices of generation $n = 2$ and to the $r_i + k$-th vertex of type i, we give $\Delta x_{r_i+k}^{i,j}$ children of type $j \in [d]$, if $j \neq i$ and $\Delta x_{r_i+k}^{i,i} + 1$ children of type i, and this procedure is repeated generation by generation. Proceeding this way, until the steps $\Delta x_{s_i}^{i,j}$, $i, j \in [d]$, we have constructed a forest of $\mathscr{F}_d^{\rm r}$. Indeed the total number of children of type j whose parent is of type $i \neq j$ is $x^{i,j}(s_i)$, hence, the total number of children of type j which is a root or whose parent is different from j is $r_j + \sum_{i\neq j} x^{i,j}(s_i)$. Moreover, each branch necessarily ends up with a leaf, since $\Delta x_{s_i+1}^{i,j} = 0$, for all $i \neq j$ and $\Delta x_{s_i+1}^{i,i} = -1$. Therefore we have constructed a forest $(\mathbf{f}, c_\mathbf{f})$ of $\mathscr{F}_d^{\rm r}$ such that $\Psi(\mathbf{f}, c_\mathbf{f}) = x$ and we have proved that Φ is onto.

Finally, let us denote the forest $(\mathbf{f}, c_{\mathbf{f}})$ which is reconstructed from x through the above procedure by $\Phi(x)$, then the application Φ is actually the inverse of Ψ. Indeed, we can check from the definition of Φ above and that of Ψ in (10) that for all $(\mathbf{f}, c_{\mathbf{f}}) \in \mathscr{F}_d^{\mathrm{r}}$, $\Phi(\Psi((\mathbf{f}, c_{\mathbf{f}}))) = (\mathbf{f}, c_{\mathbf{f}})$. Therefore Ψ is one-to-one. □

3.3 Representing the Sequence of Generation Sizes

To each $\mathbf{f} \in \mathscr{F}_d^{\mathrm{r}}$ we associate the chain $z = (z^{(1)}, \dots, z^{(d)})$ indexed by the successive generations in $\mathbf{f}$, where for each $i \in [d]$, and $n \geq 1$, $z^{(i)}(n)$ is the size of the population of type i at generation n. More formally, we say that the (index of the) generation of $u \in \mathbf{v}(\mathbf{f})$ is n if $d(r(\mathbf{t}_u), u) = n$, where $\mathbf{t}_u$ is the tree of $\mathbf{f}$ which contains u, $r(\mathbf{t}_u)$ is the root of this tree and d is the usual distance in discrete trees. In order to simplify the notation, we set $|u| = d(r(\mathbf{t}_u), u)$. Let us denote by $h(\mathbf{f})$ the index of the highest generation in $\mathbf{f}$. Then $z^{(i)}$ is defined by

$$z^{(i)}(n) = \begin{cases} \mathrm{Card}\{u \in \mathbf{v}(\mathbf{f}) : c_{\mathbf{f}}(u) = i,\ |u| = n\} & \text{if } n \leq h(\mathbf{f}), \\ 0 & \text{if } n \geq h(\mathbf{f}) + 1. \end{cases} \tag{16}$$

We also define the chains $z^{i,j}$, for $i, j \in [d]$, as follows: $z^{i,j}(0) = 0$ if $i \neq j$, $z^{i,i}(0) = r_i$, and for $n \geq 1$,

$$z^{i,j}(n) = r_i \mathbf{1}_{i=j} + \sum_{|u| \leq n-1} \mathbf{1}_{c_{\mathbf{f}}(u)=i} \left(p_j(u) - \mathbf{1}_{i=j}\right). \tag{17}$$

In words, if $i \neq j$ then $z^{i,j}(n)$ is the total number of vertices of type j whose parent is of type i in the first n generations of the forest $\mathbf{f}$. If $i = j$ then we only count the number of vertices of type i with at least one younger brother of type i and whose parent is of type i in the n first generations. To this number, we subtract the number of vertices of type i with no children of type i, whose generation is less or equal than $n-1$. Then it is not difficult to check the following relation:

$$z^{(j)}(n) = \sum_{i=1}^{d} z^{i,j}(n)\,, n \geq 0\,, \quad j \in [d]. \tag{18}$$

We end this subsection by a lemma which provides a relationship between the chains $x^{i,j}$ and $z^{i,j}$, where $x = \Psi(\mathbf{f}, c_{\mathbf{f}})$. This result is the deterministic expression of the Lamperti representation of Theorem 5 below and its continuous time counterpart in Theorems 2 and 3.

Lemma 2 *The chain $z^{i,j}$ may be obtained as the following time change of the chain $x^{i,j}$:*

$$z^{i,j}(n) = x^{i,j}(\textstyle\sum_{k=0}^{n-1} z^{(i)}(k)), \quad n \geq 1, \quad i,j \in [d], \quad i \neq j, \tag{19}$$

$$z^{i,i}(n) = r_i + x^{i,i}(\textstyle\sum_{k=0}^{n-1} z^{(i)}(k)), \quad n \geq 1, \quad i \in [d]. \tag{20}$$

In particular, we have

$$z^{(j)}(n) = r_j + \sum_{i=1}^{d} x^{i,j}(\textstyle\sum_{k=0}^{n-1} z^{(i)}(k)), \quad n \geq 1, \quad j \in [d]. \tag{21}$$

Moreover, given $x^{i,j}$, $i,j \in [d]$, the chains $z^{i,j}$, $i,j \in [d]$ are uniquely determined by Eqs. (18)–(20).

Proof It suffices to check relations (19) and (20). Then (21) will follow from (18). But (19) and (20) are direct consequences of the definition of the chains $x^{i,j}$ and $z^{i,j}$ in (11) and (17) respectively. □

3.4 Application to Discrete Time Branching Processes

Recall that $\nu = (\nu_1, \dots, \nu_d)$ is a progeny distribution, such that the ν_i's are any probability measures on $\mathbb{Z}_+^d$, such that $\nu_i(e_i) < 1$. Let $(\Omega, \mathscr{G}, P)$ be some measurable space on which, for any $\mathrm{r} \in \mathbb{Z}_+^d$ such that $\mathrm{r} > 0$, we can define a probability measure $\mathbb{P}_\mathrm{r}$ and a random variable $(F, c_F) : (\Omega, \mathscr{G}, \mathbb{P}_\mathrm{r}) \to \mathscr{F}_d^\mathrm{r}$ whose law under $\mathbb{P}_\mathrm{r}$ is this of a branching forest with progeny law ν. Then we construct from (F, c_F) the random chains, $X = (X^{(1)}, \dots, X^{(d)})$, $Z = (Z^{(1)}, \dots, Z^{(d)})$ and $Z^{i,j}$, $i,j \in [d]$, exactly as in (11), (16) and (17), respectively. In particular, $X = \Psi(F, c_F)$. We can check that under $\mathbb{P}_\mathrm{r}$, Z is a branching process with progeny distribution ν. More specifically, recall from (2) the definition of $\tilde{\nu}_i$, then the random processes X and Z satisfy the following result.

Theorem 5 *Let $\mathrm{r} \in \mathbb{Z}_+^d$ be such that $\mathrm{r} > 0$ and let (F, c_F) be an $\mathscr{F}_d^\mathrm{r}$-valued branching forest with progeny law ν under $\mathbb{P}_\mathrm{r}$. Let $N = (N_1, \dots, N_d) \in \overline{\mathbb{Z}}_+^d$, where N_i is the number of vertices of type i in F. Then,*

1. *The random variable N is almost surely the smallest solution of the system (r, X) in the sense of Definition 3. If ν is irreducible, non degenerate and (sub)critical, then almost surely, $N_i < \infty$, for all $i \in [d]$. If ν is irreducible, non degenerate and supercritical, then with some probability $p > 0$, $N_i = \infty$, for all $i \in [d]$ and with probability $1 - p$, $N_i < \infty$, for all $i \in [d]$.*
2. *On the space $(\Omega, \mathscr{G}, P)$, we can define independent random walks $\tilde{X}^{(i)}$, $i \in [d]$, with respective step distributions $\tilde{\nu}_i$, $i \in [d]$, such that $\tilde{X}_0^{(i)} = 0$ and if $\tilde{N} =$*

$(\tilde{N}_1, \dots \tilde{N}_d) \in \overline{\mathbb{Z}}_+^d$ is the smallest solution of the system $(\mathrm{r}, \tilde{X})$*, then the following identity in law*

$$(X_k^{(i)},\ 0 \le k \le N_i,\ i \in [d]) \overset{(d)}{=} (\tilde{X}_k^{(i)},\ 0 \le k \le \tilde{N}_i,\ i \in [d])$$

holds.

3. *The joint law of X_N and N is given as follows: for any integers n_i and k_{ij}, $i, j \in [d]$, such that $n_i > 0$, $k_{ij} \in \mathbb{Z}_+$, for $i \neq j$, $-k_{jj} = r_j + \sum_{i \neq j} k_{ij}$ and $n_i \ge -k_{ii}$,*

$$\mathbb{P}_{\mathrm{r}}\Big(X_{n_i}^{i,j} = k_{ij},\ i, j \in [d] \text{ and } N = \mathrm{n}\Big)$$
$$= \frac{\det(-k_{ij})}{n_1 n_2 \dots n_d} \prod_{i=1}^{d} \nu_i^{*n_i}\Big(k_{i1}, \dots, k_{i(i-1)}, n_i + k_{ii}, k_{i(i+1)}, \dots, k_{id}\Big)\,.$$

4. *The random process Z is a branching process with progeny law ν, which is related to X through the time change:*

$$Z^{(i)}(n) = \sum_{i=1}^{d} X^{i,j}\big(\textstyle\sum_{k=1}^{n-1} Z^{(i)}(k)\big)\,, \quad n \ge 1\,. \tag{22}$$

Proof The fact that N is the smallest solution of the system (r, X) is a direct consequence of Theorem 4 and the definition of X, that is $\Psi(F, c_F) = X$. Assume that ν is irreducible, non degenerate and (sub)critical. Then since the forest F contains almost surely a finite number of vertices, all coordinates of N must be finite from Theorem 4. If ν is irreducible, non degenerate and supercritical, then with probability $p > 0$ the forest F contains an infinite number of vertices of type i, for all $i \in [d]$ and with probability $1-p$ its total population is finite. Then the result also follows from Theorem 4.

In order to prove part 2, let (F_n, c_{F_n}) with $(F_1, c_{F_1}) = (F, c_F)$, be a sequence of independent and identically distributed forests. Let us define $X^n = \Psi(F_n, c_{F_n})$ and then let us concatenate the processes $X^n = (X^{n,(1)}, \dots, X^{n,(d)})$ in a process that we denote $\tilde{X}$. More specifically, let us denote by N_i^n the length of $X^{n,(i)}$, then the process obtained from this concatenation is $\tilde{X} = (\tilde{X}^{(1)}, \dots, \tilde{X}^{(d)})$, where $\tilde{X}_0^{(i)} = 0$, $N_i^0 = 0$ and

$$\tilde{X}_k^{(i)} = \tilde{X}^{(i)}_{N_i^0 + \dots + N_i^{n-1}} + X^{n,(i)}_{k-(N_i^0 + \dots + N_i^{n-1})}\,,$$
$$\text{if } N_i^0 + \dots + N_i^{n-1} \le k \le N_i^0 + \dots + N_i^n\,, \quad n \ge 1\,.$$

Note that $\tilde{X}$ is obtained by coding the forests (F_n, c_{F_n}), $n \ge 1$ successively. Then it readily follows from the construction of $\tilde{X}$ and the branching property that the coordinates $\tilde{X}^{(i)}$ are independent random walks with step distribution $\tilde{\nu}_i$. Moreover

N is a solution of the system $(\mathrm{r}, \tilde{X})$, so its smallest solution, say N', is necessarily smaller than N. This means that N' is a solution of the system (r, X), hence $N' = N$.

The third part is a direct consequence of the first part and the multivariate ballot Theorem which is proved in [5], see Theorem 3.4 therein.

Then part 4, directly follows from the definition of Z and Lemma 2. □

Conversely, from any random walk whose step distribution is given by (2), we can construct a unique branching forest with law $\mathbb{P}_{\mathrm{r}}$. The following result is a direct consequence of Theorems 4 and 5.

Theorem 6 *Let $\tilde{X}^{(i)}$, $i \in [d]$ be d independent random walks defined on $(\Omega, \mathscr{G}, P)$, whose respective step distributions are $\tilde{\nu}_i$, and set $\tilde{X} = (\tilde{X}^{(1)}, \dots, \tilde{X}^{(d)})$. Let $\mathrm{r} \in \mathbb{Z}_+^d$ such that $\mathrm{r} > 0$ and let $\tilde{N}$ be the smallest solution of the system $(\mathrm{r}, \tilde{X})$. We define the Σ_d^{r}-valued process $X = (X^{(1)}, \dots, X^{(d)})$ by $(X_k^{(i)},\, 0 \le k \le \tilde{N}_i) = (\tilde{X}_k^{(i)},\, 0 \le k \le \tilde{N}_i)$.*

Then $(F, c_F) := \Psi^{-1}(X)$ is a $\mathscr{F}_d^{\mathrm{r}}$-valued branching forest (F, c_F) with progeny distribution ν.

4 The Continuous Time Setting

4.1 *Proofs of Theorem 1 and Proposition 1*

Let $Y = (Y^{(1)}, \dots, Y^{(d)})$ be the underlying random walk of the compound Poisson process X, that is the random walk such that there are independent Poisson processes $N^{(i)}$, with parameters λ_i such that $X_t^{(i)} = Y^{(i)}(N_t^{(i)})$ and $(N^{(i)}, Y^{(j)}, i, j \in [d])$ are independent. Then from Lemma 1, there is a random time $\mathrm{s} \in \overline{\mathbb{Z}}_+^d$, such that almost surely, for all $j \in U_{\mathrm{s}}$, $x_j + \sum_{i=1}^d Y^{i,j}(s_i) = 0$. Moreover, if s' is any time satisfying this property, then $\mathrm{s}' \ge \mathrm{s}$. For $i \in [d] \setminus U_{\mathrm{s}}$, the latter equality implies that $Y^{i,j}(\infty) < \infty$. Since $Y^{i,j}$ is a renewal process, it is possible only if $Y^{i,j} \equiv 0$, a.s., so that we can write: almost surely,

$$x_j + \sum_{i,\, i \in U_{\mathrm{s}}} Y^{i,j}(s_i) = 0\,, \quad \text{for all } j \in U_{\mathrm{s}}\,.$$

Then the first part of the Theorem follows from the construction of X as a time change of Y. More formally, the coordinates of T_x are given by $T_x^{(i)} = \inf\{t : N^{(i)}(t) = \mathrm{s}_i\}$, so that in particular, $s_i = N^{(i)}(T_x^{(i)})$.

Additivity property of T_x is a consequence of Lemma 1 and time change. From this lemma, we can deduce that for all $x, y \in \mathbb{Z}_+^d$, if s is the smallest solution of $(x + y, Y)$, then conditionally to $s_i < \infty$, for all $i \in [d]$, the smallest solution $\mathrm{s}_1 = (s_{1,1}, \dots, s_{1,d})$ of (x, Y), and satisfies $\mathrm{s}_1 \le \mathrm{s}$ and $s - s_1$ is the smallest solution of the system $(y, \tilde{Y})$, where $\tilde{Y}_k^{(i)} = Y_{s_{1,i}+k}^{(i)} - Y_{s_{1,i}}^{(i)}$, $k \ge 0$. Moreover, $\tilde{Y} = (\tilde{Y}^{(i)}, i \in [d])$ has the same law as Y and is independent of $(Y_k^{(i)},\ 0 \le k \le s_{1,i})$. Using the time change, we obtain,

$$L_{x+y} \stackrel{(d)}{=} L_x + \tilde{L}_y \,,$$

where L_x has the law of T_x conditionally on $T_x^{(i)} < \infty$, for all $i \in [d]$ and $\tilde{L}_y$ is an independent copy of L_y. Then identity (4) follows.

The law of T_x on $\mathbb{R}_+^d$ follows from time change and the same result in the discrete time case obtained in [5], see Theorem 3.4 therein.

Proposition 1 is a direct consequence of part 1 of Theorem 1 and the time change.

4.2 Multitype Forests with Edge Lengths

A d type forest with edge lengths is an element $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$, where $(\mathbf{f}, c_{\mathbf{f}}) \in \mathscr{F}_d$ and $\ell_{\mathbf{f}}$ is some application $\ell_{\mathbf{f}} : \mathbf{v}(\mathbf{f}) \to (0, \infty)$. For $u \in \mathbf{v}(\mathbf{f})$, the quantity $\ell_{\mathbf{f}}(u)$ will be called the life time of u. It corresponds to the length of an edge incident to u in $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ whose color is this of u. This edge is a segment which is closed at the extremity corresponding to u and open at the other extremity. If u is not a leaf of $(\mathbf{f}, c_{\mathbf{f}})$ then $\ell_{\mathbf{f}}(u)$ corresponds to the length of the edge between u and its children in the continuous forest $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$. To each tree of $(\mathbf{f}, c_{\mathbf{f}})$ corresponds a tree of $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ which is considered as a continuous metric space, the distance being given by the Lebesgue measure along the branches. To each forest $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ we associate a time scale such that a vertex u is born at time t if the distance between u and the root of its tree in $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ is t. Time t is called the birth time of u in $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ and it is denoted by $b_{\mathbf{f}}(u)$. The death time of u in $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ is then $b_{\mathbf{f}}(u) + \ell_{\mathbf{f}}(u)$. If $s \in [b_{\mathbf{f}}(u), b_{\mathbf{f}}(u) + \ell_{\mathbf{f}}(u))$ then we say that u is alive at time s in the forest $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$. We denote by $\mathrm{h}_{\mathbf{f}}$ the smallest time when no individual is alive in $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$.

