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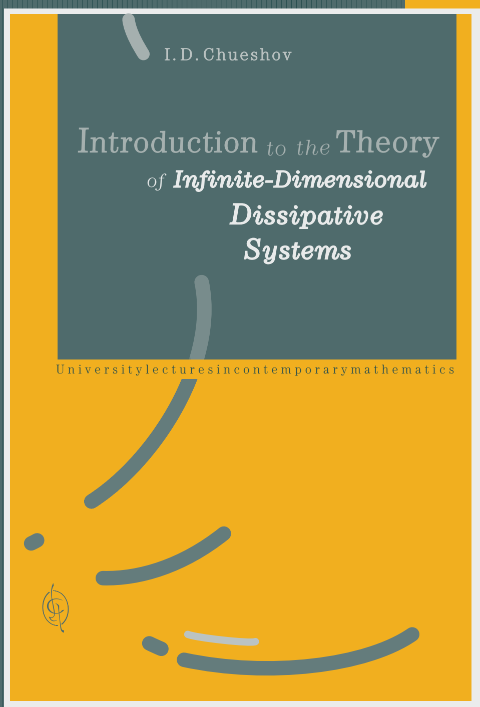

This book provides an exhaustive introduction to the scope of main ideas and methods of the theory of infinite-dimensional dissipative dynamical systems which has been rapidly developing in recent years. In the examples systems generated by nonlinear partial differential equations arising in the different problems of modern mechanics of continua are considered. The main goal of the book is to help the reader to master the basic strategies used in the study of infinite-dimensional dissipative systems and to qualify him/her for an independent scientific research in the given branch. Experts in nonlinear dynamics will find many fundamental facts in the convenient and practical form in this book.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during a number of years. This book contains a large number of exercises which make the main text more complete. It is sufficient to know the fundamentals of functional analysis and ordinary differential equations to read the book.

Translated by
Constantin I. Chueshov
from the Russian edition («ACTA», 1999)

Translation edited by
Maryna B. Khorolska

I. D. Chueshov

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Палкой щупая дорогу,
Бродит наугад слепой,
Осторожно ставит ногу
И бормочет сам с собой.
И на бельмах у слепого
Полный мир отображен:
Дом, лужок, забор, корова,
Клочья неба голубого —
Все, чего не видит он.

Вл. Ходасевич
«Слепой»

A blind man tramps at random touching the road with a stick.
He places his foot carefully and mumbles to himself.
The whole world is displayed in his dead eyes.
There are a house, a lawn, a fence, a cow
and scraps of the blue sky — everything he cannot see.

Vl. Khodasevich
«A Blind Man»

Preface

The recent years have been marked out by an evergrowing interest in the research of qualitative behaviour of solutions to nonlinear evolutionary partial differential equations. Such equations mostly arise as mathematical models of processes that take place in real (physical, chemical, biological, etc.) systems whose states can be characterized by an infinite number of parameters in general. Dissipative systems form an important class of systems observed in reality. Their main feature is the presence of mechanisms of energy reallocation and dissipation. Interaction of these two mechanisms can lead to appearance of complicated limit regimes and structures in the system. Intense interest to the infinite-dimensional dissipative systems was significantly stimulated by attempts to find adequate mathematical models for the explanation of turbulence in liquids based on the notion of strange (irregular) attractor. By now significant progress in the study of dynamics of infinite-dimensional dissipative systems have been made. Moreover, the latest mathematical studies offer a more or less common line (strategy), which when followed can help to answer a number of principal questions about the properties of limit regimes arising in the system under consideration. Although the methods, ideas and concepts from finite-dimensional dynamical systems constitute the main source of this strategy, finite-dimensional approaches require serious revaluation and adaptation.

The book is devoted to a systematic introduction to the scope of main ideas, methods and problems of the mathematical theory of infinite-dimensional dissipative dynamical systems. Main attention is paid to the systems that are generated by nonlinear partial differential equations arising in the modern mechanics of continua. The main goal of the book is to help the reader to master the basic strategies of the theory and to qualify him/her for an independent scientific research in the given branch. We also hope that experts in nonlinear dynamics will find the form many fundamental facts are presented in convenient and practical.

The core of the book is composed of the courses given by the author at the Department of Mechanics and Mathematics at Kharkov University during several years. The book consists of 6 chapters. Each chapter corresponds to a term course (34-36 hours) approximately. Its body can be inferred from the table of contents. Every chapter includes a separate list of references. The references do not claim to be full. The lists consist of the publications referred to in this book and offer additional works recommended for further reading. There are a lot of exercises in the book. They play a double role. On the one hand, proofs of some statements are presented as (or contain) cycles of exercises. On the other hand, some exercises contain an additional information on the object under consideration. We recommend that the exercises should be read at least. Formulae and statements have double indexing in each chapter (the first digit is a section number). When formulae and statements from another chapter are referred to, the number of the corresponding chapter is placed first.

It is sufficient to know the basic concepts and facts from functional analysis and ordinary differential equations to read the book. It is quite understandable for under-graduate students in Mathematics and Physics.

I.D. Chueshov

Chapter 1

Basic Concepts of the Theory of Infinite-Dimensional Dynamical Systems

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The mathematical theory of dynamical systems is based on the qualitative theory of ordinary differential equations the foundations of which were laid by Henri Poincaré (1854–1912). An essential role in its development was also played by the works of A. M. Lyapunov (1857–1918) and A. A. Andronov (1901–1952). At present the theory of dynamical systems is an intensively developing branch of mathematics which is closely connected to the theory of differential equations.

In this chapter we present some ideas and approaches of the theory of dynamical systems which are of general-purpose use and applicable to the systems generated by nonlinear partial differential equations.

§ 1 Notion of Dynamical System

In this book ***dynamical system*** is taken to mean the pair of objects (X, S_t) consisting of a complete metric space X and a family S_t of continuous mappings of the space X into itself with the properties

$$S_{t+\tau} = S_t \circ S_\tau , \quad t, \tau \in \mathbb{T}_+ , \quad S_0 = I , \tag{1.1}$$

where $\mathbb{T}_+$ coincides with either a set $\mathbb{R}_+$ of nonnegative real numbers or a set $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. If $\mathbb{T}_+ = \mathbb{R}_+$, we also assume that $y(t) = S_t y$ is a continuous function with respect to t for any $y \in X$. Therewith X is called a ***phase space***, or a state space, the family S_t is called an ***evolutionary operator*** (or semigroup), parameter $t \in \mathbb{T}_+$ plays the role of time. If $\mathbb{T}_+ = \mathbb{Z}_+$, then dynamical system is called ***discrete*** (or a system with discrete time). If $\mathbb{T}_+ = \mathbb{R}_+$, then (X, S_t) is frequently called to be dynamical system with ***continuous*** time. If a notion of dimension can be defined for the phase space X (e. g., if X is a lineal), the value $\dim X$ is called a ***dimension*** of dynamical system.

Originally a dynamical system was understood as an isolated mechanical system the motion of which is described by the Newtonian differential equations and which is characterized by a finite set of generalized coordinates and velocities. Now people associate any time-dependent process with the notion of dynamical system. These processes can be of quite different origins. Dynamical systems naturally arise in physics, chemistry, biology, economics and sociology. The notion of dynamical system is the key and uniting element in synergetics. Its usage enables us to cover a rather wide spectrum of problems arising in particular sciences and to work out universal approaches to the description of qualitative picture of real phenomena in the universe.

Let us look at the following examples of dynamical systems.

Example *1.1*

Let $f(x)$ be a continuously differentiable function on the real axis posessing the property $xf(x) \geq -C(1+x^2)$, where C is a constant. Consider the Cauchy problem for an ordinary differential equation

$$\dot{x}(t) = -f(x(t)), \qquad t > 0, \quad x(0) = x_0. \tag{1.2}$$

For any $x \in \mathbb{R}$ problem (1.2) is uniquely solvable and determines a dynamical system in $\mathbb{R}$. The evolutionary operator S_t is given by the formula $S_t x_0 = x(t)$, where $x(t)$ is a solution to problem (1.2). Semigroup property (1.1) holds by virtue of the theorem of uniqueness of solutions to problem (1.2). Equations of the type (1.2) are often used in the modeling of some ecological processes. For example, if we take $f(x) = \alpha \cdot x(x-1)$, $\alpha > 0$, then we get a logistic equation that describes a growth of a population with competition (the value $x(t)$ is the population level; we should take $\mathbb{R}_+$ for the phase space).

Example *1.2*

Let $f(x)$ and $g(x)$ be continuously differentiable functions such that

$$F(x) = \int_0^x f(\xi)\,d\xi \;\geq\; -c, \qquad g(x) \geq -c$$

with some constant c. Let us consider the Cauchy problem

$$\begin{cases} \ddot{x} + g(x)\dot{x} + f(x) = 0, \quad t > 0, \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases} \tag{1.3}$$

For any $y_0 = (x_0, x_1) \in \mathbb{R}^2$, problem (1.3) is uniquely solvable. It generates a two-dimensional dynamical system $(\mathbb{R}^2, S_t)$, provided the evolutionary operator is defined by the formula

$$S_t(x_0; x_1) = (x(t); \dot{x}(t)),$$

where $x(t)$ is the solution to problem (1.3). It should be noted that equations of the type (1.3) are known as Liénard equations in literature. The van der Pol equation:

$$g(x) = \varepsilon(x^2 - 1), \quad \varepsilon > 0, \qquad f(x) = x$$

and the Duffing equation:

$$g(x) = \varepsilon, \quad \varepsilon > 0, \qquad f(x) = x^3 - a \cdot x - b$$

which often occur in applications, belong to this class of equations.

E x a m p l e *1.3*

Let us now consider an autonomous system of ordinary differential equations

$$\dot{x}_k(t) = f_k(x_1, x_2, \ldots, x_N), \qquad k = 1, 2, \ldots, N . \tag{1.4}$$

Let the Cauchy problem for the system of equations (1.4) be uniquely solvable over an arbitrary time interval for any initial condition. Assume that a solution continuously depends on the initial data. Then equations (1.4) generate an N- dimensional dynamical system $(\mathbb{R}^N, S_t)$ with the evolutionary operator S_t acting in accordance with the formula

$$S_t y_0 = (x_1(t), \ldots, x_N(t)), \qquad y_0 = (x_{10}, x_{20}, \ldots, x_{N0}) ,$$

where $\{x_i(t)\}$ is the solution to the system of equations (1.4) such that $x_i(0) = x_{i0}$, $i = 1, 2, \ldots, N$. Generally, let X be a linear space and F be a continuous mapping of X into itself. Then the Cauchy problem

$$\dot{x}(t) = F(x(t)), \quad t > 0, \quad x(0) = x_0 \in X \tag{1.5}$$

generates a dynamical system (X, S_t) in a natural way provided this problem is well-posed, i.e. theorems on existence, uniqueness and continuous dependence of solutions on the initial conditions are valid for (1.5).

E x a m p l e *1.4*

Let us consider an ordinary retarded differential equation

$$\dot{x}(t) + \alpha x(t) = f(x(t-1)), \quad t > 0, \tag{1.6}$$

where f is a continuous function on $\mathbb{R}^1$, $\alpha > 0$. Obviously an initial condition for (1.6) should be given in the form

$$x(t)\big|_{t \in [-1, 0]} = \phi(t) . \tag{1.7}$$

Assume that $\phi(t)$ lies in the space $C[-1, 0]$ of continuous functions on the segment $[-1, 0]$. In this case the solution to problem (1.6) and (1.7) can be constructed by step-by-step integration. For example, if $0 \le t \le 1$, the solution $x(t)$ is given by

$$x(t) = e^{-\alpha t}\phi(0) + \int_0^t e^{-\alpha(t-\tau)} f(\phi(\tau-1))\,d\tau ,$$

and if $t \in [1, 2]$, then the solution is expressed by the similar formula in terms of the values of the function $x(t)$ for $t \in [0, 1]$ and so on. It is clear that the solution is uniquely determined by the initial function $\phi(t)$. If we now define an operator S_t in the space $X = C[-1, 0]$ by the formula

$$(S_t\phi)(\tau) = x(t+\tau), \qquad \tau \in [-1, 0],$$

where $x(t)$ is the solution to problem (1.6) and (1.7), then we obtain an infinite-dimensional dynamical system $(C[-1, 0], S_t)$.

Now we give several examples of discrete dynamical systems. First of all it should be noted that any system (X, S_t) with continuous time generates a discrete system if we take $t \in \mathbb{Z}_+$ instead of $t \in \mathbb{R}_+$. Furthermore, the evolutionary operator S_t of a discrete dynamical system is a degree of the mapping S_1, i. e. $S_t = S_1^t$, $t \in \mathbb{Z}_+$. Thus, a dynamical system with discrete time is determined by a continuous mapping of the phase space X into itself. Moreover, a discrete dynamical system is very often defined as a pair (X, S), consisting of the metric space X and the continuous mapping S.

Example *1.5*

Let us consider a one-step difference scheme for problem (1.5):

$$\frac{x_{n+1}-x_n}{\tau} = F(x_n), \quad n = 0, 1, 2, \ldots, \quad \tau > 0.$$

There arises a discrete dynamical system (X, S^n), where S is the continuous mapping of X into itself defined by the formula $Sx = x + \tau F(x)$.

Example *1.6*

Let us consider a nonautonomous ordinary differential equation

$$\dot{x}(t) = f(x, t), \quad t > 0, \quad x \in \mathbb{R}^1, \tag{1.9}$$

where $f(x, t)$ is a continuously differentiable function of its variables and is periodic with respect to t, i. e. $f(x, t) = f(x, t+T)$ for some $T > 0$. It is assumed that the Cauchy problem for (1.9) is uniquely solvable on any time interval. We define a ***monodromy*** operator (a period mapping) by the formula $Sx_0 = x(T)$, where $x(t)$ is the solution to (1.9) satisfying the initial condition $x(0) = x_0$. It is obvious that this operator possesses the property

$$S^k x(t) = x(t + kT) \tag{1.10}$$

for any solution $x(t)$ to equation (1.9) and any $k \in \mathbb{Z}_+$. The arising dynamical system $(\mathbb{R}^1, S^k)$ plays an important role in the study of the long-time properties of solutions to problem (1.9).

Example *1.7* (Bernoulli shift)

Let $X = \Sigma_2$ be a set of sequences $x = \{x_i,\ i \in \mathbb{Z}\}$ consisting of zeroes and ones. Let us make this set into a metric space by defining the distance by the formula

$$d(x, y) = \inf\{2^{-n} :\ x_i = y_i,\ |i| < n\}.$$

Let S be the shift operator on X, i. e. the mapping transforming the sequence $x = \{x_i\}$ into the element $y = \{y_i\}$, where $y_i = x_{i+1}$. As a result, a dynamical system (X, S^n) comes into being. It is used for describing complicated (quasirandom) behaviour in some quite realistic systems.

In the example below we describe one of the approaches that enables us to connect dynamical systems to nonautonomous (and nonperiodic) ordinary differential equations.

Example *1.8*

Let $h(x, t)$ be a continuous bounded function on $\mathbb{R}^2$. Let us define the hull L_h of the function $h(x, t)$ as the closure of a set

$$\left\{h_\tau(x, t) \equiv h(x, t+\tau), \quad \tau \in \mathbb{R}\right\}$$

with respect to the norm

$$\|h\|_C = \sup\left\{|h(x, t)| : x \in \mathbb{R}, t \in \mathbb{R}\right\}.$$

Let $g(x)$ be a continuous function. It is assumed that the Cauchy problem

$$\dot{x}(t) = g(x) + \tilde{h}(x, t), \quad x(0) = x_0 \tag{1.11}$$

is uniquely solvable over the interval $[0, +\infty)$ for any $\tilde{h} \in L_h$. Let us define the evolutionary operator S_τ on the space $X = \mathbb{R}^1 \times L_h$ by the formula

$$S_\tau(x_0, \tilde{h}) = (x(\tau), \tilde{h}_\tau),$$

where $x(t)$ is the solution to problem (1.11) and $\tilde{h}_\tau = \tilde{h}(x, t+\tau)$. As a result, a dynamical system $(\mathbb{R} \times L_h, S_t)$ comes into being. A similar construction is often used when L_h is a compact set in the space C of continuous bounded functions (for example, if $h(x, t)$ is a quasiperiodic or almost periodic function). As the following example shows, this approach also enables us to use naturally the notion of the dynamical system for the description of the evolution of objects subjected to random influences.

Example *1.9*

Assume that f_0 and f_1 are continuous mappings from a metric space Y into itself. Let Y be a state space of a system that evolves as follows: if y is the state of the system at time k, then its state at time $k+1$ is either $f_0(y)$ or $f_1(y)$ with probability $1/2$, where the choice of f_0 or f_1 does not depend on time and the previous states. The state of the system can be defined after a number of steps in time if we flip a coin and write down the sequence of events from the right to the left using 0 and 1. For example, let us assume that after 8 flips we get the following set of outcomes:

$$\ldots 10\,110010\,.$$

Here 1 corresponds to the head falling, whereas 0 corresponds to the tail falling. Therewith the state of the system at time $t = 8$ will be written in the form:

$$W = (f_1 \circ f_0 \circ f_1 \circ f_1 \circ f_0 \circ f_0 \circ f_1 \circ f_0)(y) .$$

This construction can be formalized as follows. Let Σ_2 be a set of two-sided sequences consisting of zeroes and ones (as in Example 1.7), i.e. a collection of elements of the type

$$\omega = (\ldots\ \omega_{-n} \ldots\ \omega_{-1}\, \omega_0\, \omega_1 \ldots\ \omega_n \ldots) ,$$

where ω_i is equal to either 1 or 0. Let us consider the space $X = \Sigma_2 \times Y$ consisting of pairs $x = (\omega,\ y)$, where $\omega \in \Sigma_2$, $y \in Y$. Let us define the mapping $F:\ X \to X$ by the formula:

$$F(x) \equiv F(\omega,\ y) = (S\,\omega,\ f_{\omega_0}(y)) ,$$

where S is the left-shift operator in Σ_2 (see Example 1.7). It is easy to see that the n-th degree of the mapping F actcts according to the formula

$$F^n(\omega,\ y) = (S^n\,\omega,\ (f_{\omega_{n-1}} \circ \ldots \circ f_{\omega_1} \circ f_{\omega_0})(y))$$

and it generates a discrete dynamical system $(\Sigma_2 \times Y,\ F^n)$. This system is often called a universal random (discrete) dynamical system.

Examples of dynamical systems generated by partial differential equations will be given in the chapters to follow.

Exercise *1.1* Assume that operators S_t have a continuous inverse for any t. Show that the family of operators $\{\hat{S}_t:\ t \in \mathbb{R}\}$ defined by the equality $\hat{S}_t = S_t$ for $t \ge 0$ and $\hat{S}_t = S_{|t|}^{-1}$ for $t < 0$ form a group, i.e. (1.1) holds for all $t,\ \tau \in \mathbb{R}$.

Exercise *1.2* Prove the unique solvability of problems (1.2) and (1.3) involved in Examples 1.1 and 1.2.

Exercise *1.3* Ground formula (1.10) in Example 1.6.

Exercise *1.4* Show that the mapping S_t in Example 1.8 possesses semigroup property (1.1).

Exercise *1.5* Show that the value $d(x,\ y)$ involved in Example 1.7 is a metric. Prove its equivalence to the metric

$$d^*(x,\ y) = \sum_{i=-\infty}^{\infty} 2^{-|i|}\,|x_i - y_i| .$$

§ 2 Trajectories and Invariant Sets

Let (X, S_t) be a dynamical system with continuous or discrete time. Its ***trajectory*** (or ***orbit***) is defined as a set of the type

$$\gamma = \{u(t):\ t \in \mathbb{T}\},$$

where $u(t)$ is a continuous function with values in X such that $S_\tau u(t) = u(t+\tau)$ for all $\tau \in \mathbb{T}_+$ and $t \in \mathbb{T}$. Positive (negative) ***semitrajectory*** is defined as a set $\gamma^+ = \{u(t):\ t \ge 0\}$, $(\gamma^- = \{u(t):\ t \le 0\}$, respectively), where a continuous on $\mathbb{T}_+$ ($\mathbb{T}_-$, respectively) function $u(t)$ possesses the property $S_\tau u(t) = u(t+\tau)$ for any $\tau > 0$, $t \ge 0$ $(\tau > 0,\ \ t \le 0,\ \ \tau + t \le 0$, respectively). It is clear that any positive semitrajectory γ^+ has the form $\gamma^+ = \{S_t v:\ t \ge 0\}$, i.e. it is uniquely determined by its initial state v. To emphasize this circumstance, we often write $\gamma^+ = \gamma^+(v)$. In general, it is impossible to continue this semitrajectory $\gamma^+(v)$ to a full trajectory without imposing any additional conditions on the dynamical system.

Exercise *2.1* Assume that an evolutionary operator S_t is invertible for some $t > 0$. Then it is invertible for all $t > 0$ and for any $v \in X$ there exists a negative semitrajectory $\gamma^- = \gamma^-(v)$ ending at the point v.

A trajectory $\gamma = \{u(t):\ t \in \mathbb{T}\}$ is called a ***periodic trajectory*** (or a ***cycle***) if there exists $T \in \mathbb{T}_+$, $T > 0$ such that $u(t+T) = u(t)$. Therewith the minimal number $T > 0$ possessing the property mentioned above is called a ***period*** of a trajectory. Here $\mathbb{T}$ is either $\mathbb{R}$ or $\mathbb{Z}$ depending on whether the system is a continuous or a discrete one. An element $u_0 \in X$ is called a ***fixed point*** of a dynamical system (X, S_t) if $S_t u_0 = u_0$ for all $t \ge 0$ (synonyms: ***equilibrium point***, ***stationary point***).

Exercise *2.2* Find all the fixed points of the dynamical system (R, S_t) generated by equation (1.2) with $f(x) = x(x-1)$. Does there exist a periodic trajectory of this system?

Exercise *2.3* Find all the fixed points and periodic trajectories of a dynamical system in $\mathbb{R}^2$ generated by the equations

$$\begin{cases} \dot{x} = -\alpha y - x\left[(x^2+y^2)^2 - 4(x^2+y^2) + 1\right], \\ \dot{y} = \alpha x - y\left[(x^2+y^2)^2 - 4(x^2+y^2) + 1\right]. \end{cases}$$

Consider the cases $\alpha \ne 0$ and $\alpha = 0$. *Hint*: use polar coordinates.

Exercise *2.4* Prove the existence of stationary points and periodic trajectories of any period for the discrete dynamical system described in Example 1.7. Show that the set of all periodic trajectories is dense in the phase space of this system. Make sure that there exists a trajectory that passes at a whatever small distance from any point of the phase space.

The notion of invariant set plays an important role in the theory of dynamical systems. A subset Y of the phase space X is said to be:

a) ***positively invariant***, if $S_t Y \subseteq Y$ for all $t \ge 0$;
b) ***negatively invariant***, if $S_t Y \supseteq Y$ for all $t \ge 0$;
c) ***invariant***, if it is both positively and negatively invariant, i.e. if $S_t Y = Y$ for all $t \ge 0$.

The simplest examples of invariant sets are trajectories and semitrajectories.

— Exercise *2.5* Show that γ^+ is positively invariant, γ^- is negatively invariant and γ is invariant.

— Exercise *2.6* Let us define the sets

$$\gamma^+(A) = \bigcup_{t \ge 0} S_t(A) \equiv \bigcup_{t \ge 0} \{v = S_t u : \ u \in A\}$$

and

$$\gamma^-(A) = \bigcup_{t \ge 0} S_t^{-1}(A) \equiv \bigcup_{t \ge 0} \{v : \ S_t v \in A\}$$

for any subset A of the phase space X. Prove that $\gamma^+(A)$ is a positively invariant set, and if the operator S_t is invertible for some $t > 0$, then $\gamma^-(A)$ is a negatively invariant set.

Other important example of invariant set is connected with the notions of ω-limit and α-limit sets that play an essential role in the study of the long-time behaviour of dynamical systems.

Let $A \subset X$. Then the ω-***limit set*** for A is defined by

$$\omega(A) = \bigcap_{s \ge 0} \left[\bigcup_{t \ge s} S_t(A) \right]_X ,$$

where $S_t(A) = \{v = S_t u : \ u \in A\}$. Hereinafter $[Y]_X$ is the closure of a set Y in the space X. The set

$$\alpha(A) = \bigcap_{s \ge 0} \left[\bigcup_{t \ge s} S_t^{-1}(A) \right]_X ,$$

where $S_t^{-1}(A) = \{v : \ S_t v \in A\}$, is called the α-***limit set*** for A.

Lemma 2.1

For an element y to belong to an ω-limit set $\omega(A)$, it is necessary and sufficient that there exist a sequence of elements $\{y_n\} \subset A$ and a sequence of numbers t_n, the latter tending to infinity such that

$$\lim_{n \to \infty} d(S_{t_n} y_n,\ y) = 0,$$

where $d(x,\ y)$ is the distance between the elements x and y in the space X.

Proof.

Let the sequences mentioned above exist. Then it is obvious that for any $\tau > 0$ there exists $n_0 \ge 0$ such that

$$S_{t_n} y_n \in \bigcup_{t \ge \tau} S_t(A), \quad n \ge n_0 .$$

This implies that

$$y = \lim_{n \to \infty} S_{t_n} y_n \in \left[\bigcup_{t \ge \tau} S_t(A) \right]_X$$

for all $\tau > 0$. Hence, the element y belongs to the intersection of these sets, i.e. $y \in \omega(A)$.

On the contrary, if $y \in \omega(A)$, then for all $n = 0,\ 1,\ 2,\ \ldots$

$$y \in \left[\bigcup_{t \ge n} S_t(A) \right]_X .$$

Hence, for any n there exists an element z_n such that

$$z_n \in \bigcup_{t \ge n} S_t(A), \qquad d(y,\ z_n) \le \frac{1}{n} .$$

Therewith it is obvious that $z_n = S_{t_n} y_n$, $y_n \in A$, $t_n \ge n$. This proves the lemma.

It should be noted that this lemma gives us a description of an ω-limit set but does not guarantee its nonemptiness.

Exercise *2.7* Show that $\omega(A)$ is a positively invariant set. If for any $t > 0$ there exists a continuous inverse to S_t, then $\omega(A)$ is invariant, i.e. $S_t \omega(A) = \omega(A)$.

Exercise *2.8* Let S_t be an invertible mapping for every $t > 0$. Prove the counterpart of Lemma 2.1 for an α-limit set:

$$y \in \alpha(A) \Leftrightarrow \left\{ \exists \{y_n\} \in A, \quad \exists t_n,\ t_n \to +\infty ; \quad \lim_{n \to \infty} d(S_{t_n}^{-1} y_n,\ y) = 0 \right\}.$$

Establish the invariance of $\alpha(A)$.