The set of d type forests with edge lengths will be denoted by F_d. The subset of F_d of elements $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ such that $(\mathbf{f}, c_{\mathbf{f}}) \in \mathscr{F}_d^{\mathrm{r}}$ will be denoted by F_d^{r}. Elements of F_d will be represented as in Fig. 1.

To each forest $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}}) \in F_d^{\mathrm{r}}$, we associate the multidimensional step functions $(z^{(i)}(t), t \ge 0)$ that are defined as follows:

$$z^{(i)}(t) = \mathrm{Card}\{u \in \mathbf{v}(\mathbf{f}) : c_{\mathbf{f}}(u) = i,\ u \text{ is alive at time } t \text{ in } (\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}}).\} \tag{23}$$

Then the process $(z^{i,j}(t), t \ge 0)$ introduced in Definition 1 is formally defined by $z^{i,j}(0) = 0$ if $i \ne j$, $z^{i,i}(0) = r_i$, and for $t > 0$,

$$z^{i,j}(t) = r_i \mathbf{1}_{i=j} + \sum_{b_{\mathbf{f}}(u)+\ell_{\mathbf{f}}(u)<t} \mathbf{1}_{c_{\mathbf{f}}(u)=i} \left(p_j(u) - \mathbf{1}_{i=j}\right). \tag{24}$$

It readily follows from these definitions that

$$z^{(j)}(t) = \sum_{i=1}^{d} z_t^{i,j}, \quad t \geq 0 .$$

We now define the discretisation of forests of F_d, with some span $\delta > 0$. Let $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}}) \in F_d$, then on each tree of $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}}) \in F_d$, we place new vertices at distance $n\delta$, $n \in \mathbb{Z}_+$ of the root along all the branches. A vertex which is placed along an edge with color i has also color i. Then we define a forest in $\mathscr{F}_d$ as follows. A new vertex v is the child of a new vertex u if and only if both vertices belong to the same branch of $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$, and there is $n \geq 0$ such that u and v are respectively at distance $n\delta$ and $(n+1)\delta$ from the root. This transformation defines an application which we will denote by

$$D_\delta : F_d \to \mathscr{F}_d$$
$$(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}}) \mapsto D_\delta(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}}) .$$

Note that with this definition, the roots of the three forests $(\mathbf{f}, c_{\mathbf{f}})$, $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ and $D_\delta(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ are the same and more generally, a vertex of $D_\delta(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ corresponds to a vertex u of $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ if and only if u is at a distance equal to $n\delta$ from the root, for some integer $n \geq 0$. It is also equivalent to the fact that u is at generation n in $D_\delta(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$. The definition of the discretisation of a forest with edge lengths should be obvious from Fig. 3.

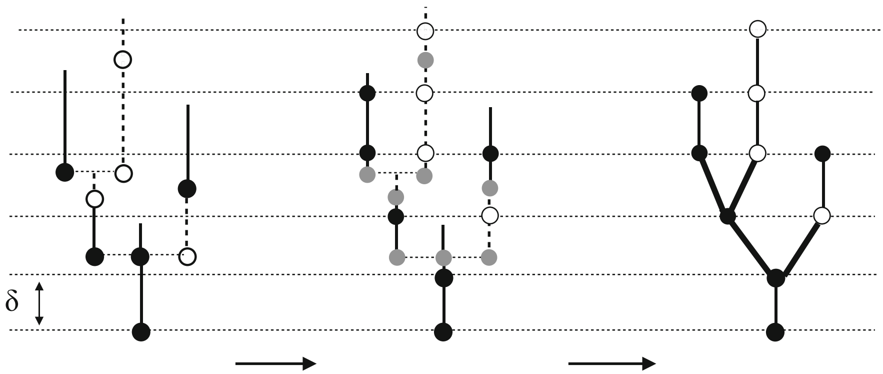

Fig. 3 Discretisation of a two type tree with edge lengths: from $(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$ to $D_\delta(\mathbf{f}, c_{\mathbf{f}}, \ell_{\mathbf{f}})$

4.3 Multitype Branching Forests with Edge Lengths

Recall from Sect. 2 that $\lambda_1, \dots, \lambda_d$ are positive, finite and constant rates, and that $\nu = (\nu_1, \dots, \nu_d)$, where ν_i, $i \in [d]$ are any distributions in $\mathbb{Z}_+^d$ such that $\nu_i(e_i) < 1$. From the setting established in Sect. 4.2, we can define a branching forest with edge lengths as a random variable $(F, c_F, \ell_F) : (\Omega, \mathscr{G}, \mathbb{P}_{\mathrm{r}}) \to (F_d, \mathscr{H}_d)$, where $\mathscr{H}_d$ is the sigma field of the Borel sets of F_d endowed with the Gromov-Hausdorff topology (see Sect. 2.1 in [11]) and where the law of $(F, c_F) : (\Omega, \mathscr{G}, \mathbb{P}_{\mathrm{r}}) \to \mathscr{F}_d^{\mathrm{r}}$ under $\mathbb{P}_{\mathrm{r}}$ is this of a discrete branching forest with progeny distribution ν, as defined in Sect. 3.4. Besides, let N_i be the number of vertices of type i in (F, c_F), then for all $\mathrm{n} = (n_1, \dots, n_d) \in \overline{\mathbb{Z}}_+^d$, conditionally on $N_i = n_i$, $i \in [d]$, $(\ell_F(u_n^{(i)}))_{0 \le n \le n_i}$ are sequences of i.i.d. exponentially distributed random variables with respective parameters λ_i, and $[(\ell_F(u_n^{(i)}))_{0 \le n \le n_i}, i \in [d], (F, c_F)]$ are independent.

Then we have the following result which is straightforward from the above definitions.

Proposition 2 *Let (F, c_F, ℓ_F) be a branching forest with edge lengths with progeny distribution $\nu = (\nu_1, \dots, \nu_d)$ on $\mathbb{Z}_+^d$ and reproduction rates $\lambda_1, \dots, \lambda_d \in (0, \infty)$. Then for all* r*, under $\mathbb{P}_{\mathrm{r}}$, the process $Z = (Z^{(i)}(t),\ t \ge 0,\ i \in [d])$ which is defined as in (23) with respect to (F, c_F, ℓ_F) is a continuous time, $\mathbb{Z}_+^d$-valued branching process starting at $Z_0 = \mathrm{r}$, with progeny distribution $\nu = (\nu_1, \dots, \nu_d)$ and reproduction rates $\lambda_1, \dots, \lambda_d$.*

The law of a discretised branching forest with edge lengths is given by the following lemma.

Lemma 3 *Let (F, c_F, ℓ_F) be a branching forest with edge lengths, with progeny distribution $\nu = (\nu_1, \dots, \nu_d)$ on $\mathbb{Z}_+^d$ and life time rates $\lambda_1, \dots, \lambda_d \in (0, \infty)$. For $\delta > 0$, the forest $D_\delta(F, c_F, \ell_F)$ is a (discrete) branching forest with progeny distribution:*

$$\nu_i^{(\delta)}(\mathrm{k}) = \mathbb{P}_{e_i}(Z_\delta = \mathrm{k}), \quad \mathrm{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d,$$

where e_i is the i-th unit vector of $\mathbb{Z}_+^d$.

Proof The fact that $D_\delta(F, c_F, \ell_F)$ is a discrete branching forest is a direct consequence of its construction. Indeed, at generation n, that is at time $n\delta$, the vertices of this forest inherits from the (time homogeneous) branching property of (F, c_F, ℓ_F) the fact they will give birth to some progeny, independently of each other and with some distribution which only depends on their type and δ. Then it remains to determine the progeny distribution $\nu^{(\delta)}$. But it is obvious from the construction of $D_\delta(F, c_F, \ell_F)$ that $\nu_i^{(\delta)}$ is the law of the total offspring at time δ of a root of type i in the forest (F, c_F, ℓ_F). This is precisely the expression which is given in the statement. □

4.4 Proof of Theorem 2

Let (F, c_F, ℓ_F) be a branching forest issued from x, with edge lengths, with progeny distribution $\nu = (\nu_1, \dots, \nu_d)$ on $\mathbb{Z}_+^d$ and life time rates $\lambda_1, \dots, \lambda_d \in (0, \infty)$. Then from Proposition 2, the process $Z = (Z^{(i)}(t),\ t \ge 0,\ i \in [d])$ which is defined as in (23) with respect to (F, c_F, ℓ_F) is a continuous time, $\mathbb{Z}_+$-valued branching process with progeny distribution $\nu = (\nu_1, \dots, \nu_d)$ and life time rates $\lambda_1, \dots, \lambda_d$.

Let $\delta > 0$ and consider the discrete forest $D_\delta(F, c_F, \ell_F)$ whose progeny distribution is given by Lemma 3. Let $\mathrm{Z}^\delta = (\mathrm{Z}^{\delta,(1)}, \dots, \mathrm{Z}^{\delta,(d)})$ be the associated (discrete time) branching process and let $\mathrm{Z}^{\delta,i} := (\mathrm{Z}^{\delta,i,1}, \dots, \mathrm{Z}^{\delta,i,d})$, $i \in [d]$ be the decomposition of Z^δ, as it is defined in (17). Then it is straightforward that

$$(\mathrm{Z}^{\delta,i,j}([\delta^{-1}t])\,, t \ge 0) \to (Z_t^{i,j}\,, t \ge 0)\,, \tag{25}$$

almost surely on the Skohorod's space of càdlàg paths toward the process $Z^{i,j}$, as δ tends to 0, for all $i, j \in [d]$, where $Z^{i,j}$ is the decomposition of Z as it is defined in (24).

Now, let $\mathrm{X}^\delta = (\mathrm{X}^{\delta,(i)},\ i \in [d])$ be the coding random walk associated to $D_\delta(F, c_F, \ell_F)$, as in Theorem 5. We will first assume that X^δ is actually the coding random walk of a sequence of i.i.d. discrete forests with the same distribution as $D_\delta(F, c_F, \ell_F)$ in the same manner as in the proof of part 2 of Theorem 5, so that in particular, X^δ is not a stopped random walk.

For $i \in [d]$, let $\tau_{i,n}^{\mathrm{X}^\delta}$ and $\tau_{i,n}^{\mathrm{Z}^\delta}$ be the sequences of jump times of the discrete time processes $\mathrm{X}^{\delta,(i)}$ and $\mathrm{Z}^{\delta,i}$. That is the ordered sequences of times such that $\tau_{i,0}^{\mathrm{X}^\delta} = \tau_{i,0}^{\mathrm{Z}^\delta} = 0$ and $\Delta \mathrm{X}_n^{\delta,(i)} := \mathrm{X}^{\delta,(i)}(\tau_{i,n}^{\mathrm{X}^\delta}) - \mathrm{X}^{\delta,(i)}(\tau_{i,n-1}^{\mathrm{X}^\delta}) \neq 0$ and $\Delta \mathrm{Z}_n^{\delta,i} := \mathrm{Z}^{\delta,i}(\tau_{i,n}^{\mathrm{Z}^\delta}) - \mathrm{Z}^{\delta,i}(\tau_{i,n-1}^{\mathrm{Z}^\delta}) \neq 0$, for $n \ge 1$. Fix $k \ge 1$, then since two jumps of the process Z almost surely never occur at the same time, there is δ_0, sufficiently small such that for all $0 < \delta \le \delta_0$, the sequences $(\Delta \mathrm{X}_n^{\delta,(i)},\ 0 \le n \le k)$ and $(\Delta \mathrm{Z}_n^{\delta,i},\ 0 \le n \le k)$ are a.s. identical, for all $i \in [d]$. Moreover, from Lemma 3 and Theorem 5, the jumps $\Delta \mathrm{X}_n^{\delta,(i)}$ have law

$$\mu_i^{(\delta)}(\mathrm{k}) = \frac{\tilde{\nu}_i^{(\delta)}(\mathrm{k})}{1 - \tilde{\nu}_i^{(\delta)}(0)}\,,\quad \mathrm{k} \neq 0\,,\quad \mu_i^{(\delta)}(0) = 0\,,$$

where $\tilde{\nu}_i^{(\delta)}(\mathrm{k}) = \mathbb{P}_{e_i}(Z_\delta = (k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_d))$. The measure $\mu_i^{(\delta)}$ converges weakly to μ_i defined in (1), as $\delta \to 0$. Hence from (25), the sequence $(\Delta \mathrm{Z}_n^{\delta,i},\ n \ge 0)$ converges almost surely toward the sequence $(\Delta Z_n^i,\ n \ge 0)$ of jumps of the process $(Z_t^i,\ t \ge 0) := (Z_t^{i,j},\ j \in [d]\,, t \ge 0)$, which is therefore a sequence of i.i.d. random variables, with law μ_i.

On the other hand, it follows from (19) and (20) in Lemma 2 and the fact that two jumps of the process Z almost surely never occur at the same time, that for all

n_1, there is $\delta_1 > 0$ sufficiently small, such that for all $n \le n_1$ and $0 < \delta \le \delta_1$,

$$\tau^{X^\delta}_{i,n} - \tau^{X^\delta}_{i,n-1} = \sum_{k=\tau^{Z^\delta}_{i,n-1}}^{\tau^{Z^\delta}_{i,n}} Z^{\delta,(i)}_k \,, \quad n \ge 1\,.$$

From Lemma 3 , the latter is a sequence of i.i.d. geometrically distributed random variables with parameter $1 - \mathbb{P}_{e_i}(Z_\delta = e_i)$. Hence from (25), the sequence

$$\delta \cdot \sum_{k=\delta^{-1}\tau^{Z^\delta}_{i,n-1}}^{\delta^{-1}\tau^{Z^\delta}_{i,n}} Z^{\delta,(i)}_k \,, \quad n \ge 1$$

converges almost surely toward the sequence

$$\int_{\tau^Z_{i,n-1}}^{\tau^Z_{i,n}} Z^{(i)}_t \, dt\,, \quad n \ge 1\,,$$

as $\delta \to 0$, where $(\tau^Z_{i,n})$ is the sequence of jump times of Z^i. Moreover the variables of this sequence are i.i.d. and exponentially distributed with parameter $\lim_{\delta\to 0} \delta^{-1}(1 - \mathbb{P}_{e_i}(Z_\delta = e_i)) = \lambda_i$.

Then since $(X^{\delta,(i)}_n)$ is a random walk, the sequences $(\Delta X^{\delta,(i)}_n)_{n\ge 0}$, $(\tau^{X^\delta}_{i,n})_{n\ge 0}$, $i \in [d]$ are independent. Therefore, from the convergences proved above, the sequences $(\Delta Z^i_n,\, n \ge 0)$ and $(\int_{\tau^Z_{i,n-1}}^{\tau^Z_{i,n}} Z^{(i)}_t \, dt\,, n \ge 1)$ are independent. Then we have proved that the process

$$X^{(i)} := (Z^i(\tau^{(i)}_t),\, t \ge 0)\,, \quad \text{where } \tau^{(i)}_t = \inf\{s : \textstyle\int_0^s Z^{(i)}_u \, du > t\}, \tag{26}$$

is a compound Poisson process with parameter λ_i and jump distribution μ_i. Moreover, it follows from the independence between the random walks $(X^{\delta,(i)}_n)$, $i \in [d]$ that the processes $(Z^i(\tau^{(i)}_t),\, t \ge 0)$, $i \in [d]$ are independent.

Now from part 1 of Theorem 5, if N^δ_i is the total population of type $i \in [d]$ in the forest $D_\delta(F, c_F, \ell_F)$, then $N^\delta = (N^\delta_1, \dots, N^\delta_d)$ is the smallest solution of the system (x, X^δ). Moreover it follows from the construction of $D_\delta(F, c_F, \ell_F)$, that $\lim_{\delta\to 0} \delta N^\delta_i = \int_0^\infty Z^{(i)}_t \, dt := T^{(i)}_x$, almost surely. Note that $T^{(i)}_x$ represents the total length of the branches of type i in the forest (F, c_F, ℓ_F). Then from the definition of $X^{(i)}$ in (26) it appears the these compound Poisson processes are stopped at $T^{(i)}_x$ and that $T_x = (T^{(1)}_x, \dots, T^{(d)}_x)$ satisfies (3).

The fact that (5) is invertible is a direct consequence the first part of the following lemma.

Lemma 4 *Let* $x = (x_1, \dots, x_d) \in \mathbb{Z}_+^d$ *and* $\{(x_t^{i,j},\ t \ge 0),\ i,j \in [d]\}$ *be a family of càdlàg* $\mathbb{Z}$*-valued, step functions such that for* $i \ne j$, $x^{i,j} = 0$, $x^{i,j}$ *are increasing,* $x_0^{i,i} = x_i \ge 0$ *and* $x^{i,i}$ *are downward skip free, i.e.* $x_t^{i,i} - x_{t-}^{i,i} \ge -1$, *for all* $t \ge 0$, *with* $x_{0-}^{i,i} = x_i$. *Then there exists a (unique) time* $t_x = (t_x^{(1)} \dots, t_x^{(d)}) \in \overline{\mathbb{R}}_+^d$ *such that*

$$x_j + \sum_{i=1}^{d} x^{i,j}(t_x^{(i)}) = 0\,, \quad \text{for all } j \text{ such that } t_x^{(j)} < \infty\,, \tag{27}$$

and if t'_x *is any time satisfying (27), then* $t'_x \ge t_x$.

Moreover, the system

$$z_t^{i,j} = \begin{cases} x^{i,j}\left(\int_0^t z_s^{(i)}\,ds\right)\,, & t \ge 0\,, \quad \text{if } i \ne j \\ x_i + x^{i,i}\left(\int_0^t z_s^{(i)}\,ds\right)\,, & t \ge 0\,, \quad \text{if } i = j \end{cases}$$

admits a unique solution $\{(z_t^{i,j},\ t \ge 0),\ i,j \in [d]\}$ *and the system*

$$z_t^{(j)} = x_j + \sum_{i=1}^{d} x^{i,j}\left(\int_0^t z_s^{(i)}\,ds\right)\,, \quad t \ge 0, j \in [d]$$

admits a unique solution $(z_t^{(i)},\ t \ge 0,\ i \in [d])$. *These solutions are càdlàg step functions which are functionals of the stopped functions* $\{(x_t^{i,j},\ 0 \le t \le t_x^{(i)}),\ i,j \in [d]\}$.

Proof The proof of the first part of the lemma can be done by applying Lemma 1 to the discrete time skeleton of the functions $\{(x_t^{i,j},\ t \ge 0),\ i,j \in [d]\}$, exactly as for the proof of the first part of Theorem 1, see Sect. 4.1.

Then the proof of the existence and uniqueness of the solutions of both systems is straightforward. Let τ_n, $n \ge 1$ be the discrete, ordered sequence of jump times of the processes $\{(x_t^{i,j},\ t \ge 0),\ i,j \in [d]\}$ (note that two functions $x^{i,j}$ can jump simultaneously). Then for each of these two systems the solution can be constructed in between the times τ_n and τ_{n+1} in a unique way. □

4.5 Proof of Theorem 3

Let $(\theta_{k,n},\ k, n \ge 1)$ be a family of independent random variables, such that for each k, $\theta_{k,n}$ is uniformly distributed on $[n]$. We assume moreover that the family $(\theta_{k,n},\ k, n \ge 1)$ is independent of the compound Poisson process $X = (X^{(1)}, \dots, X^{(d)})$.

Then let us construct a multitype branching forest with edge lengths in the following way. Let $\tau_n^{(i)}$, $n \geq 1$ be the sequence of ordered jump times of the process $X^{(i)}$. We first start with x_i vertices of type $i \in [d]$. Let i be such that $x_i^{-1}\tau_1^{(i)} = \min\{x_j^{-1}\tau_1^{(j)} : x_j^{-1}\tau_1^{(j)} \leq T_x^{(j)}, j \in [d]\}$. Then we grow the branches issued from each of the $x_1 + \cdots + x_d$ vertices in the same time scale and at time $x_i^{-1}\tau_1^{(i)}$, we choose among the x_i vertices of type i according to θ_{1,x_i} the vertex who gives birth.

Then the construction is done recursively. Let y_j be the number of vertices of type $j \in [d]$ in the forest at time $x_i^{-1}\tau_1^{(i)}$ and let $Y = (Y^{(1)}, \ldots, Y^{(d)})$ be such that $Y^{(j)}$ corresponds to $X^{(j)}$ shifted at time $s_j = x_j x_i^{-1}\tau_1^{(i)}$, i.e. $Y_t^{(j)} = X_{s_j+t}^{(j)}$. Then let $\tau_{Y,n}^{(j)}$, $n \geq 1$ be the sequence of ordered jump times of the process $Y^{(j)}$, and let k such that $y_k^{-1}\tau_{Y,1}^{(k)} = \min\{y_j^{-1}\tau_{Y,1}^{(j)} : x_i^{-1}\tau_1^{(i)} + y_j^{-1}\tau_{Y,1}^{(j)} \leq T_x^{(j)}, j \in [d]\}$. Then we continue the construction of the branches issued from each vertex of type $j \in [d]$ and at time $x_i^{-1}\tau_1^{(i)} + y_k^{-1}\tau_{Y,1}^{(k)}$, we choose among the y_k vertices of type k according to θ_{2,y_k} the vertex who gives birth. This construction is performed until all processes $X^{(i)}$ are stopped at time $T_x^{(i)}$.

It is clear from this construction that the forest which is obtained is a multitype branching forest with edge lengths, with the required distribution and such that the processes $Z^{i,j}$ defined as in Definition 1 with respect to this forest satisfy Eq. (5).

Finally, the fact that equation,

$$(Z_t^{(1)}, \ldots, Z_t^{(d)}) = x + \left(\sum_{i=1}^{d} X^{i,1}_{\int_0^t Z_s^{(i)}\,ds}, \ldots, \sum_{i=1}^{d} X^{i,d}_{\int_0^t Z_s^{(i)}\,ds} \right), \quad t \geq 0$$

admits a unique solution is a direct consequence of the second part of Lemma 4. □

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Copulas with Prescribed Correlation Matrix

Luc Devroye and Gérard Letac

Abstract Consider the convex set R_n of semi positive definite matrices of order n with diagonal $(1, \ldots, 1)$. If μ is a distribution in $\mathbb{R}^n$ with second moments, denote by $R(\mu) \in R_n$ its correlation matrix. Denote by C_n the set of distributions in $[0, 1]^n$ with all margins uniform on $[0, 1]$ (called copulas). The paper proves that $\mu \mapsto R(\mu)$ is a surjection from C_n on R_n if $n \leq 9$. It also studies the Gaussian copulas μ such that $R(\mu) = R$ for a given $R \in R_n$.

1 Foreword

Marc Yor was also an explorer in the jungle of probability distributions, either in discovering a new species, or in landing on an unexpected simple law after a difficult trip on stochastic calculus: we remember his enthousiam after proving that $\left(\int_0^\infty \exp(2B(t) - 2st)dt\right)^{-1}$ is gamma distributed with shape parameter s ('The first natural occurrence of a gamma distribution which is not a chi square!'). Although the authors have been rather inclined to deal with discrete time, common discussions with Marc were about laws in any dimension. Here are some remarks—actually initially coming from financial mathematics—where the beta-gamma algebra (a term coined by Marc) has a role.

2 Introduction

The set of symmetric positive semi-definite matrices $(r_{ij})_{1 \leq i,j \leq n}$ of order n such that the diagonal elements r_{ii} are equal to 1 for all $i = 1, \ldots, n$ is denoted by $\mathcal{R}_n$. Given a random variable $(X_1, \ldots, X_n)$ on $\mathbb{R}^n$ with distribution μ such that the second

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moments of the X_i's exist, its correlation matrix

$$R(\mu) = (r_{ij})_{1\le i,j\le n} \in \mathcal{R}_n$$

is defined by r_{ij} as the correlation of X_i and X_j if $i < j$, and $r_{ii} = 1$. A copula is a probability μ on $[0,1]^n$ such that X_i is uniform on $[0,1]$ for $i = 1,\dots,n$ when $(X_1,\dots,X_n) \sim \mu$. We consider the following problem: given $R \in \mathcal{R}_n$, does there exist a copula μ such that $R(\mu) = R$? The aim of this note is to show that the answer is yes if $n \le 9$. The present authors believe that this limit $n = 9$ is a real obstruction and that for $n \ge 10$ there exists $R \in \mathcal{R}_n$ such that there is no copula μ such that $R(\mu) = R$.

Section 3 gives some general facts about the convex set $\mathcal{R}_n$. Section 4 proves that if $k \ge 1/2$, if $2 \le n \le 5$ and if $R \in \mathcal{R}_n$ there exists a distribution μ on $[0,1]^n$ such that

$$X_i \sim \beta_{k,k}(dx) = \frac{1}{B(k,k)} x^{k-1}(1-x)^{k-1}\mathbf{1}_{(0,1)}(x)dx \tag{1}$$

if $(X_1,\dots,X_n) \sim \mu$. This is an extension of the previous statement since $\beta_{k,k}$ is the uniform distribution if $k = 1$. Section 5 proves the remainder of the theorem, namely for $6 \le n \le 9$. Section 6 considers the useful and classical Gaussian copulas and explains why there are $R \in \mathcal{R}_n$ that cannot be the correlation matrix of any Gaussian copula. The present paper is both a simplification and an extension of the arXiv paper [1].

3 Extreme Points of $\mathcal{R}_n$

The set $\mathcal{R}_n$ is a convex part of the linear space of symmetric matrices of order n. It is clearly closed and if $R = (r_{ij})_{1\le i,j\le n} \in Rc_n$ we have $|r_{ij}| \le 1$: this shows that $\mathcal{R}_n$ is compact. More specifically, $\mathcal{R}_n$ is in the affine subspace of dimension $n(n-1)/2$ of the symmetric matrices of order n with diagonal $(1,\dots,1)$. Its extreme points have been described in [8]. In particular we have

Theorem 1 *If an extreme point of $\mathcal{R}_n$ has rank r then $r(r+1)/2 \le n$.*

We vizualize this statement:

r	1	2	3	4	5	...
$\frac{r(r+1)}{2}$	1	3	6	10	15	...

- Case $n = 2$. As a consequence the extreme points of $\mathcal{R}_2$ are of rank one. They are nothing but the two matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

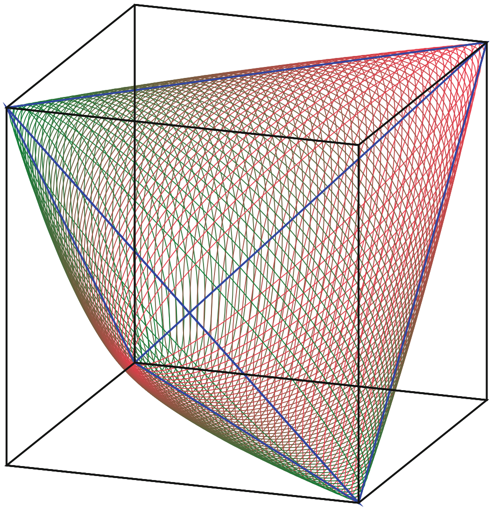

Fig. 1 The space of the three off-diagonal correlation coefficients of a correlation matrix is a convex subset of $[0, 1]^3$

This comes from the fact $R \in \mathcal{R}_2$ of rank one has the form $R = AA^t$ where $A^t = (a_1, a_2)$: since $r_{ii} = 1$ this implies that $a_1^2 = a_2^2 = 1$.

- Case $n \geq 3$. Figure 1 below displays the acceptable values of (x, y, z) when

$$R(x, y, z) = \begin{bmatrix} 1 & z & y \\ z & 1 & x \\ y & x & 1 \end{bmatrix} \tag{2}$$

is positive definite. Its boundary is the part in $|x|, |y|, |z| \leq 1$ of the Steiner surface $1 - x^2 - y^2 - z^2 + 2xyz = 0$.

Proposition 1 *Let $n \geq 3$. Then $R = (r_{ij})_{1 \leq i,j \leq n} \in \mathcal{R}_n$ has rank 2 if and only if there exists n distinct numbers $\alpha_1, \ldots, \alpha_n$ such that $r_{ij} = \cos(\alpha_i - \alpha_j)$.*

Proof $\Rightarrow$: Since R has rank 2 there are two independent vectors A and B of $\mathbb{R}^n$ such that $R = AA^t + BB^t$. Writing $A^t = (a_1, \ldots, a_n)$ and $B^t = (b_1, \ldots, b_n)$ the fact that $r_{ii} = 1$ implies that $a_i^2 + b_i^2 = 1$. Taking $a_i = \cos\alpha_i$ and $b_i = \sin\alpha_i$ gives $r_{ij} = \cos(\alpha_i - \alpha_j)$. $\Leftarrow$: Since only differences $\alpha_i - \alpha_j$ appear in $r_{ij} = \cos(\alpha_i - \alpha_j)$ without loss of generality we take $\alpha_n = 0$ we define $A^t = (\cos\alpha_1, \ldots, \cos\alpha_{n-1}, 1)$ and $B^t = (\sin\alpha_1, \ldots, \sin\alpha_{n-1}, 1)$ and $R = AA^t + BB^t$ is easily checked. $\square$

- Case $n \geq 6$.

Proposition 2 *Let $n \geq 6$. Then $R = (r_{ij})_{1 \leq i,j \leq n} \in \mathcal{R}_n$ has rank 3 if and only if there exist $v_1, \ldots, v_n$ on the unit sphere S_2 of $\mathbb{R}^3$ such that for all $i < j$ we have $r_{ij} = \langle v_i, v_j \rangle$ and such that the system $v_1, \ldots, v_n$ generates $\mathbb{R}^3$.*

Proof The direct proof is quite analogous to Proposition 1: there exist $A, B, C \in \mathbb{R}^n$ such that $R = AA^t + BB^t + CC^t$. and such that A, B, C are independent. Writing

$$[A, B, C] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \dots & \dots & \dots \\ a_n & b_n & c_n \end{bmatrix} \tag{3}$$

the desired vectors are $v_i^t = (a_i, b_i, c_i)$. The converse is similar. □

The following proposition explains the importance of the extreme points of $\mathcal{R}_n$ for our problem.

Proposition 3 *Let $X = (X_1, \dots, X_n) \sim \mu$ and $Y = (Y_1, \dots, Y_n) \sim \nu$ be two random variables of $\mathbb{R}^n$ such that for all $i = 1, \dots, n$ we have $X_i \sim Y_i$ and X_i has second moments and are not Dirac. Then for all $\lambda \in (0, 1)$ we have*

$$R(\lambda\mu + (1 - \lambda)\nu) = \lambda R(\mu) + (1 - \lambda)R(\nu).$$

Proof $X_i \sim Y_i$ implies that the mean m_i and the dispersion σ_i of X_i and Y_i are the same. Denote $D = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$. Since the X_i are not Dirac, D is invertible. Denote by

$$\Sigma(\mu) = (\mathbb{E}((X_i - m_i)(X_j - m_j))_{1 \le i,j \le n} = DR(\mu)D$$

the covariance matrix of μ. Define $Z = (Z_1, \dots, Z_n)$ by $Z = X$ with probability λ and $Z = Y$ with probability $(1 - \lambda)$. Thus $Z \sim \lambda\mu + (1 - \lambda)\nu$. Here again the mean and the dispersion of Z_i are m_i and σ_i. Finally the covariance matrix of Z is $\Sigma(\lambda\mu + (1 - \lambda)\nu) = \lambda\Sigma(\mu) + (1 - \lambda)\Sigma(\nu)$ which gives

$$\begin{aligned} R(\lambda\mu + (1 - \lambda)\nu) &= D^{-1}\Sigma(\lambda\mu + (1 - \lambda)\nu)D^{-1} \\ &= \lambda D^{-1}\Sigma(\mu)D^{-1} + (1 - \lambda)D^{-1}\Sigma(\nu)D^{-1} \\ &= \lambda R(\mu) + (1 - \lambda)R(\nu). \end{aligned}$$

□

Corollary 1 *Let $\nu_1, \dots, \nu_n$ a sequence of probabilities on $\mathbb{R}$ having second moments and denote by M the set of probabilities μ on $\mathbb{R}^n$ such that for all $i = 1, \dots, n$ we have $X_i \sim \nu_i$, with $(X_1, \dots, X_n) \sim \mu$. Then the map from M to $\mathcal{R}_n$ defined by $\mu \mapsto R(\mu)$ is surjective if and only if for any extreme point R of $\mathcal{R}_n$ there exists a $\mu \in M$ such that $R = R(\mu)$.*

Proof $\Rightarrow$ comes from the definition. $\Leftarrow$: Since the convex set $\mathcal{R}_n$ has dimension $N = n(n - 1)/2$, the Caratheodory theorem implies that if $R \in \mathcal{R}_n$ then there exists

$N+1$ extreme points $R_0, \ldots, R_N$ of $\mathcal{R}_n$ and non negative numbers $(\lambda_i)_{i=0}^N$ of sum 1 such that

$$R = \lambda_0 R_0 + \cdots + \lambda_N R_N.$$

From the hypothesis, for $j = 0, \ldots, N$ there exists $\mu_j \in M$ such that $R(\mu_j) = R_j$. Define finally

$$\mu = \lambda_0 \mu_0 + \cdots + \lambda_N \mu_N$$

and apply Proposition 3, we get that $R = R(\mu)$ as desired. □

Comments: With the notation of Corollary 1 and the result of Proposition 3, the map $\mu \mapsto R(\mu)$ from M to $\mathcal{R}_n$ is affine. Consider now the case where for all $i = 1, \ldots, n$, the probability ν_i is concentrated on a finite number of atoms. In this particular case M is a polytope, and therefore its image $R(M)$ is a polytope contained in $\mathcal{R}_n$. For $n = 3$ clearly $\mathcal{R}_3$ is not a polytope (see Fig. 1) and therefore there exists a $R \in \mathcal{R}_3$ which is not in $R(M)$: with discrete margins, you cannot reach an arbitrary correlation matrix.

4 The Case $3 \leq n \leq 5$ and the Gasper Distribution

In this section we prove (Proposition 5) that if $\nu_1 = \ldots = \nu_n = \beta_{kk}$ as defined by (1) and with $k \geq 1/2$, if M is defined as in Corollary 1 and if $R \in \mathcal{R}_n$ has rank 2 one can find $\mu \in M$ such that $R = R(\mu)$. The corollary of this Proposition 1 will be that for any $R \in \mathcal{R}_n$ with $3 \leq n \leq 5$ one can find μ such that $R(\mu) = R$ and such that the margins of μ are β_{kk}. Proposition 4 relies on the existence of a special distribution $\Phi_{k,r}$ called the Gasper distribution in the plane that we are going to describe.

Definition Let $k \geq 1/2$. Let $D > 0$ such that $D^2 \sim \beta_{1,k-\frac{1}{2}}$ (if $k > \frac{1}{2}$) and $D \sim \delta_1$ if $k = \frac{1}{2}$. We assume that D is independent of Θ, uniformly distributed on $(0, 2\pi)$. Let $r \in (-1, 1)$ and $\alpha \in (0, \pi)$ such that $r = \cos\alpha$. The Gasper distribution $\Phi_{k,r}$ is the distribution of $(D\cos\Theta, D\cos(\Theta - \alpha))$.