Exercise 2.9 Let $\gamma = \{u(t): -\infty < t < \infty\}$ be a periodic trajectory of a dynamical system. Show that $\gamma = \omega(u) = \alpha(u)$ for any $u \in \gamma$.

Exercise 2.10 Let us consider the dynamical system $(\mathbb{R}, S_t)$ constructed in Example 1.1. Let a and b be the roots of the function $f(x)$: $f(a) = f(b) = 0$, $a < b$. Then the segment $I = \{x: a \le x \le b\}$ is an invariant set. Let $F(x)$ be a primitive of the function $f(x)$ ($F'(x) = f(x)$). Then the set $\{x: F(x) \le c\}$ is positively invariant for any c.

Exercise 2.11 Assume that for a continuous dynamical system (X, S_t) there exists a continuous scalar function $V(y)$ on X such that the value $V(S_t y)$ is differentiable with respect to t for any $y \in X$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(S_t y)) + \alpha V(S_t y) \le \rho, \qquad (\alpha > 0, \ \rho > 0, \ y \in X).$$

Then the set $\{y: V(y) \le R\}$ is positively invariant for any $R \ge \ge \rho/\alpha$.

§ 3 Definition of Attractor

Attractor is a central object in the study of the limit regimes of dynamical systems. Several definitions of this notion are available. Some of them are given below. From the point of view of infinite-dimensional systems the most convenient concept is that of the global attractor.

A bounded closed set $A_1 \subset X$ is called a ***global attractor*** for a dynamical system (X, S_t), if

1) A_1 is an invariant set, i.e. $S_t A_1 = A_1$ for any $t > 0$;
2) the set A_1 uniformly attracts all trajectories starting in bounded sets, i.e. for any bounded set B from X

$$\lim_{t \to \infty} \sup \left\{ \mathrm{dist}(S_t y, A_1): y \in B \right\} = 0.$$

We remind that the distance between an element z and a set A is defined by the equality:

$$\mathrm{dist}(z, A) = \inf\{d(z, y): y \in A\},$$

where $d(z, y)$ is the distance between the elements z and y in X.

The notion of a weak global attractor is useful for the study of dynamical systems generated by partial differential equations.

Let X be a complete linear metric space. A bounded weakly closed set A_2 is called a ***global weak attractor*** if it is invariant $(S_t A_2 = A_2,\ t > 0)$ and for any weak vicinity $\mathcal{O}$ of the set A_2 and for every bounded set $B \subset X$ there exists $t_0 = t_0(\mathcal{O},\ B)$ such that $S_t B \subset \mathcal{O}$ for $t \ge t_0$.

We remind that an open set in weak topology of the space X can be described as finite intersection and subsequent arbitrary union of sets of the form

$$U_{l,\ c} = \{x \in X:\ l(x) < c\},$$

where c is a real number and l is a continuous linear functional on X.

It is clear that the concepts of global and global weak attractors coincide in the finite-dimensional case. In general, a global attractor A is also a global weak attractor, provided the set A is weakly closed.

— Exercise *3.1* Let A be a global or global weak attractor of a dynamical system $(X,\ S_t)$. Then it is uniquely determined and contains any bounded negatively invariant set. The attractor A also contains the ω-limit set $\omega(B)$ of any bounded $B \subset X$.

— Exercise *3.2* Assume that a dynamical system $(X,\ S_t)$ with continuous time possesses a global attractor A_1. Let us consider a discrete system $(X,\ T^n)$, where $T = S_{t_0}$ with some $t_0 > 0$. Prove that A_1 is a global attractor for the system $(X,\ T^n)$. Give an example which shows that the converse assertion does not hold in general.

If the global attractor A_1 exists, then it contains a ***global minimal attractor*** A_3 which is defined as a minimal closed positively invariant set possessing the property

$$\lim_{t \to \infty} \operatorname{dist}(S_t y,\ A_3) = 0 \quad \text{for every} \quad y \in X.$$

By definition minimality means that A_3 has no proper subset possessing the properties mentioned above. It should be noted that in contrast with the definition of the global attractor the uniform convergence of trajectories to A_3 is not expected here.

— Exercise *3.3* Show that $S_t A_3 = A_3$, provided A_3 is a compact set.

— Exercise *3.4* Prove that $\omega(x) \in A_3$ for any $x \in X$. Therewith, if A_3 is a compact, then $A_3 = \bigcup\{\omega(x):\ x \in X\}$.

By definition the attractor A_3 contains limit regimes of each individual trajectory. It will be shown below that $A_3 \ne A_1$ in general. Thus, a set of real limit regimes (states) originating in a dynamical system can appear to be narrower than the global attractor. Moreover, in some cases some of the states that are unessential from the point of view of the frequency of their appearance can also be "removed" from A_3, for example, such states like absolutely unstable stationary points. The next two definitions take into account the fact mentioned above. Unfortunately, they require additional assumptions on the properties of the phase space. Therefore, these definitions are mostly used in the case of finite-dimensional dynamical systems.

Let a Borel measure μ such that $\mu(X) < \infty$ be given on the phase space X of a dynamical system (X, S_t). A bounded set A_4 in X is called a ***Milnor attractor*** (with respect to the measure μ) for (X, S_t) if A_4 is a minimal closed invariant set possessing the property

$$\lim_{t \to \infty} \operatorname{dist}(S_t y, A_4) = 0$$

for almost all elements $y \in X$ with respect to the measure μ. The Milnor attractor is frequently called a probabilistic global minimal attractor.

At last let us introduce the notion of a statistically essential global minimal attractor suggested by Ilyashenko. Let U be an open set in X and let $X_U(x)$ be its characteristic function: $X_U(x) = 1$, $x \in U$; $X_U(x) = 0$, $x \notin U$. Let us define the average time $\tau(x, U)$ which is spent by the semitrajectory $\gamma^+(x)$ emanating from x in the set U by the formula

$$\tau(x, U) = \overline{\lim_{T \to \infty}} \frac{1}{T} \int_0^T X_U(S_t x)\, dt .$$

A set U is said to be unessential with respect to the measure μ if

$$M(U) \equiv \mu\{x:\ \tau(x, U) > 0\} = 0 .$$

The complement A_5 to the maximal unessential open set is called an ***Ilyashenko attractor*** (with respect to the measure μ).

It should be noted that the attractors A_4 and A_5 are used in cases when the natural Borel measure is given on the phase space (for example, if X is a closed measurable set in $\mathbb{R}^N$ and μ is the Lebesgue measure).

The relations between the notions introduced above can be illustrated by the following example.

E x a m p l e *3.1*

Let us consider a quasi-Hamiltonian system of equations in $\mathbb{R}^2$:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} - \mu H \dfrac{\partial H}{\partial q}, \\[2ex] \dot{p} = -\dfrac{\partial H}{\partial q} - \mu H \dfrac{\partial H}{\partial p}, \end{cases} \tag{3.1}$$

where $H(p, q) = (1/2)p^2 + q^4 - q^2$ and μ is a positive number. It is easy to ascertain that the phase portrait of the dynamical system generated by equations (3.1) has the form represented on *Fig. 1*.

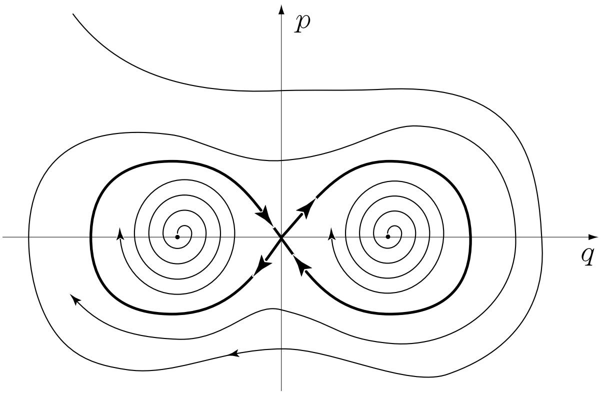

Fig. 1. Phase portrait of system (3.1)

A separatrix ("eight curve") separates the domains of the phase plane with the different qualitative behaviour of the trajectories. It is given by the equation $H(p,q)=0$. The points $(p,\ q)$ inside the separatrix are characterized by the equation $H(p,\ q)<0$. Therewith it appears that

$$A_1 = A_2 = \{(p,\ q):\ H(p,\ q) \le 0\},$$

$$A_3 = \left\{(p,\ q):\ H(p,\ q)=0\right\} \cup \left\{(p,\ q):\ \frac{\partial}{\partial p}H(p,\ q)=\frac{\partial}{\partial q}H(p,\ q)=0\right\},$$

$$A_4 = \{(p,\ q):\ H(p,\ q)=0\}.$$

Finally, the simple calculations show that $A_5 = \{0,\ 0\}$, i.e. the Ilyashenko attractor consists of a single point. Thus,

$$A_1 = A_2 \supset A_3 \supset A_4 \supset A_5,$$

all inclusions being strict.

Exercise *3.5* Display graphically the attractors A_j of the system generated by equations (3.1) on the phase plane.

Exercise *3.6* Consider the dynamical system from Example 1.1 with $f(x) = x(x^2-1)$. Prove that $A_1 = \{x:\ -1 \le x \le 1\}$, $A_3 = \{x=0;\ x=\pm 1\}$, and $A_4 = A_5 = \{x=\pm 1\}$.

Exercise *3.7* Prove that $A_4 \subset A_3$ and $A_5 \subset A_3$ in general.

Exercise *3.8* Show that all positive semitrajectories of a dynamical system which possesses a global minimal attractor are bounded sets.

In particular, the result of the last exercise shows that the global attractor can exist only under additional conditions concerning the behaviour of trajectories of the system at infinity. The main condition to be met is the dissipativity discussed in the next section.

§ 4 Dissipativity and Asymptotic Compactness

From the physical point of view dissipative systems are primarily connected with irreversible processes. They represent a rather wide and important class of the dynamical systems that are intensively studied by modern natural sciences. These systems (unlike the conservative systems) are characterized by the existence of the accented direction of time as well as by the energy reallocation and dissipation. In particular, this means that limit regimes that are stationary in a certain sense can arise in the system when $t \to +\infty$. Mathematically these features of the qualitative behaviour of the trajectories are connected with the existence of a bounded absorbing set in the phase space of the system.

A set $B_0 \subset X$ is said to be ***absorbing*** for a dynamical system (X, S_t) if for any bounded set B in X there exists $t_0 = t_0(B)$ such that $S_t(B) \subset B_0$ for every $t \ge t_0$. A dynamical system (X, S_t) is said to be ***dissipative*** if it possesses a bounded absorbing set. In cases when the phase space X of a dissipative system (X, S_t) is a Banach space a ball of the form $\{x \in X\colon \|x\|_X \le R\}$ can be taken as an absorbing set. Therewith the value R is said to be a ***radius of dissipativity***.

As a rule, dissipativity of a dynamical system can be derived from the existence of a Lyapunov type function on the phase space. For example, we have the following assertion.

Theorem 4.1.

Let the phase space of a continuous dynamical system (X, S_t) be a Banach space. Assume that:

(a) *there exists a continuous function $U(x)$ on X possessing the properties*

$$\varphi_1(\|x\|) \le U(x) \le \varphi_2(\|x\|)\ , \tag{4.1}$$

where $\varphi_j(r)$ are continuous functions on $\mathbb{R}_+$ and $\varphi_1(r) \to +\infty$ when $r \to \infty$;

(b) *there exist a derivative $\frac{d}{dt}U(S_t y)$ for $t \ge 0$ and positive numbers α and ρ such that*

$$\frac{d}{dt}U(S_t y) \le -\alpha \quad \textit{for} \quad \|S_t y\| > \rho. \tag{4.2}$$

Then the dynamical system (X, S_t) is dissipative.

Proof.

Let us choose $R_0 \ge \rho$ such that $\varphi_1(r) > 0$ for $r \ge R_0$. Let

$$l = \sup\{\varphi_2(r)\colon\ r \le 1 + R_0\}$$

and $R_1 > R_0 + 1$ be such that $\varphi_1(r) > l$ for $r > R_1$. Let us show that

$$\|S_t y\| \le R_1 \quad \text{for all} \quad t \ge 0 \quad \text{and} \quad \|y\| \le R_0 . \tag{4.3}$$

Assume the contrary, i.e. assume that for some $y \in X$ such that $\|y\| \le R_0$ there exists a time $\bar{t} > 0$ possessing the property $\|S_{\bar{t}}\, y\| > R_1$. Then the continuity of $S_t y$ implies that there exists $0 < t_0 < \bar{t}$ such that $\rho < \|S_{t_0} y\| \le R_0 + 1$. Thus, equation (4.2) implies that

$$U(S_t y) \le U(S_{t_0} y), \qquad t \ge t_0 ,$$

provided $\|S_t y\| > \rho$. It follows that $U(S_t y) \le l$ for all $t \ge t_0$. Hence, $\|S_t y\| \le R_1$ for all $t \ge t_0$. This contradicts the assumption. Let us assume now that B is an arbitrary bounded set in X that does not lie inside the ball with the radius R_0. Then equation (4.2) implies that

$$U(S_t y) \le U(y) - \alpha t \le l_B - \alpha t , \quad y \in B , \tag{4.4}$$

provided $\|S_t y\| > \rho$. Here

$$l_B = \sup\{U(x):\ x \in B\}.$$

Let $y \in B$. If for a time $t^* < (l_B - l)/\alpha$ the semitrajectory $S_t y$ enters the ball with the radius ρ, then by (4.3) we have $\|S_t y\| \le R_1$ for all $t \ge t^*$. If that does not take place, from equation (4.4) it follows that

$$\varphi_1(\|S_t y\|) \le U(S_t y) \le l \quad \text{for} \quad t \ge \frac{l_B - l}{\alpha} ,$$

i.e. $\|S_t y\| \le R_1$ for $t \ge \alpha^{-1}(l_B - l)$. Thus,

$$S_t B \subset \{x:\ \|x\| \le R_1\}, \qquad t \ge \frac{l_B - l}{\alpha} .$$

This and (4.3) imply that the ball with the radius R_1 is an absorbing set for the dynamical system (X, S_t). Thus, **Theorem 4.1 is proved**.

Exercise *4.1* Show that hypothesis (4.2) of Theorem 4.1 can be replaced by the requirement

$$\frac{d}{dt} U(S_t y) + \gamma U(S_t y) \le C ,$$

where γ and C are positive constants.

Exercise *4.2* Show that the dynamical system generated in $\mathbb{R}$ by the differential equation $\dot{x} + f(x) = 0$ (see Example 1.1) is dissipative, provided the function $f(x)$ possesses the property: $x f(x) \ge \delta x^2 - C$, where $\delta > 0$ and C are constants (*Hint*: $U(x) = x^2$). Find an upper estimate for the minimal radius of dissipativity.

Exercise *4.3* Consider a discrete dynamical system $(\mathbb{R}, f^n)$, where f is a continuous function on $\mathbb{R}$. Show that the system $(\mathbb{R}, f)$ is dissipative, provided there exist $\rho > 0$ and $0 < \alpha < 1$ such that $|f(x)| < \alpha |x|$ for $|x| > \rho$.

Exercise 4.4 Consider a dynamical system $(\mathbb{R}^2,\ S_t)$ generated (see Example 1.2) by the Duffing equation

$$\ddot{x} + \varepsilon\dot{x} + x^3 - ax = b\,,$$

where a and b are real numbers and $\varepsilon > 0$. Using the properties of the function

$$U(x,\ \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4 - \frac{a}{2}x^2 + \nu\left(x\dot{x} + \frac{\varepsilon}{2}x^2\right)$$

show that the dynamical system $(\mathbb{R}^2,\ S_t)$ is dissipative for $\nu > 0$ small enough. Find an upper estimate for the minimal radius of dissipativity.

Exercise 4.5 Prove the dissipativity of the dynamical system generated by (1.4) (see Example 1.3), provided

$$\sum_{k=1}^{N} x_k f_k(x_1,\ x_2,\ \dots,\ x_N) \le -\delta\sum_{k=1}^{N} x_k^2 + C\,, \qquad \delta > 0\,.$$

Exercise 4.6 Show that the dynamical system of Example 1.4 is dissipative if $f(z)$ is a bounded function.

Exercise 4.7 Consider a cylinder Ц with coordinates $(x,\ \varphi)$, $x \in \mathbb{R}$, $\varphi \in [0,\ 1)$ and the mapping T of this cylinder which is defined by the formula $T(x,\ \varphi) = (x',\ \varphi')$, where

$$x' = \alpha x + k\sin 2\pi\varphi\,,$$

$$\varphi' = \varphi + x' \pmod 1.$$

Here α and k are positive parameters. Prove that the discrete dynamical system $(Ц,\ T^n)$ is dissipative, provided $0 < \alpha < 1$. We note that if $\alpha = 1$, then the mapping T is known as the Chirikov mapping. It appears in some problems of physics of elementary particles.

Exercise 4.8 Using Theorem 4.1 prove that the dynamical system $(\mathbb{R}^2,\ S_t)$ generated by equations (3.1) (see Example 3.1) is dissipative. (*Hint*: $U(x) = [H(p,\ q)]^2$).

In the proof of the existence of global attractors of infinite-dimensional dissipative dynamical systems a great role is played by the property of asymptotic compactness. For the sake of simplicity let us assume that X is a closed subset of a Banach space. The dynamical system $(X,\ S_t)$ is said to be ***asymptotically compact*** if for any $t > 0$ its evolutionary operator S_t can be expressed by the form

$$S_t = S_t^{(1)} + S_t^{(2)}\,, \tag{4.5}$$

where the mappings $S_t^{(1)}$ and $S_t^{(2)}$ possess the properties:

a) for any bounded set B in X

$$r_B(t) = \sup_{y \in B} \left\| S_t^{(1)} y \right\|_X \to 0\,, \qquad t \to +\infty\,;$$

b) for any bounded set B in X there exists t_0 such that the set

$$[\gamma_{t_0}^{(2)}(B)] = \left[\bigcup_{t \geq t_0} S_t^{(2)} B \right] \tag{4.6}$$

is compact in X, where $[\gamma]$ is the closure of the set γ.

A dynamical system is said to be ***compact*** if it is asymptotically compact and one can take $S_t^{(1)} \equiv 0$ in representation (4.5). It becomes clear that any finite-dimensional dissipative system is compact.

Exercise *4.9* Show that condition (4.6) is fulfilled if there exists a compact set K in H such that for any bounded set B the inclusion $S_t^{(2)} B \subset K$, $t \geq t_0(B)$ holds. In particular, a dissipative system is compact if it possesses a compact absorbing set.

Lemma 4.1.

The dynamical system (X, S_t) *is asymptotically compact if there exists a compact set* K *such that*

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(S_t u,\, K) :\ u \in B\} = 0 \tag{4.7}$$

for any set B *bounded in* X.

Proof.

The distance to a compact set is reached on some element. Hence, for any $t > 0$ and $u \in X$ there exists an element $v \equiv S_t^{(2)} u \in K$ such that

$$\operatorname{dist}(S_t u,\, K) = \left\| S_t u - S_t^{(2)} u \right\|.$$

Therefore, if we take $S_t^{(1)} u = S_t u - S_t^{(2)} u$, it is easy to see that in this case decomposition (4.5) satisfies all the requirements of the definition of asymptotic compactness.

Remark 4.1.

In most applications Lemma 4.1 *plays a major role in the proof of the property of asymptotic compactness. Moreover, in cases when the phase space* X *of the dynamical system* (X, S_t) *does not possess the structure of a linear space it is convenient to define the notion of the asymptotic compactness using equation* (4.7). *Namely, the system* (X, S_t) *is said to be asymptotically compact if there exists a compact* K *possessing property* (4.7) *for any bounded set* B *in* X. *For one more approach to the definition of this concept see Exercise* 5.1 *below.*

Exercise *4.10* Consider the infinite-dimensional dynamical system generated by the retarded equation

$$\dot{x}(t) + \alpha x(t) = f(x(t-1)) \ ,$$

where $\alpha > 0$ and $f(z)$ is bounded (see Example 1.4). Show that this system is compact.

Exercise *4.11* Consider the system of Lorentz equations arising as a three-mode Galerkin approximation in the problem of convection in a thin layer of liquid:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \,, \\ \dot{y} = r x - y - x z \,, \\ \dot{z} = -b z + x y \,. \end{cases}$$

Here σ, r, and b are positive numbers. Prove the dissipativity of the dynamical system generated by these equations in $\mathbb{R}^3$.
Hint: Consider the function

$$V(x,\ y,\ z) \ = \ \frac{1}{2}(x^2 + y^2 + (z - r - \sigma)^2)$$

on the trajectories of the system.

§ 5 *Theorems on Existence of Global Attractor*

For the sake of simplicity it is assumed in this section that the phase space X is a Banach space, although the main results are valid for a wider class of spaces (see, e. g., Exercise 5.8). The following assertion is the main result.

Theorem 5.1.

Assume that a dynamical system (X, S_t) is dissipative and asymptotically compact. Let B be a bounded absorbing set of the system (X, S_t). Then the set $A = \omega(B)$ is a nonempty compact set and is a global attractor of the dynamical system (X, S_t). The attractor A is a connected set in X.

In particular, this theorem is applicable to the dynamical systems from Exercises 4.2–4.11. It should also be noted that Theorem 5.1 along with Lemma 4.1 gives the following criterion: a dissipative dynamical system possesses a compact global attractor if and only if it is asymptotically compact.

The proof of the theorem is based on the following lemma.

Lemma 5.1.

Let a dynamical system (X, S_t) be asymptotically compact. Then for any bounded set B of X the ω-limit set $\omega(B)$ is a nonempty compact invariant set.

Proof.

Let $y_n \in B$. Then for any sequence $\{t_n\}$ tending to infinity the set $\{S^{(2)}_{t_n} y_n,\ n = 1, 2, \dots\}$ is relatively compact, i.e. there exist a sequence n_k and an element $y \in X$ such that $S^{(2)}_{t_{n_k}} y_{n_k}$ tends to y as $k \to \infty$. Hence, the asymptotic compactness gives us that

$$\left\|y - S_{t_{n_k}} y_{n_k}\right\| \le \left\|S^{(1)}_{t_{n_k}} y_{n_k}\right\| + \left\|y - S^{(2)}_{t_{n_k}} y_{n_k}\right\| \to 0 \quad \text{as} \quad k \to \infty .$$

Thus, $y = \lim_{k \to \infty} S_{t_{n_k}} y_{n_k}$. Due to Lemma 2.1 this indicates that $\omega(B)$ is nonempty.

Let us prove the invariance of ω-limit set. Let $y \in \omega(B)$. Then according to Lemma 2.1 there exist sequences $\{t_n\}$, $t_n \to \infty$, and $\{z_n\} \subset B$ such that $S_{t_n} z_n \to y$. However, the mapping S_t is continuous. Therefore,

$$S_{t + t_n} z_n = S_t \circ S_{t_n} z_n \to S_t y , \qquad n \to \infty .$$

Lemma 2.1 implies that $S_t y \in \omega(B)$. Thus,

$$S_t\, \omega(B) \subset \omega(B) , \qquad t > 0 .$$

Let us prove the reverse inclusion. Let $y \in \omega(B)$. Then there exist sequences $\{v_n\} \subset B$ and $\{t_n:\ t_n \to \infty\}$ such that $S_{t_n} v_n \to y$. Let us consider the sequence $y_n = S_{t_n - t} v_n$, $t_n \ge t$. The asymptotic compactness implies that there exist a subsequence t_{n_k} and an element $z \in X$ such that

$$z = \lim_{k \to \infty} S^{(2)}_{t_{n_k} - t}\ y_{n_k} .$$

As stated above, this gives us that

$$z = \lim_{k \to \infty} S_{t_{n_k} - t}\ y_{n_k} .$$

Therefore, $z \in \omega(B)$. Moreover,

$$S_t z = \lim_{k \to \infty} S_t \circ S_{t_{n_k} - t} v_{n_k} = \lim_{k \to \infty} S_{t_{n_k}} v_{n_k} = y .$$

Hence, $y \in S_t\, \omega(B)$. Thus, the invariance of the set $\omega(B)$ is proved.

Let us prove the compactness of the set $\omega(B)$. Assume that $\{z_n\}$ is a sequence in $\omega(B)$. Then Lemma 2.1 implies that for any n we can find $t_n \ge n$ and $y_n \in B$ such that $\left\|z_n - S_{t_n} y_n\right\| \le 1/n$. As said above, the property of asymptotic compactness enables us to find an element z and a sequence $\{n_k\}$ such that

$$\left\|S_{t_{n_k}} y_{n_k} - z\right\| \to 0 , \quad k \to \infty .$$

This implies that $z \in \omega(B)$ and $z_{n_k} \to z$. This means that $\omega(B)$ is a closed and compact set in H. Lemma 5.1 is proved completely.

Now we establish Theorem 5.1. Let B be a bounded absorbing set of the dynamical system. Let us prove that $\omega(B)$ is a global attractor. It is sufficient to verify that $\omega(B)$ uniformly attracts the absorbing set B. Assume the contrary. Then the value $\sup\{\text{dist}(S_t y,\ \omega(B)):\ y \in B\}$ does not tend to zero as $t \to \infty$. This means that there exist $\delta > 0$ and a sequence $\{t_n:\ t_n \to \infty\}$ such that

$$\sup\left\{\text{dist}(S_{t_n} y,\ \omega(B)):\ y \in B\right\} \ge 2\delta .$$

Therefore, there exists an element $y_n \in B$ such that

$$\text{dist}(S_{t_n} y_n,\ \omega(B)) \ge \delta, \quad n = 1,\ 2,\ \dots . \tag{5.1}$$

As before, a convergent subsequence $\{S_{t_{n_k}} y_{n_k}\}$ can be extracted from the sequence $\{S_{t_n} y_n\}$. Therewith Lemma 2.1 implies

$$z \equiv \lim_{k \to \infty} S_{t_{n_k}} y_{n_k} \in \omega(B)$$

which contradicts estimate (5.1). Thus, $\omega(B)$ is a global attractor. Its compactness follows from the easily verifiable relation

$$A \equiv \omega(B) = \bigcap_{\tau > 0} \Big[\bigcap_{t \ge \tau} S_t^{(2)} B \Big].$$

Let us prove the connectedness of the attractor by reductio ad absurdum. Assume that the attractor A is not a connected set. Then there exists a pair of open sets U_1 and U_2 such that

$$U_i \cap A \ne \varnothing, \qquad i = 1,\ 2, \quad A \subset U_1 \cup U_2, \quad U_1 \cap U_2 = \varnothing .$$

Let $A^c = \text{conv}(A)$ be a convex hull of the set A, i.e.

$$A^c = \left[\left\{ \sum_{i=1}^{N} \lambda_i v_i:\ v_i \in A,\ \lambda_i \ge 0,\ \sum_{i=1}^{N} \lambda_i = 1, \quad N = 1,\ 2,\ \dots \right\}\right].$$

It is clear that A^c is a bounded connected set and $A^c \supset A$. The continuity of the mapping S_t implies that the set $S_t A^c$ is also connected. Therewith $A = S_t A \subset S_t A^c$. Therefore, $U_i \cap S_t A^c \ne \varnothing$, $i = 1,\ 2$. Hence, for any $t > 0$ the pair $U_1,\ U_2$ cannot cover $S_t A^c$. It follows that there exists a sequence of points $x_n = S_n y_n \in S_n A^c$ such that $x_n \notin U_1 \cup U_2$. The asymptotic compactness of the dynamical system enables us to extract a subsequence $\{n_k\}$ such that $x_{n_k} = S_{n_k} y_{n_k}$ tends in X to an element y as $k \to \infty$. It is clear that $y \notin U_1 \cup U_2$ and $y \in \omega(A^c)$. These equations contradict one another since $\omega(A^c) \subset \omega(B) = A \subset U_1 \cup U_2$. Therefore, **Theorem 5.1 is proved** completely.