Proposition 4 *If* $(X_1, X_2) \sim \Phi_{k,r}$ *then* X_1 *and* X_2 *have distribution* $\nu_k(dx) = \frac{1}{B(k,k)}(1-x^2)^{k-1}1_{(-1,1)}(x)dx$ *and correlation* r.

Proof Clearly $X_1 \sim -X_1$ and for seeing that $X_1 \sim \nu_k$ enough is to prove that

$$\mathbb{E}(X_1^{2s}) = \frac{2^{1-2k}}{B(k,k)} \int_{-1}^{1} x^{2s}(1-x^2)^{k-1}dx \tag{4}$$

The right-hand side of (4) is

$$\frac{2^{2-2k}}{B(k,k)}\int_0^1 x^{2s}(1-x^2)^{k-1}dx = 2^{1-2k}\frac{\Gamma(s+\frac{1}{2})\Gamma(2k)}{\Gamma(s+\frac{1}{2}+k)\Gamma(k)}.$$

The left-hand side of (4) is

$$\mathbb{E}(D^{2s})\mathbb{E}((\cos^2\Theta)^s) = \frac{\Gamma(s+1)\Gamma(k+\frac{1}{2})}{\Gamma(s+k+\frac{1}{2})}\times\frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\Gamma(s+1)}.$$

Using the duplication formula $\Gamma(k)\Gamma(k+\frac{1}{2}) = 2^{1-2k}\sqrt{\pi}\,\Gamma(2k)$ proves (4). Since Θ is uniform one has $\cos(\Theta-\alpha)\sim\cos\Theta$ and $X_1\sim X_2$. For showing that the correlation of (X_1,X_2) is $r=\cos\alpha$ we observe that

$$\mathbb{E}(X_1^2) = \mathbb{E}(D^2)\mathbb{E}(\cos^2\Theta) = \frac{1}{2k+1}$$

$$\mathbb{E}(X_1X_2) = \mathbb{E}(D^2)\mathbb{E}(\cos\Theta\cos(\Theta-\alpha)) = \frac{\cos\alpha}{2k+1}.\ \Box$$

Comments: It is worthwhile to say a few things about this Gasper distribution. It is essentially considered in two celebrated papers by George Gasper [3] and [4]. If $k=\frac{1}{2}$ then $\Phi_{\frac{1}{2},r}$ is concentrated on the ellipse $E_r = E_{\cos\alpha}$ parameterized by the circle as

$$\theta\mapsto(x(\theta),y(\theta)) = (\cos\theta,\cos(\theta-\alpha))$$

$$E_r = \{(x,y);(y-xr)^2 = (1-x^2)(1-r^2)\} = \{(x,y);\Delta(x,y,z)=0\}$$

where

$$\Delta(x,y,r) = \det\begin{bmatrix}1 & r & y\\ r & 1 & x\\ y & x & 1\end{bmatrix} = 1-x^2-y^2-r^2+2xyr$$

(Compare with (2)). Now denote by $U_r = \{(x,y);\Delta(x,y,r)>0\}$ the interior of the convex hull of E_r and assume that $k>\frac{1}{2}$. Then Gasper shows that

$$\Phi_{r,k}(dx,dy) = \frac{2k-1}{2\pi}(1-r^2)^{\frac{1}{4}-\frac{k}{2}}\Delta(x,y,r)^{k-\frac{3}{2}}\mathbf{1}_{U_r}(x,y)\,dxdy.$$

The Gasper distribution $\phi_{k,r}$ appears as a Lancaster distribution (see [7]) for the pair (ν_k,ν_k). More specifically consider the sequence $(Q_n)_{n=0}^{\infty}$ of the orthonormal polynomials for the weight ν_k. Thus Q_n is the Jacobi polynomial $P_n^{k-1,k-1}$ normalized such that

$$\int_{-1}^{1} Q_n^2(x)\nu_k(dx) = 1.$$

For $1/2 < k$ denote

$$K(x,y,z) = \sum_{n=0}^{\infty} \frac{Q_n(x)Q_n(y)Q_n(z)}{Q_n(1)}.$$

This series converges if $|x|, |y|, |z| < 1$ and its sum is zero when (x, y) is not in the interior U_r of the ellipse E_r. With this notation we have

$$\phi_{k,r}(dx, dy) = K(x, y, r)\nu_k(dx)\nu_k(dy).$$

This result is essentially due to [3] (with credits to Sonine, Gegenbauer and Moller). See [5, 6] for details.

Proposition 5 *Let $\alpha_1, \ldots, \alpha_n$ which are distinct modulo π. Let*

$$R = (\cos(\alpha_i - \alpha_j)_{1\le i,j\le n} \in \mathcal{R}_n$$

and consider the two-dimensional plane $H \subset \mathbb{R}^n$ generated by $c = (\cos\alpha_1, \ldots, \cos\alpha_n)$ and $s = (\sin\alpha_1, \ldots, \sin\alpha_n)$. Consider the random variable $X = (X_1, \ldots, X_n)$ concentrated on H such that $(X_1, X_2) \sim \Phi_{k,\cos(\alpha_1-\alpha_2)}$ and denote by μ the distribution of X. Then

- *For $1 \le i < j \le n$ we have $(X_i, X_j) \sim \Phi_{k,\cos(\alpha_i-\alpha_j)}$*
- *$R = R(\mu)$.*

Proof Recall that $R \in \mathcal{R}_n$ from Proposition 1. Since $X \in H$ there exists A, B such that for all $i = 1, \ldots, n$ one has $X_i = A\cos\alpha_i + B\sin\alpha_i$. From the fact that $(X_1, X_2) \sim \Phi_{k,\cos(\alpha_1-\alpha_2)}$ we can claim the existence of a (Θ, D) such that Θ is uniform on the circle and is independent of $D > 0$ such that $D^2 \sim \beta_{1,k-\frac{1}{2}}$ and such that

$$(X_1, X_2) \sim D\cos(\Theta - \alpha_1), D\cos(\Theta - \alpha_2)).$$

From an elementary calculation this leads to saying that $(A, B) \sim (D\cos\Theta, D\sin\Theta)$ and finally that

$$(X_1, \ldots, X_n) \sim (D\cos(\Theta - \alpha_1), \ldots, D\cos(\Theta - \alpha_n)).$$

From Proposition 4 this proves the results. □

Conclusion: The previous proposition has shown that for $k \geq \frac{1}{2}$ and for any extremal point R of $\mathcal{R}_n$ there exists a distribution μ_R in $(-1,1)^n$ with margins ν_k and correlation matrix R. From Corollary 1 above, since an arbitrary $R \in \mathcal{R}_n$ is a convex combination $R = \lambda_0 R_0 + \cdots + \lambda_n R_n$ of extreme points R_i of $\mathcal{R}_n$ the distribution $\mu = \lambda_0 \mu_{R_0} + \cdots + \lambda_n \mu_{R_n}$ has margins ν_k and correlation R.

Since ν_k is the affine transformation of $\beta_{k,k}$ by $u \mapsto x = 2u - 1$ this implies that there exists also a distribution in $(0,1)^k$ with margins $\beta_{k,k}$ and correlation matrix R. Since $\beta_{1,1}$ is the uniform distribution on $(0,1)$ a corollary is the existence of a copula with arbitrary correlation matrix R.

Example To illustrate Proposition 5 consider the case $n = 3$ and $R \in \mathcal{R}_3$ defined by

$$R = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

which is an extreme point corresponding to $\alpha_1 = 0, \alpha_2 = 2\pi/3 = -\alpha_3$. This example is important since, as we are going to observe in Sect. 6, it is not possible to find a Gaussian copula having R as correlation matrix. Recall now a celebrated result:

Archimedes Theorem: If X is uniformly distributed on the unit sphere S of the three-dimensional Euclidean space E and if Π is an orthogonal projection of E on a one-dimensional line $F \subset E$ then $\Pi(X)$ is uniform on the diameter with end points $S \cap F$.

Proof While we learnt a different proof in 'classe de Première' in the middle of the fifties, here is a computational proof: let $Z \sim N(0, \mathrm{id}_E)$. Then $X \sim Z/\|Z\|$. Choose orthonormal coordinates (x_1, x_2, x_3) such that F is the x_1 axis. As a consequence of $Z = (Z_1, Z_2, Z_3)$ we have $X_1^2 \sim Z_1^2/(Z_1^1 + Z_2^2 + Z_3^2)$ and since the Z_i^2 are chi square independent with one degree of freedom, this implies that $X_1^2 \sim \beta_{1/2,1}$ which leads quickly to X_1 uniformly distributed on $(-1,1)$since $X_1 \sim -X_1$. □

Proposition 5 offers a construction (see Fig. 2) of a distribution in $C = [-1,1]^3$ with uniform margins ν_1 on $(-1,1)$ as a distribution concentrated on the plane P of equation $x + y + z = 1$. The intersection $C \cap P$ is a regular hexagon. Introduce the disc D inscribed in the hexagon $C \cap P$ and the sphere S admitting the boundary of D as one of its grand circles. Now consider the uniform distribution on S. Denote by μ its orthogonal projection μ on D. Actually any orthogonal projection of μ on a diameter of D is uniform on this diameter, from Archimedes Theorem. Apply this to the three diagonals of the hexagon $C \cap P$: this proves that the three margins of μ are the uniform measure ν_1.

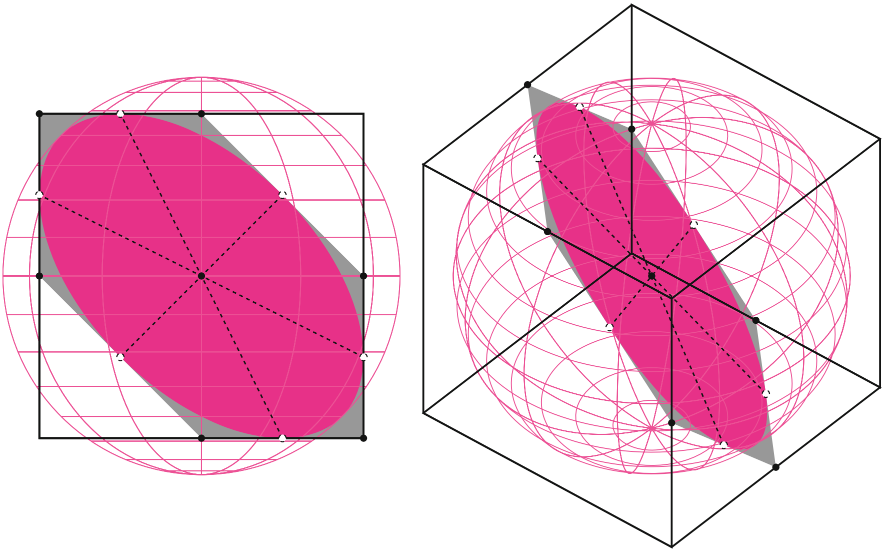

Fig. 2 Illustration of our construction. First take a point uniformly on the surface of the ball. Project it to the plane shown (so that it falls in the *circle*). The three coordinates of that point are each uniformly distributed on $[-1, 1]$

5 The Case $6 \leq n \leq 9$

Proposition 6 *Let $n \geq 6$ and let A, B, C be three independent vectors of $\mathbb{R}^n$ such that $R = [A, B, C][A^t, B^t, C^t]^t = AA^t + BB^t + CC^t$ is a correlation matrix. Let $Y = (U, V, W)$ be uniformly distributed on the unit sphere $S_2 \subset \mathbb{R}^3$ and let μ be the distribution of $X = AU + BV + CW$ in $\mathbb{R}^n$. Then $R(\mu) = R$ and the marginal distributions of μ are ν_1, the uniform distribution in* $(-1, 1)$.

Proof From Archimedes Theorem, U, V and W have distribution ν_1. Furthermore, since the distribution of (U, V) is invariant by rotation, then $(U, V) \sim (D\cos\Theta, D\sin\Theta)$ where $D = \sqrt{U^2 + V^2}$ is independent of Θ uniform on the circle. This implies that $\mathbb{E}(UV) = 0$. Since $\mathbb{E}(U^2) = 1/3$ the covariance matrix of (U, V, W) is $I_3/3$. From this remark, and using the fact that $AU + BV + CW$ is centered, the covariance matrix of $AU + BV + CW$ is

$$\mathbb{E}((AU + BV + CW)(AU + BV + CW)^t) = R/3$$

and this proves $R(\mu) = R$. Finally, using the representation (4) of the matrix $[A, B, C]$ and denoting $v_i = (a_i, b_i, c_i)$ we see that the component X_i of $AU + BV + CW$ is $a_iU + b_iV + c_iW = \langle v_i, Y\rangle$. Since $\|v_i\|^2 = 1$ the random variable X_i is the orthogonal projection of Y on $\mathbb{R}v_i$ and is uniform on $(-1, 1)$ from Archimedes Theorem. □

Comments: The above proposition finishes the proof of the fact that for $n \leq 9$, and if R is an extreme point of $\mathcal{R}_n$ then it is the correlation of some copula. From Proposition 3 this completes the proof that any $R \in \mathcal{R}_n$ is the correlation of a copula for $n \leq 9$. The fact that this result can be extended to $n \geq 10$ is doubtful, since there are $R \in \mathcal{R}_{10}$ of the form $AA^t + BB^t + CC^t + DD^t$ where $A, B, C, D \in \mathbb{R}^{10}$ and the technique of the proof of Proposition 6 seems to indicate that it is impossible. A similar phenomenon seems to occur if we want to construct a distribution μ in $\mathbb{R}^6$ such that $R(\mu)$ has rank 3 and such that the margins of μ are $\beta_{1/2,1/2}$.

Accordingly, we conjecture the existence of $R \in \mathcal{R}_{10}$ which cannot be the correlation of a copula, and we conjecture the existence of $R \in \mathcal{R}_6$ which cannot be the correlation of a distribution whose margins are the arsine distribution.

6 Gaussian Copulas

In this section, we explore the simplest idea for building a copula on $[0, 1]^n$ with a non trivial variance: select a Gaussian random variable $(X_1, \ldots, X_n) \sim N(0, R)$ where $R \in \mathcal{R}_n$, introduce the distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

of $N(0, 1)$ and observe that the law μ of $(U_1, \ldots, U_n) = (\Phi(X_1), \ldots, \Phi(X_n))$ is a copula. A μ which can be obtained in that way is called a Gaussian copula. However its correlation $R^* = R(\mu)$ is not equal to R except in trivial cases.

Therefore this section considers the map from $\mathcal{R}_n$ to itself defined by $R \mapsto R^*$. This map is not surjective: in particular, in comments following Proposition 7 we exhibit a correlation matrix which cannot be the correlation of a Gaussian copula. First we compute R^* by brute force (Proposition 7), getting a result of [2]. We make also two remarks about the expectation of $f_1(X)f_2(Y)$ when (X, Y) is centered Gaussian (Propositions 8 and 9). Proposition 10 leads to a more elegant proof of Proposition 7 by using Hermite polynomials.

Proposition 7 *Let $R = (r_{ij})_{1 \leq i,j \leq n}$ be a correlation matrix, let*

$$(X_1, \ldots, X_n) \sim N(0, R)$$

and let μ be the law of $(U_1, \ldots, U_n) = (\Phi(X_1), \ldots, \Phi(X_n))$. Then

$$R(\mu) = R^* = (g(r_{ij}))_{1 \leq i,j \leq n}$$

where

$$g(r) = \frac{6}{\pi} \arcsin \frac{r}{2}. \tag{5}$$

Proof We begin with a standard calculation. We start with (X, Y) centered Gaussian with covariance

$$\Sigma_r = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}. \tag{6}$$

We now compute the quadruple integral

$$f(r) = \mathbb{E}(\Phi(X)\Phi(Y)) = \int_{\mathbb{R}^4} e^{-\frac{1}{2}(u^2+v^2+\frac{1}{1-r^2}(x^2-2rxy+y^2))}\mathbf{1}_{u<x,v<y}\frac{dxdydudv}{(2\pi)^2\sqrt{1-r^2}}.$$

Performing the change of variables $(x, y, u, v) \mapsto (x, y, x-u, y-v) = (x, y, t, s)$ we get

$$f(r) = \frac{1}{\sqrt{4-r^2}} \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(t^2+s^2)} g(r, t, s) \frac{dtds}{2\pi}$$

with

$$g(r, t, s) = \sqrt{\frac{4-r^2}{1-r^2}} \int_{\mathbb{R}^2} e^{xt+ys-\frac{1}{2(1-r^2)}((2-r^2)x^2-2rxy+(2-r^2)y^2)} \frac{dxdy}{2\pi}.$$

Consider

$$A = \frac{1}{1-r^2} \begin{bmatrix} 2-r^2 & -r \\ -r & 2-r^2 \end{bmatrix}, \; B = \frac{1}{4-r^2} \begin{bmatrix} 2-r^2 & r \\ r & 2-r^2 \end{bmatrix}.$$

Then $B = A^{-1}$, $\det A = \frac{4-r^2}{1-r^2}$ and $\det B = \frac{1-r^2}{4-r^2}$. Therefore $g(r, t, s)$ is the Laplace transform of a centered random Gaussian random variable with covariance matrix B. We get

$$g(r, t, s) = e^{\frac{1}{2(4-r^2)}((2-r^2)t^2+2rts+(2-r^2)s^2)}$$

and therefore

$$f(r) = \frac{1}{\sqrt{4-r^2}} \int_0^\infty \int_0^\infty e^{-\frac{1}{2(4-r^2)}(2t^2-2rts+2s^2)} \frac{dtds}{2\pi}.$$