It should be noted that the connectedness of the global attractor can also be proved without using the linear structure of the phase space (do it yourself).

— Exercise *5.1* Show that the assumption of asymptotic compactness in Theorem 5.1 can be replaced by the Ladyzhenskaya assumption: the sequence $\{S_{t_n} u_n\}$ contains a convergent subsequence for any bounded sequence $\{u_n\} \subset X$ and for any increasing sequence $\{t_n\} \subset \mathbb{T}_+$ such that $t_n \to +\infty$. Moreover, the Ladyzhenskaya assumption is equivalent to the condition of asymptotic compactness.

— Exercise *5.2* Assume that a dynamical system (X, S_t) possesses a compact global attractor A. Let A^* be a minimal closed set with the property

$$\lim_{t \to \infty} \operatorname{dist}(S_t y, A^*) = 0 \quad \text{for every} \quad y \in X.$$

Then $A^* \subset A$ and $A^* = \bigcup\{\omega(x) : x \in X\}$, i.e. A^* coincides with the global minimal attractor (cf. Exercise 3.4).

— Exercise *5.3* Assume that equation (4.7) holds. Prove that the global attractor A possesses the property $A = \omega(K) \subset K$.

— Exercise *5.4* Assume that a dissipative dynamical system possesses a global attractor A. Show that $A = \omega(B)$ for any bounded absorbing set B of the system.

The fact that the global attractor A has the form $A = \omega(B)$, where B is an absorbing set of the system, enables us to state that the set $S_t B$ not only tends to the attractor A, but is also uniformly distributed over it as $t \to \infty$. Namely, the following assertion holds.

Theorem 5.2.

Assume that a dissipative dynamical system (X, S_t) ***possesses a compact global attractor*** A. ***Let*** B ***be a bounded absorbing set for*** (X, S_t). ***Then***

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(a, S_t B) : a \in A\} = 0. \tag{5.2}$$

Proof.

Assume that equation (5.2) does not hold. Then there exist sequences $\{a_n\} \subset A$ and $\{t_n : t_n \to \infty\}$ such that

$$\operatorname{dist}(a_n, S_{t_n} B) \geq \delta \quad \text{for some} \quad \delta > 0. \tag{5.3}$$

The compactness of A enables us to suppose that $\{a_n\}$ converges to an element $a \in A$. Therewith (see Exercise 5.4)

$$a = \lim_{m \to \infty} S_{\tau_m} y_m, \qquad \{y_m\} \subset B,$$

where $\{\tau_m\}$ is a sequence such that $\tau_m \to \infty$. Let us choose a subsequence $\{m_n\}$ such that $\tau_{m_n} \ge t_n + t_B$ for every $n = 1, 2, \ldots$. Here t_B is chosen such that $S_t B \subset \subset B$ for all $t \ge t_B$. Let $z_n = S_{\tau_{m_n} - t_n} y_{m_n}$. Then it is clear that $\{z_n\} \subset B$ and

$$a = \lim_{n \to \infty} S_{\tau_{m_n}} y_{m_n} = \lim_{n \to \infty} S_{t_n} z_n .$$

Equation (5.3) implies that

$$\operatorname{dist}(a_n, S_{t_n} z_n) \ge \operatorname{dist}(a_n, S_{t_n} B) \ge \delta .$$

This contradicts the previous equation. **Theorem 5.2 is proved**.

For a description of convergence of the trajectories to the global attractor it is convenient to use the ***Hausdorff metric*** that is defined on subsets of the phase space by the formula

$$\rho(C, D) = \max\{h(C, D); \quad h(D, C)\}, \tag{5.4}$$

where $C, D \in X$ and

$$h(C, D) = \sup\{\operatorname{dist}(c, D): \ c \in C\}. \tag{5.5}$$

Theorems 5.1 and 5.2 give us the following assertion.

Corollary 5.1.

Let (X, S_t) be an asymptotically compact dissipative system. Then its global attractor A possesses the property $\lim_{t \to \infty} \rho(S_t B, A) = 0$ for any bounded absorbing set B of the system (X, S_t).

In particular, this corollary means that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for every $t > t_\varepsilon$ the set $S_t B$ gets into the ε-vicinity of the global attractor A; and vice versa, the attractor A lies in the ε-vicinity of the set $S_t B$. Here B is a bounded absorbing set.

The following theorem shows that in some cases we can get rid of the requirement of asymptotic compactness if we use the notion of the global weak attractor.

Theorem 5.3.

Let the phase space H of a dynamical system (H, S_t) be a separable Hilbert space. Assume that the system (H, S_t) is dissipative and its evolutionary operator S_t is weakly closed, i.e. for all $t > 0$ the weak convergence $y_n \to y$ and $S_t y_n \to z$ imply that $z = S_t y$. Then the dynamical system (H, S_t) possesses a global weak attractor.

The proof of this theorem basically repeats the reasonings used in the proof of Theorem 5.1. The weak compactness of bounded sets in a separable Hilbert space plays the main role instead of the asymptotic compactness.

Lemma 5.2.

Assume that the hypotheses of Theorem 5.3 *hold. For* $B \subset H$ *we define the weak* ω*-limit set* $\omega_w(B)$ *by the formula*

$$\omega_w(B) = \bigcap_{s \geq 0} \Big[\bigcup_{t \geq s} S_t(B) \Big]_w , \tag{5.6}$$

where $[Y]_w$ *is the weak closure of the set* Y. *Then for any bounded set* $B \subset H$ *the set* $\omega_w(B)$ *is a nonempty weakly closed bounded invariant set.*

Proof.

The dissipativity implies that each of the sets $\gamma_w^s(B) = [\bigcup_{t \geq s} S_t(B)]_w$ is bounded and therefore weakly compact. Then the Cantor theorem on the collection of nested compact sets gives us that $\omega_w(B) = \bigcap_{s \geq 0} \gamma_w^s(B)$ is a nonempty weakly closed bounded set. Let us prove its invariance. Let $y \in \omega_w(B)$. Then there exists a sequence $y_n \in \bigcup_{t \geq n} S_t(B)$ such that $y_n \to y$ weakly. The dissipativity property implies that the set $\{S_t y_n\}$ is bounded when t is large enough. Therefore, there exist a subsequence $\{y_{n_k}\}$ and an element z such that $y_{n_k} \to y$ and $S_t y_{n_k} \to z$ weakly. The weak closedness of S_t implies that $z = S_t y$. Since $S_t y_{n_k} \in \gamma_w^s(B)$ for $n_k \geq s$, we have that $z \in \gamma_w^s(B)$ for all s. Hence, $z \in \omega_w(B)$. Therefore, $S_t \omega_w(B) \subset \omega_w(B)$. The proof of the reverse inclusion is left to the reader as an exercise.

For the proof of Theorem 5.3 it is sufficient to show that the set

$$A_w = \omega_w(B), \tag{5.7}$$

where B is a bounded absorbing set of the system (H, S_t), is a global weak attractor for the system. To do that it is sufficient to verify that the set B is uniformly attracted to $A_w = \omega_w(B)$ in the weak topology of the space H. Assume the contrary. Then there exist a weak vicinity $\mathcal{O}$ of the set A_w and sequences $\{y_n\} \subset B$ and $\{t_n:\ t_n \to \to \infty\}$ such that $S_{t_n} y_n \notin \mathcal{O}$. However, the set $\{S_{t_n} y_n\}$ is weakly compact. Therefore, there exist an element $z \notin \mathcal{O}$ and a sequence $\{n_k\}$ such that

$$z = w - \lim_{k \to \infty} S_{t_{n_k}} y_{n_k} .$$

However, $S_{t_{n_k}} y_{n_k} \in \gamma_w^s(B)$ for $t_{n_k} \geq s$. Thus, $z \in \gamma_w^s(B)$ for all $s \geq 0$ and $z \in \in \omega_w(B)$, which is impossible. **Theorem 5.3 is proved**.

— Exercise *5.5* Assume that the hypotheses of Theorem 5.3 hold. Show that the global weak attractor A_w is a connected set in the weak topology of the phase space H.

— Exercise *5.6* Show that the global weak minimal attractor $A_w^* = \bigcup\{\omega_w(x):\ x \in H\}$ is a strictly invariant set.

— Exercise *5.7* Prove the existence and describe the structure of global and global minimal attractors for the dynamical system generated by the equations

$$\begin{cases} \dot{x} = \mu x - y - x(x^2+y^2), \\ \dot{y} = x + \mu y - y(x^2+y^2) \end{cases}$$

for every real μ.

— Exercise *5.8* Assume that X is a metric space and (X, S_t) is an asymptotically compact (in the sense of the definition given in Remark 4.1) dynamical system. Assume also that the attracting compact K is contained in some bounded connected set. Prove the validity of the assertions of Theorem 5.1 in this case.

In conclusion to this section, we give one more assertion on the existence of the global attractor in the form of exercises. This assertion uses the notion of the asymptotic smoothness (see [3] and [9]). The dynamical system (X, S_t) is said to be ***asymptotically smooth*** if for any bounded positively invariant $(S_t B \subset B,\ t \ge 0)$ set $B \subset X$ there exists a compact K such that $h(S_t B, K) \to 0$ as $t \to \infty$, where the value $h(\cdot, \cdot)$ is defined by formula (5.5).

— Exercise *5.9* Prove that every asymptotically compact system is asymptotically smooth.

— Exercise *5.10* Let (X, S_t) be an asymptotically smooth dynamical system. Assume that for any bounded set $B \subset X$ the set $\gamma^+(B) = \bigcup_{t \ge 0} S_t(B)$ is bounded. Show that the system (X, S_t) possesses a global attractor A of the form

$$A = \bigcup \{ w(B) : \ B \subset X,\ B \text{ is bounded} \}.$$

— Exercise *5.11* In addition to the assumptions of Exercise 5.10 assume that (X, S_t) is pointwise dissipative, i.e. there exists a bounded set $B_0 \subset X$ such that $\mathrm{dist}_X(S_t y, B_0) \to 0$ as $t \to \infty$ for every point $y \in X$. Prove that the global attractor A is compact.

§ 6 *On the Structure of Global Attractor*

The study of the structure of global attractor of a dynamical system is an important problem from the point of view of applications. There are no universal approaches to this problem. Even in finite-dimensional cases the attractor can be of complicated structure. However, some sets that undoubtedly belong to the attractor can be pointed out. It should be first noted that every stationary point of the semigroup S_t belongs to the attractor of the system. We also have the following assertion.

Lemma 6.1.

Assume that an element z lies in the global attractor A of a dynamical system (X, S_t). Then the point z belongs to some trajectory γ that lies in A wholly.

Proof.

Since $S_t A = A$ and $z \in A$, then there exists a sequence $\{z_n\} \subset A$ such that $z_0 = z$, $S_1 z_n = z_{n-1}$, $n = 1, 2, \ldots$. Therewith for discrete time the required trajectory is $\gamma = \{u_n : n \in \mathbb{Z}\}$, where $u_n = S_n z$ for $n \ge 0$ and $u_n = z_{-n}$ for $n \le 0$. For continuous time let us consider the value

$$u(t) = \begin{cases} S_t z, & t \ge 0, \\ S_{t+n} z_n, & -n \le t \le -n+1, \quad n = 1, 2, \ldots \end{cases}$$

Then it is clear that $u(t) \in A$ for all $t \in \mathbb{R}$ and $S_\tau u(t) = u(t+\tau)$ for $\tau \ge 0$, $t \in \mathbb{R}$. Therewith $u(0) = z$. Thus, the required trajectory is also built in the continuous case.

Exercise *6.1* Show that an element z belongs to a global attractor if and only if there exists a bounded trajectory $\gamma = \{u(t) : -\infty < t < \infty\}$ such that $u(0) = z$.

Unstable sets also belong to the global attractor. Let Y be a subset of the phase space X of the dynamical system (X, S_t). Then the ***unstable set emanating from*** Y is defined as the set $\mathbb{M}_+(Y)$ of points $z \in X$ for every of which there exists a trajectory $\gamma = \{u(t) : t \in \mathbb{T}\}$ such that

$$u(0) = z, \qquad \lim_{t \to -\infty} \operatorname{dist}(u(t), Y) = 0.$$

Exercise *6.2* Prove that $\mathbb{M}_+(Y)$ is invariant, i.e. $S_t \mathbb{M}_+(Y) = \mathbb{M}_+(Y)$ for all $t > 0$.

Lemma 6.2.

Let $\mathcal{N}$ be a set of stationary points of the dynamical system (X, S_t) possessing a global attractor A. Then $\mathbb{M}_+(\mathcal{N}) \subset A$.

Proof.

It is obvious that the set $\mathcal{N} = \{z : S_t z = z, \quad t > 0\}$ lies in the attractor of the system and thus it is bounded. Let $z \in \mathbb{M}_+(\mathcal{N})$. Then there exists a trajectory $\gamma_z = \{u(t), t \in \mathbb{T}\}$ such that $u(0) = z$ and

$$\operatorname{dist}(u(\tau), \mathcal{N}) \to 0, \qquad \tau \to -\infty.$$

Therefore, the set $B_s = \{u(\tau):\ \tau \le -s\}$ is bounded when $s > 0$ is large enough. Hence, the set $S_t B_s$ tends to the attractor of the system as $t \to +\infty$. However, $z \in S_t B_s$ for $t \ge s$. Therefore,

$$\operatorname{dist}(z,\ A) \le \sup\{\operatorname{dist}(S_t y,\ A):\ y \in B_s\} \to 0, \qquad t \to +\infty .$$

This implies that $z \in A$. The lemma is proved.

Exercise *6.3* Assume that the set $\mathcal{N}$ of stationary points is finite. Show that

$$\mathbb{M}_+(\mathcal{N}) = \bigcup_{k=1}^{l} \mathbb{M}_+(z_k),$$

where z_k are the stationary points of S_t (the set $\mathbb{M}_+(z_k)$ is called an unstable manifold emanating from the stationary point z_k).

Thus, the global attractor A includes the unstable set $\mathbb{M}_+(\mathcal{N})$. It turns out that under certain conditions the attractor includes nothing else. We give the following definition. Let Y be a positively invariant set of a semigroup S_t: $S_t Y \subset Y$, $t > 0$. The continuous functional $\Phi(y)$ defined on Y is called the ***Lyapunov function*** of the dynamical system $(X,\ S_t)$ on Y if the following conditions hold:

a) for any $y \in Y$ the function $\Phi(S_t y)$ is a nonincreasing function with respect to $t \ge 0$;

b) if for some $t_0 > 0$ and $y \in X$ the equation $\Phi(y) = \Phi(S_{t_0} y)$ holds, then $y = S_t y$ for all $t \ge 0$, i.e. y is a stationary point of the semigroup S_t.

Theorem 6.1.

Let a dynamical system $(X,\ S_t)$ possess a compact attractor A. Assume also that the Lyapunov function $\Phi(y)$ exists on A. Then $A = \mathbb{M}_+(\mathcal{N})$, where $\mathcal{N}$ is the set of stationary points of the dynamical system.

Proof.

Let $y \in A$. Let us consider a trajectory γ passing through y (its existence follows from Lemma 6.1). Let

$$\gamma = \{u(t):\ t \in \mathbb{T}\} \quad \text{and} \quad \gamma_\tau^- = \{u(t):\ t \le \tau\}.$$

Since $\gamma_\tau^- \subset A$, the closure $[\gamma_\tau^-]$ is a compact set in X. This implies that the α-limit set

$$\alpha(\gamma) = \bigcap_{\tau < 0} [\gamma_\tau^-]$$

of the trajectory γ is nonempty. It is easy to verify that the set $\alpha(\gamma)$ is invariant: $S_t \alpha(\gamma) = \alpha(\gamma)$. Let us show that the Lyapunov function $\Phi(y)$ is constant on $\alpha(\gamma)$. Indeed, if $u \in \alpha(\gamma)$, then there exists a sequence $\{t_n\}$ tending to $-\infty$ such that

$$\lim_{t_n \to -\infty} u(t_n) = u .$$

Consequently,

$$\Phi(u) = \lim_{n \to \infty} \Phi(u(t_n)) .$$

By virtue of monotonicity of the function $\Phi(u)$ along the trajectory we have

$$\Phi(u) = \sup\{\Phi(u(\tau)) : \ \tau < 0\} .$$

Therefore, the function $\Phi(u)$ is constant on $\alpha(\gamma)$. Hence, the invariance of the set $\alpha(\gamma)$ gives us that $\Phi(S_t u) = \Phi(u)$, $t > 0$ for all $u \in \alpha(\gamma)$. This means that $\alpha(\gamma)$ lies in the set $\mathcal{N}$ of stationary points. Therewith (verify it yourself)

$$\lim_{t \to -\infty} \operatorname{dist}(u(t), \ \alpha(\gamma)) = 0 .$$

Hence, $y \in \mathbb{M}_+(\mathcal{N})$. **Theorem 6.1 is proved**.

— Exercise *6.4* Assume that the hypotheses of Theorem 6.1 hold. Then for any element $y \in A$ its ω-limit set $\omega(y)$ consists of stationary points of the system.

Thus, the global attractor coincides with the set of all full trajectories connecting the stationary points.

— Exercise *6.5* Assume that the system (X, S_t) possesses a compact global attractor and there exists a Lyapunov function on X. Assume that the Lyapunov function is bounded below. Show that any semitrajectory of the system tends to the set $\mathcal{N}$ of stationary points of the system as $t \to +\infty$, i.e. the global minimal attractor coincides with the set $\mathcal{N}$.

In particular, this exercise confirms the fact realized by many investigators that the global attractor is a too wide object for description of actually observed limit regimes of a dynamical system.

— Exercise *6.6* Assume that $(\mathbb{R}, S_t)$ is a dynamical system generated by the logistic equation (see Example 1.1): $\dot{x} + \alpha x(x-1) = 0$, $\alpha > 0$. Show that $V(x) = x^3/3 - x^2/2$ is a Lyapunov function for this system.

— Exercise *6.7* Show that the total energy

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4 - \frac{a}{2}x^2 - bx$$

is a Lyapunov function for the dynamical system generated (see Example 1.2) by the Duffing equation

$$\ddot{x} + \varepsilon\dot{x} + x^3 - ax = b , \quad \varepsilon > 0 .$$

If in the definition of a Lyapunov functional we omit the second requirement, then a minor modification of the proof of Theorem 6.1 enables us to get the following assertion.

Theorem 6.2.

Assume that a dynamical system (X, S_t) ***possesses a compact global attractor*** A ***and there exists a continuous function*** $\Psi(y)$ ***on*** X ***such that*** $\Psi(S_t y)$ ***does not increase with respect to*** t ***for any*** $y \in X$***. Let*** $\mathcal{L}$ ***be a set of elements*** $u \in A$ ***such that*** $\Psi(u(t)) = \Psi(u)$ ***for all*** $-\infty < t < \infty$***. Here*** $\{u(t)\}$ ***is a trajectory of the system passing through*** u $(u(0) = u)$***. Then*** $\mathbb{M}_+(\mathcal{L}) = A$ ***and*** $\mathcal{L}$ ***contains the global minimal attractor*** $A^* = \bigcup_{x \in X} \omega(x)$.

Proof.

In fact, the property $\mathbb{M}_+(\mathcal{L}) = A$ was established in the proof of Theorem 6.1. As to the property $A^* \subset \mathcal{L}$, it follows from the constancy of the function $\Psi(u)$ on the ω-limit set $\omega(x)$ of any element $x \in X$.

— Exercise *6.8* Apply Theorem 6.2 to justify the results of Example 3.1 (see also Exercise 4.8).

If the set $\mathcal{N}$ of stationary points of a dynamical system (X, S_t) is finite, then Theorem 6.1 can be extended a little. This extension is described below in Exercises 6.9–6.12. In these exercises it is assumed that the dynamical system (X, S_t) is continuous and possesses the following properties:

(a) there exists a compact global attractor A ;

(b) there exists a Lyapunov function $\Phi(x)$ on A ;

(c) the set $\mathcal{N}=\{z_1, \dots z_N\}$ of stationary points is finite, therewith $\Phi(z_i) \neq \Phi(z_j)$ for $i \neq j$ and the indexing of z_j possesses the property

$$\Phi(z_1) < \Phi(z_2) < \dots < \Phi(z_N) . \tag{6.1}$$

We denote

$$A_j = \bigcup_{k=1}^{j} \mathbb{M}_+(z_k) , \qquad j = 1, 2, \dots, N \ , \quad A_0 = \varnothing .$$

— Exercise *6.9* Show that $S_t A_j = A_j$ for all $j = 1, 2, \dots N$.

— Exercise *6.10* Assume that $B \subset A_j \backslash \{z_j\}$. Then

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(S_t y, A_{j-1}) : y \in B\} = 0 . \tag{6.2}$$

— Exercise *6.11* Assume that the function Φ is defined on the whole X. Then (6.2) holds for any bounded set $B \subset \{x : \Phi(x) < \Phi(z_j) - \delta\}$, where δ is a positive number.

— Exercise *6.12* Assume that $[\mathbb{M}_+(z_j)]$ is the closure of the set $\mathbb{M}_+(z_j)$ and $\partial\mathbb{M}_+(z_j) = [\mathbb{M}_+(z_j)] \setminus \mathbb{M}_+(z_j)$ is its boundary. Show that $\partial\mathbb{M}_+ z_j \subset \subset A_{j-1}$ and

$$S_t [\mathbb{M}_+(z_j)] = [\mathbb{M}_+(z_j)], \qquad S_t \partial\mathbb{M}_+(z_j) = \partial\mathbb{M}_+(z_j).$$

It can also be shown (see the book by A. V. Babin and M. I. Vishik [1]) that under some additional conditions on the evolutionary operator S_t the unstable manifolds $\mathbb{M}_+(z_j)$ are surfaces of the class C^1, therewith the facts given in Exercises 6.9–6.12 remain true if strict inequalities are substituted by nonstrict ones in (6.1). It should be noted that a global attractor possessing the properties mentioned above is frequently called ***regular***.

Let us give without proof one more result on the attractor of a system with a finite number of stationary points and a Lyapunov function. This result is important for applications.

At first let us remind several definitions. Let S be an operator acting in a Banach space X. The operator S is called ***Frechét differentiable at a point*** $x \in X$ provided that there exists a linear bounded operator $S'(x)\colon X \to X$ such that

$$\|S(y) - S(x) - S'(x)(y - x)\| \le \gamma(\|x - y\|)\|x - y\|$$

for all y from some vicinity of the point x, where $\gamma(\xi) \to 0$ as $\xi \to 0$. Therewith, the operator S is said to belong to the class $C^{1+\alpha}$, $0 < \alpha < 1$, on a set Y if it is ***differentiable at every point*** $x \in Y$ and

$$\|S'(x) - S'(y)\|_{L(X,\,X)} \le C\|x - y\|^{\alpha}$$

for all y from some vicinity of the point $x \in Y$. A stationary point z of the mapping S is called ***hyperbolic*** if $S \in C^{1+\alpha}$ in some vicinity of the point z, the spectrum of the linear operator $S'(z)$ does not cross the unit circle $\{\lambda\colon |\lambda| = 1\}$ and the spectral subspace of the operator corresponding to the set $\{\lambda\colon |\lambda| > 1\}$ is finite-dimensional.

Theorem 6.3.

Let X be a Banach space and let a continuous dynamical system (X, S_t) possess the properties:

1) *there exists a global attractor A;*
2) *there exists a vicinity Ω of the attractor A such that*

$$\|S_t x - S_t y\| \le C e^{\alpha(t-\tau)} \|S_\tau x - S_\tau y\|$$

for all $t \ge \tau \ge 0$, provided $S_t x$ and $S_t y$ belong to Ω for all $t \ge 0$;

3) *there exists a Lyapunov function continuous on X;*
4) *the set $\mathcal{N} = \{z_1, \ldots, z_N\}$ of stationary points is finite and all the points are hyperbolic;*

5) ***the mapping*** $(t, u) \to S_t u$ ***is continuous.***
Then for any compact set B ***in*** X ***the estimate***

$$\sup\left\{\operatorname{dist}(S_t y, A): \ y \in B\right\} \le C_B e^{-\eta t} \tag{6.3}$$

holds for all $t \ge 0$***, where*** $\eta > 0$ ***does not depend on*** B.

The proof of this theorem as well as other interesting results on the asymptotic behaviour of a dynamical system possessing a Lyapunov function can be found in the book by A. V. Babin and M. I. Vishik [1].

To conclude this section, we consider a finite-dimensional example that shows how the Lyapunov function method can be used to prove the existence of periodic trajectories in the attractor.

Example *6.1* (on the theme by E. Hopf)

Studying Galerkin approximations in a model suggested by E. Hopf for the description of possible mechanisms of turbulence appearence, we obtain the following system of ordinary differential equations

$$\begin{cases} \dot u + \mu u + v^2 + w^2 = 0, & (6.4)\\ \dot v + \nu v - v u - \beta w = 0, & (6.5)\\ \dot w + \nu w - w u + \beta v = 0. & (6.6)\end{cases}$$

Here μ is a positive parameter, ν and β are real parameters. It is clear that the Cauchy problem for (6.4)–(6.6) is solvable, at least locally for any initial condition. Let us show that the dynamical system generated by equations (6.4)–(6.6) is dissipative. It will also be sufficient for the proof of global solvability. Let us introduce a new unknown function $u^* = u + \mu/2 - \nu$. Then equations (6.4)–(6.6) can be rewritten in the form

$$\begin{cases} \dot u^* + \mu u^* + v^2 + w^2 = \mu\left(\frac{\mu}{2} - \nu\right),\\ \dot v + \frac{1}{2}\mu v - v u^* - \beta w = 0,\\ \dot w + \frac{1}{2}\mu w - w u^* + \beta v = 0.\end{cases}$$

These equations imply that

$$\frac{1}{2}\frac{d}{dt}\left(|u^*|^2 + |v|^2 + |w|^2\right) + \mu|u^*|^2 + \frac{\mu}{2}\left(|v|^2 + |w|^2\right) = \mu\left(\frac{\mu}{2} - \nu\right)u^*$$

on any interval of existence of solutions. Hence,

$$\frac{d}{dt}\left(\left(|u^*|^2 + |v|^2 + |w|^2\right)\right) + \mu\left(|u^*|^2 + |v|^2 + |w|^2\right) \le \mu\left(\frac{\mu}{2} - \nu\right)^2 .$$

Thus,

$$|u^*(t)|^2 + |v(t)|^2 + |w(t)|^2 \le$$
$$\le \left(|u^*(0)|^2 + |v(0)|^2 + |w(0)|^2\right) e^{-\mu t} + \left(\frac{\mu}{2} - \nu\right)^2 (1 - e^{-\mu t}) \ .$$

Firstly, this equation enables us to prove the global solvability of problem (6.4)–(6.6) for any initial condition and, secondly, it means that the set

$$B_0 = \left\{(u,\ v,\ w):\ \left(u + \frac{\mu}{2} - \nu\right)^2 + v^2 + w^2 \le 1 + \left(\frac{\mu}{2} - \nu\right)^2\right\}$$

is absorbing for the dynamical system $(\mathbb{R}^3,\ S_t)$ generated by the Cauchy problem for equations (6.4)–(6.6). Thus, Theorem 5.1 guarantees the existence of a global attractor A. It is a connected compact set in $\mathbb{R}^3$.