Now we use the fact that if (T, S) is a Gaussian centered random variable with correlation coefficient $\cos\alpha$ with $0 < \alpha < \pi$ then $\Pr(T > 0, S > 0)$ is explicit. For computing it, just introduce $S' \sim N(0,1)$ independent of T observe that $(T,S) \sim (T, T\cos\alpha + S'\sin\alpha)$ and finally write $(T,S') = (D\cos\Theta, D\sin\Theta)$ where $D > 0$ and Θ are independent and where Θ is uniform on $(0, 2\pi)$. This leads to

$$\Pr(T > 0, S > 0) = \Pr(\cos\Theta > 0,\ \cos(\Theta - \alpha) > 0) = \frac{\pi - \alpha}{2\pi}.$$

We apply this principle to the above integral which can be seen as

$$\Pr(T > 0, S > 0) = f(r)$$

when $(T,S) \sim N(0, \begin{bmatrix} 2 & r \\ r & 2 \end{bmatrix})$. The correlation coefficient of (T,S) is here $\cos\alpha = \frac{r}{2}$ and we finally get

$$f(r) = \frac{1}{2\pi}(\pi - \arg\cos\frac{r}{2}) = \frac{1}{2} - \frac{1}{2\pi}\arg\cos\frac{r}{2}.$$

Now we consider the function

$$g(r) = 12\mathbb{E}((\Phi(X) - 1/2)(\Phi(Y) - 1/2)) = 12f(r) - 3 = \frac{6}{\pi}\arg\sin\frac{r}{2}$$

and the function $T(x) = 2\sqrt{3}(\Phi(x) - 1/2)$. Thus the random variables $T(X)$ and $T(Y)$ are uniform on $(-\sqrt{3}, \sqrt{3})$ with mean 0, variance 1 and correlation $g(r)$. This implies that the correlation between $\Phi(X)$ and $\Phi(Y)$ is $g(r)$. Coming back to the initial $(X_1, \dots, X_n)$ the correlation between $\Phi(X_i)$ and $\Phi(X_j)$ is $g(r)$. □

Comments: The function g is odd and increasing since $g'(r) = \frac{6}{\pi\sqrt{4-r^2}}$. Thus we have $|g(r)| < r < 1$. It satisfies $g(0) = 0$, $g(\pm 1) = \pm 1$, $g'(0) = \frac{3}{\pi}$ and $g'(1) = \frac{2\sqrt{3}}{\pi}$. Finally for $-1 < \rho < 1$ we have

$$\rho = g(r) \Leftrightarrow r = 2\sin\frac{\pi\rho}{6}.$$

Calculation shows that for $-1 < \rho < 1$ we have $0 \leq |2\sin\frac{\pi\rho}{6} - \rho| \leq 0.0180\dots$ therefore the two functions are quite close. It is useful to picture g and its inverse function in Fig. 3. Observe also that if $\rho = -1/2$ we get

$$r = -2\sin\frac{\pi}{12} = -\frac{\sqrt{3}-1}{\sqrt{2}} = -0.51\dots < -1/2.$$

An important consequence is the fact that since $r < -1/2$ the matrix $R(r,r,r)$ of (2) is not a correlation matrix and therefore the correlation matrix $R(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ cannot be the correlation matrix of a Gaussian copula. Falk [2] makes essentially a similar observation.

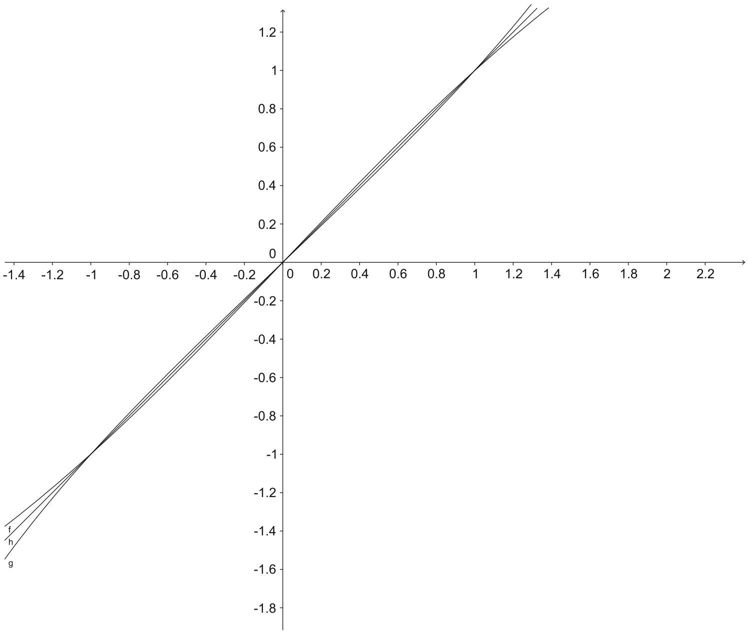

Fig. 3 Graphs of $\rho = g(r) = \frac{6}{\pi} \arg\sin \frac{r}{2}$ and $r = g^{-1}(\rho) = 2\sin \frac{\pi\rho}{6}$

In the sequel, we proceed to a more general study of the correlation between $f_1(Y_1)$ and $f_2(Y_2)$ when $(Y_1, Y_2) \sim N(0, \Sigma_r)$ as defined in (6). We thank Ivan Nourdin for a shorter proof of the following proposition:

Proposition 8 *Given any $r \in [-1, 1]$ consider the Gaussian random variable $(Y_1, Y_2) \sim N(0, \Sigma_r)$. Consider two probabilities ν_1 and ν_2 on $\mathbb{R}$ with respective distribution functions G_1 and G_2. Then the correlation of $G_1(Y_1)$ and $G_2(Y_2)$ is a continuous increasing function of r.*

Proof We use the fact that if $f \in C^2(\mathbb{R}^2)$ then

$$\frac{d}{dr}\mathbb{E}(f(Y_1, Y_2)) = \mathbb{E}(\frac{\partial^2}{\partial y_1 \partial y_2} f(Y_1, Y_2)) \tag{7}$$

To see this recall that if $X \sim N(0, 1)$ then an integration by parts gives

$$\mathbb{E}(X\varphi(X)) = \mathbb{E}(\varphi'(X)). \tag{8}$$

Writing $Y_2 = rY_1 + \sqrt{1-r^2}Y_3$ where Y_1 and Y_3 are independent $N(0,1)$ we get

$$\frac{d}{dr}\mathbb{E}(f(Y_1,Y_2)) = \mathbb{E}((Y_1 - \frac{r}{\sqrt{1-r^2}}Y_3)\frac{\partial}{\partial y_2}f(Y_1,Y_2)) \tag{9}$$

$$= \mathbb{E}(Y_1\frac{\partial}{\partial y_2}f(Y_1,Y_2)) - \frac{r}{\sqrt{1-r^2}}\mathbb{E}(Y_3\frac{\partial}{\partial y_2}f(Y_1,Y_2))$$

$$= \mathbb{E}(Y_1\frac{\partial}{\partial y_2}f(Y_1,Y_2)) - r\mathbb{E}(\frac{\partial^2}{\partial y_2^2}f(Y_1,Y_2)) \tag{10}$$

$$= \mathbb{E}(\frac{\partial^2}{\partial y_1 \partial y_2}f(Y_1,Y_2)) \tag{11}$$

In this sequence of equalities (9) is derivation inside an integral, (10) is the application of (8) to $\varphi(Y_3) = \frac{\partial}{\partial y_2}f(Y_1, rY_1 + \sqrt{1-r^2}Y_3))$ and (11) is the application of (8) to $\varphi(Y_1) = \frac{\partial}{\partial y_2}f(Y_1, rY_1 + \sqrt{1-r^2}Y_3))$ which satisfies

$$\varphi'(Y_1) = \frac{\partial^2}{\partial y_1 \partial y_2}f(Y_1,Y_2) + r\frac{\partial^2}{\partial y_2^2}f(Y_1,Y_2).$$

The application of (7) to the proof of Proposition 1 is clear: if G_1 and G_2 are smooth enough, we take $f(y_1,y_2)$ as $G_1(y_1)G_2(y_2)$. If not we use an approximation. □.

Corollary 2 *Given two probability distributions μ_1 and μ_2 on the real line having second moments with respective distribution functions F_1 and F_2. Given any $r \in [-1,1]$ consider the Gaussian random variable $(Y_1,Y_2) \sim N(0,\Sigma_r)$. Then $(X_1,X_2) = F_1^{-1}(\Phi(Y_1)), F_1^{-1}(\Phi(Y_2))$ has a correlation*

$$\rho = g_{\mu_1,\mu_2}(r)$$

which is a continuous increasing function on $[-1,1]$. In particular if $g_{\mu_1,\mu_2}(-1) = a$ and $g_{\mu_1,\mu_2}(1) = b$ and if $a \le \rho \le b$ there exists a unique $r = f_{\mu_1,\mu_2}(\rho) \in [-1,1]$ such that (X_1,X_2) has correlation ρ.

Proposition 9 *Let (X,Y) be a centered Gaussian variable of $\mathbb{R}^2$ with covariance matrix Σ_r and let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{E}(f(X)) = 0$ and $\mathbb{E}(f(X)^2) = 1$. Then $\mathbb{E}(f(X)f(Y)) = r$ for all $-1 \le r \le 1$ if and only if $f(x) = \pm x$.*

Proof Write $r = \cos\alpha$ with $0 \le \alpha \le \pi$. If X, Z are independent centered real Gaussian random variables with variance 1, then $Y = X\cos\alpha + Z\sin\alpha$ is centered with variance 1, (X,Y) is Gaussian and $\mathbb{E}(XY) = \cos\alpha$. Therefore we rewrite this as

$$\cos\alpha = \int_{\mathbb{R}^2} f(x)f(x\cos\alpha + z\sin\alpha)e^{-\frac{1}{2}(x^2+z^2)}\frac{dxdz}{2\pi} \tag{12}$$

$$rangle = \int_0^\infty \rho e^{-\frac{\rho^2}{2}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho\cos\theta)f(\rho\cos(\alpha-\theta))d\theta\right)d\rho \tag{13}$$

where we have used polar coordinates $x = \rho\cos\theta$ and $z = \rho\sin\theta$ for the second equality. This equality is established for $0 \le \alpha \le \pi$ but it is still correct when we change α into $-\alpha$. Now we introduce the Fourier coefficients for n in the set $\mathbb{Z}$ of relative integers:

$$\hat{f}_n(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho\cos\theta)e^{-in\theta}d\theta.$$

Since f is real we have the Hermitian symmetry $\hat{f}_{-n}(\rho) = \overline{\hat{f}_n(\rho)}$. Expanding the periodic function (13) in Fourier series and considering the Fourier coefficients of $\alpha \mapsto \cos\alpha$ we get for $n \neq \pm 1$

$$\int_0^\infty \rho e^{-\frac{\rho^2}{2}}\hat{f}_n^2(\rho)d\rho = 0 \tag{14}$$

and $\int_0^\infty \rho e^{-\frac{\rho^2}{2}}\hat{f}_{\pm 1}^2(\rho)d\rho = \frac{1}{2}$. Hermitian symmetry implies that $\hat{f}_0^2(\rho)$ is real and since $\int_0^\infty \rho e^{-\frac{\rho^2}{2}}\hat{f}_0^2(\rho)d\rho = 0$ we get that $\hat{f}_0^2(\rho) = 0$ for almost all $\rho > 0$. This is saying that for almost all $\rho > 0$ we have

$$\int_{-\pi}^{\pi} f(\rho\cos\theta)d\theta = 0. \tag{15}$$

Since $\theta \mapsto f(\rho\cos\theta)$ is a real even function we have

$$f(\rho\cos\theta) \sim \sum_{n=1}^{\infty} a_n(\rho)\cos n\theta$$

and the real number $a_n(\rho)$ is equal to $2\hat{f}_n(\rho)$ and to $2\hat{f}_{-n}(\rho)$ which are therefore real numbers. Using (14) they are zero for all $n \neq \pm 1$ and we get almost everywhere that $f(\rho\cos\theta) = a_1(\rho)\cos\theta$ or $f(\rho u) = a_1(\rho)u$ for all $-1 \le u \le 1$. To conclude we write

$$a_1(\rho)u = f(\rho u) = f(\rho_1\frac{\rho}{\rho_1}u) = a_1(\rho_1)\frac{\rho}{\rho_1}u$$

where u is small enough such that $|\frac{\rho}{\rho_1}u| \le 1$. This implies $\frac{a_1(\rho)}{\rho} = \frac{a_1(\rho_1)}{\rho_1}$ which is a constant c by the principle of separation of variables. Therefore $f(x) = cx$ almost everywhere and $\mathbb{E}(f(X)^2) = 1$ implies that $c = \pm 1$. □

For computing expressions like $\mathbb{E}(f_1(Y_1)f_2(Y_2))$ when $(Y_1, Y_2) \sim N(0, \Sigma_r)$ we use the classical fact below:

Proposition 10 *Let $(Y_1, Y_2) \sim N(0, \Sigma_r)$. Let f_1 and f_2 be real measurable functions such that $\mathbb{E}(f_i(Y_i)^2)$ is finite for $i = 1, 2$. Consider the Hermite polynomials $(H_k)_{k=0}^{\infty}$*

defined by the generating function

$$e^{xt-\frac{t^2}{2}} = \sum_{k=0}^{\infty} H_k(x)\frac{t^k}{k!}$$

and the expansions

$$f_1(x) = \sum_{k=1}^{\infty} a_k \frac{H_k(x)}{\sqrt{k!}},\ f_2(x) = \sum_{k=1}^{\infty} b_k \frac{H_k(x)}{\sqrt{k!}}.$$

Then for all $-1 \le r \le 1$

$$\mathbb{E}(f_1(Y_1)f_2(Y_2)) = \sum_{k=1}^{\infty} a_k b_k r^k.$$

Proof Let us compute

$$\mathbb{E}(e^{Y_1 t-\frac{t^2}{2}} e^{Y_2 s-\frac{s^2}{2}}) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^k}{k!}\frac{s^m}{m!}\mathbb{E}(H_k(Y_1)H_m(Y_2)).$$

For this, we use the usual procedure and first write $r = \cos\theta$ with $0 \le \theta \le \pi$. If Y_1, Y_3 are independent centered real Gaussian random variables with variance 1, then $Y_2 = Y_1\cos\theta + Y_3\sin\theta$ is centered with variance 1, (Y_1, Y_2) is Gaussian and $\mathbb{E}(Y_1Y_2) = \cos\theta$. Furthermore a simple calculation using the definition of Y_2 gives

$$\mathbb{E}(e^{Y_1 t-\frac{t^2}{2}} e^{Y_2\, s-\frac{s^2}{2}}) = e^{ts\cos\theta}$$

This shows that $\mathbb{E}(H_k(Y_1)H_m(Y_2)) = 0$ if $k \ne m$ and that $\mathbb{E}(H_k(Y_1)H_k(Y_2)) = k!\cos^k\theta$. From this we get the result. □

Corollary 3 *Let* $p_n \ge 0$ *such that* $\sum_{n=1}^{\infty} p_n = 1$ *and consider the generating function* $g(r) = \sum_{n=1}^{\infty} p_n r^n$. *Let* $R = (r_{ij})_{1\le i,j\le d}$ *in* $\mathcal{R}_n$. *Then* $R^* = (g(r_{ij}))_{1\le i,j\le d}$ *is the covariance matrix of the random variable* $(f(X_1),\dots,f(X_d))$ *where* $(X_1,\dots,X_d)$ *is centered Gaussian with covariance R and where*

$$f(x) = \sum_{n=1}^{\infty} \epsilon_n \sqrt{p_n}\frac{H_n(x)}{\sqrt{n!}}$$

with fixed $\epsilon_n = \pm 1$.

Example We have seen an example of such a function f with $f(x) = T(x) = 2\sqrt{3}(\Phi(x) - 1/2)$ and

$$g(r) = \frac{6}{\pi}\arg\sin\frac{r}{2} = \frac{3}{\pi}\sum_{n=0}^{\infty}(\frac{1}{2})_n\frac{1}{4^n n!}\frac{r^{2n+1}}{2n+1}.$$

Thus $p_{2n+1} = \frac{3}{\pi}(\frac{1}{2})_n \frac{1}{4^n n!}\frac{1}{2n+1}$ and $p_{2n} = 0$. For computing ϵ_n we have really to compute

$$\epsilon_n \frac{\sqrt{p_n}}{\sqrt{n!}} = \mathbb{E}(T(X)\frac{H_n(X)}{n!})$$

For this we watch the coefficient of t^n in the power expansion of

$$\mathbb{E}(T(X)e^{Xt-\frac{t^2}{2}})$$

For this we need

$$\mathbb{E}(\Phi(X)e^{Xt-\frac{t^2}{2}}) = 1 - \Phi(-\frac{t}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n n!}\frac{t^{2n+1}}{2n+1}$$

$$\mathbb{E}(T(X)e^{Xt-\frac{t^2}{2}}) = \sqrt{\frac{3}{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n n!}\frac{t^{2n+1}}{2n+1}$$

Therefore

$$\epsilon_{2n+1}\frac{\sqrt{p_{2n+1}}}{\sqrt{(2n+1)!}} = \sqrt{\frac{3}{\pi}}\frac{(-1)^n}{4^n n!}\frac{1}{2n+1}$$

which shows that $\epsilon_{2n+1} = (-1)^n$.

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Remarks on the HRT Conjecture

Daniel W. Stroock

Abstract Motivated by a conjecture about time-frequency translations of functions, several properties of the Bargmann–Fock space $\mathscr{H}$ and the Segal–Bargmann transform $\mathscr{S}$ are investigated in this note. In particular, a characterization is given of those square integrable functions φ on $\mathbb{R}$ such that $z \in \mathbb{C} \longmapsto \mathscr{S}\varphi(z + \zeta) \in \mathbb{C}$ is in $\mathscr{H}$ for all $\zeta \in \mathbb{C}$.

1 Marc Yor

This article is dedicated to the memory of Marc Yor, a man whom I knew, liked, and admired for more than forty five years. Although this is not an entirely appropriate venue for reminiscences, I cannot resist the temptation to relate a vignette that epitomizes Marc for me. He and I were attending a conference in Japan. The conference was funded by the Taniguchi foundation and, by mathematical standards, quite lavish. Toward the end of the conference, the participants were invited to a formal dinner at a French restaurant in Kyoto. Having not anticipated such an occasion, Marc had failed to bring a tie and told me that he therefore could not attend the dinner. He was quite distraught about this and much relieved when I offered to lend him one of mine. Out of sartorial pride, nearly anyone else would have refused my offer, but Marc's habidashery was just as dowdy as my own and he had no reservations about accepting. At the end of the meal, the chef appeared and introduced himself, asking whether we had enjoyed the food. Apparently he had spent two years at a Parisian culinary school, and, when he discovered that there were French diners in our group, he began talking to us in what he considered to be French. However, none of the French present could understand a word of what he said and sat dumbfounded while he held forth. Sensing that this could become an embarrassing situation, Marc took it upon himself to respond and ended up accompanying the chef back into the kitchen.

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I met Marc at the Paris VI Laboratoire de Probabilités, where he held a position that provided him lots of time to do research and barely enough money to support his family. Like most young French probabilists at that time, Marc was deeply under the influence of P.A. Meyer and his Strasbourg school. However, although he understood Meyer's work as well as or better than anyone else whom I have known, he was launched on a research program that would take him in a quite different direction. Instead of the tomes of articles and books about abstract martingale theory, Marc had on his desk a large paperback book containing page after page of formulae, most of which he had already assimilated and the rest of which he soon would. He was spending more and more of his time mastering special functions and other material related to hard computation. As a result, Marc developed computational powers that very few probabilists possess, and these, combined with his thorough understanding of stochastic processes, became the tools that enabled him to make the remarkable contributions for which he is renowned.

The topic of this note has scant, if any, relationship to probability theory. Indeed, the only connection is the appearance of the Gaussian distribution and Hermite functions in the otherwise basic classical analysis that follows. Nonetheless, although I cannot pretend to share Marc's skills, I think that he might have found the computations here amusing if not profound, even though the conclusions drawn do not make a major contribution to topic under consideration. At best, this note will introduce a new audience to the topic and suggest an alternative approach. However, like Marc's life, it is sadly incomplete.

2 The HRT Conjecture

Let $\lambda_{\mathbb{R}}$ denote Lebesgue measure on $\mathbb{R}$. Given $(\xi, \eta) \in \mathbb{R}^2$, define the unitary map $\tau_{(\xi,\eta)} : L^2(\lambda_{\mathbb{R}}; \mathbb{C}) \longrightarrow L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ by[1]

$$\tau_{(\xi,\eta)}\varphi(t) = e^{i\xi\eta}e^{i\eta t}\varphi(t + 2\xi).$$

The HRT (Heil, Ramanathan, Topiwala) conjecture in [3] states that for distinct points $(\xi_0, \eta_0), \ldots, (\xi_n, \eta_n)$ in $\mathbb{R}^2$ and $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}) \setminus \{0\}$, $\tau_{(\xi_0,\eta_0)}\varphi, \ldots, \tau_{(\xi_n,\eta_n)}\varphi$ are linearly independent elements of $L^2(\lambda_{\mathbb{R}}; \mathbb{C})$. Experts (of which I am not one) in time-frequency analysis, wavelets, and Gabor bases have expended a great deal of energy in attempts to prove this conjecture, and there is good deal of evidence that it is true.[2] However, as yet, only special cases have been verified, and the present

[1]It should be clear that there is nothing sacrosanct about the choice of either the constant factor or scaling in the definition of $\tau_{(\xi,\eta)}$. I made the choice that I did because it simplifies some of the expressions below.

[2]Evidence of their interest in the conjecture can be found by doing a Google search for "HRT conjecture".

article does little to lengthen the list of those cases. My own interest in the conjecture is its intimate relationship to questions about Weyl operators and Fock space.

To provide a brief introduction to the subject, I will begin by presenting some cases in which the conjecture is easily verified. Observe that

$$\tau_{(\xi,\eta)} \circ \tau_{(\xi',\eta')} = e^{i(\xi\eta'-\xi'\eta)} \tau_{(\xi+\xi',\eta+\eta')}. \tag{1}$$

Using this it is easy to check that the conjecture is true when $n = 1$. Indeed, linear dependence would imply the existence of $(c_0, c_1) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $c_0\tau_{(\xi_0,\eta_0)}\varphi = c_1\tau_{(\xi_1,\eta_1)}\varphi$. Clearly $\varphi = 0$ if either c_0 or c_1 is 0. Thus, by (1), we may assume that there exist a $c \neq 0$ and a $(\xi, \eta) \neq (0, 0)$ such that $\varphi(t) = ce^{i\eta t}\varphi(t + \xi)$. If $\xi = 0$, then it is clear that this is possible only if $\varphi = 0$. On the other hand, if $\xi \neq 0$, then we would have that $|\varphi(t)|^2 = |c||\varphi(t + \xi)|^2$. Thus, either $\varphi = 0$ or $|c| = 1$. But if $|c| = 1$, then $|\varphi(t)|^2$ is a periodic, integrable function, and as such must be 0.

Using the following elementary lemma, another case for which one can easily verify their conjecture is when the (ξ_m, η_m)'s lie on either a vertical or a horizontal line.

Lemma 2.1 *Let $z_0, \dots, z_n \in \mathbb{C}$ be distinct and $c_0, \dots, c_n \in \mathbb{C}$ not all 0. Set $\psi(t) = \sum_{m=1}^{n} c_m e^{z_m t}$ for $t \in \mathbb{R}$. Then, for each $t \in \mathbb{R}$, $\varphi^{(k)}(t) \neq 0$ for some $0 \le k \le n$. In particular, the zeroes of φ are isolated and therefore φ vanishes at most countably often.*

Proof Suppose that $\varphi^{(k)}(t) = 0$ for $0 \le k \le n$, and set $c'_m = c_m e^{z_m t}$. Then $\sum_{m=0}^{n} c'_m z_m^k = 0$ for $0 \le k \le n$. But this would mean that the matrix $\big((z_m^k)\big)_{0\le k,m\le n}$ is degenerate, and therefore there must exist $b_0, \dots, b_n \in \mathbb{C}$, not all 0, such that the at most nth order polynomial $p(\zeta) = \sum_{m=0}^{n} b_k\zeta^k$ vanishes at the $(n + 1)$ points $z_0, \dots, z_n$. □

To check the conjecture when the (ξ_m, η_m)'s lie on a vertical line, first note that, by (1), it suffices to handle the case when $\xi_m = 0$ for $0 \le m \le n$. In this case, linear dependence would imply the existence of $(c_0, \dots, c_n) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$ that such that $\varphi(t)\sum_{m=0}^{n} c_m e^{i\eta_m t} = 0$, and since $\sum_{m=0}^{n} c_m e^{i\eta_m t}$ can vanish only countably often, this would mean that $\varphi = 0$ almost everywhere. When the (ξ_m, η_m)'s lie on a horizontal line, again one can use (1) to reduce this time to the case when the η_m's are zero. But then linear dependence implies $\sum_{m=0}^{n} c_m\varphi(t + \xi_m) = 0$ for some $(c_0, \dots, c_n) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$. Since this would mean that $\hat{\varphi}(t)\sum_{m=0}^{n} c_m e^{-i\xi_m t} = 0$, we know that linear dependence implies $\varphi = 0$ in this case also.

In the next section I will develop some machinery that will allow us to recast the HRT in terms of Fock space.

3 Fock Space

Larry Baggett, who introduced me to the HRT conjecture, suggested that it would be helpful to reformulate the conjecture in Fock space, and, as we will see, his was a good suggestion because it allows us to introduce rotations into our arguments. What follows is a brief introduction to Fock space and an isometric, isomorphism between it and $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

Let $\gamma_{\mathbb{C}}$ be the Gaussian measure on the complex plane $\mathbb{C}$ given by $\gamma_{\mathbb{C}}(dz) = (\pi)^{-1}e^{-|z|^2}\,dz$, where dz here and elsewhere denotes Lebesgue measure on $\mathbb{C}$, and let $\mathscr{H}$ be the space of analytic functions f on $\mathbb{C}$ in $L^2(\gamma_{\mathbb{C}};\mathbb{C})$. Using the mean-value property for analytic functions, it is easy to show that $\mathscr{H}$ is closed in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$, and the Hilbert space $\mathscr{H}$ with the inner product structure it inherits from $L^2(\gamma_{\mathbb{C}};\mathbb{C})$ is a called the either *Bargmann-Fock space*[3] or just *Fock space*.

In connection with his program to construct quantum fields, I.M. Segal [4] introduced an isometric isomorphism that takes $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ onto $\mathscr{H}$. To describe the transformation, first observe that $\{z^n : n \geq 0\}$ is an orthogonal basis in $\mathscr{H}$. To check this, use polar coordinates to see that

$$\int z^m\bar{z}^n\,\gamma_{\mathbb{C}}(dz) = \delta_{m,n}2\int_0^\infty r^{2m+1}e^{-r^2}\,dr = \delta_{m,n}\int_0^\infty \rho^m e^{-\rho}\,d\rho = \delta_{m,n}m!,$$

which proves orthogonality and that $\|z^m\|_{\mathscr{H}} = \sqrt{m!}$. Next, define the operator $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, and note that $\partial_z^m e^{-|z|^2} = (-\bar{z})^m e^{-|z|^2}$. Hence, after integrating by parts, one sees that if $f \in \mathscr{H}$, then, by the mean value property for analytic functions,

$$\begin{aligned}\left(f,z^m\right)_{\mathscr{H}} &= \frac{1}{\pi}\int f^{(m)}(z)e^{-|z|^2}\,dz = \frac{1}{\pi}\int_0^\infty re^{-r^2}\left(\int_0^{2\pi} f^{(m)}\left(re^{i\theta}\right)d\theta\right)dr\\ &= 2f^{(m)}(0)\int_0^\infty re^{-r^2}\,dr = f^{(m)}(0),\end{aligned}$$

which means that

$$\sum_{m=0}^\infty \frac{(f,z^m)_{\mathscr{H}}}{m!}z^m = \sum_{m=0}^\infty \frac{f^{(m)}(0)}{m!}z^m = f(z).$$

Next define the Hermite function $h_n(t) = (-1)^n(2\pi)^{-\frac{1}{4}}e^{\frac{t^2}{4}}\partial_t^n e^{-\frac{t^2}{2}}$ for $n \geq 0$. Then it is a familiar fact that $\{h_m : m \geq 0\}$ is an orthogonal basis in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. In addition, $\|h_m\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} = \sqrt{m!}$. The *Segal-Bargmann transform* is the isometric,

[3]Before Segal's paper appeared in print, V. Bargmann discussed the same isomorphism in [1] without, in Segal's opinion, giving sufficient credit to its provenance. Segal never forgave Bargmann. For a more complete account of these ideas, see [2].

isomorphism $\mathscr{S}$ from $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ onto $\mathscr{H}$ that takes $h_m \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ to z^m. That is,

$$\mathscr{S}\varphi(z) = \sum_{m=0}^{\infty} \frac{(\varphi, h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}}{m!} z^m \quad \text{for } \varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$$

and

$$\mathscr{S}^{-1} f = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} h_m \quad \text{for } f \in \mathscr{H}.$$

There is another representation of $\mathscr{S}$. Namely, by Taylor's theorem one knows that

$$\sum_{m=0}^{\infty} \frac{h_m(t)}{m!} z^m = e^{-\frac{z^2}{2}} e^{zt-\frac{t^2}{4}}.$$

Hence,

$$\mathscr{S}\varphi(z) = (2\pi)^{-\frac{1}{4}} \int k(z,t)\varphi(t)\, dt \quad \text{where } k(z,t) = (2\pi)^{-\frac{1}{4}} e^{-\frac{z^2}{2}} e^{zt-\frac{t^2}{4}}. \tag{2}$$

Using this representation for $\mathscr{S}$ it is easy to check that if $(\xi,\eta) \in \mathbb{R}^2$ and $\zeta = \xi + i\eta$, then

$$\mathscr{S} \circ \tau_{(\xi,\eta)} = \mathscr{U}_\zeta \circ \mathscr{S} \quad \text{where } \mathscr{U}_\zeta f(z) = e^{-\frac{|\zeta|^2}{2} - \bar{\zeta} z} f(z+\zeta) \quad \text{for } f \in \mathscr{H}. \tag{3}$$

We will make use of the following elementary application of complex analysis.

Lemma 3.1 *Suppose that $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function, and for each $\zeta \in \mathbb{C}$ define $\varphi^\zeta : \mathbb{R} \longrightarrow \mathbb{C}$ by $\varphi^\zeta(t) = \Phi(t+\zeta)$. If*

$$\int_{\mathbb{R}\times[-R,R]} |\Phi(x+iy)|^2\, dxdy < \infty \quad \textit{for all } R \in (0,\infty),$$

then $\varphi^\zeta \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ and $\widehat{\varphi^\zeta}(\tau) = e^{-i\zeta\tau}\widehat{\varphi^0}(\tau)$ for all $\zeta \in \mathbb{C}$.

Proof Clearly the result for all $\zeta \in \mathbb{C}$ follows as soon as one proves it for purely imaginary ζ's. Thus we will restrict our attention to $\zeta = i\eta$ for $\eta \in \mathbb{R}$.

First observe that $\varphi^{i\eta} \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ for (Lebesgue) almost every $\eta \in \mathbb{R}$. Next, by the mean-value property for analytic functions,

$$|\Phi(z)| \le \frac{1}{\pi r^2} \int_{B(z,r)} |\Phi(x+iy)|\, dxdy \le \frac{1}{\sqrt{\pi} r} \left(\int_{B(z,r)} |\Phi(x+iy)|^2\, dxdy \right)^{\frac{1}{2}},$$

and so Φ is uniformly bounded on $\mathbb{R} \times [-R,R]$ for all $R > 0$.

Now set $\Phi_\epsilon(z) = e^{-\frac{\epsilon z^2}{2}}\Phi(z)$ for $\epsilon > 0$, and define $\varphi_\epsilon^{i\eta}(t) = \Phi_\epsilon(t + i\eta)$. Given $\eta_1 < \eta_2$ and $\tau \in \mathbb{R}$, apply Cauchy's formula to $z \rightsquigarrow e^{i\tau z}\Phi_\epsilon(z)$ on regions of the form $\{x + i\eta : x \in [-r, r] \ \& \ \eta \in [\eta_1, \eta_2]\}$ to obtain that $\widehat{\varphi_\epsilon^{i\eta_2}}(\tau) = e^{(\eta_2-\eta_1)\tau}\widehat{\varphi_\epsilon^{i\eta_1}}(\tau)$ after letting $r \to \infty$. In particular,

$$\widehat{\varphi^{i\eta_2}}(\tau) = e^{(\eta_2-\eta_1)\tau}\widehat{\varphi^{i\eta_1}}(\tau) \quad \text{if } \varphi^{i\eta_1}, \varphi^{i\eta_2} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}). \tag{*}$$

Thus, all that remains is to show that $\varphi^{i\eta} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ for all $\eta \in \mathbb{R}$. To this end, choose $\eta_- \leq \eta + 2$, $\eta_+ \geq \eta + 2$, and and $\{\eta_k : k \geq 1\} \subseteq (\eta - 1, \eta + 1)$ so that $\varphi^{i\eta\pm} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$, $\eta_k \longrightarrow \eta$, and $\varphi^{i\eta_k} \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ for each $k \geq 1$. By (*),

$$\big(e^{(\eta_+-\eta_k)\tau} + e^{(\eta_--\eta_k)\tau}\big)\widehat{\varphi^{i\eta_k}} = \widehat{\varphi^{i\eta_+}} + \widehat{\varphi^{i\eta_-}},$$

and so $|\widehat{\varphi^{i\eta_k}}| \leq |\widehat{\varphi^{i\eta_+}}| + |\widehat{\varphi^{i\eta_-}}|$. But, by Parseval's identity, this shows that

$$\sup_{k\geq 1} \|\varphi^{i\eta_k}\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} < \infty$$

and therefore, by Fatou's lemma, that $\|\varphi^{i\eta}\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} < \infty$.

Our first application of this lemma provides another way of recovering φ from $\mathscr{S}\varphi$.