— Exercise *6.13* Verify that B_0 is a positively invariant set for $(\mathbb{R}^3,\ S_t)$.

In order to describe the structure of the global attractor A we introduce the polar coordinates

$$v(t) = r(t)\cos\varphi(t), \qquad w(t) = r(t)\sin\varphi(t)$$

on the plane of the variables $\{v;\ w\}$. As a result, equations (6.4)–(6.6) are transformed into the system

$$\begin{cases} \dot{u} + \mu u + r^2 = 0, & (6.7) \\ \dot{r} + \nu r - u r = 0, & (6.8) \end{cases}$$

therewith, $\varphi(t) = -\beta t + \varphi_0$. System (6.7) and (6.8) has a stationary point $\{u = 0,\ r = 0\}$ for all $\mu > 0$ and $\nu \in \mathbb{R}$. If $\nu < 0$, then one more stationary point $\{u = \nu,\ r = \sqrt{-\mu\nu}\}$ occurs in system (6.7) and (6.8). It corresponds to a periodic trajectory of the original problem (6.4)–(6.6).

— Exercise *6.14* Show that the point $(0;\ 0)$ is a stable node of system (6.7) and (6.8) when $\nu > 0$ and it is a saddle when $\nu < 0$.

— Exercise *6.15* Show that the stationary point $\{u = \nu,\ r = \sqrt{-\mu\nu}\}$ is stable ($\nu < 0$) being a node if $-\mu/8 < \nu < 0$ and a focus if $\nu < -\dfrac{\mu}{8}$.

If $\nu > 0$, then (6.7) and (6.8) imply that

$$\frac{1}{2}\frac{d}{dt}(u^2 + r^2) + \min(\mu,\ \nu)(u^2 + r^2) \le 0 .$$

Therefore,

$$|u(t)|^2 + |r(t)|^2 \le |u(0)|^2 + |r(0)|^2\, e^{-2\min(\mu,\ \nu)t} .$$

Hence, for $\nu > 0$ the global attractor A of the system $(\mathbb{R}^3, S_t)$ consists of the single stationary exponentially attracting point

$$\{u = 0,\ v = 0,\ w = 0\}.$$

Exercise *6.16* Prove that for $\nu = 0$ the global attractor of problem (6.4)–(6.6) consists of the single stationary point $\{u = 0,\ v = 0,\ w = 0\}$. Show that it is not exponentially attracting.

Now we consider the case $\nu < 0$. Let us again refer to problem (6.7) and (6.8). It is clear that the line $r = 0$ is a stable manifold of the stationary point $\{u = 0,\ r = 0\}$. Moreover, it is obvious that if $r(t_0) > 0$, then the value $r(t)$ remains positive for all $t > t_0$. Therefore, the function

$$V(u,\ r) = \frac{1}{2}(u - \nu)^2 + \frac{1}{2}r^2 + \mu\nu \ln r \tag{6.9}$$

is defined on all the trajectories, the initial point of which does not lie on the line $\{r = 0\}$. Simple calculations show that

$$\frac{d}{dt}(V(u(t),\ r(t))) + \mu(u(t) - \nu)^2 = 0 \tag{6.10}$$

and

$$V(u,\ r) \ \ge\ V(\nu,\ \sqrt{-\mu\nu}) + \frac{1}{2}\left(|u - \nu|^2 + |r - \sqrt{-\mu\nu}|^2\right); \tag{6.11}$$

therewith, $V(\nu,\ \sqrt{-\mu\nu}) = (1/2)\,\mu|\nu|\,\ln(e/(\mu|\nu|))$. Equation (6.10) implies that the function $V(u,\ r)$ does not increase along the trajectories. Therefore, any semi-trajectory $\{(u(t);\ r(t)),\ t \in \mathbb{R}_+\}$ emanating from the point $\{u_0,\ r_0;\ \ r_0 \ne 0\}$ of the system $(\mathbb{R} \times \mathbb{R}_+,\ S_t)$ generated by equations (6.7) and (6.8) possesses the property $V(u(t),\ r(t)) \le V(u_0,\ r_0)$ for $t \ge 0$. Therewith, equation (6.9) implies that this semitrajectory can not approach the line $\{r = 0\}$ at a distance less then $\exp\{[1/(\mu\nu)] \cdot V(u_0,\ r_0)\}$. Hence, this semitrajectory tends to $\bar{y} = \{u = \nu,\ r = \sqrt{-\mu\nu}\}$. Moreover, for any $\xi \in \mathbb{R}$ the set

$$B_\xi = \{y = (u,\ r):\ V(u,\ r) \le \xi\}$$

is uniformly attracted to $\bar{y}$, i.e. for any $\varepsilon > 0$ there exists $t_0 = t_0(\xi,\ \varepsilon)$ such that

$$S_t B_\xi \subset \{y:\ |y - \bar{y}| \le \varepsilon\}.$$

Indeed, if it is not true, then there exist $\varepsilon_0 > 0$, a sequence $t_n \to +\infty$, and $z_n \in B_\xi$ such that $\left|S_{t_n} z_n - \bar{y}\right| > \varepsilon_0$. The monotonicity of $V(y)$ and property (6.11) imply that

$$V(S_t z_n) \ \ge\ V(S_{t_n} z_n) \ \ge\ V(\nu,\ \sqrt{-\mu\nu}) + \frac{1}{2}\varepsilon_0^2$$

for all $0 \le t \le t_n$. Let z be a limit point of the sequence $\{z_n\}$. Then after passing to the limit we find out that

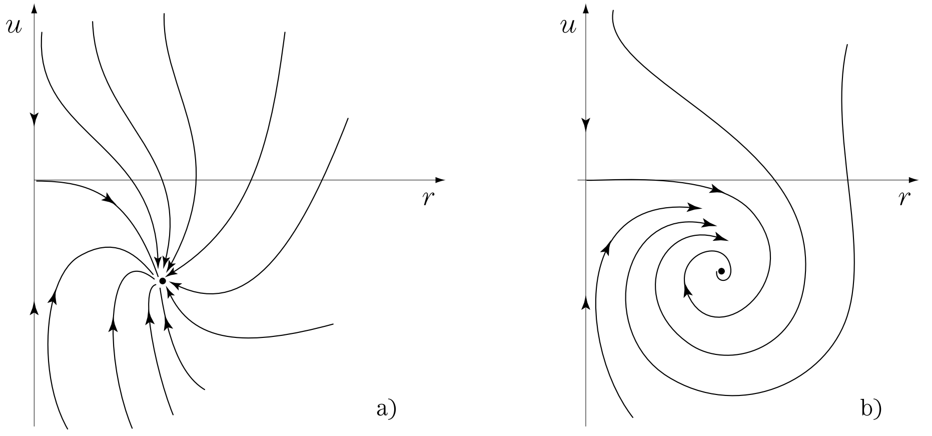

Fig. 2. Qualitative behaviour of solutions to problem (6.7), (6.8):
a) $-\mu/8 < \nu < 0$, b) $\nu < -\mu/8$

$$V(S_t z) \geq V(\nu, \sqrt{-\mu\nu}) + \frac{1}{2}\varepsilon_0^2, \qquad t \geq 0$$

with $z \notin \{r = 0\}$. Thus, the last inequality is impossible since $S_t z \to \bar{y} = \{u = \nu, r = \sqrt{-\mu\nu}\}$. Hence

$$\lim_{t\to\infty} \sup\{\operatorname{dist}(S_t y, \bar{y}):\ y \in B_\xi\} = 0. \tag{6.12}$$

The qualitative behaviour of solutions to problem (6.7) and (6.8) on the semiplane is shown on *Fig. 2*.

In particular, the observations above mean that the global minimal attractor $A_{\min}$ of the dynamical system $(\mathbb{R}^3, S_t)$ generated by equations (6.4)–(6.6) consists of the saddle point $\{u = 0,\ v = 0,\ w = 0\}$ and the stable limit cycle

$$C_\nu = \{u = \nu, \quad v^2 + w^2 = -\mu\nu\} \tag{6.13}$$

for $\nu < 0$. Therewith, equation (6.12) implies that the cycle C_ν uniformly attracts all bounded sets B in $\mathbb{R}^3$ possessing the property

$$d \equiv \inf\{v^2 + w^2:\ (u, v, w) \in B\} > 0, \tag{6.14}$$

i.e. which lie at a positive distance from the line $\{v = 0,\ w = 0\}$.

Exercise *6.17* Using the structure of equations (6.7) and (6.8) near the stationary point $\{u = \nu,\ r = \sqrt{-\mu\nu}\}$, prove that a bounded set B possessing property (6.14) is uniformly and exponentially attracted to the cycle C_ν, i.e.

$$\sup\{\operatorname{dist}(S_t y, C_\nu),\quad y \in B\} \leq C e^{-\gamma(t - t_B)}$$

for $t \geq t_B$, where γ is a positive constant.

Now let $y_0 = (u_0,\ v_0,\ w_0)$ lie in the global attractor A of the system $(\mathbb{R}^3,\ S_t)$. Assume that $r_0 \neq 0$ and $r_0^2 = v_0^2 + w_0^2 \neq -\mu\nu$. Then (see Lemma 6.1) there exists a trajectory $\gamma = \{y(t) = (u(t);\ v(t);\ w(t)),\quad t \in \mathbb{R}\}$ lying in A such that $y(0) = y_0$. The analysis given above shows that $y(t) \to C_\nu$ as $t \to +\infty$. Let us show that $y(t) \to 0$ when $t \to -\infty$. Indeed, the function $V(u(t),\ r(t))$ is monotonely nondecreasing as $t \to -\infty$. If we argue by contradiction and use the fact that $|y(t)|$ is bounded we can easily find out that

$$\lim_{t \to -\infty} V(u(t),\ r(t)) = \infty$$

and therefore

$$r(t) = \left(|v(t)|^2 + |w(t)|^2\right)^{1/2} \to 0 \quad \text{as} \quad t \to -\infty. \tag{6.16}$$

Equation (6.7) gives us that

$$u(t) = e^{-\mu(t-\tau)}u(s) - \int_s^t e^{-\mu(t-\tau)}[r(\tau)]^2\, d\tau. \tag{6.17}$$

Since $u(s)$ is bounded for all $s \in \mathbb{R}$, we can get the equation

$$u(t) = -\int_{-\infty}^t e^{-\mu(t-\tau)}[r(\tau)]^2\, d\tau$$

by tending $s \to -\infty$ in (6.17). Therefore, by virtue of (6.16) we find that $u(t) \to 0$ as $t \to -\infty$. Thus, $y(t) \to 0$ as $t \to -\infty$. Hence, for $\nu < 0$ the global attractor A

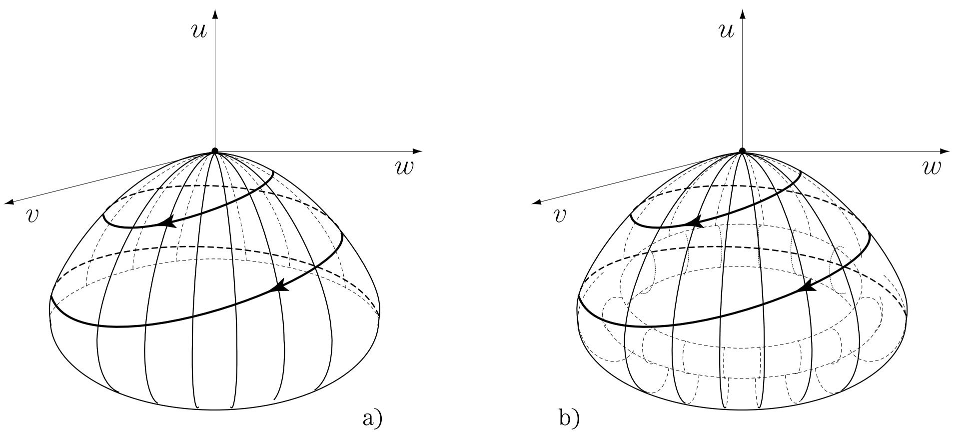

Fig. 3. Attractor of the system (6.4)–(6.6);
a) $-\mu/8 < \nu < 0$, b) $\nu < -\mu/8$

of the system $(\mathbb{R}^3, S_t)$ coincides with the union of the unstable manifold $\mathbb{M}_+(0)$ emanating from the point $\{u = 0,\ v = 0,\ w = 0\}$ and the limit cycle (6.13). The attractor is shown on *Fig. 3*.

§ 7 Stability Properties of Attractor and Reduction Principle

A positively invariant set M in the phase space of a dynamical system (X, S_t) is said to be ***stable (in Lyapunov's sense)*** in X if its every vicinity $\mathcal{O}$ contains some vicinity $\mathcal{O}'$ such that $S_t(\mathcal{O}') \subset \mathcal{O}$ for all $t \ge 0$. Therewith, M is said to be ***asymptotically stable*** if it is stable and $S_t y \to M$ as $t \to \infty$ for every $y \in \mathcal{O}'$. A set M is called ***uniformly asymptotically stable*** if it is stable and

$$\lim_{t \to \infty} \sup\{\operatorname{dist}(S_t y, M) :\ y \in \mathcal{O}'\} = 0. \tag{7.1}$$

The following simple assertion takes place.

Theorem 7.1.

Let A be the compact global attractor of a continuous dynamical system (X, S_t). Assume that there exists its bounded vicinity U such that the mapping $(t, u) \to S_t u$ is continuous on $\mathbb{R}_+ \times U$. Then A is a stable set.

Proof.

Assume that $\mathcal{O}$ is a vicinity of A. Then there exists $T > 0$ such that $S_t U \subset \mathcal{O}$ for $t \ge T$. Let us show that there exists a vicinity $\mathcal{O}'$ of the attractor A such that $S_t \mathcal{O}' \subset \mathcal{O}$ for all $t \in [0, T]$. Assume the contrary. Then there exist sequences $\{u_n\}$ and $\{t_n\}$ such that $\operatorname{dist}(u_n, A) \to 0$, $\{t_n\} \in [0, T]$ and $S_{t_n} u_n \notin \mathcal{O}$. The set A being compact, we can choose a subsequence $\{n_k\}$ such that $u_{n_k} \to u \in A$ as $t_{n_k} \to$ $\to t \in [0, T]$. Therefore, the continuity property of the function $(t, u) \to S_t u$ gives us that $S_{t_{n_k}} u_{n_k} \to S_t u \in A$. This contradicts the equation $S_{t_n} u_n \notin \mathcal{O}$. Thus, there exists $\mathcal{O}'$ such that $S_t \mathcal{O}' \subset \mathcal{O}$ for $t \in [0, T]$. We can choose T such that $S_t(\mathcal{O}' \cap U) \subset \mathcal{O}$ for all $t \ge 0$. Therefore, the attractor A is stable. **Theorem 7.1 is proved**.

It is clear that the stability of the global attractor implies its uniform asymptotic stability.

Exercise *7.1* Assume that M is a positively invariant set of a system (X, S_t). Prove that if there exists an element $y \notin M$ such that its α-limit set $\alpha(y)$ possesses the property $\alpha(y) \cap M \ne \varnothing$, then M is not stable.

In particular, the result of this exercise shows that the global minimal attractor can appear to be an unstable set.

— Exercise *7.2* Let us return to Example 3.1 (see also Exercises 4.8 and 6.8). Show that:

(a) the global attractor A_1 and the Milnor attractor A_4 are stable;

(b) the global minimal attractor A_3 and the Ilyashenko attractor A_5 are unstable.

Now let us consider the question concerning the stability of the attractor with respect to perturbations of a dynamical system. Assume that we have a family of dynamical systems (X, S_t^λ) with the same phase space X and with an evolutionary operator S_t^λ depending on a parameter λ which varies in a complete metric space Λ. The following assertion was proved by L. V. Kapitansky and I. N. Kostin [6].

Theorem 7.2.

Assume that a dynamical system (X, S_t^λ) ***possesses a compact global attractor*** A^λ ***for every*** $\lambda \in \Lambda$. ***Assume that the following conditions hold:***

(a) ***there exists a compact*** $K \subset X$ ***such that*** $A^\lambda \subset K$ ***for all*** $\lambda \in \Lambda$;

(b) ***if*** $\lambda_k \to \lambda_0$, $x_k \in A^{\lambda_k}$ ***and*** $x_k \to x_0$, ***then*** $S_{t_0}^{\lambda_k} x_k \to S_{t_0} x_0$ ***for some*** $t_0 > 0$.

Then the family of attractors A^λ ***is upper semicontinuous at the point*** λ_0, ***i.e.***

$$h(A^{\lambda_k}, A^{\lambda_0}) \equiv \sup\{\operatorname{dist}(y, A^{\lambda_0}) : \; y \in A^{\lambda_k}\} \to 0 \tag{7.2}$$

as $\lambda_k \to \lambda_0$.

Proof.

Assume that equation (7.2) does not hold. Then there exist a sequence $\lambda_k \to \to \lambda_0$ and a sequence $x_k \in A^{\lambda_k}$ such that $\operatorname{dist}(x_k, A^{\lambda_0}) \ge \delta$ for some $\delta > 0$. But the sequence x_k lies in the compact K. Therefore, without loss of generality we can assume that $x_k \to x_0 \in K$ for some $x_0 \in K$ and $x_0 \notin A^{\lambda_0}$. Let us show that this result leads to contradiction. Let $\gamma_k = \{u_k(t) : \; -\infty < t < \infty\}$ be a trajectory of the dynamical system $(X, S_t^{\lambda_k})$ passing through the element x_k ($u_k(0) = x_k$). Using the standard diagonal process it is easy to find that there exist a subsequence $\{k(n)\}$ and a sequence of elements $\{u_m\} \subset K$ such that

$$\lim_{n \to \infty} u_{k(n)}(-m t_0) = u_m \quad \text{for all} \quad m = 0, 1, 2, \dots ,$$

where $u_0 = x_0$. Here $t_0 > 0$ is a fixed number. Sequential application of condition (b) gives us that

$$u_{m-l} = \lim_{n\to\infty} u_{k(n)}(-(m-l)t_0) = \lim_{n\to\infty} S_{lt_0}^{\lambda_{k(n)}} u_{k(n)}(-mt_0) = S_{lt_0}^{\lambda_0} u_m$$

for all $m = 1, 2, \ldots$ and $l = 1, 2, \ldots, m$. It follows that the function

$$u(t) = \begin{cases} S_t^{\lambda_0} u_0, & t \ge 0, \\ S_{t+t_0 m}^{\lambda_0} u_m, & -t_0 m \le t < -t_0(m-1), \quad m = 1, 2, \ldots \end{cases}$$

gives a full trajectory γ passing through the point x_0. It is obvious that the trajectory γ is bounded. Therefore (see Exercise 6.1), it wholly belongs to A^{λ_0}, but that contradicts the equation $x_0 \notin A^{\lambda_0}$. **Theorem 7.2 is proved**.

Exercise *7.3* Following L. V. Kapitansky and I. N. Kostin [6], for $\lambda \to \lambda_0$ define the upper limit $A(\lambda_0; \Lambda)$ of the attractors A^λ along Λ by the equality

$$A(\lambda_0, \Lambda) = \bigcap_{\delta > 0} \left[\bigcup \{A^\lambda : \lambda \in \Lambda, \quad 0 < \operatorname{dist}(\lambda, \lambda_0) < \delta\} \right],$$

where $[\,\cdot\,]$ denotes the closure operation. Prove that if the hypotheses of Theorem 7.2 hold, then $A(\lambda_0, \Lambda)$ is a nonempty compact invariant set lying in the attractor A^{λ_0}.

Theorem 7.2 embraces only the upper semicontinuity of the family of attractors $\{A^\lambda\}$. In order to prove their continuity (in the Hausdorff metric defined by equation (5.4)), additional conditions should be imposed on the family of dynamical systems (X, S_t^λ). For example, the following assertion proved by A. V. Babin and M. I. Vishik concerning the power estimate of the deviation of the attractors A^λ and A^{λ_0} in the Hausdorff metric holds.

Theorem 7.3.

Assume that a dynamical system (X, S_t^λ) ***possesses a global attractor*** A^λ ***for every*** $\lambda \in \Lambda$. ***Let the following conditions hold:***

(a) ***there exists a bounded set*** $B_0 \subset X$ ***such that*** $A^\lambda \subset B_0$ ***for all*** $\lambda \in \Lambda$ ***and***

$$h(S_t^\lambda B_0, A^\lambda) \le C_0 e^{-\eta t}, \qquad \lambda \in \Lambda, \tag{7.3}$$

with constants $C_0 > 0$ ***and*** $\eta > 0$ ***independent of*** λ ***and with***

$$h(B, A) = \sup\{\operatorname{dist}(b, A) : b \in B\};$$

(b) ***for any*** $\lambda_1, \lambda_2 \in \Lambda$ ***and*** $u_1, u_2 \in B_0$ ***the estimate***

$$\operatorname{dist}(S_t^{\lambda_1} u_1, S_t^{\lambda_2} u_2) \le C_1 e^{\alpha t}(\operatorname{dist}(u_1, u_2) + \operatorname{dist}(\lambda_1, \lambda_2)) \tag{7.4}$$

holds, with constants C_1 ***and*** α ***independent of*** λ.

Then there exists $C_2 > 0$ ***such that***

$$\rho(A^{\lambda_1}, A^{\lambda_2}) \le C_2[\operatorname{dist}(\lambda_1, \lambda_2)]^q, \qquad q = \frac{\eta}{\eta+\alpha}. \tag{7.5}$$

Here $\rho(\cdot\,,\cdot)$ ***is the Hausdorff metric defined by the formula***

$$\rho(B, A) = \max\{h(B, A);\ \ h(A, B)\}.$$

Proof.

By virtue of the symmetry of (7.5) it is sufficient to find out that

$$h(A^{\lambda_1}, A^{\lambda_2}) \le C_2[\operatorname{dist}(\lambda_1, \lambda_2)]^q. \tag{7.6}$$

Equation (7.3) implies that for any $\varepsilon > 0$

$$S_t^{\lambda} B_0 \subset \mathcal{O}_\varepsilon(A^\lambda) \quad \text{for all} \quad \lambda \in \Lambda \tag{7.7}$$

when $t \ge t^*(\varepsilon, C_0) \equiv \eta^{-1}(\ln 1/\varepsilon + \ln C_0)$. Here $\mathcal{O}_\varepsilon(A^\lambda)$ is an ε-vicinity of the set A^λ. It follows from equation (7.4) that

$$h(S_t^{\lambda_1} B_0, S_t^{\lambda_2} B_0) = \sup_{x \in B_0} \inf_{y \in B_0} \operatorname{dist}(S_t^{\lambda_1} x, S_t^{\lambda_2} y) \le$$

$$\le \sup_{x \in B_0} \operatorname{dist}(S_t^{\lambda_1} x, S_t^{\lambda_2} x) \le C_1 e^{\alpha t} \operatorname{dist}(\lambda_1, \lambda_2). \tag{7.8}$$

Since $A^\lambda \subset B_0$, we have $A^\lambda = S_t^\lambda A^\lambda \subset S_t^\lambda B_0$. Therefore, with $t \ge t^*(\varepsilon, C_0)$, equation (7.7) gives us that

$$A^\lambda \subset S_t^\lambda B_0 \subset \mathcal{O}_\varepsilon(A^\lambda). \tag{7.9}$$

For any $x, z \in X$ the estimate

$$\operatorname{dist}(x, A^\lambda) \le \operatorname{dist}(x, z) + \operatorname{dist}(z, A^\lambda)$$

holds. Hence, we can find that

$$\operatorname{dist}(x, A^\lambda) \le \operatorname{dist}(x, z) + \varepsilon$$

for all $x \in X$ and $z \in \mathcal{O}_\varepsilon(A^\lambda)$. Consequently, equation (7.9) implies that

$$\operatorname{dist}(x, A^\lambda) \le \operatorname{dist}(x, S_t^\lambda B_0) + \varepsilon, \qquad x \in X$$

for $t \ge t^*(\varepsilon, C_0)$. It means that

$$h(A^{\lambda_1}, A^{\lambda_2}) = \sup_{x \in A^{\lambda_1}} \operatorname{dist}(x, A^{\lambda_2}) \le$$

$$\le \sup_{x \in A^{\lambda_1}} \operatorname{dist}(x, S_t^{\lambda_2} B_0) + \varepsilon \le h(S_t^{\lambda_1} B_0, S_t^{\lambda_2} B_0) + \varepsilon.$$

Thus, equation (7.8) gives us that for any $\varepsilon > 0$

$$h(A^{\lambda_1}, A^{\lambda_2}) \le C_1 e^{\alpha t} \operatorname{dist}(\lambda_1, \lambda_2) + \varepsilon$$

for $t \ge t^*(\varepsilon, C_0)$. By taking $\varepsilon = [\operatorname{dist}(\lambda_1, \lambda_2)]^q$, $q = \dfrac{\eta}{\eta + \alpha}$ and $t = t^*(\varepsilon, C_0) \equiv \equiv \eta^{-1}(\ln 1/\varepsilon + \ln C_0)$ in this formula we find estimate (7.6). **Theorem 7.3 is proved**.

It should be noted that condition (7.3) in Theorem 7.3 is quite strong. It can be verified only for a definite class of systems possessing the Lyapunov function (see Theorem 6.3).

In the theory of dynamical systems an important role is also played by the notion of the Poisson stability. A trajectory $\gamma = \{u(t):\ -\infty < t < \infty\}$ of a dynamical system (X, S_t) is said to be ***Poisson stable*** if it belongs to its ω-limit set $\omega(\gamma)$. It is clear that stationary points and periodic trajectories of the system are Poisson stable.

— Exercise *7.4* Show that any Poisson stable trajectory is contained in the global minimal attractor if the latter exists.

— Exercise *7.5* A trajectory γ is Poisson stable if and only if any point x of this trajectory is recurrent, i.e. for any vicinity $\mathcal{O} \ni x$ there exists $t > 0$ such that $S_t x \in \mathcal{O}$.

The following exercise testifies to the fact that not only periodic (and stationary) trajectories can be Poisson stable.

— Exercise *7.6* Let $C_b(\mathbb{R})$ be a Banach space of continuous functions bounded on the real axis. Let us consider a dynamical system $(C_b(\mathbb{R}), S_t)$ with the evolutionary operator defined by the formula

$$(S_t f)(x) = f(x+t), \quad f(x) \in C_b(\mathbb{R}).$$

Show that the element $f_0(x) = \sin\omega_1 x + \sin\omega_2 x$ is recurrent for any real ω_1 and ω_2 (in particular, when ω_1/ω_2 is an irrational number). Therewith the trajectory $\gamma = \{f_0(x+t):\ -\infty < t < \infty\}$ is Poisson stable.

In conclusion to this section we consider a theorem that is traditionally associated with the stability theory. Sometimes this theorem enables us to significantly decrease the dimension of the phase space, this fact being very important for the study of infinite-dimensional systems.

Theorem 7.4. (reduction principle).

Assume that in a dissipative dynamical system (X, S_t) ***there exists a positively invariant locally compact set*** M ***possessing the property of uniform attraction, i.e. for any bounded set*** $B \subset X$ ***the equation***

$$\lim_{t \to \infty} \sup_{y \in B} \operatorname{dist}(S_t y, M) = 0 \tag{7.10}$$

***holds. Let** A **be a global attractor of the dynamical system** (M, S_t). **Then** A **is also a global attractor of** (X, S_t).*

Proof.

It is sufficient to verify that

$$\lim_{t \to \infty} \sup_{y \in B} \operatorname{dist}(S_t y, A) = 0 \tag{7.11}$$

for any bounded set $B \subset X$. Assume that there exists a set B such that (7.11) does not hold. Then there exist sequences $\{y_n\} \subset B$ and $\{t_n\colon\ t_n \to \infty\}$ such that

$$\operatorname{dist}(S_{t_n} y_n, A) \ge \delta \tag{7.12}$$

for some $\delta > 0$. Let B_0 be a bounded absorbing set of (X, S_t). We choose a moment t_0 such that

$$\sup\left\{\operatorname{dist}(S_{t_0} y, A)\colon\ y \in M \cap B_0\right\} \le \frac{\delta}{2}. \tag{7.13}$$

This choice is possible because A is a global attractor of (M, S_t). Equation (7.10) implies that

$$\operatorname{dist}(S_{t_n - t_0} y_n, M) \to 0, \qquad t_n \to \infty.$$

The dissipativity property of (X, S_t) gives us that $S_{t_n - t_0} y_n \in B_0$ when n is large enough. Therefore, local compactness of the set M guarantees the existence of an element $z \in M \cap B_0$ and a subsequence $\{n_k\}$ such that

$$z = \lim_{k \to \infty} S_{t_{n_k} - t_0} y_{n_k}.$$

This implies that $S_{t_{n_k}} y_{n_k} \to S_{t_0} z$. Therefore, equation (7.12) gives us that $\operatorname{dist}(S_{t_0} z, A) \ge \delta$. By virtue of the fact that $z \in M \cap B_0$ this contradicts equation (7.13). **Theorem 7.4 is proved**.

E x a m p l e *7.1*

We consider a system of ordinary differential equations

$$\begin{cases} \dot{y} + y^3 - y = y z^2, & y|_{t=0} = y_0, \\ \dot{z} + z(1 + y^2) = 0, & z|_{t=0} = z_0. \end{cases} \tag{7.14}$$

It is obvious that for any initial condition (y_0, z_0) problem (7.14) is uniquely solvable over some interval $(0, t^*(y_0, z_0))$. If we multiply the first equation by y and the second equation by z and if we sum the results obtained, then we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(y^2+z^2)+y^4-y^2+z^2=0\,, \qquad t\in(0,\ t^*(y_0,\ z_0))\,.$$

This implies that the function $V(y,\ z)=y^2+z^2$ possesses the property

$$\frac{\mathrm{d}}{\mathrm{d}t}V(y(t),\ z(t))+2\,V(y(t),\ z(t))\le 2\,, \qquad t\in[0,\ t^*(y_0,\ z_0))\,.$$

Therefore,

$$V(y(t),\ z(t))\ \le\ V(y_0,\ z_0)e^{-2t}+1\,, \qquad t\in[0,\ t^*(y_0,\ z_0))\,.$$

This implies that any solution to problem (7.14) can be extended to the whole semiaxis $\mathbb{R}_+$ and the dynamical system $(\mathbb{R}^2,\ S_t)$ generated by equation (7.14) is dissipative. Obviously, the set $M=\{(y,\ 0)\colon\ y\in\mathbb{R}\}$ is positively invariant. Therewith the second equation in (7.14) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}z^2+z^2\ \le\ 0\,, \qquad t>0\,.$$

Hence, $|z(t)|^2\le z_0^2e^{-2t}$. Thus, the set M exponentially attracts all the bounded sets in $\mathbb{R}^2$. Consequently, Theorem 7.4 gives us that the global attractor of the dynamical system $(M,\ S_t)$ is also the attractor of the system $(\mathbb{R}^2,\ S_t)$. But on the set M system of equations (7.14) is reduced to the differential equation

$$\dot{y}+y^3-y=0, \qquad y\big|_{t=0}=y_0\,. \tag{7.15}$$

Thus, the global attractors of the dynamical systems generated by equations (7.14) and (7.15) coincide. Therewith the study of dynamics on the plane is reduced to the investigation of the properties of the one-dimensional dynamical system.

Exercise *7.7* Show that the global attractor A of the dynamical system $(\mathbb{R}^2,\ S_t)$ generated by equations (7.14) has the form

$$A=\{(y,\ z)\colon\ -1\le y\le 1,\quad z=0\}\,.$$

Figure the qualitative behaviour of the trajectories on the plane.

Exercise *7.8* Consider the system of ordinary differential equations

$$\begin{cases}\dot{y}-y^5+y^3(1+2z)-y(1+z^2)=0\\[2ex] \dot{z}+z(1+4y^4)-2y^2(z^2+y^4-y^2+3/2)=0\,.\end{cases} \tag{7.16}$$

Show that these equations generate a dissipative dynamical system in $\mathbb{R}^2$. Verify that the set $M=\{(y,\ z)\colon\ z=y^2\,,\ y\in\mathbb{R}\}$ is invariant and exponentially attracting. Using Theorem 7.4, prove that the global attractor A of problem (7.16) has the form

$$A=\{(y,\ z)\colon\ z=y^2,\quad -1\le y\le 1\}\,.$$

Hint: Consider the variable $w=z-y^2$ instead of the variable z.

§ 8 Finite Dimensionality of Invariant Sets

Finite dimensionality is an important property of the global attractor which can be established in many situations interesting for applications. There are several approaches to the proof of this property. The simplest of them seems to be the one based on Ladyzhenskaya's theorem on the finite dimensionality of the invariant set. However, it should be kept in mind that the estimates of dimension based on Ladyzhenskaya's theorem usually turn out to be too overstated. Stronger estimates can be obtained on the basis of the approaches developed in the books by A. V. Babin and M. I. Vishik, and by R. Temam (see the references at the end of the chapter).

Let M be a compact set in a metric space X. Then its ***fractal dimension*** is defined by

$$\dim_f M = \overline{\lim_{\varepsilon \to 0}} \frac{\ln n(M, \varepsilon)}{\ln(1/\varepsilon)} ,$$

where $n(M, \varepsilon)$ is the minimal number of closed balls of the radius ε which cover the set M.

Let us illustrate this definition with the following examples.

Example *8.1*

Let M be a segment of the length l. It is evident that

$$\frac{l}{2\varepsilon} - 1 \le n(M, \varepsilon) \le \frac{l}{2\varepsilon} + 1 .$$

Therefore,

$$\ln\frac{1}{\varepsilon} + \ln\frac{l - 2\varepsilon}{2} \le \ln n(M, \varepsilon) \le \ln\frac{1}{\varepsilon} + \ln\frac{l + 2\varepsilon}{2} .$$

Hence, $\dim_f M = 1$, i.e. the fractal dimension coincides with the value of the standard geometric dimension.

Example *8.2*

Let M be the Cantor set obtained from the segment $[0, 1]$ by the sequential removal of the centre thirds. First we remove all the points between $1/3$ and $2/3$. Then the centre thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the two remaining segments $[0, 1/3]$ and $[2/3, 1]$ are deleted. After that the centre parts $(1/27, 2/27)$, $(7/27, 8/27)$, $(19/27, 20/27)$ and $(25/27, 26/27)$ of the four remaining segments $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$ and $[8/9, 1]$, respectively, are deleted. If we continue this process to infinity, we obtain the Cantor set M. Let us calculate its fractal dimension. First of all it should be noted that

$$M = \bigcap_{k=0}^{\infty} J_k\,,$$

$$J_0 = [0,\ 1], \quad J_1 = [0,\ 1/3] \cup [2/3,\ 1],$$

$$J_2 = [0,\ 1/9] \cup [2/9,\ 1/3] \cup [2/3,\ 7/9] \cup [8/9,\ 1]$$

and so on. Each set J_k can be considered as a union of 2^k segments of the length 3^{-k}. In particular, the cardinality of the covering of the set M with the segment of the length 3^{-k} equals to 2^k. Therefore,

$$\dim_f M = \lim_{k \to \infty} \frac{\ln 2^k}{\ln(2 \cdot 3^k)} = \frac{\ln 2}{\ln 3}\,.$$

Thus, the fractal dimension of the Cantor set is not an integer (if a set possesses this property, it is called fractal).

It should be noted that the fractal dimension is often referred to as the metric order of a compact. This notion was first introduced by L. S. Pontryagin and L. G. Shnirelman in 1932. It can be shown that any compact set with the finite fractal dimension is homeomorphic to a subset of the space $\mathbb{R}^d$ when $d > 0$ is large enough.

To obtain the estimates of the fractal dimension the following simple assertion is useful.

Lemma 8.1.

The following equality holds:

$$\dim_f M = \overline{\lim_{\varepsilon \to 0}} \frac{\ln N(M,\ \varepsilon)}{\ln(1/\varepsilon)}\,,$$

where $N(M,\ \varepsilon)$ is the cardinality of the minimal covering of the compact M with closed sets diameter of which does not exceed 2ε (the diameter of a set X is defined by the value $d(X) = \sup\{\|x-y\|:\ x,\ y \in X\}$).

Proof.

It is evident that $N(M,\ \varepsilon) \le n(M,\ \varepsilon)$. Since any set of the diameter d lies in a ball of the radius d, we have that $n(M,\ 2\varepsilon) \le N(M,\ \varepsilon)$. These two inequalities provide us with the assertion of the lemma.

All the sets are expected to be compact in Exercises 8.1–8.4 given below.

— Exercise *8.1* Prove that if $M_1 \subseteq M_2$, then $\dim_f M_1 \le \dim_f M_2$.

— Exercise *8.2* Verify that $\dim_f (M_1 \cup M_2) \le \max\{\dim_f M_1;\ \dim_f M_2\}$.

— Exercise *8.3* Assume that $M_1 \times M_2$ is a direct product of two sets. Then

$$\dim_f (M_1 \times M_2) \le \dim_f M_1 + \dim_f M_2\,.$$

— Exercise *8.4* Let g be a Lipschitzian mapping of one metric space into another. Then $\dim_f g(M) \le \dim_f M$.

The notion of the dimension by Hausdorff is frequently used in the theory of dynamical systems along with the fractal dimension. This notion can be defined as follows. Let M be a compact set in X. For positive d and ε we introduce the value

$$\mu(M, d, \varepsilon) = \inf \sum (r_j)^d ,$$

where the infimum is taken over all the coverings of the set M with the balls of the radius $r_j \le \varepsilon$. It is evident that $\mu(M, d, \varepsilon)$ is a monotone function with respect to ε. Therefore, there exists

$$\mu(M, d) = \lim_{\varepsilon \to 0} \mu(M, d, \varepsilon) = \sup_{\varepsilon > 0} \mu(M, d, \varepsilon) .$$

The ***Hausdorff dimension*** of the set M is defined by the value

$$\dim_H M = \inf\{d : \mu(M, d) = 0\} .$$

— Exercise *8.5* Show that the Hausdorff dimension does not exceed the fractal one.

— Exercise *8.6* Show that the fractal dimension coincides with the Hausdorff one in Example 8.1, the same is true for Example 8.2.

— Exercise *8.7* Assume that $M = \{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, where a_n monotonically tends to zero. Prove that $\dim_H M = 0$ (*Hint*: $\mu(M, d, \varepsilon) \le \le a_{n+1}^d + n\,2^{-dn}$ when $a_{n+1} \le \varepsilon \le a_n$).

— Exercise *8.8* Let $M = \{1/n\}_{n=1}^{\infty} \subset \mathbb{R}$. Show that $\dim_f M = 1/2$. $\Big($Hint: $n < n(M, \varepsilon) < n+1+\dfrac{1}{(n+1)\varepsilon}$ when $\dfrac{1}{(n+1)(n+2)} \le \varepsilon < < \dfrac{1}{n(n+1)}\Big)$.

— Exercise *8.9* Let $M = \left\{\dfrac{1}{\ln n}\right\}_{n=2}^{\infty} \subset \mathbb{R}$. Prove that $\dim_f M = 1$.

— Exercise *8.10* Find the fractal and Hausdorff dimensions of the global minimal attractor of the dynamical system in $\mathbb{R}$ generated by the differential equation

$$\dot{y} + y \sin\frac{1}{|y|} = 0 .$$

The facts presented in Exercises 8.7–8.9 show that the notions of the fractal and Hausdorff dimensions do not coincide. The result of Exercise 8.5 enables us to restrict ourselves to the estimates of the fractal dimension when proving the finite dimensionality of a set.

The main assertion of this section is the following variant of Ladyzhenskaya's theorem. It will be used below in the proof of the finite dimensionality of global attractors of a number of infinite-dimensional systems generated by partial differential equations.

Theorem 8.1.

Assume that M is a compact set in a Hilbert space H. Let V be a continuous mapping in H such that $V(M) \supset M$. Assume that there exists a finite-dimensional projector P in the space H such that

$$\|P(Vv_1 - Vv_2)\| \le l \|v_1 - v_2\|, \qquad v_1, v_2 \in M, \tag{8.1}$$

$$\|(1-P)(Vv_1 - Vv_2)\| \le \delta \|v_1 - v_2\|, \qquad v_1, v_2 \in M, \tag{8.2}$$

where $\delta < 1$. We also assume that $l \ge 1 - \delta$. Then the compact M possesses a finite fractal dimension and

$$\dim_f M \le \dim P \cdot \ln \frac{9l}{1-\delta} \cdot \left[\ln \frac{2}{1+\delta} \right]^{-1}. \tag{8.3}$$

We remind that a projector in a space H is defined as a bounded operator P with the property $P^2 = P$. A projector P is said to be finite-dimensional if the image PH is a finite-dimensional subspace. The dimension of a projector P is defined as a number $\dim P \equiv \dim PH$.

The following lemmata are used in the proof of Theorem 8.1.

Lemma 8.2.

Let B_R be a ball of the radius R in $\mathbb{R}^d$. Then

$$N(B_R, \varepsilon) \le n(B_R, \varepsilon) \le \left(1 + \frac{2R}{\varepsilon}\right)^d. \tag{8.4}$$

Proof.

Estimate (8.4) is self-evident when $\varepsilon \ge R$. Assume that $\varepsilon < R$. Let $\{\xi_1, \dots, \xi_l\}$ be a maximal set in B_R with the property $|\xi_i - \xi_j| > \varepsilon$, $i \ne j$. By virtue of its maximality for every $x \in B_R$ there exists ξ_i such that $|x - \xi_i| \le \varepsilon$. Hence, $n(B_R, \varepsilon) \le l$. It is clear that

$$B_{\varepsilon/2}(\xi_i) \subset B_{R+\varepsilon/2}, \qquad B_{\varepsilon/2}(\xi_i) \cap (B_{\varepsilon/2}(\xi_j)) = \varnothing, \qquad i \ne j.$$

Here $B_r(\xi)$ is a ball of the radius r centred at ξ. Therefore,

$$l \,\mathrm{Vol}(B_{\varepsilon/2}) = \sum_{i=1}^{l} \mathrm{Vol}(B_{\varepsilon/2}(\xi_i)) \le \mathrm{Vol}(B_{R+\varepsilon/2}).$$

This implies the assertion of the lemma.

— Exercise *8.11* Show that

$$n(B_R, \varepsilon) \ge \left(\frac{R}{\varepsilon}\right)^d, \qquad \dim_f B_R = d.$$

Lemma 8.3.

Let $\mathcal{F}$ be a closed subset in H such that equations (8.1) *and* (8.2) *hold for all its elements. Then for any $q > 0$ and $\varepsilon > 0$ the following estimate holds:*

$$N(V\mathcal{F},\ \varepsilon(q+\delta)) \le \left(1+\frac{4l}{q}\right)^n N(\mathcal{F},\ \varepsilon)\ , \tag{8.5}$$

where $n = \dim P$ is the dimension of the projector P.

Proof.

Let $\{\mathcal{F}_i\}$ be a minimal covering of the set $\mathcal{F}$ with its closed subsets the diameter of which does not exceed 2ε. Equation (8.1) implies that in PH there exist balls B_i with radius $2l\varepsilon$ such that $PV\mathcal{F}_i \subset B_i$. By virtue of Lemma 8.1 there exists a covering $\{B_{ij}\}_{j=1}^{N_i}$ of the set $PV\mathcal{F}_i$ with the balls of the diameter $2q\varepsilon$, where $N_i \le (1+(4l/q))^n$. Therefore, the collection

$$\left\{G_{ij} = B_{ij} + (1-P)V\mathcal{F}_i:\ i = 1,\ 2,\ \ldots,\ N(\mathcal{F},\ \varepsilon),\quad j = 1,\ 2,\ \ldots,\ N_i\right\}$$

is a covering of the set $V\mathcal{F}$. Here the sum of two sets A and B is defined by the equality

$$A + B = \{a+b:\ a \in A,\ b \in B\}.$$

It is evident that

$$\operatorname{diam} G_{ij} \le \operatorname{diam} B_{ij} + \operatorname{diam}(1-P)V\mathcal{F}_i\,.$$

Equation (8.2) implies that $\operatorname{diam}(1-P)V\mathcal{F}_i \le 2\delta\varepsilon$. Therefore, $\operatorname{diam} G_{ij} \le$ $\le 2(q+\delta)\varepsilon$. Hence, estimate (8.5) is valid. Lemma 8.3 is proved.

Let us return to the proof of Theorem 8.1. Since $M \subset VM$, Lemma 8.3 gives us that

$$N(M,\ \varepsilon(q+\delta)) \le N(M,\ \varepsilon)\cdot\left(1+\frac{4l}{q}\right)^n .$$

It follows that

$$N(M,\ (q+\delta)^m) \le N(M,\ 1)\cdot\left(1+\frac{4l}{q}\right)^{nm}, \qquad m = 1,\ 2,\ \ldots\ .$$

We choose q and $m = m(\varepsilon)$ such that

$$\delta + q < 1, \qquad (\delta+q)^m \le \varepsilon\,,$$

where $0 < \varepsilon < 1$. Then

$$N(M,\ \varepsilon) \ \le\ N(M,\ (\delta+q)^m) \ \le\ N(M,\ 1)\cdot\left(1+\frac{4l}{q}\right)^{nm(\varepsilon)} .$$

Consequently,

$$\dim_f M = \varlimsup_{\varepsilon\to 0} \frac{\ln N(M,\ \varepsilon)}{\ln(1/\varepsilon)} \ \le\ n\cdot\ln\left(1+\frac{4l}{q}\right)\cdot \varlimsup_{\varepsilon\to 0} \frac{m(\varepsilon)}{\ln(1/\varepsilon)}\ .$$

Obviously, the choice of $m(\varepsilon)$ can be made to fulfil the condition

$$m(\varepsilon) \le \frac{\ln\varepsilon}{\ln(q+\delta)} + 1 .$$

Thus,

$$\dim_f M \le n \ln\left(1 + \frac{4l}{q}\right)\left[\ln\frac{1}{q+\delta}\right]^{-1} .$$

By taking $q = 1/2(1-\delta)$ we obtain estimate (8.3). **Theorem 8.1 is proved**.

Exercise *8.12* Assume that the hypotheses of Theorem 8.1 hold and $l < 1-\delta$. Prove that $\dim_f M = 0$.

Of course, in the proof of Theorem 8.1 a principal role is played by equations (8.1) and (8.2). Roughly speaking, they mean that the mapping V squeezes sets along the space $(1-P)H$ while it does not stretch them too much along PH. Negative invariance of M gives us that $M \subset V^k M$ for all $k = 1, 2, \dots$. Therefore, the set M should be initially squeezed. This property is expressed by the assertion of its finite dimensionality. As to positively invariant sets, their finite dimensionality is not guaranteed by conditions (8.1) and (8.2). However, as the next theorem states, they are attracted to finite-dimensional compacts at an exponential velocity.

Theorem 8.2.

Let V ***be a continuous mapping defined on a compact set*** M ***in a Hilbert space*** H ***such that*** $VM \subset M$. ***Assume that there exists a finite-dimensional projector*** P ***such that equations*** (8.1) ***and*** (8.2) ***hold with*** $0 < \delta < 1/2$ ***and*** $l + \delta \ge 1$. ***Then for any*** $\theta \in (\delta, 1)$ ***there exists a positively invariant closed set*** $A_\theta \subset M$ ***such that***

$$\sup\{\operatorname{dist}(V^k y, A_\theta):\ y \in M\} \le \theta^k , \qquad k = 1, 2, \dots \tag{8.6}$$

and

$$\dim_f A_\theta \le \dim P \cdot \max\left\{\frac{\ln\left(1 + \frac{4l}{\theta-\delta}\right)}{\ln\frac{1}{\theta}},\ \frac{\ln\left(1 + \frac{4l}{q}\right)}{\ln\frac{1}{2(q+\delta)}}\right\}, \tag{8.7}$$

where q ***is an arbitrary number from the interval*** $(0,\ 1/2 - \delta)$.

Proof.

The pair (M, V^k) is a discrete dynamical system. Since M is compact, Theorem 5.1 gives us that there exists a global attractor $M_0 = \bigcap_{k \ge 0} V^k M$ with the properties $VM_0 = M_0$ and

$$h(V^k M, M_0) \equiv \sup\{\operatorname{dist}(V^k y, M_0):\ y \in M\} \to 0 . \tag{8.8}$$

We construct a set A_θ as an extension of M_0. Let E_j be a maximal set in V^jM possessing the property $\mathrm{dist}(a, b) \ge \theta^j$ for $a, b \in E_j$, $a \ne b$. The existence of such a set follows from the compactness of V^jM. It is obvious that

$$L_j \equiv \mathrm{Card}\, E_j = N\Big(E_j, \tfrac{1}{3}\theta^j\Big) \le N\Big(V^jM, \tfrac{1}{3}\theta^j\Big).$$

Lemma 8.3 with $\mathscr{F} = M$, $q = \theta - \delta$, and $\varepsilon = (1/3)\,\theta^{j-1}$ gives us that

$$N\Big(V^jM, \tfrac{1}{3}\theta^j\Big) \le \Big(1 + \frac{4l}{\theta-\delta}\Big)^n N\Big(V^{j-1}M, \tfrac{1}{3}\theta^{j-1}\Big)$$

with $\theta > \delta$. Hereinafter $n = \dim P$. Therefore,

$$L_j \equiv \mathrm{Card}E_j \le \Big(1 + \frac{4l}{\theta-\delta}\Big)^{nj} N\Big(M, \tfrac{1}{3}\Big), \quad \theta > \delta. \tag{8.9}$$

Let us prove that the set

$$A_\theta = M_0 \cup \{\cup\{V^kE_j:\ j = 1, 2, \ldots,\quad k = 0, 1, 2, \ldots\}\} \tag{8.10}$$

possesses the properties required. It is evident that $VA_\theta \subset A_\theta$. Since $V^kE_j \subset \subset V^{k+j}M$, by virtue of (8.8) all the limit points of the set

$$\bigcup\{V^kE_j:\ j = 1, 2, \ldots,\quad k = 0, 1, 2, \ldots\}$$

lie in M_0. Thus, A_θ is a closed subset in M. The evident inequality

$$h(V^kM, A_\theta) \le h(V^kM, E_k) \le \theta^k \tag{8.11}$$

implies (8.6). Here and below $h(X, Y) = \sup\{\mathrm{dist}(x, Y):\ x \in X\}$. Let us prove (8.7). It is clear that

$$A_\theta = VA_\theta \cup \{\cup\{E_j:\ j = 1, 2, \ldots\}\}. \tag{8.12}$$

Let $\{F_i\}$ be a minimal covering of the set A_θ with the closed sets the diameter of which is not greater than 2ε. By virtue of Lemma 8.3 there exists a covering $\{G_i\}$ of the set VA_θ with closed subsets of the diameter $2\varepsilon(q+\delta)$. The cardinality of this covering can be estimated as follows

$$N(\varepsilon, q, \delta) \equiv N(VA_\theta, \varepsilon(q+\delta)) \le \Big(1 + 4\frac{l}{q}\Big)^n N(A_\theta, \varepsilon). \tag{8.13}$$

Using the covering $\{G_i\}$, we can construct a covering of the same cardinality of the set VA_θ with the balls $B(x_i, 2\varepsilon(q+\delta))$ of the radius $2\varepsilon(q+\delta)$ centered at the points x_i, $\quad i = 1, 2, \ldots, N(\varepsilon, q, \delta)$. We increase the radius of every ball up to the value $2\varepsilon(q+\delta+\gamma)$. The parameter $\gamma > 0$ will be chosen below. Thus, we consider the covering

$$\{B(x_i, 2\varepsilon(q+\delta+\gamma)),\quad i = 1, 2, \ldots, N(\varepsilon, q, \delta)\}$$

of the set VA_θ. It is evident that every point $x \in VA_\theta$ belongs to this covering together with the ball $B(x, 2\gamma\varepsilon)$. If $j \ge 2$, the inequalities

$$h(E_j,\ VA_\theta)\ \le\ h(V^jM,\ VA_\theta)\ \le\ h(V^jM,\ VE_{j-1})$$

hold. By virtue of equation (8.11) with the help of (8.1) and (8.2) we have that

$$h(V^jM,\ VE_{j-1})\ \le\ (l+\delta)h(V^{j-1}M,\ E_{j-1})\ \le\ (l+\delta)\,\theta^{j-1}\,.$$