Lemma 3.2 *Given $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$, set $f = \mathscr{S}\varphi$ and define $\Psi(z) = e^{-\frac{z^2}{2}}f(z)$ for $z \in \mathbb{C}$. If ψ is the restriction of Ψ to $\mathbb{R}$, then $\psi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$,*

$$e^{\frac{\tau^2}{4}}\hat{\psi}(\tau) = \left(\frac{\pi}{2}\right)^{\frac{1}{4}}\hat{\varphi}\left(\tfrac{\tau}{2}\right)$$

and so

$$\int e^{\frac{\tau^2}{2}}|\hat{\psi}(\tau)|^2\,d\tau = (2\pi)^{\frac{3}{2}}\|f\|_{\mathscr{H}}^2.$$

Proof Since

$$\int e^{-2y^2}|\Psi(x + iy)|^2\,dxdy = \pi\|f\|_{\mathscr{H}}^2,$$

Lemma 3.1 applies to Ψ and guarantees that $\psi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$. Next observe that

$$\begin{aligned}\int e^{i\tau\xi}\hat{\psi}(\xi)\,d\xi &= (2\pi)^{-\frac{1}{4}}\int e^{-\frac{t^2}{4}}\varphi(t)\left(\int e^{-\xi^2}e^{\xi(t+i\tau)}\,d\xi\right)dt\\ &= \left(\frac{\pi}{2}\right)^{\frac{1}{4}}e^{-\frac{\tau^2}{4}}\int e^{\frac{i\tau t}{2}}\varphi(t)\,dt = \left(\frac{\pi}{2}\right)^{\frac{1}{4}}e^{-\frac{\tau^2}{4}}\hat{\varphi}\left(\tfrac{\tau}{2}\right).\end{aligned}$$

Finally, the concluding assertion follows easily from this when one applies Parseval's identity and remembers that $\|f\|_{\mathscr{H}} = \|\varphi\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}$. □

Let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$, and for $\omega \in \mathbb{D}$ define $M_\omega f$ for $f \in \mathscr{H}$ by $M_\omega f(z) = f(\omega z)$. Then it is obvious that M_ω takes $\mathscr{H}$ into itself, M_ω is a contraction all $\omega \in \mathbb{D}$, and that M_ω is unitary if and only if $\omega \in \partial\mathscr{H}$. In addition, the operation H_ω on $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ for which $M_\omega \circ \mathscr{S} = \mathscr{S} \circ H_\omega$ is

$$H_\omega\varphi = \sum_{m=0}^{\infty} \frac{\omega^m(\varphi, h_m)_{L^2(\lambda_{\mathbb{R}};\mathbb{C})}}{m!} h_m \quad \text{for } \omega \in \mathbb{D} \text{ and } \varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C}).$$

To develop an expression for H_ω as an integral operator, define

$$H(\omega, t, s) = \frac{1}{\sqrt{2\pi(1-\omega^2)}} \exp\left(-\frac{(1+\omega^2)t^2 - 4\omega ts + (1+\omega^2)s^2}{4(1-\omega^2)}\right) \tag{4}$$

for $(\omega, t, s) \in \operatorname{int}(\mathbb{D}) \times \mathbb{R} \times \mathbb{R}$. One can then check that $H_\omega h_n = \omega^n h_n$. Perhaps the easiest way to do this is to note that, for each $t \in \mathbb{R}$, both $H_\omega h_n(t)$ and $\omega^n h_n(t)$ are analytic functions on $\operatorname{int}(\mathbb{D})$. Hence, it suffices to check the equation when $\omega \in (0, 1)$. To this end, set $u(\tau, t) = \int H\big(e^{-\tau}, t, s\big)h_n(s)\,ds$. Then $u(\tau, t) \longrightarrow h_n(t)$ as $\tau \searrow 0$, and a straight forward computation shows that $\partial_\tau u = \big(\partial_t^2 - \frac{t^2}{2} + \frac{1}{2}\big)u$. At the same time, it is well known that $\big(\partial_t^2 - \frac{t^2}{2} + \frac{1}{2}\big)h_n = -nh_n$, and therefore $e^{-n\tau}h_n(t)$ is another solution to this initial value problem. Hence, by standard uniqueness results for solutions to the Cauchy initial value problem for parabolic equations, $u(\tau, t) = e^{-n\tau}h_n(t)$. When $\omega \in \partial\mathbb{D} \setminus \{1\}$, one can again express H_ω as an integral operator, although the integrand will no longer be integrable in general. Indeed, if

$$H(\omega, t, s) = \frac{1}{\sqrt{2\pi(1-\omega^2)}} \exp\left(-i\frac{\Im(\omega^2)t^2 - 4\Im(\omega)st + \Im(\omega^2)s^2}{2|1-\omega^2|^2}\right), \tag{5}$$

then, by the same argument as one uses to define the Fourier transform, one can show that

$$H_\omega\varphi(t) = \lim_{R\to\infty} \int_{-R}^{R} H(\omega, t, s)\varphi(s)\,ds,$$

where the limit is in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$.

4 Rotations and the HRT Conjecture

Suppose $\omega \in \partial\mathbb{D}$. Then $M_\omega \circ \mathscr{U}_\zeta = \mathscr{U}_{\bar\omega\zeta} \circ M_\omega$, and so, because $M_\omega \circ \mathscr{S} = \mathscr{S} \circ H_\omega$, (3) implies that $H_\omega \circ \tau_{(\xi,\eta)} = \tau_{(\Re(\bar\omega\zeta),\Im(\bar\omega\zeta))} \circ H_\omega$, or, equivalently, that $H_\omega \circ \tau_{(\xi,\eta)} = \tau_{\rho_\omega(\xi,\eta)} \circ H_\omega$ where ρ_ω is the rotation given by the matrix $\begin{pmatrix} \Re(\omega) & \Im(\omega) \\ -\Im(\omega) & \Re(\omega) \end{pmatrix}$.

Lemma 4.1 *Let $(\xi_0, \eta_0), \dots, (\xi_\ell, \eta_\ell)$ be distinct points in $\mathbb{R}^2$, and assume that $(\xi_1, \eta_1), \dots, (\xi_\ell, \eta_\ell)$ lie on a line. Then there are real numbers $\alpha_0, \dots, \alpha_\ell$ and β_0 and an $\omega \in \partial\mathbb{D}$ such that, for any $\varphi \in L^2(\lambda_\mathbb{R}; \mathbb{C})$, $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ are linearly dependent if and only if $\tau_{(\beta_0,\alpha_0)}H_\omega\varphi, \tau_{(0,\alpha_1)}H_\omega\varphi, \dots, \tau_{(0,\alpha_\ell)}H_\omega\varphi$ are.*

Proof Under the stated conditions, there exist $(\xi, \eta) \in \mathbb{R}^2$, $\alpha_0, \dots, \alpha_\ell \in \mathbb{R}$, $\beta_0 \in \mathbb{R}$, and $\theta \in [0, 2\pi)$ such that

$$\begin{aligned} (\xi_0, \eta_0) &= (\xi, \eta) + \alpha_0(-\sin\theta, \cos\theta) + \beta_0(\cos\theta, \sin\theta) \\ (\xi_k, \eta_k) &= (\xi, \eta) + \alpha_k(-\sin\theta, \cos\theta) \quad \text{for } 1 \le k \le \ell. \end{aligned} \tag{6}$$

Using (1), one can replace the (ξ_k, η_k)'s by $(\xi_k, \eta_k) - (\xi, \eta)$, and so we will assume that $(\xi, \eta) = (0, 0)$. Now take $\omega = \cos\theta + i\sin\theta$, and apply the preceding to see that $H_\omega \circ \tau_{(\xi_0,\eta_0)}\varphi = \tau_{(\beta_0,\alpha_0)}H_\omega\varphi$ and $H_\omega \circ \tau_{(\xi_k,\eta_k)}\varphi = \tau_{(0,\alpha_k)}H_\omega\varphi$ for $1 \le k \le \ell$. □

These simple observations allow us to prove the following two results about the HRT conjecture.

Theorem 4.2 *Refer to Lemma 4.1, and assume that, for each $1 \le k \le \ell$, $(\alpha_k - \alpha_0)\beta_0$ is a rational multiple of π. Then $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ are linearly independent for any $\varphi \in L^2(\lambda_\mathbb{R}; \mathbb{C}) \setminus \{0\}$.*

Proof Assume that $\tau_{(\xi_0,\eta_0)}\varphi, \dots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ are linearly dependent. Then, by Lemma 4.1,

$$a_0 H_\omega\varphi(t + 2\beta_0) = \left(\sum_{k=1}^{\ell} a_k e^{i\alpha_k t}\right) H_\omega\varphi(t)$$

for some choice of $\omega \in \partial\mathbb{D}$ and $a_0, \dots, a_\ell \in \mathbb{C}$ that are not all 0. Set $\varphi_\omega = H_\omega\varphi$. If $a_0 = 0$ or $\beta_0 = 0$, then, by Lemma 2.1, φ_ω must vanish off a countable set, which means that, as an element of $L^2(\lambda_\mathbb{R}; \mathbb{C})$, it, and therefore φ itself, is 0. Thus, we can assume that $a_0 = 1$ and that $\beta_0 \ne 0$. In fact, because the argument is essentially the same when $\beta_0 < 0$, we will assume that $\beta_0 > 0$. In this case we have that

$$\varphi_\omega(t + 2\beta_0) = \chi(t)\varphi_\omega(t), \quad \text{where } \chi(t) = \sum_{k=1}^{\ell} a_k e^{i(\alpha_k - \alpha_0)t}. \tag{7}$$

Using induction on $n \geq 1$, it follows that

$$\varphi_\omega(t + 2n\beta_0) = \chi_n(t)\varphi_\omega(t) \quad \text{where } \chi_n(t) = \prod_{m=0}^{n-1} \chi\big(t + 2m\beta_0\big).$$

Thus, for any $n \geq 1$,

$$\int_{\mathbb{R}} |\varphi_\omega(t)|^2\,dt = \int_{\mathbb{R}} |\varphi_\omega(t + 2n\beta_0)|^2\,dt = \int_{\mathbb{R}} |\chi_n(t)|^2|\varphi_\omega(t)|^2\,dt. \tag{*}$$

Because of the rationality hypothesis, there exists a positive integer q such that $q(\alpha_k - \alpha_0)\beta_0$ is an integer multiple of π, and therefore $\chi_{nq}(t) = \chi_q(t)^n$. Hence, for any $m \in \mathbb{Z}$,

$$\begin{aligned}\int_{2mq\beta_0}^{\infty} |\varphi_\omega(t)|^2\,dt &= \sum_{n=0}^{\infty} \int_{2mq\beta_0}^{2(m+1)q\beta_0} |\varphi_\omega(t + 2nq\beta_0)|^2\,dt \\ &= \int_{2mq\beta_0}^{2(m+1)q\beta_0} \left(\sum_{n=0}^{\infty} |\chi_q(t)|^{2n}\right) |\varphi_\omega(t)|^2\,dt,\end{aligned}$$

which is possible only if $|\chi_q(t)| < 1$ for almost every $t \in \mathbb{R}$ at which $\varphi_\omega(t) \neq 0$. Plugging this into (*) and applying Lebesgue's dominated theorem, we conclude that

$$\int_{\mathbb{R}} |\varphi_\omega(t)|^2\,dt = \int_{\mathbb{R}} |\chi_q(t)|^{2n}|\varphi_\omega(t)|^2\,dt \longrightarrow 0 \text{ as } n \to \infty.$$

□

Theorem 4.3 *Again refer to Lemma 4.1, let* $(\xi_0, \eta_0), \ldots, (\xi_\ell, \eta_\ell)$ *be given by* (6) *with some* $\beta_0 \neq 0$, *and determine* $(\tilde{\xi}_0, \tilde{\eta}_0)$ *by the first line of* (6) *with* $-\beta_0$ *replacing* β_0. *Assume that, for each* $1 \leq j, k \leq \ell$ *and* $1 \leq j', k' \leq \ell$, $\alpha_j + \alpha_k = \alpha_{j'} + \alpha_{k'}$ *if and only if* $\{j, k\} = \{j', k'\}$. *Then, for each* $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C}) \setminus \{0\}$, *either* $\tau_{(\xi_0,\eta_0)}\varphi, \ldots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ *are linearly independent or* $\tau_{(\tilde{\xi}_0,\tilde{\eta}_0)}\varphi, \tau_{(\xi_1,\eta_1)}\varphi, \ldots, \tau_{(\xi_\ell,\eta_\ell)}\varphi$ *are linearly independent.*

Proof The only case needing comment is when, after taking the steps used in the proof of Theorem 4.2, one arrives at

$$\varphi_\omega(t + 2\beta_0) = \chi(t)\varphi_\omega(t) \quad \text{and} \quad \varphi_\omega(t - 2\beta_0) = \tilde{\chi}(t)\varphi_\omega(t),$$

where $\chi(t) = \sum_{k=1}^{\ell} a_k e^{i(\alpha_k - \alpha_0)t}$ and $\tilde{\chi}(t) = \sum_{k=1}^{\ell} \tilde{a}_k e^{i(\alpha_k - \alpha_0)t}$. But then

$$\varphi_\omega(t) = \chi(t - 2\beta_0)\varphi_\omega(t - 2\beta_0) = \chi(t - 2\beta_0)\tilde{\chi}(t)\varphi_\omega(t),$$

and, since $\varphi_\omega(t) \neq 0$ for uncountably many t's, one can apply Lemma 2.1 to conclude that $\chi(t-2\beta_0)\tilde{\chi}(t) = 1$ for all $t \in \mathbb{R}$. Taking $b_k = a_k e^{-i2(\alpha_k+\alpha_0)\beta_0}$ and $\tilde{b}_k = \tilde{a}_k$, we therefore have that

$$\sum_{k=1}^{\ell} b_k\tilde{b}_k e^{i2(\alpha_k-\alpha_0)t} + \frac{1}{2}\sum_{1\le j\neq k\le \ell}(b_j\tilde{b}_k + b_k\tilde{b}_j)e^{i(\alpha_j+\alpha_k-2\alpha_0)t} = 1. \qquad (*)$$

By Lemma 2.1, (*) cannot hold unless either $\alpha_m = \alpha_0$ for some $1 \le m \le \ell$ or $\alpha_m + \alpha_n = 2\alpha_0$ for some $1 \le m \neq n \le \ell$.

Suppose $\alpha_m = \alpha_0$. Then, by Lemma 2.1 and the distinctness of the $\alpha_1, \dots, \alpha_\ell$ and of the sums $\alpha_j + \alpha_k$, $b_k\tilde{b}_k = 0$ for $k \neq m$ and $b_j\tilde{b}_k = -b_k\tilde{b}_j$ for all $j \neq k$. Thus, if $b_m = \tilde{b}_m = 0$ but $\tilde{b}_k \neq 0$ for some $k \neq m$, then $b_k = 0$ and $b_j = -\frac{b_k\tilde{b}_j}{\tilde{b}_k} = 0$ for $j \neq k$, which means that χ must vanish. That is, either $\tilde{b}_k = 0$ for all k, in which case $\tilde{\chi}$ vanishes, or χ vanishes. Similarly, if b_m or $\tilde{b}_m$ is 0, then the equation $b_k\tilde{b}_m = -b_m\tilde{b}_k$ implies either χ or $\tilde{\chi}$ must vanish also. But if either χ or $\tilde{\chi}$ vanishes, then so does φ.

Finally, suppose that $\alpha_k \neq \alpha_0$ for any $1 \le k \le \ell$ but that $\alpha_m + \alpha_n = 2\alpha_0$ for some $m \neq n$. Then $b_k\tilde{b}_k = 0$ for all $1 \le k \le \ell$ and $b_j\tilde{b}_k = -b_k\tilde{b}_j$ if $j \neq k$ and $\{j,k\} \neq \{m,n\}$. Now assume that $\tilde{b}_k \neq 0$ for some k. Then, $b_j = 0$ for all j if $k \notin \{m,n\}$, $b_j = 0$ for $j \neq n$ if $k = m$, and $b_j = 0$ for $j \neq m$ if $k = n$. Hence, either $b_k \neq 0$ or $|\chi|$ would have to be constant, which would lead quickly to the conclusion that $\varphi = 0$. When $b_k \neq 0$, the same reasoning shows that $|\tilde{\chi}|$ is constant and therefore that $\varphi = 0$. □

It may be of some interest to point out that, under the conditions on the α_k's in Theorem 4.3, the conclusion can be formulated as the statement that not both the families $\tau_{(\beta_0,\alpha_0)}\varphi_\omega, \tau_{(0,\alpha_1)}\varphi_\omega \dots, \tau_{(0,\alpha_\ell)}\varphi_\omega$ and $\tau_{(\beta_0,\alpha_0)}\overline{\varphi_\omega}, \tau_{(0,\alpha_1)}\overline{\varphi_\omega} \dots, \tau_{(0,\alpha_\ell)}\overline{\varphi_\omega}$ can be linearly dependent unless $\varphi = 0$.

5 Baggett's Idea

The use to which Baggett put Fock space in connection with the HRT conjecture is the following. Suppose that $\zeta_0, \dots, \zeta_n$ are distinct elements of $\mathbb{C}$ and that $\mathscr{U}_{\zeta_0}f, \dots, \mathscr{U}_{\zeta_n}f$ are linearly dependent for some $f \in \mathscr{H}$. Then, without loss in generality, we may assume that, for some choice of $a_1, \dots, a_n \in \mathbb{C}$, $\mathscr{U}_{\zeta_0}f = \sum_{m=1}^n a_m\mathscr{U}_{\zeta_m}f$, where $\Re(\zeta_0) \le \Re(\zeta_m)$ and $\Re(\zeta_m) = \Re(\zeta_0) \implies \Im(\zeta_m) > \Im(\zeta_0)$ for each $1 \le m \le n$. Hence, if $\tilde{f} = \mathscr{U}_{-\zeta_0}f$, $\tilde{a}_m = e^{i\Im(\zeta_0\bar{\zeta}_m)}a_m$, and $\tilde{\zeta}_m = \zeta_m - \zeta_0$ for $1 \le m \le n$, then $\tilde{f} = \sum_{m=1}^n \tilde{a}_m\mathscr{U}_{\tilde{\zeta}_m}f$ where $\tilde{\zeta}_m = \xi_m + i\eta_m$ have the property that

$$\xi_m \ge 0 \quad \text{and} \quad \xi_m = 0 \implies \eta_m > 0.$$

In particular, there exist $\alpha > 0$ and $\epsilon > 0$ such that $\alpha\xi_m + \eta_m \geq \epsilon$ for all $1 \leq m \leq n$. Now choose $\beta > 0$ so that $e^{\beta\epsilon} \geq 2\sum_{m=1}^{n} |\tilde{a}_m|$, and set $\zeta = \beta(\alpha + i)$. Then, the translate $T_\zeta \tilde{f}$ given by $T_\zeta \tilde{f}(z) = \tilde{f}(z + \zeta)$ satisfies

$$T_\zeta \tilde{f} = \sum_{m=1}^{n} e^{-\zeta\overline{\tilde{\zeta}_m}} \tilde{a}_m \mathscr{U}_{\tilde{\zeta}_m} T_\zeta \tilde{f}.$$

Since

$$\sum_{m=1}^{n} \left| e^{-\zeta\overline{\tilde{\zeta}_m}} \tilde{a}_m \right| \leq \frac{1}{2},$$

it follows that $\|T_\zeta \tilde{f}\|_{\mathscr{H}} \leq \frac{1}{2}\|T_\zeta \tilde{f}\|_{\mathscr{H}}$ and therefore that $f = 0$ if $T_\zeta \circ \mathscr{U}_{-\zeta_0} f \in \mathscr{H}$, which, since

$$\begin{aligned}\int |T_\zeta \circ \mathscr{U}_{-\zeta_0} f|^2 e^{-|z|^2}\, dz &= e^{2\Re(\bar{\zeta}_0\zeta)} \int |f(z - \zeta_0 + \zeta)|^2 e^{-|z-\zeta_0|^2}\, dz \\ &= e^{2\Re(\bar{\zeta}_0\zeta)} \int |f(z + \zeta)|^2 e^{-|z|^2}\, dz\end{aligned}$$

is tantamount to the assumption that $T_\zeta f \in \mathscr{H}$. Thus Baggett's argument proves that the Fock space formulation of HRT conjecture holds for $f \in \mathscr{H}$ with the property that, for each $\zeta \in \mathbb{C}$, the translate $T_\zeta f$ of f is again in $\mathscr{H}$.