Therefore, $h(E_j,\ VA_\theta)\le 2\gamma\varepsilon$, provided $2\gamma\varepsilon\le(l+\delta)\,\theta^{j-1}$, i.e. if

$$j\ge j_0\equiv 2+\left[\frac{\ln\dfrac{l+\delta}{2\gamma\varepsilon}}{\ln\dfrac{1}{\theta}}\right].$$

Here $[z]$ is an integer part of the number z. Consequently,

$$VA_\theta\cup\left\{\bigcup_{j\ge j_0}E_j\right\}\subset\bigcup_{i=1}^{N(\varepsilon,\ q,\ \delta)}B(x_i,\ 2\varepsilon(q+\delta+\gamma))\,.$$

Therefore, equation (8.12) gives us that

$$N(A_\theta,\ \varepsilon\xi)\le N(\varepsilon,\ q,\ \delta)+\sum_{j=0}^{j_0-1}\mathrm{Card}\,E_j\,,$$

where $\xi=2(q+\delta+\gamma)$. Using (8.9) and (8.13) we find that

$$N(A_\theta,\ \varepsilon\xi)\le\ \eta N(A_\theta,\ \varepsilon)+N\Big(M,\ \frac{1}{3}\Big)\sum_{j=1}^{j_0-1}\Big(1+\frac{4l}{\theta-\delta}\Big)^{nj}$$

for $\theta>\delta$. Here and further $\eta=\Big(1+\dfrac{4l}{q}\Big)^n$. Since

$$j_0\le\frac{\ln\dfrac{1}{\varepsilon}}{\ln\dfrac{1}{\theta}}+C(l,\ \delta,\ \gamma,\ \theta)\,,$$

it is easy to find that

$$\sum_{j=0}^{j_0-1}\Big(1+\frac{4l}{\theta-\delta}\Big)^{nj}\le\beta\cdot\Big(\frac{1}{\varepsilon}\Big)^{\alpha}\quad\text{for}\quad\alpha=\frac{n}{\ln(1/\theta)}\cdot\ln\Big(1+\frac{4l}{\theta-\delta}\Big),$$

where the constant $\beta>0$ does not depend on ε (its value is unessential further). Therefore,

$$N(A_\theta,\ \varepsilon\xi)\le\ \eta N(A_\theta,\ \varepsilon\xi)+\beta\varepsilon^{-\alpha}\,.$$

If we take $\varepsilon=\xi^{m-1}$, then after iterations we get

$$N(A_\theta,\ \xi^m)\le\ \eta N(A_\theta,\ \xi^{m-1})+\beta\xi^{-(m-1)\alpha}\le$$

$$\le\ \eta^m N(A_\theta,1)+\beta\eta^{m-1}\sum_{i=0}^{m-1}(\xi^\alpha\eta)^{-i}\,.$$

Let us fix $\delta \in (0, 1/2)$, $\theta \in (\delta, 1)$ and $q \in (0, 1/2 - \delta)$ and choose $\gamma > 0$ such that $\xi = 2(q + \delta + \gamma) < 1$ and $\xi^{\alpha}\eta \neq 1$. Then summarizing the geometric progression we obtain

$$N(A_\theta, \xi^m) \le \eta^m N(A_\theta, 1) + \beta \frac{\xi^{-\alpha m} - \eta^m}{\xi^{-\alpha} - \eta} \le$$

$$\le \eta^m \left(N(A_\theta, 1) + \frac{\beta}{|\xi^{-\alpha} - \eta|} \right) + \frac{\beta}{|\xi^{-\alpha} - \eta|} \xi^{-\alpha m} \quad . \tag{8.14}$$

Let $\varepsilon > 0$ be small enough and

$$m(\varepsilon) = 1 + \left[\frac{\ln(1/\varepsilon)}{\ln(1/\xi)} \right],$$

where, as mentioned above, $[z]$ is an integer part of the number z. Since $\varepsilon \le \xi^{m(\varepsilon)}$, equation (8.14) gives us that

$$N(A_\theta, \varepsilon) \le N(A_\theta, \xi^m) \le a_1 \left(1 + \frac{4l}{q} \right)^{n\, m(\varepsilon)} + a_2 \left(\frac{1}{\xi \alpha} \right)^{m(\varepsilon)},$$

where a_1 and a_2 are positive numbers which do not depend on ε. Therefore,

$$\dim_f A_\theta = \overline{\lim_{\varepsilon \to 0}} \frac{\ln N(A_\theta, \varepsilon)}{\ln \frac{1}{\varepsilon}} \le$$

$$\le \overline{\lim_{\varepsilon \to 0}} \frac{m(\varepsilon)}{\ln \frac{1}{\varepsilon}} \lim_{m \to \infty} \frac{1}{m} \left\{ a_1 \left(1 + \frac{4e}{q}\right)^{nm} + a_2 \left(\frac{1}{\xi^{\alpha}} \right)^m \right\} .$$

Simple calculations give us that

$$\dim_f A_\theta \le \frac{1}{\ln \frac{1}{\xi}} \cdot \ln \left\{ \max \left(\left(1 + \frac{4e}{q} \right)^n, \ \xi^{-\alpha} \right) \right\} .$$

This easily implies estimate (8.7). Thus, **Theorem 8.2 is proved**.

— Exercise *8.13* Show that for $\delta < \theta < 1/2$ formula (8.7) for the dimension of the set A_θ can be rewritten in the form

$$\dim_f A_\theta = \dim P \cdot \frac{\ln\left(1 + \frac{4l}{\theta - \delta}\right)}{\ln \frac{1}{2\theta}} \, . \tag{8.15}$$

If the hypotheses of Theorem 8.2 hold, then the discrete dynamical system (M, V^k) possesses a finite-dimensional global attractor M_0. This attractor uniformly attracts all the trajectories of the system. Unfortunately, the speed of its convergence to the attractor cannot be estimated in general. This speed can appear to be small. However, Theorem 8.2 implies that the global attractor is contained in a finite-dimensional positively invariant set possessing the property of uniform exponential attraction. From the applied point of view the most interesting corollary of this fact is that the dynamics of a system becomes ***finite-dimensional*** exponentially fast independent of the speed of convergence of the trajectories to the global attractor. Moreover, the reduction principle (see Theorem 7.4) is applicable in this case. Thus, finite-dimensional invariant exponentially attracting sets can be used to describe the qualitative behaviour of infinite-dimensional systems. These sets are frequently referred to as ***inertial sets***, or ***fractal exponential attractors***. In some cases they turn out to be surfaces in the phase space. In contrast with the global attractor, the inertial set of a dynamical system can not be uniquely determined. The construction in the proof of Theorem 8.2 shows it.

§ 9 Existence and Properties of Attractors of a Class of Infinite-Dimensional Dissipative Systems

The considerations given in the previous sections are mainly of general character. They are related to a dissipative dynamical system of the generic structure. Therewith, we inevitably make additional assumptions on the behaviour of trajectories of these systems (e.g., the asymptotic compactness, the existence of a Lyapunov function, the squeezing property along a subspace, etc.). Thereby it is natural to ask what properties of the original objects of a particular dynamical system guarantee the fulfilment of the assumptions mentioned above. In this section we discuss this question in terms of the dynamical system generated by a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} y + A y = B(y), \qquad y\big|_{t=0} = y_0 \tag{9.1}$$

in a separable Hilbert space $\mathcal{H}$, where A is a linear operator and B is a nonlinear mapping which is coordinated with A in some sense. Our main goal is to demonstrate the *generic line of arguments* as well as to describe those properties of the operators A and B which provide the applicability of general theorems proved in the previous sections. The main attention is paid to the questions of existence and finite dimensionality of a global attractor. Nowadays the presented line of arguments (or a modification of it) is one of the main components of a great number of works on global attractors.

It is assumed below that the following conditions are fulfilled.

(A) There exists a strongly continuous semigroup S_t of continuous mappings in $\mathcal{H}$ such that $y(t) = S_t y_0$ is a solution to problem (9.1) in the sense that the following identity holds:

$$S_t y_0 = T_t y_0 + G(t, y_0) \equiv T_t y_0 + \int_0^t T_{t-\tau} B(S_\tau y_0) \mathrm{d}\tau , \tag{9.2}$$

where $T_t = \exp(-At)$ (see condition **(B)** below). The semigroup S_t is dissipative, i.e. there exists $R > 0$ such that for any B from the collection $\mathcal{B}(\mathcal{H})$ of all bounded subsets of the space $\mathcal{H}$ the estimate $\|S_t y\| < R$ holds when $y \in B$ and $t \geq t_0(B)$. We also assume that the set $\gamma^+(B) = \bigcup_{t \geq 0} S_t B$ is bounded for any $B \in \mathcal{B}(\mathcal{H})$.

(B) The linear closed operator A generates a semigroup $T_t = \exp(-At)$ which admits the estimate $\|T_t\| \leq L_1 \exp(\omega t)$ (L_1 and ω are some constants). There exists a sequence of finite-dimensional projectors $\{P_n\}$ which strongly converges to the identity operator such that

1) A commutes with P_n, i.e. $P_n A \subset A P_n$ for any n;
2) there exists n_0 such that $\|T_t (1-P_n)\| \leq L_2 \exp(-\varepsilon t)$ for $n > n_0$, where $\varepsilon, L_2 > 0$;
3) $r_n = \|A^{-1}(1-P_n)\| \to 0$ as $n \to \infty$.

(C) For any $R' > 0$ the nonlinear operator $B(u)$ possesses the properties:

1) $\|B(u_1) - B(u_2)\| \leq C_1(R')\|u_1 - u_2\|$ if $\|u_i\| \leq R'$, $i = 1, 2$;
2) for $n \geq n_0$, $\|u_i\| \leq R'$, $i = 1, 2$, and for some $\sigma > 0$ the following equations hold:

$$\|A^\sigma (1-P_n) B(u_1)\| \leq C_2(R'),$$

$$\|A^\sigma (1-P_n)(B(u_1) - B(u_2))\| \leq C_3(R')\|u_1 - u_2\|$$

(the existence of the operator $A^\sigma(1-P_n)$ follows from **(B2)**).

It should be noted that although conditions **(A)–(C)** seem a little too lengthy, they are valid for a class of problems of the theory of nonlinear oscillations as well as for a number of systems generated by parabolic partial differential equations.

The following assertion should be mainly interpreted as a principal result which testifies to the fact that the asymptotic behaviour of the system is determined by a finite set of parameters.

Theorem 9.1.

If conditions **(A)–(C)** *are fulfilled, then the semigroup S_t possesses a compact global attractor $\mathcal{A}$. The attractor has a finite fractal dimension which can be estimated as follows:*

$$\dim_f \mathcal{A} = a_1 (1 + \ln\|P_n\| L_1 L_2)(1 + D(R)\varepsilon^{-1}) \dim P_n , \tag{9.3}$$

where $D(R) = \omega + L_1 C_1(R)$ ***and*** $n \geq n_0$ ***is determined from the condition***

$$r_n^\sigma \leq a_2 D(R)[L_1 L_2 C_3(R)]^{-1} \cdot \exp\{-a_3 D(R)(1+\ln L_2)\varepsilon^{-1}\}. \qquad (9.4)$$

Here a_1, a_2, ***and*** a_3 ***are some absolute constants.***

When proving the theorem, we mainly rely on decomposition (9.2) and the lemmata below.

Lemma 9.1.

Let K_n *be a set of elements which for some* $t \geq 0$ *have the form* $v = u_0 + (1-P_n)G(t, u)$, *where* $u_0 \in P_n \mathcal{H}$, $\|u_0\| \leq c_0 R$ *with the constant* c_0 *determined by the condition* $\|P_n\| \leq c_0$ *for* $n = 1, 2, \ldots$. *Here the value* $G(t, u)$ *is the same as in* (9.2) *with the element* $u \in \mathcal{H}$ *being such that* $\|S_t u\| \leq R$ *for all* $t \geq 0$. *Then the set* K_n *is precompact in* $\mathcal{H}$ *for* $n \geq n_0$.

Proof.

Properties **(B2)** and **(C2)** imply that

$$\|A^\sigma(1-P_n)G(t, u)\| \leq L_2 \int_0^t e^{-\varepsilon(t-\tau)} \|A^\sigma(1-P_n) B(S_\tau u)\| \, d\tau \leq \frac{C_2(R)L_2}{\varepsilon}$$

when $\|S_\tau u\| \leq R$ for $\tau \geq 0$. Therefore, the set

$$\{v: \ v = (1-P_n)G(t, u), \ t > 0\}, \qquad (9.5)$$

where $\|S_t u\| \leq R$ for all $t \geq 0$, is bounded in the space $D(A^\sigma \upharpoonright (1-P_n)\mathcal{H})$ with the norm $\|A^\sigma \cdot\|$. The symbol $\upharpoonright$ denotes the restriction of an operator on a subspace. However, property **(B3)** implies that

$$\lim_{m \to \infty} \|P_m A^{-1}(1-P_n) - A^{-1}(1-P_n)\| = 0.$$

Therefore, the operator $A^{-1} \upharpoonright (1-P_n)\mathcal{H}$ is compact. Hence, $D(A^\sigma \upharpoonright (1-P_n)\mathcal{H})$ is compactly embedded into $(1-P_n)\mathcal{H}$. It means that the set (9.5) is precompact in $(1-P_n)\mathcal{H}$. This implies the precompactness of K_n.

Lemma 9.2.

There exists a compact set $\overline{K}$ *in the space* $\mathcal{H}$ *such that*

$$h(S_t B, \overline{K}) \equiv \sup\{\mathrm{dist}(S_t y, \overline{K}): \ y \in B\} \leq L_2 R e^{-\varepsilon(t-t_0)} \qquad (9.6)$$

for any bounded set $B \subset \mathcal{H}$ *and* $t \geq t_0 \equiv t_0(B)$.

Proof.

Let $u \in B$, where B is a bounded set in $\mathcal{H}$. Then $\|S_t u\| \leq R$ for $t \geq t_0 = t_0(B)$. By virtue of (9.2) we have that

$$S_t u = (1-P_n)T_{t-t_0} S_{t_0} u + W_n(t, t_0, u),$$

where

$$W_n(t, t_0, u) = P_n S_t u + (1-P_n) G(t-t_0, S_{t_0} u).$$

It is evident that $W_n(t, t_0, u) \in K_n$ for $t \geq t_0$. Therefore,

$$\operatorname{dist}(S_t u, K_n) \leq \left\| (1-P_n)T_{t-t_0} S_{t_0} u \right\| \leq L_2 R e^{-\varepsilon(t-t_0)}.$$

This implies (9.6) with $\overline{K} = [K_n]$, where $[K_n]$ is the closure in $\mathcal{H}$ of the set K_n described in Lemma 9.1.

Exercise *9.1* Show that $\overline{K} = [K_n]$ lies in the set

$$K_{n,\sigma} = \{ v_1 + v_2 :\ v_1 \in P_n \mathcal{H},\ v_2 \in (1-P_n)\mathcal{H},$$

$$\|v_1\| \leq C_1,\ \|A^\sigma v_2\| \leq C_2 \}, \tag{9.7}$$

where C_1 and C_2 are some constants.

In particular, Lemma 9.2 means that the system $(\mathcal{H}, S_t)$ is asymptotically compact. Therefore, we can use Theorem 5.1 (see also Exercise 5.3) to guarantee the existence of the global attractor $\mathcal{A}$ lying in $\overline{K} = [K_n]$.

Let us use Theorem 8.1 to prove the finite dimensionality of the attractor. Verification of the hypotheses of the theorem is based on the following assertion.

Lemma 9.3.

Let $\|S_t u_i\| \leq R$, $t \geq 0$, $i = 1, 2$. *Then*

$$\|S_t u_1 - S_t u_2\| \leq L_1 \exp(D(R)t) \|u_1 - u_2\| \tag{9.8}$$

and

$$\|(1-P_n)(S_t u_1 - S_t u_2)\| \leq L_2 e^{-\varepsilon t} \cdot \left(1 + C_0 r_n^\sigma \frac{L_1 C_3(R)}{D(R)} e^{\alpha t}\right) \|u_1 - u_2\| \tag{9.9}$$

for $n \geq n_0$ *and* $\alpha = \varepsilon + D(R)$.

Proof.

Decomposition (9.2) and condition **(C1)** imply that

$$\|S_t u_1 - S_t u_2\| \leq L_1 \left(\|u_1 - u_2\| + C_1(R) \int_0^t e^{-w\tau} \|S_\tau u_1 - S_\tau u_2\| \, d\tau \right) e^{wt}.$$

With the help of Gronwall's lemma we obtain (9.8).

To prove (9.9) it should be kept in mind that decomposition (9.2) and equations **(B2)** and **(C2)** imply that for $n \geq n_0$

$$\|(1-P_n)(S_t u_1 - S_t u_2)\| \le L_2 e^{-\varepsilon t}\left(\|u_1-u_2\| + \right.$$

$$\left. + C_0 r_n^{\sigma} C_3(R) \int_0^t e^{\varepsilon\tau} \|S_\tau u_1 - S_\tau u_2\| \, d\tau \right). \tag{9.10}$$

Here the inequality $\|A^{-\sigma}(1-P_n)\| \le C_0 r_n^{\sigma}$ is used . If we put (9.8) in the right-hand side of formula (9.10), we obtain estimate (9.9).

The following simple argument completes the proof of Theorem 9.1. Let us fix an arbitrary number $0 < \delta < 1$ and choose t_0 and n such that

$$L_2 e^{-\varepsilon t_0} = \frac{\delta}{2} \quad \text{and} \quad C_0 r_n^{\sigma} L_1 \frac{C_3(R)}{D(R)} \, e^{\alpha t_0} \le 1 \, .$$

Then the hypotheses of Theorem 8.1 with $M = \mathcal{A}$, $V = S_{t_0}$, $P = P_n$, and $l = L\|P_n\| \exp(D(R)t_0)$ hold for the attractor $\mathcal{A}$. Hence, it is finite-dimensional with estimate (9.3) holding for its fractal dimension. **Theorem 9.1 is proved**.

Exercise *9.2* Prove that the global attractor $\mathcal{A}$ of problem (9.1) is stable (*Hint*: verify that the hypotheses of Theorem 7.1 hold).

Properties **(A)–(C)** also enable us to prove that the system generated by equation (9.1) possesses an inertial set. A compact set $A_{\exp}$ in the phase space $\mathcal{H}$ is said to be an ***inertial set*** (or a fractal exponential attractor) if it is positively invariant $(S_t A_{\exp} \subset A_{\exp})$, its fractal dimension is finite $(\dim_f A_{\exp} < \infty)$ and it possesses the property

$$h(S_t B, A_{\exp}) \equiv \sup\{\operatorname{dist}(S_t y, A_{\exp}) : \; y \in B\} \le C_B \, e^{-\gamma(t-t_0)} \tag{9.11}$$

for any bounded set $B \subset \mathcal{H}$ and for $t \ge t_0 \ge t_0(B)$, where C_B and γ are positive numbers. (The importance of this notion for the theory of infinite-dimensional dynamical systems has been discussed at the end of Section 8).

Lemma 9.4.

Assume that properties **(A)–(C)** *hold. Then the dynamical system* $(\mathcal{H}, S_t)$ *generated by equation* (9.1) *possesses the following properties:*

1) *there exist a compact positively invariant set* K *and constants* $C, \gamma > 0$ *such that*

$$\sup\{\operatorname{dist}(S_t y, K) : \; y \in B\} \le C e^{-\gamma(t-t_B)} \tag{9.12}$$

for any bounded set B *in* $\mathcal{H}$ *and for* $t \ge t_B > 0$*;*

2) *there exist a vicinity* $\mathcal{O}$ *of the compact* K *and numbers* Δ_1 *and* $\alpha_1 > 0$ *such that*

$$\|S_t y_1 - S_t y_2\| \le \Delta_1 \, e^{\alpha_1 t} \|y_1 - y_2\| \,, \tag{9.13}$$

provided that for all $t \ge 0$ *the semitrajectories* $S_t y_i$ *lie in the closure* $[\mathcal{O}]$ *of the set* $\mathcal{O}$*;*

3) *there exist a sequence of finite-dimensional projectors* $\{P_n\}$ *in the space* $\mathcal{H}$*, constants* Δ_2, α_2, $\beta > 0$, *and a sequence of positive numbers* $\{\rho_n\}$ *tending to zero as* $n \to \infty$ *such that*

$$\|(1-P_n)(S_t y_1 - S_t y_2)\| \le \Delta_2 \, e^{-\beta t}(1 + \rho_n e^{\alpha_2 t})\|y_1 - y_2\| \tag{9.14}$$

for any $y_1, \; y_2 \in K$.

Proof.

Let $\overline{K}$ be a compact set from Lemma 9.2. Let

$$K^* = \gamma^+(\overline{K}) \equiv \bigcup_{t \ge 0} S_t \, \overline{K} \,.$$

It is clear that $S_t K^* \subset K^*$ and equation (9.12) holds for $K = K^*$ with $C = L_2 R$ and $\gamma = \varepsilon$. Let us prove that K^* is a compact set. Let $\{z_n\}$ be a sequence of elements of $K^* = \gamma^+(\overline{K})$. Then $z_n = S_{t_n} y_n$ for some $t_n > 0$ and $y_n \in \overline{K}$. If there exists an infinitely increasing subsequence $\{t_{n_k}\}$, then equation (9.6) gives us that

$$\lim_{k \to \infty} \operatorname{dist}\Big(S_{t_{n_k}} y_{n_k}, \; \overline{K}\Big) = 0 \,.$$

Therefore, the sequence $\{z_n\}$ possesses a limit point in $\overline{K} \subset K^*$. If $\{t_n\}$ is a bounded sequence, then by virtue of the compactness of $\overline{K}$ there exist a number $t_0 > 0$, an element $y \in \overline{K}$ and a sequence $\{n_k\}$ such that $y_{n_k} \to y$ and $t_{n_k} \to t$. Therewith

$$\Big\|S_{t_{n_k}} y_{n_k} - S_{t_0} y\Big\| \le \Big\|S_{t_n} y - S_{t_0} y\Big\| + \Big\|S_{t_{n_k}} y_{n_k} - S_{t_{n_k}} y\Big\| \,.$$

The first term in the right-hand side of this inequality evidently tends to zero. As for the second term, our argument is the same as in the proof of formula (9.8). We use the boundedness of the set $\gamma^+(\overline{K})$ (see property **(A)**) and properties **(B)** and **(C2)** to obtain the estimate

$$\|S_t y_1 - S_t y_2\| \le C \, e^{C_{\overline{K}} t} \|y_1 - y_2\| \,, \qquad y_1, \; y_2 \in \overline{K} \,. \tag{9.15}$$

It follows that

$$\lim_{k \to \infty} \Big\|S_{t_{n_k}} y_{n_k} - S_{t_{n_k}} y\Big\| = 0 \,.$$

Therefore,

$$S_{t_{n_k}} y_{n_k} \to S_{t_0} y \in K^* = \gamma^+(\overline{K}).$$

The closedness of the set K^* can be established with the help of similar arguments. Thus, property (9.12) is proved for $K = K^*$. Now we suppose that $K = S_{t_0} K^*$, where t_0 is chosen such that $\|y\| < R$ for all $y \in K$. It is obvious that K is a compact positively invariant set. As it is proved above, it is easy to find the estimate of form (9.15) for all y_1 and y_2 from an arbitrary bounded set B. Here an important role is played by the boundedness of the set $\gamma^+(B)$ (see property **(A)**). Therefore, for any $B \in \mathcal{B}(\mathcal{H})$ there exists a constant $C_0 = C(B, K^*, t_0) > 0$ such that

$$\left\|S_{t_0} y_1 - S_{t_0} y_2\right\| \le C_0 \|y_1 - y_2\|, \qquad y_1, y_2 \in B \cup K^*.$$

Hence, for $y \in B$ we have that

$$\operatorname{dist}(S_t y, K) = \operatorname{dist}(S_{t_0} \cdot S_{t-t_0} y, S_{t_0} K^*) \le C_0 \operatorname{dist}(S_{t-t_0} y, K^*)$$

for $t > t_0$. This implies estimate (9.12) with the constant C depending on K and B. However, if we change the moment t_B in equation (9.12), we can presume that, for example, $C = 1$. Therewith $\gamma = \varepsilon$. Thus, the first assertion of the lemma is proved.

Since the set K lies in the ball of dissipativity $\{z \in \mathcal{H} : \|z\| \le R\}$, estimates (9.13) and (9.14) follow from Lemma 9.3. Moreover,

$$\Delta_1 = L_1, \quad \alpha_1 = D(R), \qquad \Delta_2 = L_2, \quad \alpha_2 = \varepsilon + D(R),$$

$$\beta = \gamma = \varepsilon, \quad \rho_n = C_0 r_n^{\sigma} C_3(R) \cdot D(R)^{-1}. \tag{9.16}$$

Thus, Lemma 9.4 is proved.

Lemma 9.4 along with the theorem given below enables us to verify the existence of an inertial set for the dynamical system generated by equation (9.1).

Theorem 9.2.

Let the phase space $\mathcal{H}$ ***of a dynamical system*** $(\mathcal{H}, S_t)$ ***be a Hilbert space. Assume that in*** $\mathcal{H}$ ***there exists a compact positively invariant set*** K ***possessing properties*** (9.12)–(9.14). ***Then for any*** $\nu > \ln 2$ ***there exists an inertial set*** $A^{\nu}_{\exp}$ ***of the dynamical system*** $(\mathcal{H}, S_t)$ ***such that***

$$h(S_t B, A^{\nu}_{\exp}) \le C(B, \nu) \cdot \exp\left\{-\gamma\left(1 - \frac{\gamma + \alpha_1}{\nu + \gamma + \alpha_1}\right)(t - t_B)\right\} \tag{9.17}$$

for any bounded set B ***and*** $t \ge t_B$. ***Here, as above,*** $h(X, Y) = \sup\{\operatorname{dist}(x, Y) : x \in X\}$. ***Moreover,***

$$\dim_f A^{\nu}_{\exp} \le C_0 \cdot (1 + \ln\|P_n\|) \dim P_n, \tag{9.18}$$

where the number n is determined from the condition

$$\rho_n \le (4\Delta_2)^{\frac{\alpha_2}{\beta}} \cdot \exp\left\{-\nu\frac{\alpha_2}{\beta}\right\} \tag{9.19}$$

and constant C_0 does not depend on ν and n.

The proof of the theorem is based on the following preliminary assertions.

Lemma 9.5.