Unfortunately, although this property holds for lots of $f \in \mathscr{H}$, it definitely does hold for all. For example, if $f(z) = e^{\frac{z^2}{2}} \frac{\sin z}{z}$, then $f \in \mathscr{H}$ but $T_\zeta f \notin \mathscr{H}$ unless $\Re(\zeta) = 0$. On the other hand, if $f \in \mathscr{H} \cap L^{2p}(\gamma_{\mathbb{C}}; \mathbb{C})$ for some $p > 1$, then $T_\zeta f \in \mathscr{H}$ for all $\zeta \in \mathbb{C}$. Indeed, writing $z = x + iy$ and $\zeta = \xi + i\eta$, one has that

$$\begin{aligned}\int |f(z+\zeta)|^2 e^{-|z|^2}\, dz &= e^{-|\zeta|^2} \int |f(z)|^2 e^{2(x\xi - y\eta)} e^{-|z|^2}\, dz \\ &\leq e^{-|\zeta|^2} \left(\int |f(z)|^{2p} e^{-|z|^2}\, dz \right)^{\frac{1}{p}} \left(\int e^{2p'(x\xi - y\eta)} e^{-|z|^2}\, dz \right)^{\frac{1}{p'}} \\ &= e^{(p'-1)|\zeta|^2} \|f\|^2_{L^p(\gamma_{\mathbb{C}};\mathbb{C})},\end{aligned}$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. In this connection, notice that if $\theta \in (0, 1)$, then $M_\theta f \in L^{2p}(\mu_{\mathbb{C}}; \mathbb{C})$ for $1 \leq p < \theta^{-2}$. Thus, since $M_\theta f \longrightarrow f$ in $\mathscr{H}$ as $\theta \nearrow 1$ and $M_\theta \circ \mathscr{S}\varphi = \mathscr{S} \circ H_\theta \varphi$, we know that the HRT conjecture holds for a dense set of $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$.

In view of the preceding considerations, it would be interesting to characterize those $\varphi \in L^2(\lambda_{\mathbb{R}}; \mathbb{C})$ for which $T_\zeta \circ \mathscr{S}\varphi \in \mathscr{H}$ for all $\zeta \in \mathscr{H}$, and the rest of this article is devoted to finding such a characterization.

6 Translation in $\mathscr{H}$

Suppose that $f \in \mathscr{H}$, and set $\varphi = \mathscr{S}^{-1}f$. Then, after some elementary manipulations, one sees that

$$T_\zeta f(z) = (2\pi)^{-\frac{1}{4}} e^{\frac{\zeta^2}{4}} e^{-\frac{z^2}{2}} \int e^{z(t-\zeta)-\frac{(t-\zeta)^2}{4}} e_{\frac{\zeta}{2}}(t-\zeta)\varphi(t)\,dt, \tag{8}$$

where $e_\zeta(t) = e^{\zeta t}$. When $\zeta = \xi \in \mathbb{R}$, (8) implies that

$$T_\xi f(z) = e^{\frac{\xi^2}{4}} \int k(z,t) e_{\frac{\xi}{2}}(t)\varphi(t+\xi)\,dt,$$

and so $T_\xi f \in \mathscr{H}$ if and only if $e_{\frac{\xi}{2}}\varphi \in L^2(\lambda_\mathbb{R};\mathbb{C})$, in which case one has that

$$T_\xi f = e^{\frac{\xi^2}{4}} \mathscr{S}\big(e_{\frac{\xi}{2}}\varphi^\xi\big) \quad \text{where } \varphi^\xi(t) = \varphi(\xi+t). \tag{9}$$

When ζ is not real, we will use Lemma 3.1 to justify the preceding change of variables.

Notice that the function $H(\omega,t,s)$ in (4) extends as an analytic function on $\mathrm{int}(\mathbb{D}) \times \mathbb{C} \times \mathbb{C}$ and that

$$\int_\mathbb{R} |H(\theta,w,s)|^2\,ds = \frac{1}{\sqrt{1+\theta^2}} \exp\left(\frac{1+\theta^2}{2(1-\theta^2)}v^2 - \frac{1-\theta^2}{2(1+\theta^2)}u^2\right) \quad \text{for } \theta \in (0,1) \text{ and } w = u+iv. \tag{10}$$

Hence, if

$$H_\theta\varphi(w) = \int H(\theta,w,s)\varphi(s)\,ds \quad \text{for } \varphi \in L^2(\lambda_\mathbb{R};\mathbb{C}), \theta \in (0,1), \text{ and } w \in \mathbb{C},$$

then, for each $\theta \in (0,1)$, Lemma 3.1 applies to the function

$$\Phi(w) = e^{z(w-\zeta)-\frac{(w-\zeta)^2}{4}} e_{\frac{\zeta}{2}}(w-\zeta) H_\theta \circ \mathscr{S}^{-1}f(w)$$

and, together with (8), allows us to conclude that

$$T_\zeta \circ M_\theta f = e^{\frac{\zeta^2}{4}} \mathscr{S}\big(e_{\frac{\zeta}{2}}(H_\theta \circ \mathscr{S}^{-1}f\big) \tag{11}$$

for $f \in \mathscr{H}$ and $\theta \in (0,1)$.

Theorem 6.1 *Let $f \in \mathscr{H}$, and set $\varphi = \mathscr{S}^{-1}f$. Then $T_\zeta f \in \mathscr{H}$ if and only if*

$$\varliminf_{\theta \nearrow 1} \left\| e_{\frac{\zeta}{2}} (H_\theta \varphi)^\zeta \right\|_{L^2(\lambda_{\mathbb{R}};\mathbb{C})} < \infty. \tag{12}$$

Moreover, if (12) holds, then $e_{\frac{\zeta}{2}}(H_\theta\varphi)^\zeta \longrightarrow e^{-\frac{\zeta^2}{4}} \mathscr{S}^{-1} \circ T_\zeta f$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ as $\theta \nearrow 1$.

Proof Set $f_\theta = M_\theta f$, $\varphi = \mathscr{S}^{-1}f$, and $\varphi_\theta = H_\theta\varphi$. By (11),

$$T_\zeta f_\theta = e^{\frac{\zeta^2}{4}} \mathscr{S}\big(e_{\frac{\zeta}{2}} \varphi_\theta^\zeta\big). \tag{*}$$

First suppose that $T_\zeta f \in \mathscr{H}$. If we show that $T_\zeta f_\theta \longrightarrow T_\zeta f$ in $\mathscr{H}$, then, by (*) we will know that $(e_{\frac{\zeta}{2}}\varphi_\theta)^\zeta \longrightarrow e^{-\frac{\zeta^2}{4}} \mathscr{S}^{-1} \circ T_\zeta f$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Because $T_\zeta f_\theta(z) = f(\theta z + \theta\zeta) \longrightarrow T_\zeta f(z)$ for all $z \in \mathbb{C}$, $T_\zeta f_\theta \longrightarrow T_\zeta f$ in $\mathscr{H}$ will follow once we show that $\|T_\zeta f_\theta\|_{\mathscr{H}} \longrightarrow \|T_\zeta f\|_{\mathscr{H}}$. To this end, observe that

$$\int e^{-|z|^2} |f(\theta z + \theta\zeta)|^2 \, dz = \theta^{-2} \int e^{-|\theta^{-1}(z+(1-\theta)\zeta)|^2} |T_\zeta f(z)|^2 \, dz.$$

Thus, since

$$|\theta^{-1}(z + (1-\theta)\zeta)|^2 \geq |z|^2 - \frac{1-\theta}{1+\theta}|\zeta|^2,$$

the desired convergence follows from Lebesgue's dominated convergence theorem.

Finally, assume that (12) holds. Then, since $T_\zeta f_\theta \longrightarrow T_\zeta f$ pointwise, Fatou's lemma says that $T_\zeta f \in \mathscr{H}$.

□

Corollary 6.2 *Let $f \in \mathscr{H}$, and set $\varphi = \mathscr{S}^{-1}f$. If $\xi \in \mathbb{R}$, then $T_\xi f \in \mathscr{H}$ if and only if $e_{\frac{\xi}{2}}\varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$, in which case (9) holds. If $\eta \in \mathbb{R}$, then $T_{i\eta} f \in \mathscr{H}$ if and only if*

$$\int e^{2\eta\tau} |\hat{\varphi}(\tau)|^2 \, d\tau < \infty,$$

in which case $T_{i\eta} f = e^{-\frac{\eta^2}{4}} \mathscr{S}\big(e_{\frac{i\eta}{2}} \varphi^{i\eta}\big)$, where

$$\varphi^{i\eta}(t) = \frac{1}{2\pi} \int e^{-it\tau} e^{\eta\tau} \hat{\varphi}(\tau) \, d\tau$$

is the inverse Fourier transform of $e_\eta\hat{\varphi}$. *In particular, if* $\zeta = \xi + i\eta$, *then both* $T_\xi f$ *and* $T_\zeta f$ *are in* $\mathscr{H}$ *if and only if* $e_{\frac{\xi}{2}}\varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ *and*

$$\int e^{2\eta\tau}\left|\widehat{e_{\frac{\xi}{2}}\varphi}(\tau)\right|^2 d\tau < \infty,$$

in which case $T_\zeta f = e^{\frac{\zeta^2}{4}}\mathscr{S}\big(e_{\frac{\zeta}{2}}\varphi^\zeta\big)$ *where*

$$\varphi^\zeta(t) = \frac{1}{2\pi}\int e^{-i(t+\zeta)\tau}\hat{\varphi}(\tau)\,d\tau$$

is the inverse Fourier transform of $e_{-i\zeta}\hat{\varphi}$.

Proof The first assertion was covered in the discussion leading to (9).

To prove the second assertion, first assume that $T_{i\eta}f \in \mathscr{H}$. Set $\Phi_\theta = H_\theta\varphi$ and $\varphi^\zeta_\theta(t) = \Phi_\theta(\zeta + t)$ for $\zeta \in \mathbb{C}$. Then, by Lemma 3.1, $\widehat{\varphi^\zeta_\theta}(\tau) = e^{-i\zeta\tau}\widehat{\varphi_\theta}(\tau)$, and so

$$\varphi^\zeta_\theta(t) = \frac{1}{2\pi}\int e^{-i(t+\zeta)\tau}\widehat{\varphi_\theta}(\tau)\,d\tau.$$

Hence, by (11) and Parseval's identity,

$$\|T_{i\eta}\circ M_\theta f\|^2_{\mathscr{H}} = \frac{e^{-\frac{\eta^2}{2}}}{2\pi}\int e^{2\eta\tau}|\widehat{\varphi_\theta}(\tau)|^2\,d\tau,$$

and so, by Fatou's lemma and Theorem 6.1,

$$\int e^{2\eta\tau}|\hat{\varphi}(\tau)|^2\,d\tau \le \varliminf_{\theta\nearrow 1}\int e^{2\eta\tau}|\widehat{\varphi_\theta}(\tau)|^2\,d\tau = 2\pi e^{\frac{\eta^2}{2}}\varliminf_{\theta\nearrow 1}\|T_{i\eta}\circ M_\theta f\|_{\mathscr{H}} < \infty.$$

Finally, assume that $\eta \in (0,\infty)$ and that $\int e^{2\eta\tau}|\hat{\varphi}(\tau)|^2\,d\tau < \infty$. Because

$$\left|e^{-i\omega\tau}\hat{\varphi}(\tau)\right| \le \mathbf{1}_{[0,\infty)}(\tau)e^{|\Im(\omega)|\tau}|\hat{\varphi}(\tau)| + \mathbf{1}_{(-\infty,0]}(\tau)|\hat{\varphi}(\tau)|, \tag{*}$$

it is clear that, for each $0 < \delta < \frac{\eta}{2}$, $\tau \rightsquigarrow e^{-i\omega\tau}\hat{\varphi}(\tau)$ is uniformly integrable for ω with $\Im(\omega) \in [\delta, \eta - \delta]$ and therefore that

$$\omega \rightsquigarrow \Phi(\omega) = \frac{1}{2\pi}\int e^{-i\omega\tau}\hat{\varphi}(\tau)\,d\tau \quad \text{for } \omega \in \mathbb{C}\big([0,\eta)\big).$$

is analytic for ω with $\Im(\omega) \in [0,\eta)$. Furthermore, if $\varphi^\zeta(t) = \Phi(\zeta + t)$ for ζ with $\Im(\zeta) \in (0,\eta)$, then $\widehat{\varphi^{i\eta'}}(\tau) = e^{\eta'\tau}\hat{\varphi}(\tau)$ for $\eta' \in (0,\eta)$, and so by (*), Lebesgue's dominated convergence theorem, and Parseval's identity, $\varphi^{i\eta'} \longrightarrow \varphi^{i\eta}$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$

as $\eta' \nearrow \eta$ and $\varphi^{i\eta'} \longrightarrow \varphi$ in $L^2(\lambda_{\mathbb{R}};\mathbb{C})$ as $\eta' \searrow 0$. Finally, if $0 < \eta_1 < \eta_2 < \eta$, then, by Lemma 3.1,

$$\int e^{z(t-i(\eta_2-\eta_1))-\frac{(t-i(\eta_2-\eta_1))^2}{4}} e_{\frac{i\eta}{2}}\big(t-i(\eta_2-\eta_1)\big)\varphi^{i\eta_1}(t)\,dt = \int e^{zt-\frac{t^2}{4}} e_{\frac{i\eta}{2}}(t)\varphi^{i\eta_2}(t)\,dt.$$

Thus, by (8), we get the desired result by letting $\eta_1 \searrow 0$ and $\eta_2 \nearrow \eta$. When $\eta < 0$, one can apply the preceding to $\omega \rightsquigarrow \Psi(-\omega)$.

□

The following corollary follows easily from the preceding and the fact that translates of $f \in \mathscr{H}$ are also in $\mathscr{H}$ if $f \in L^{2p}(\gamma_{\mathbb{C}};\mathbb{C})$ for some $p > 1$.

Corollary 6.3 *Suppose that $\varphi \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$. Then $T_\zeta \circ \mathscr{S}\varphi \in \mathscr{H}$ for all $\zeta \in \mathbb{C}$ if and only if there exists an analytic function $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that (cf. the notation in Lemma 3.1) $\zeta \in \mathbb{C} \longmapsto e_{\frac{\zeta}{2}}\varphi^\zeta \in L^2(\lambda_{\mathbb{R}};\mathbb{C})$ is continuous and $\varphi = \varphi^0$, in which case $T_\zeta \circ \mathscr{S}\varphi = e^{\frac{\zeta^2}{4}}\mathscr{S}\big(e_{\frac{\zeta}{2}}\varphi^\zeta\big)$. In particular, if $\mathscr{S}\varphi \in L^{2p}(\gamma_{\mathbb{C}};\mathbb{C})$ for some $p > 1$, then such a Φ exists.*

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Addendum to Séminaire de Probabilités XLV

Michel Émery

On page 160 of Séminaire XLV, Lemma 1 of *A Planar Borel Set...* deals with two subsets H and K of the set $\Omega = \{0, 1\}^n$ endowed with a product probability p. The hypothesis says that $\sum_{i \in I} x_i y_i$ is even for all $x \in H$ and $y \in K$, and the conclusion is a majoration of $p(H)\, p(K)$. The method of proof is linear-algebraic: when Ω is considered as an n-dimensional vector space over the field $\mathbb{Z}/2\mathbb{Z}$ of integers mod 2, the hypothesis means that every vector in H is orthogonal to every vector in K.

Of course, this kind of argument is not new, but I was not able to locate it in the literature, and the combinatorists I questioned did not know any reference.

I recently hit upon the site (in French)

http://images.math.cnrs.fr/Villes-paires-et-impaires-Oddtown-2470.html

where this argument is used, and called *the linear algebra method.* Two references are given:

S. Jukna, Extremal combinatorics. With applications in computer science, Texts in Theoretical Computer Science. Springer, Heidelberg, 2011.

L. Babai, P. Frankl, Linear Algebra Methods in Combinatorics, manuscript accessible on the internet.

I have not been able to access these references, but there is also a nice set of lecture notes from a course given at ETHZ in Fall 2012 by Paolo Penna and Katerina Böhmova. They can be found at

http://www.cadmo.ethz.ch/education/lectures/HS12/LAMC

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