Let (K, S_t) be a dynamical system, its phase space being a compact in a Hilbert space $\mathcal{H}$. Assume that for all $(y_1, y_2) \in K$ equations (9.13) *and* (9.14) *are valid. Then for any $\nu > \ln 2$ there exists an inertial set $A^\nu_{\exp}$ of the system (K, S_t) such that*

$$h(S_t K, A^\nu_{\exp}) = \sup\{\operatorname{dist}(S_t y, A^\nu_{\exp}): \; y \in K\} \le C_\nu e^{-\nu t}. \tag{9.20}$$

Moreover, estimate (9.18) *holds for the value* $\dim_f A^\nu_{\exp}$.

Proof.

We use Theorem 8.2 with $M = K$, $V = S_{t_0}$, and $\delta = \frac{1}{2}e^{-\nu}$, where t_0 and n are chosen to fulfil

$$\Delta_2\, e^{-\beta t_0} = \frac{\delta}{2} = \frac{1}{4}e^{-\nu} \quad \text{and} \quad \rho_n\, e^{\alpha_2 t_0} \le 1.$$

In this case conditions (8.1) and (8.2) are valid for $V = S_{t_0}$ with $\delta = \frac{1}{2}e^{-\nu}$ and $l = \|P_n\|\Delta_1\, e^{\alpha_1 t_0}$. Therefore, there exists a bounded closed positively invariant set A_θ with $\delta < \theta < 1/2$ such that (see (8.6) and (8.15))

$$\sup\{\operatorname{dist}(V^m y, A_\theta): \; y \in K\} \le \theta^m, \quad m = 1, 2, \ldots \tag{9.21}$$

and

$$\dim_f A_\theta \le \ln\left(1 + \frac{4l}{\theta - \delta}\right)\left[\ln\frac{1}{2\theta}\right]^{-1} \cdot \dim P_n \tag{9.22}$$

Assume that $\theta = 2\delta = e^{-\nu}$ and consider the set

$$A^\nu_{\exp} = \bigcup\{S_t A_\theta: \; 0 \le 0 \le t_0\}.$$

Here $\nu = \ln\frac{1}{\theta} > \ln 2$. It is easy to see that

$$\dim_f A^\nu_{\exp} \le 1 + \dim_f A^\nu_{\exp}.$$

Therefore, equations (9.20) and (9.18) follow from (9.21) and (9.20) after some simple calculations.

Lemma 9.6.

Assume that in the phase space $\mathcal{H}$ of a dynamical system $(\mathcal{H}, S_t)$ there exist compact sets K and K_0 such that (a) $K_0 \subset K$; (b) *properties*

(9.12) *and* (9.13) *are valid for* K; *and* (c) *the set* K_0 *possesses the property*

$$h(S_t K,\, K_0) \le C e^{-\gamma_0 t}, \tag{9.23}$$

where $h(X, Y) = \sup\{\operatorname{dist}(x, Y) \colon x \in X\}$. *Then for any bounded set* $B \subset \mathcal{H}$ *and* $t \ge t_B$ *the following inequality holds*

$$h(S_t B,\, K_0) \le C_B \exp\left\{ -\frac{\gamma\gamma_0}{\gamma + \gamma_0 + \alpha_1} t \right\}. \tag{9.24}$$

Proof.

By virtue of (9.12) every bounded set B reaches the vicinity $\mathcal{O}$ in finite time and stays in it. Therefore, it is sufficient to prove the lemma for a set $B \in \mathcal{B}(\mathcal{H})$ such that $S_t B \subset [\mathcal{O}]$ for $t \ge 0$, where $[\mathcal{O}]$ denotes the closure of $\mathcal{O}$. Let $k_0 \in K_0$ and $y \in B$. Evidently,

$$\|S_t y - k_0\| \le \|S_{\varkappa t} S_{(1-\varkappa)t} y - S_{\varkappa t} k\| + \|S_{\varkappa t} k - k_0\|$$

for any $0 \le \varkappa \le 1$ and $k \in K$. With the help of (9.13) we have that

$$\|S_t y - k_0\| \le \Delta_1 e^{\alpha_1 \varkappa t} \|S_{(1-\varkappa)t} y - k\| + \|S_{\varkappa t} k - k_0\|.$$

Therefore, for any $0 < \varkappa < 1$ and $k \in K$ we have that

$$\operatorname{dist}(S_t y,\, K_0) \le \Delta_1 e^{\alpha_1 \varkappa t} \|S_{(1-\varkappa)t}\, y - k\| + \operatorname{dist}(S_{\varkappa t} k,\, K_0) \le$$

$$\le \Delta_1 e^{\alpha_1 \varkappa t} \|S_{(1-\varkappa)t} y - k\| + h(S_{\varkappa t} K,\, K_0).$$

If we take an infimum over $k \in K$ and a supremum over $y \in B$, we find that

$$h(S_t B,\, K_0) \le \Delta_1 e^{\alpha_1 \varkappa t} h(S_{(1-\varkappa)t} B,\, K) + h(S_{\varkappa t} K,\, K_0)$$

for all $0 < \varkappa < 1$. Hence, equations (9.12) and (9.23) give us that

$$h(S_t B,\, K_0) \le C_B e^{(\alpha_1 \varkappa - \gamma(1-\varkappa))t} + C_K e^{-\varkappa \gamma_0 t}$$

for $t \ge t_B$. If we choose $\varkappa = \gamma(\gamma + \gamma_0 + \alpha_1)^{-1}$, we obtain (9.24). Lemma 9.6 is proved.

If we now use Lemma 9.6 with $K_0 = A^{\nu}_{\exp}$ and estimate (9.20), we get equation (9.17). This **completes the proof of Theorem 9.2**.

Thus, by virtue of Lemma 9.4 and Theorem 9.2 the dynamical system $(\mathcal{H},\, S_t)$ generated by equation (9.1) possesses an inertial set $A^{\nu}_{\exp}$ for which equations (9.17)–(9.19) hold with relations (9.16).

It should be noted that a slightly different approach to the construction of inertial sets is developed in the book by A. Eden, C. Foias, B. Nicolaenko, and R. Temam (see the list of references). This book contains further developments and applications of the theory of inertial sets.

To conclude this section, we outline the results on the behaviour of the projection onto the finite-dimensional subspace $P_n \mathcal{H}$ of the trajectories of the system $(\mathcal{H}, S_t)$ generated by equation (9.1).

Assume that an element y_0 belongs to the global attractor $\mathcal{A}$ of a dynamical system $(\mathcal{H}, S_t)$. Lemma 6.1 implies that there exists a trajectory $\gamma = \{y(t),\ t \in \mathbb{R}\}$ lying in $\mathcal{A}$ wholly such that $y(0) = y_0$. Therewith the following assertion is valid.

Lemma 9.7.

Assume that properties **(A)–(C)** *are fulfilled and let* $y_0 \in \mathcal{A}$. *Then the following equation holds:*

$$(1-P_n)y_0 = \int_{-\infty}^{0} (1-P_n)T_{-\tau}B(y(\tau))\,\mathrm{d}\tau\ , \qquad n \ge n_0\ , \tag{9.25}$$

where $\{y(\tau)\}$ *is a trajectory passing through* y_0, *the number* n_0 *can be found from* **(B2)** *and the integral in* (9.25) *converges in the norm of the space* $\mathcal{H}$.

Proof.

Since $y_0 = S_t\, y(-t)$, equation (9.2) gives us that

$$(1-P_n)y_0 = (1-P_n)\left(T_t\, y(-t) + \int_{-t}^{0} (1-P_n)T_{-\tau}B(y(\tau))\,\mathrm{d}\tau \right). \tag{9.26}$$

A trajectory in the attractor possesses the property $\|y(t)\| \le R$, $t \in (-\infty, \infty)$. Therefore, property **(B2)** implies that

$$\|(1-P_n)T_t\, y(-t)\| \le L_2 \cdot R\, e^{-\varepsilon t} \quad \text{and} \quad \|(1-P_n)T_{-\tau}B(y(\tau))\| \le L_2 C_R\, e^{-\varepsilon|\tau|}\ .$$

These estimates enable us to pass to the limit in (9.26) as $t \to -\infty$. Thereupon we obtain (9.25).

The following assertion is valid under the hypotheses of Theorem 9.1.

Theorem 9.3.

There exists $N_0 \ge n_0$ ***such that for all*** $N \ge N_0$ ***the following assertions are valid:***

1) ***for any two trajectories*** $y_1(t)$ ***and*** $y_2(t)$ ***lying in the attractor of the system generated by equation*** (9.1) ***the equality*** $P_N y_1(t) = P_N y_2(t)$ ***for all*** $t \in \mathbb{R}$ ***implies that*** $y_1(t) \equiv y_2(t)$***;***
2) ***for any two solutions*** $u_1(t)$ ***and*** $u_2(t)$ ***of the system*** (9.1) ***the equation***

$$\lim_{t \to -\infty} P_N(u_1(t) - u_2(t)) = 0 \tag{9.27}$$

implies that $\|u_1(t) - u_2(t)\| \to 0$ ***as*** $t \to \infty$.

We can also obtain an upper estimate of the number N_0 from the inequality $r_{N_0}^{\sigma} \le a_0 \varepsilon (L_2 C_3(R))^{-1}$.

Proof.

Equation (9.25) implies that for any trajectory $y_i(t)$ lying in the attractor of system (9.1) the equation

$$(1-P_N)\, y_i(t) = \int_{-\infty}^{t} (1-P_N) T_{t-\tau} B(y_i(\tau))\, \mathrm{d}\tau , \quad i = 1,\ 2 \ ,$$

holds. Therefore, if $P_N y_1(t) = P_N y_2(t)$, then properties **(B2)**, **(B3)**, and **(C2)** give us that

$$\| y_1(t) - y_2(t) \| \le C_0 r_N^{\sigma} L_2 C_3(R) \int_{-\infty}^{t} e^{-\varepsilon(t-\tau)} \| y_1(\tau) - y_2(\tau) \| \mathrm{d}\tau .$$

It follows that the estimate

$$\| y_1(t) - y_2(t) \| \le A_N \exp\{-\varepsilon t + A_N (t - t_0)\} \int_{-\infty}^{t_0} e^{\varepsilon \tau} \| y_1(\tau) - y_2(\tau) \| \mathrm{d}\tau$$

holds for $t \ge t_0$, where $A_N = C_0 r_N^{\sigma} L_2 C_3(R)$. If we tend $t_0 \to -\infty$, we obtain the first assertion, provided $A_N < \varepsilon$.

Now let us prove the second assertion of the theorem. Let

$$\alpha_N(t) = \| P_N (u_1(t) - u_2(t)) \| .$$

Then

$$\| u_1(t) - u_2(t) \| \le \alpha_N(t) + \| (1-P_N)(u_1(t) - u_2(t)) \| .$$

Therefore, equation (9.10) for the function $\psi(t) = \| u_1 - u_2 \| \exp(\varepsilon t)$ gives us that

$$\psi(t) \le \alpha_N(t)\, e^{\varepsilon t} + L_2 \| u_1(0) - u_2(0) \| + A_N \int_0^t \psi(\tau) \mathrm{d}\tau .$$

This and Gronwall's lemma imply that

$$\| u_1(t) - u_2(t) \| \le \alpha_N(t) + L_2 \exp(-(\varepsilon - A_N) t) \| u_1(0) - u_2(0) \| +$$

$$+ A_N \int_0^t \alpha_N(\tau) \exp(-(\varepsilon - A_N)(t - \tau)) \mathrm{d}\tau .$$

Therefore, if $A_N < \varepsilon$, then equation (9.27) gives us that $\| u_1(t) - u_2(t) \| \to 0$. Thus, the second assertion of **Theorem 9.3 is proved**.

Theorem 9.3 can be presented in another form. Let $\{e_k: \; k = 1, \ldots \; , d_N\}$ be a basis in the space $P_N \mathscr{H}$. Let us define linear functionals $l_j(u) = (u\,, e_j)$ on $\mathscr{H}$, $j = 1, \ldots, d_N$. Theorem 9.3 implies that the asymptotic behaviour of trajectories of the system $(\mathscr{H}, S_t)$ is uniquely determined by its values on the functionals l_j. Therefore, it is natural that the family of functionals $\{l_j\}$ is said to be the determining collection. At present some general approaches have been worked out which enable us to define whether a particular set of functionals is determining. Chapter 5 is devoted to the exposition of these approaches. It should be noted that for the first time Theorem 9.3 was proved for the two-dimensional Navier-Stokes system by C. Foias and D. Prodi (the second assertion) and by O. A. Ladyzhenskaya (the first assertion).

Concluding the chapter, we would like to note that the list of references given below does not claim to be full. It contains only references to some monographs and reviews devoted to the developments of the questions touched on here and comprising intensive bibliography.

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Chapter 2

Long-Time Behaviour of Solutions to a Class of Semilinear Parabolic Equations

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In this chapter we study well-posedness and the asymptotic behaviour of solutions to a class of abstract nonlinear parabolic equations. A typical representative of this class is the nonlinear heat equation

$$\frac{\partial u}{\partial t} = \nu \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} + f(x,\, u)$$

considered in a bounded domain Ω of $\mathbb{R}^d$ with appropriate boundary conditions on the border $\partial\Omega$. However, the class also contains a number of nonlinear partial differential equations arising in Mechanics and Physics that are interesting from the applied point of view. The main feature of this class of equations lies in the fact that the corresponding dynamical systems possess a compact absorbing set.

The first three sections of this chapter are devoted to the questions of existence and uniqueness of solutions and a brief description of examples. They are independent of the results of Chapter 1. In the other sections containing the discussion of asymptotic properties of solutions we use general results on the existence and properties of global attractors proved in Chapter 1. In Sections 6 and 7 we present two quite simple infinite-dimensional systems for which the asymptotic behaviour of the trajectories can be explicitly described. In Section 8 we consider a class of systems generated by infinite-dimensional retarded equations.

The list of references at the end of the chapter consists only of the books recommended for further reading.

§ 1 Positive Operators with Discrete Spectrum

This section contains some auxiliary facts that play an important role in the subsequent considerations related to the study of the asymptotic properties of solutions to abstract semilinear parabolic equations.

Assume that H is a separable Hilbert space with the inner product $(\cdot,\,\cdot)$ and the norm $\|\cdot\|$. Let A be a selfadjoint positive linear operator with the domain $D(A)$. An operator A is said to have a ***discrete spectrum*** if in the space H there exists an orthonormal basis $\{e_k\}$ of the eigenvectors:

$$(e_k,\, e_j) = \delta_{kj}, \quad A e_k = \lambda_k e_k, \qquad k,\, j = 1,\, 2,\, \dots\,, \tag{1.1}$$

such that

$$0 < \lambda_1 \le \lambda_2 \le \dots\,, \qquad \lim_{k \to \infty} \lambda_k = \infty\,. \tag{1.2}$$

The following exercise contains a simple example of an operator with discrete spectrum.

Exercise *1.1* Let $H = L^2(0, 1)$ and let A be an operator defined by the equation $Au = -u''$ with the domain $D(A)$ which consists of continuously differentiable functions $u(x)$ such that (a) $u(0)=u(1)=0$, (b) $u'(x)$ is absolutely continuous and (c) $u'' \in L^2(0, 1)$. Show that A is a positive operator with discrete spectrum. Find its eigenvectors and eigenvalues.

The above-mentioned structure of the operator A enables us to define an operator $f(A)$ for a wide class of functions $f(\lambda)$ defined on the positive semiaxis. It can be done by supposing that

$$D(f(A)) = \left\{ h = \sum_{k=1}^{\infty} c_k e_k \in H : \sum_{k=1}^{\infty} c_k^2 [f(\lambda_k)]^2 < \infty \right\},$$

$$f(A)h = \sum_{k=1}^{\infty} c_k f(\lambda_k) e_k, \qquad h \in D(f(A)). \tag{1.3}$$

In particular, one can define operators A^α with $\alpha \in \mathbb{R}$. For $\alpha = -\beta < 0$ these operators are bounded. However, in this case it is also convenient to introduce the lineals $D(A^\alpha)$ if we regard $D(A^{-\beta})$ as a completion of the space H with respect to the norm $\|A^{-\beta} \cdot\|$.

Exercise *1.2* Show that the space $\mathcal{F}_{-\beta} = D(A^{-\beta})$ with $\beta > 0$ can be identified with the space of formal series $\sum c_k e_k$ such that

$$\sum_{k=1}^{\infty} c_k^2 \lambda_k^{-2\beta} < \infty .$$

Exercise *1.3* Show that for any $\beta \in \mathbb{R}$ the operator A^β can be defined on every space $D(A^\alpha)$ as a bounded mapping from $D(A^\alpha)$ into $D(A^{\alpha-\beta})$ such that

$$A^\beta D(A^\alpha) = D(A^{\alpha-\beta}), \qquad A^{\beta_1+\beta_2} = A^{\beta_1} \cdot A^{\beta_2}. \tag{1.4}$$

Exercise *1.4* Show that for all $\alpha \in \mathbb{R}$ the space $\mathcal{F}_\alpha \equiv D(A^\alpha)$ is a separable Hilbert space with the inner product $(u, v)_\alpha = (A^\alpha u, A^\alpha v)$ and the norm $\|u\|_\alpha = \|A^\alpha u\|$.

Exercise *1.5* The operator A with the domain $\mathcal{F}_{1+\sigma}$ is a positive operator with discrete spectrum in each space $\mathcal{F}_\sigma$.

Exercise *1.6* Prove the continuity of the embedding of the space $\mathcal{F}_\alpha$ into $\mathcal{F}_\beta$ for $\alpha > \beta$, i.e. verify that $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ and $\|u\|_\beta \le C\|u\|_\alpha$.

Exercise *1.7* Prove that $\mathcal{F}_\alpha$ is dense in $\mathcal{F}_\beta$ for any $\alpha > \beta$.

— Exercise *1.8* Let $f \in \mathcal{F}_\sigma$ for $\sigma > 0$. Show that the linear functional $F(g) \equiv \equiv (f, g)$ can be continuously extended from the space H to $\mathcal{F}_{-\sigma}$ and $|(f, g)| \le \|f\|_\sigma \cdot \|g\|_{-\sigma}$ for any $f \in \mathcal{F}_\sigma$ and $g \in \mathcal{F}_{-\sigma}$.

— Exercise *1.9* Show that any continuous linear functional F on $\mathcal{F}_\sigma$ has the form: $F(f) = (f, g)$, where $g \in \mathcal{F}_{-\sigma}$. Thus, $\mathcal{F}_{-\sigma}$ is the space of continuous linear functionals on $\mathcal{F}_\sigma$.

The collection of Hilbert spaces with the properties mentioned in Exercises 1.7–1.9 is frequently called a ***scale*** of Hilbert spaces. The following assertion on the compactness of embedding is valid for the scale of spaces $\{\mathcal{F}_\sigma\}$.

Theorem 1.1.

Let $\sigma_1 > \sigma_2$. Then the space $\mathcal{F}_{\sigma_1}$ is compactly embedded into $\mathcal{F}_{\sigma_2}$, i.e. every sequence bounded in $\mathcal{F}_{\sigma_1}$ is compact in $\mathcal{F}_{\sigma_2}$.

Proof.

It is well known that every bounded set in a separable Hilbert space is weakly compact, i.e. it contains a weakly convergent sequence. Therefore, it is sufficient to prove that any sequence weakly tending to zero in $\mathcal{F}_{\sigma_1}$ converges to zero with respect to the norm of the space $\mathcal{F}_{\sigma_2}$. We remind that a sequence $\{f_n\}$ in $\mathcal{F}_\sigma$ weakly converges to an element $f \in \mathcal{F}_\sigma$ if for all $g \in \mathcal{F}_\sigma$

$$\lim_{n \to \infty} (f_n, g)_\sigma = (f, g)_\sigma .$$

Let the sequence $\{f_n\}$ be weakly convergent to zero in $\mathcal{F}_{\sigma_1}$ and let

$$\|f_n\|_{\sigma_1} \le C, \qquad n = 1, 2, \dots . \tag{1.5}$$

Then for any N we have

$$\|f_n\|^2_{\sigma_2} \le \sum_{k=1}^{N-1} \lambda_k^{2\sigma_2} (f_n, e_k)^2 + \frac{1}{N^{2(\sigma_1 - \sigma_2)}} \sum_{k=N}^{\infty} \lambda_k^{2\sigma_1} (f_n, e_k)^2 . \tag{1.6}$$

Here we applied the fact that for $k \ge N$

$$\lambda_k^{2\sigma_2} \le \frac{1}{\lambda_N^{2(\sigma_1 - \sigma_2)}} \lambda_k^{2\sigma_1} .$$

Equations (1.5) and (1.6) imply that

$$\|f_n\|^2_{\sigma_2} \le \sum_{k=1}^{N-1} \lambda_k^{2\sigma_2} (f_n, e_k)^2 + C \lambda_N^{-2(\sigma_1 - \sigma_2)} .$$

We fix $\varepsilon > 0$ and choose N such that

$$\|f_n\|^2_{\sigma_2} \le \sum_{k=1}^{N-1} \lambda_k^{2\sigma_2} (f_n, e_k)^2 + \varepsilon . \tag{1.7}$$

Let us fix the number N. The weak convergence of f_n to zero gives us

$$\lim_{n \to \infty} (f_n, e_k) = 0, \qquad k = 1, 2, \dots, N-1.$$

Therefore, it follows from (1.7) that

$$\overline{\lim_{n \to \infty}} \|f_n\|_{\sigma_2} \le \varepsilon.$$

By virtue of the arbitrariness of ε we have

$$\lim_{n \to \infty} \|f_n\|_{\sigma_2} = 0.$$

Thus, **Theorem 1.1 is proved**.

— Exercise *1.10* Show that the resolvent $R_\lambda(A) = (A-\lambda)^{-1}$, $\lambda \ne \lambda_k$, is a compact operator in each space $\mathcal{F}_\sigma$.

We point out several properties of the scale of spaces $\{\mathcal{F}_\sigma\}$ that are important for further considerations.

— Exercise *1.11* Show that in each space $\mathcal{F}_\sigma$ the equation

$$P_l u = \sum_{k=1}^{l} (u, e_k) e_k, \qquad u \in \mathcal{F}_\sigma, \quad -\infty < \sigma < \infty$$

defines an orthoprojector onto the finite-dimensional subspace generated by the set of elements $\{e_k, k = 1, 2, \dots, l\}$. Moreover, for each σ we have

$$\lim_{l \to \infty} \|P_l u - u\|_\sigma = 0.$$

— Exercise *1.12* Using the Hölder inequality

$$\sum_k a_k b_k \le \left(\sum_k a_k^p\right)^{1/p} \left(\sum_k b_k^q\right)^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad a_k, b_k > 0,$$

prove the interpolation inequality

$$\|A^\theta u\| \le \|Au\|^\theta \cdot \|u\|^{1-\theta}, \qquad 0 \le \theta \le 1, \quad u \in D(A).$$

— Exercise *1.13* Relying on the result of the previous exercise verify that for any $\sigma_1, \sigma_2 \in \mathbb{R}$ the following interpolation estimate holds:

$$\|u\|_{\sigma(\theta)} \le \|u\|_{\sigma_1}^\theta \|u\|_{\sigma_2}^{1-\theta},$$

where $\sigma(\theta) = \theta\sigma_1 + (1-\theta)\sigma_2$, $0 \le \theta \le 1$, and $u \in \mathcal{F}_{\max(\sigma_1, \sigma_2)}$. Prove the inequality

$$\|u\|_{\sigma(\theta)}^2 \le \varepsilon \|u\|_{\sigma_1}^2 + C_\theta \varepsilon^{-\frac{\theta}{1-\theta}} \|u\|_{\sigma_2}^2,$$

where $0 < \theta < 1$ and ε is a positive number.

Equations (1.3) enable us to define an exponential operator $\exp(-tA)$, $t \ge 0$, in the scale $\{\mathcal{F}_\sigma\}$. Some of its properties are given in exercises 1.14–1.17.

— Exercise *1.14* For any $\alpha \in \mathbb{R}$ and $t > 0$ the linear operator $\exp(-tA)$ maps $\mathcal{F}_\alpha$ into $\bigcap_{\sigma \ge 0} \mathcal{F}_\sigma$ and possesses the property $\|e^{-tA}u\|_\alpha \le$ $\le e^{-t\lambda_1}\|u\|_\alpha$.

— Exercise *1.15* The following semigroup property holds:

$$\exp(-t_1 A)\cdot \exp(-t_2 A) = \exp(-(t_1+t_2)A), \qquad t_1,\ t_2 \ge 0 .$$

— Exercise *1.16* For any $u \in \mathcal{F}_\sigma$ and $\sigma \in \mathbb{R}$ the following equation is valid:

$$\lim_{t \to \tau} \|e^{-tA}u - e^{-\tau A}u\|_\sigma = 0 . \tag{1.8}$$

— Exercise *1.17* For any $\sigma \in \mathbb{R}$ the exponential operator e^{-tA} defines a dissipative compact dynamical system $(\mathcal{F}_\sigma, e^{-tA})$. What can you say about its global attractor?

Let us introduce the following notations. Let $C(a, b; \mathcal{F}_\alpha)$ be the space of strongly continuous functions on the segment $[a, b]$ with the values in $\mathcal{F}_\alpha$, i.e. they are continuous with respect to the norm $\|\cdot\|_\alpha = \|A^\alpha \cdot\|$. In particular, Exercise 1.16 means that $e^{-tA}u \in C(\mathbb{R}_+, \mathcal{F}_\alpha)$ if $u \in \mathcal{F}_\alpha$. By $C^1(a, b; \mathcal{F}_\alpha)$ we denote the subspace of $C(a, b; \mathcal{F}_\alpha)$ that consists of the functions $f(t)$ which possess strong (in $\mathcal{F}_\alpha$) derivatives $f'(t)$ lying in $C(a, b; \mathcal{F}_\alpha)$. The space $C^k(a, b; \mathcal{F}_\alpha)$ is defined similarly for any natural k. We remind that the strong derivative (in $\mathcal{F}_\alpha$) of a function $f(t)$ at a point $t = t_0$ is defined as an element $v \in \mathcal{F}_\alpha$ such that

$$\lim_{h \to 0} \left\| \frac{1}{h}(f(t_0+h) - f(t_0)) - v \right\|_\alpha = 0 .$$

— Exercise *1.18* Let $u_0 \in \mathcal{F}_\sigma$ for some σ. Show that

$$e^{-tA}u_0 \in C^k(\delta, +\infty; \mathcal{F}_\alpha)$$

for all $\delta > 0$, $\alpha \in \mathbb{R}$, and $k = 1, 2, \ldots$. Moreover,

$$\frac{d^k}{dt^k} e^{-tA}u_0 = (-A)^k e^{-tA}u_0, \qquad k = 1, 2, \ldots .$$

Let $L^2(a, b; \mathcal{F}_\alpha)$ be the space of functions on the segment $[a, b]$ with the values in $\mathcal{F}_\alpha$ for which the integral

$$\|u\|^2_{L^2(a,\, b;\, \mathcal{F}_\alpha)} = \int_a^b \|u(t)\|^2_\alpha \, dt$$

exists. Let $L^\infty(a, b; \mathcal{F}_\alpha)$ be the space of essentially bounded functions on $[a, b]$ with the values in $\mathcal{F}_\alpha$ and the norm

$$\|u\|_{L^\infty(a,\, b;\; D(A^\alpha))} = \operatorname*{ess\,sup}_{t\, \in\, [a,\, b]} \|A^\alpha u(t)\| \quad .$$

We consider the Cauchy problem

$$\frac{dy}{dt} + Ay = f(t), \quad t \in (a,\, b); \qquad y(a) = y_0, \tag{1.9}$$

where $y \in \mathcal{F}_\alpha$ and $f(t) \in L^2(a,\, b;\; F_{\alpha - 1/2})$. The ***weak solution*** (in $\mathcal{F}_\alpha$) to this problem on the segment $[a,\, b]$ is defined as a function

$$y(t) \in C(a,\, b;\; \mathcal{F}_\alpha) \cap L^2(a,\, b;\; \mathcal{F}_{\alpha + 1/2}) \tag{1.10}$$

such that $dy/dt \in L^2(a,\, b;\; F_{\alpha - 1/2})$ and equalities (1.9) hold. Here the derivative $y'(t) \equiv dy/dt$ is considered in the generalized sense, i.e. it is defined by the equality

$$\int_a^b \varphi(t) y'(t) dt = -\int_a^b \varphi'(t) y(t) dt, \qquad \varphi \in C_0^\infty(a,\, b),$$

where $C_0^\infty(a,\, b)$ is the space of infinitely differentiable scalar functions on $(a,\, b)$ vanishing near the points a and b.

Exercise *1.19* Show that every weak solution to problem (1.9) possesses the property

$$\|y(t)\|_\alpha^2 + \int_a^t \|y(\tau)\|_{\alpha + \frac{1}{2}}^2 d\tau \;\le\; \|y_0\|_\alpha + \int_a^t \|f(\tau)\|_{\alpha - \frac{1}{2}}^2 d\tau. \tag{1.11}$$

(*Hint*: first prove the analogue of formula (1.11) for $y_n(t) = P_n y(t)$, then use Exercise 1.11).

Exercise *1.20* Prove the theorem on the existence and uniqueness of weak solutions to problem (1.9). Show that a weak solution $y(t)$ to this problem can be represented in the form

$$y(t) = e^{-(t-a)A} y_0 + \int_a^t e^{-(t-\tau)A} f(\tau) d\tau. \tag{1.12}$$

Exercise *1.21* Let $y_0 \in \mathcal{F}_\alpha$ and let a function $f(t)$ possess the property

$$\|f(t_1) - f(t_2)\|_\alpha \le C|t_1 - t_2|^\theta$$

for some $0 < \theta \le 1$. Then formula (1.12) gives us a solution to problem (1.9) belonging to the class

$$C([a,\, b];\; \mathcal{F}_\alpha) \cap C^1(]a,\, b];\; \mathcal{F}_\alpha) \cap C(]a,\, b];\; \mathcal{F}_{\alpha+1}).$$

Such a solution is said to be strong in $\mathcal{F}_\alpha$.

The following properties of the exponential operator e^{-tA} play an important part in the further considerations.

Lemma 1.1.

Let Q_N be the orthoprojector onto the closure of the span of elements $\{e_k,\ k \geq N+1\}$ in H and let $P_N = I - Q_N$, $N = 0,\ 1,\ 2,\ \dots$. Then

1) *for all $h \in H$, $\beta \geq 0$ and $t \in \mathbb{R}$ the following inequality holds:*

$$\left\| A^{\beta} P_N e^{-tA} h \right\| \leq \lambda_N^{\beta} e^{\lambda_N |t|} \|h\| ; \tag{1.13}$$

2) *for all $h \in D(A^{\beta})$, $t > 0$ and $\alpha \geq \beta$ the following estimate is valid:*

$$\left\| A^{\alpha} Q_N e^{-tA} h \right\| \leq \left[\left(\frac{\alpha - \beta}{t} \right)^{\alpha - \beta} + \lambda_{N+1}^{\alpha - \beta} \right] e^{-t \lambda_{N+1}} \| A^{\beta} h \| , \tag{1.14}$$

in the case $\alpha - \beta = 0$ we supose that $0^0 = 0$ in (1.14).

Proof.

Estimate (1.13) follows from the equation

$$\left\| A^{\beta} P_N e^{-tA} h \right\|^2 = \sum_{k=1}^{N} \lambda_k^{2\beta} e^{-2t\lambda_k} (h,\ e_k)^2 .$$

In the proof of (1.14) we similarly have that

$$\left\| A^{\alpha} Q_N e^{-tA} h \right\|^2 \leq \max_{\lambda \geq \lambda_{N+1}} (\lambda^{\alpha - \beta} e^{-t\lambda})^2 \sum_{k=N+1}^{\infty} \lambda_k^{2\beta} (h,\ e_k)^2 .$$

This gives us the inequality

$$\left\| A^{\alpha} Q_N e^{-tA} h \right\| \leq \frac{1}{t^{\alpha - \beta}} \max_{\mu \geq t\lambda_{N+1}} (\mu^{\alpha - \beta} e^{-\mu}) \| A^{\beta} h \| .$$

Since $\max\{ \mu^{\gamma} e^{-\mu};\ \mu \geq 0 \}$ is attained when $\mu = \gamma$, we have that

$$\max_{\mu \geq \lambda_{N+1} t} (\mu^{\gamma} e^{-\mu}) = \begin{cases} (\lambda_{N+1} t)^{\gamma}\, e^{-\lambda_{N+1} t}, & \text{if} \quad \lambda_{N+1} t \geq \gamma ; \\ \gamma^{\gamma} e^{-\gamma}, & \text{if} \quad \lambda_{N+1} t < \gamma . \end{cases}$$

Therefore,

$$\max_{\mu \geq \lambda_{N+1} t} (\mu^{\gamma} e^{-\mu}) \leq (\gamma^{\gamma} + (\lambda_{N+1} t)^{\gamma})\, e^{-\lambda_{N+1} t} .$$

This implies estimate (1.14). Lemma 1.1 is proved.

In particular, we note that it follows from (1.14) that

$$\| A^{\alpha} e^{-tA} h \| \leq \left[\left(\frac{\alpha - \beta}{t} \right)^{\alpha - \beta} + \lambda_1^{\alpha - \beta} \right] \cdot e^{-t\lambda_1} \| A^{\beta} h \| , \qquad \alpha \geq \beta . \tag{1.15}$$

— Exercise *1.22* Using estimate (1.15) and the equation

$$e^{-tA}u - e^{-sA}u = -\int_s^t A e^{-\tau A} u \, d\tau, \qquad t \ge s, \quad u \in \mathcal{F}_\beta,$$

prove that

$$\|e^{-tA}u - e^{-sA}u\|_\theta \le C_{\theta,\sigma}|t-s|^{\sigma-\theta}\|u\|_\sigma; \quad t, s > 0, \quad (1.16)$$

provided $\theta < \sigma \le 1 + \theta$, where a constant $C_{\theta,\sigma}$ does not depend on t and s (cf. Exercise 1.16).

— Exercise *1.23* Show that

$$\|A^\alpha e^{-tA}\| \le \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha}, \qquad t > 0, \quad \alpha > 0. \quad (1.17)$$

Lemma 1.2.

Let $f(t) \in L^\infty(\mathbb{R}, \mathcal{F}_{\alpha-\gamma})$ for $0 \le \gamma < 1$. Then there exists a unique solution $v(t) \in C(\mathbb{R}, \mathcal{F}_\alpha)$ to the nonhomogeneous equation

$$\frac{dv}{dt} + Av = f(t), \quad t \in \mathbb{R}, \quad (1.18)$$

that is bounded in $\mathcal{F}_\alpha$ on the whole axis. This solution can be represented in the form

$$v(t) = \int_{-\infty}^t e^{-(t-\tau)A} f(\tau) d\tau. \quad (1.19)$$

We understand the solution to equation (1.18) on the whole axis as a function $v(t) \in C(\mathbb{R}, \mathcal{F}_\alpha)$ such that for any $a < b$ the function $v(t)$ is a weak solution (in $\mathcal{F}_{\alpha-1/2}$) to problem (1.9) on the segment $[a, b]$ with $y_0 = v(a)$.

Proof.

If there exist two bounded solutions to problem (1.18), then their difference $w(t)$ is a solution to the homogeneous equation. Therefore, $w(t) = \exp\{-(t-t_0)A\} w(t_0)$ for $t \ge t_0$ and for any t_0. Hence,

$$\|A^\alpha w(t)\| \le e^{-(t-t_0)\lambda_1}\|A^\alpha w(t_0)\| \le Ce^{-(t-t_0)\lambda_1}.$$

If we tend $t_0 \to -\infty$ here, then we obtain that $w(t) = 0$. Thus, the bounded solution to problem (1.18) is unique. Let us prove that the function $v(t)$ defined by formula (1.19) is the required solution. Equation (1.15) implies that

$$\|A^\alpha e^{-(t-\tau)A}\| \le \left[\left(\frac{\gamma}{t-\tau}\right)^\gamma + \lambda_1^\gamma\right] e^{-(t-\tau)\lambda_1} \operatorname*{ess\,sup}_{\tau \in \mathbb{R}} \|A^{\alpha-\gamma} f(\tau)\|$$

for $t > \tau$ and $0 < \gamma < 1$. Therefore, integral (1.19) exists and it can be uniformly estimated with respect to t as follows:

$$\|A^{\alpha} v(t)\| \le \frac{1+k}{\lambda_1^{1-\gamma}} \operatorname*{ess\,sup}_{\tau \in \mathbb{R}} \|A^{\alpha-\gamma} f(\tau)\|,$$

where $k = 0$ for $\gamma = 0$ and

$$k = \gamma^{\gamma} \int_0^{\infty} s^{-\gamma} e^{-s} \mathrm{d}s \quad \text{for} \quad 0 < \gamma < 1 .$$

The continuity of the function $v(t)$ in $\mathcal{F}_{\alpha}$ follows from the following equation that can be easily verified:

$$v(t) = e^{-(t-t_0)A} v(t_0) + \int_{t_0}^{t} e^{-(t-\tau)A} g(\tau) \mathrm{d}\tau .$$

This also implies (see Exercise 1.18) that $v(t)$ is a solution to equation (1.18). Lemma 1.2 is proved.

§ 2 *Semilinear Parabolic Equations in Hilbert Space*

In this section we prove theorems on the existence and uniqueness of solutions to an evolutionary differential equation in a separable Hilbert space H of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = B(u, t), \qquad u\big|_{t=s} = u_0, \tag{2.1}$$

where A is a positive operator with discrete spectrum and $B(\cdot, \cdot)$ is a nonlinear continuous mapping from $D(A^{\theta}) \times \mathbb{R}$ into H, $0 \le \theta < 1$, possessing the property

$$\|B(u_1, t) - B(u_2, t)\| \le M(\rho) \|A^{\theta}(u_1 - u_2)\| \tag{2.2}$$

for all u_1 and u_2 from the domain $\mathcal{F}_{\theta} = D(A^{\theta})$ of the operator A^{θ} and such that $\|A^{\theta} u_j\| \le \rho$. Here $M(\rho)$ is a nondecreasing function of the parameter ρ that does not depend on t and $\|\cdot\|$ is a norm in the space H.

A function $u(t)$ is said to be a ***mild solution*** (in $\mathcal{F}_{\theta}$) to problem (2.1) on the half-interval $[s, s+T)$ if it lies in $C(s, s+T'; \mathcal{F}_{\theta})$ for every $T' < T$ and for all $t \in [s, s+T)$ satisfies the integral equation

$$u(t) = e^{-(t-s)A} u_0 + \int_s^t e^{-(t-\tau)A} B(u(\tau), \tau) \mathrm{d}\tau . \tag{2.3}$$

The fixed point method enables us to prove the following assertion on the local existence of mild solutions.

Theorem 2.1.

Let $u_0 \in \mathcal{F}_\theta$. Then there exists T^* depending on θ and $\|u_0\|_\theta$ such that problem (2.1) possesses a unique mild solution on the half-interval $[s,\ s+T^*)$. Moreover, either $T^* = \infty$ or the solution cannot be continued in $\mathcal{F}_\theta$ up to the moment $t = s + T^*$.

Proof.

On the space $C_{s,\,\theta} \equiv C(s,\ s+T;\ \mathcal{F}_\theta)$ we define the mapping

$$G[u](t) \;=\; e^{-(t-s)A}u_0 + \int_s^t e^{-(t-\tau)A}B(u(\tau),\ \tau)\mathrm{d}\tau\,.$$

Let us prove that $G[u](t) \in C(s,\ s+T;\ \mathcal{F}_\theta)$ for any $T > 0$. Assume that $t_1,\ t_2 \in$ $\in [s,\ s+T]$ and $t_1 < t_2$. It is evident that

$$G[u](t_2) \;=\; e^{-(t_2-t_1)A}G[u](t_1) + \int_{t_1}^{t_2} e^{-(t_2-\tau)A}B(u(\tau),\ \tau)\mathrm{d}\tau\,. \qquad (2.4)$$

By virtue of (1.8) we have that if $t_2 \to t_1$, then

$$\left\| G[u](t_1) - e^{-(t_2-t_1)A}G[u](t_1)\right\|_\theta \to 0\,.$$

Therefore, it is sufficient to estimate the second term in (2.4). Equation (1.15) implies that

$$\left\|\int_{t_1}^{t_2} e^{-(t_2-\tau)A}B(u(\tau),\ \tau)\mathrm{d}\tau\right\|_\theta \;\le$$

$$\le\; \int_{t_1}^{t_2}\left[\left(\frac{\theta}{t_2-\tau}\right)^\theta + \lambda_1^\theta\right]\mathrm{d}\tau\cdot \max_{\tau\,\in\,[s,\ s+T]}\|B(u(\tau),\ \tau)\| \;\le$$

$$\le\; |t_2-t_1|^{1-\theta}\left\{\frac{\theta^\theta}{1-\theta} + \lambda_1^\theta|t_2-t_1|^\theta\right\}\max_{\tau\,\in\,[s,\ s+T]}\|B(u(\tau),\ \tau)\| \qquad (2.5)$$

(if $\theta = 0$, then the coefficient in the braces should be taken to be equal to 1). Thus, G maps $C_{s,\,\theta} = C(s,\ s+T;\ \mathcal{F}_\theta)$ into itself. Let $v_0(t) = e^{-(t-s)A}u_0$. In $C_{s,\,\theta}$ we consider a ball of the form

$$U \;=\; \left\{u(t) \in C_{s,\,\theta}:\quad |u - v_0|_{C_{s,\,\theta}} \;\equiv\; \max_{[s,\ s+T]}\|u(t) - v_0(t)\|_\theta \;\le\; 1\right\}.$$

Let us show that for T small enough the operator G maps U into itself and is contractive. Since $|u|_{C_{s,\,\theta}} \le 1 + \|u_0\|_\theta$ for $u \in U$, equation (2.2) gives

$$\max_{\tau\in[s,\ s+T]} \|B(u(\tau),\ \tau)\| \le \max_{\tau\in[s,\ s+T]} \|B(0,\ \tau)\| +$$

$$+\ (1+\|u_0\|_\theta)M(1+\|u_0\|_\theta) \equiv C(T_0,\ \|u_0\|_\theta)$$

for all $T \le T_0$, where T_0 is a fixed number. Therefore, with the help of (2.5) we find that

$$|G[u]-v_0|_{C_{s,\ \theta}} \le T^{1-\theta}\cdot C_1(T_0,\ \theta,\ \|u_0\|_\theta).$$

Similarly we have

$$|G[u]-G[v]|_{C_{s,\ \theta}} \le T^{1-\theta}\cdot C_2(T_0,\ \theta)M(1+\|u\|_\theta)|u-v|_{C_{s,\ \theta}}$$

for $u,\ v\in U$. Consequently, if we choose T_1 such that

$$T_1^{1-\theta}C_1(T_0,\ \theta,\ \|u_0\|_\theta) \le 1 \quad \text{and} \quad T_1^{1-\theta}C_2(T_0,\ \theta)M(1+\|u_0\|_\theta) < 1,$$

we obtain that G is a contractive mapping of U into itself. Therefore, G possesses a unique fixed point in $U \subset C_{s,\ \theta}$. Thus, we have constructed a solution on the segment $[s,\ s+T_1]$. Taking $s+T_1$ as an initial moment, we can construct a solution on the segment $[s+T_1,\ s+T_1+T_2]$ with the initial condition $u_0 = u(s+T_1)$. If we continue our reasoning, then we can construct a solution on some maximal half-interval $[s,\ s+T^*)$. Moreover, it is possible that $T^* = \infty$. **Theorem 2.1 is proved**.

— Exercise *2.1* Let $u_0\in\mathscr{F}_\theta$ and let $T^* = T^*(\theta,\ u_0)$ be such that $[s,\ s+T^*)$ is the maximal half-interval of the existence of the mild solution $u(t)$ to problem (2.1). Then we have either $T^* < \infty$ and $\overline{\lim}_{t\to s+T^*}\|u(t)\|_\theta = \infty$, or $T^* = \infty$.

— Exercise *2.2* Using equations (1.16) and (2.5), prove that for any mild solution $u(t)$ to problem (2.1) on $[s,\ s+T^*)$ the estimate

$$\|u(t)-u(\tau)\|_\alpha \le C(\theta,\ \alpha,\ T)|t-\tau|^{\theta-\alpha}, \quad t,\ \tau\in[s,\ s+T], \quad (2.6)$$

is valid, provided $u_0\in\mathscr{F}_\theta$, $0\le\alpha\le\theta$, and $T \le T^*$.

— Exercise *2.3* Let $u_0\in\mathscr{F}_\theta$ and let $u(t)$ be a mild solution to problem (2.1) on the half-interval $[s,\ s+T^*)$. Then

$$u(t) \in C(s,\ s+T,\ \mathscr{F}_\theta)\cap C(s+\delta,\ s+T,\ \mathscr{F}_{1-\sigma})\cap$$

$$\cap\ C^1(s+\delta,\ s+T,\ \mathscr{F}_{-\sigma})$$

for any $\sigma > 0$, $0 < \delta < T$, and $T < T^*$. Moreover, equations (2.1) are valid if they are understood as the equalities in $\mathscr{F}_{-\sigma}$ and $\mathscr{F}_\theta$, respectively.

It is frequently convenient to use the Galerkin method in the study of properties of mild solutions to the problem of the type (2.1). Let P_m be the orthoprojector in H

onto the span of elements $\{e_1, e_2, \ldots, e_m\}$. ***Galerkin approximate solution of the order*** m with respect to the basis $\{e_k\}$ is defined as a continuously differentiable function

$$u_m(t) = \sum_{k=1}^{m} g_k(t)\, e_k \tag{2.7}$$

with the values in the finite-dimensional space $P_m H$ that satisfies the equations

$$\frac{d}{dt} u_m(t) + A u_m(t) = P_m B(u_m(t),\ t)\,, \qquad t > s\,, \quad u_m\big|_{t=s} = P_m u_0\,. \tag{2.8}$$

It is clear that (2.8) can be rewritten as a system of ordinary differential equations for the functions $g_k(t)$.

— Exercise *2.4* Show that problem (2.8) is equivalent to the problem of finding a continuous function $u_m(t)$ with the values in $P_m H$ that satisfies the integral equation

$$u_m(t) = e^{-(t-s)A} P_m u_0 + \int_s^t e^{-(t-\tau)A} P_m B(u_m(\tau),\ \tau)\, d\tau\,. \tag{2.9}$$

— Exercise *2.5* Using the method of the proof of Theorem 2.1, prove the local solvability of problem (2.9) on a segment $[s,\ s+T]$, where the parameter $T > 0$ can be chosen to be independent of m. Moreover, the following uniform estimate is valid:

$$\max_{[s,\ s+T]} \|u_m(t)\|_\theta < R\,, \qquad m = 1,\ 2,\ 3,\ \ldots\,, \tag{2.10}$$

where $R > 0$ is a constant.

The following assertion on the convergence of approximate functions to exact ones holds.

Theorem 2.2.

Let $u_0 \in \mathcal{F}_\theta$. ***Assume that there exists a sequence of approximate solutions*** $u_m(t)$ ***on a segment*** $[s,\ s+T]$ ***for which estimate*** (2.10) ***is valid. Then there exists a mild solution*** $u(t)$ ***to problem*** (2.1) ***on the segment*** $[s,\ s+T]$ ***and***

$$\max_{[s,\ s+T]} \|u(t) - u_m(t)\|_\theta \le C\left(\|(1-P_m)u_0\|_\theta + \frac{1}{\lambda_{m+1}^{1-\theta}}\right), \tag{2.11}$$

where $C = C(\theta,\ R,\ T)$ ***is a positive constant independent of*** s.

Proof.

Let $n > m$. We use (2.9), (1.14) and (1.17) to find that for $\theta > 0$ we have

$$\|u_n(t) - u_m(t)\|_\theta \le \|(P_n - P_m)u_0\|_\theta +$$

$$+\int_s^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right] e^{-\lambda_{m+1}(t-\tau)} \|B(u_n(\tau),\ \tau)\| d\tau +$$

$$+ e^{-\theta}\int_s^t \left(\frac{\theta}{t-\tau}\right)^\theta \|B(u_n(\tau),\ \tau) - B(u_m(\tau),\ \tau)\| d\tau\ .$$

Therefore, equations (2.2) and (2.10) give us that

$$\|u_n(t) - u_m(t)\|_\theta \le \|(P_n - P_m)u_0\|_\theta +$$

$$+\left(\max_{[s,\ s+T]}\|B(0,\ \tau)\| + M(R)R\right) J_m(t,\ s) +$$

$$+M(R)\theta^\theta e^{-\theta}\int_s^t (t-\tau)^{-\theta}\|u_n(\tau) - u_m(\tau)\|_\theta d\tau\ , \tag{2.12}$$

where

$$J_m(t,\ s) = \int_s^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right] e^{-\lambda_{m+1}(t-\tau)} d\tau\,.$$

It is evident that $J_m(t,\ s) \le J_m(t,\ -\infty)$. By changing the variable in the integral $\xi = \lambda_{m+1}(t-\tau)$, we obtain

$$J_m(t,\ -\infty) = \lambda_{m+1}^{-1+\theta}\left(1 + \theta^\theta\int_0^\infty \xi^{-\theta} e^{-\xi} d\xi\right) \equiv \lambda_{m+1}^{-1+\theta}(1+k)\,.$$

Thus, equation (2.12) implies

$$\|u_n(t) - u_m(t)\|_\theta \le \|(P_n - P_m)u_0\|_\theta + \frac{a_1(\theta,\ R)}{\lambda_{m+1}^{1-\theta}} +$$

$$+a_2(\theta,\ R)\int_s^t (t-\tau)^{-\theta}\|u_n(\tau) - u_m(\tau)\|_\theta d\tau\ .$$

Hence, if we use Lemma 2.1 which is given below, we can find that

$$\|u_n(t) - u_m(t)\|_\theta \le C\left(\|(P_n - P_m)u_0\|_\theta + \frac{1}{\lambda_{m+1}^{1-\theta}}\right) \tag{2.13}$$

for all $t \in [s, s+T]$, where $C > 0$ is a constant depending on θ, R, and T. It is also evident that estimate (2.13) remains true for $\theta = 0$. It means that the sequence of approximate solutions $\{u_m(t)\}$ is a Cauchy sequence in the space $C(s, s+T; \mathcal{F}_\theta)$. Therefore, there exists an element $u(t) \in C(s, s+T; \mathcal{F}_\theta)$ such that equation (2.11) holds. Estimates (2.10) and (2.11) enable us to pass to the limit in (2.9) and to obtain equation (2.3) for $u(t)$. **Theorem 2.2 is proved**.

Exercise *2.6* Show that if the hypotheses of Theorem 2.2 hold, then the estimate

$$\max_{[s,\, s+T]} \|u(t) - u_m(t)\|_\alpha \le \frac{a_1}{\lambda_{m+1}^{\theta-\alpha}} \|(1-P_m)u_0\|_\theta + \frac{a_2}{\lambda_{m+1}^{1-\alpha}}$$

is valid with $0 \le \alpha \le \theta$. Here a_1 and a_2 are constants independent of θ, R, and T.

The following assertion provides a simple sufficient condition of the global solvability of problem (2.1).

Theorem 2.3.

Assume that the constant $M(\rho)$ in (2.2) does not depend on ρ, i.e. the mapping $B(u, t)$ satisfies the global Lipschitz condition

$$\|B(u_1, t) - B(u_2, t)\| \le M\|u_1 - u_2\|_\theta \tag{2.14}$$

for all $u_j \in \mathcal{F}_\theta$ with some constant $M > 0$. Then problem (2.1) has a unique mild solution on the half-interval $[s, +\infty)$, provided $u_0 \in \mathcal{F}_\theta$. Moreover, for any two solutions u_1 and u_2 the estimate

$$\|u_1(t) - u_2(t)\|_\theta \le a_1 e^{a_2(t-s)} \|u_1(s) - u_2(s)\|_\theta , \qquad t \ge s , \tag{2.15}$$

holds, where a_1 and a_2 are constants that depend on θ, λ_1, and M only.

The proof of this theorem is based on the following lemma (see the book by Henry [3], Chapter 7).

Lemma 2.1.

Assume that $\varphi(t)$ is a continuous nonnegative function on the interval $(0, T)$ such that

$$\varphi(t) \le c_0 t^{-\gamma_0} + c_1 \int_0^t (t-\tau)^{-\gamma_1} \varphi(\tau)\, d\tau, \qquad t \in (0, T), \tag{2.16}$$

where $c_0, c_1 \ge 0$ and $0 \le \gamma_0, \gamma_1 < 1$. Then there exists a constant $K = K(\gamma_1, c_1, T)$ such that

$$\varphi(t) \le \frac{c_0}{1-\gamma_0} t^{-\gamma_0} K(\gamma_1, c_1, T) . \tag{2.17}$$

Proof of Theorem 2.3.

Let $u(t)$ be a solution to problem (2.1) on the maximal half-interval of its existence $[s,\ s+T)$. Assume that $T<\infty$. Condition (2.14) gives us that

$$\|B(u,\ t)\| \ \le\ \|B(0,\ t)\|+M\|u\|_\theta \ \le\ M_0(T)+M\|u\|_\theta$$

for all $u\in\mathcal{F}_\theta$ and $t\in[0,\ T]$. Therefore, from (2.3) and (1.15) we find that

$$\|u(t)\|_\theta \ \le\ \|u_0\|_\theta+\int_s^t\left[\left(\frac{\theta}{t-\tau}\right)^\theta+\lambda_1^\theta\right](M_0(T)+M\|u(\tau)\|_\theta)\,d\tau\ .$$

Hence, for $t\in[s,\ s+T]$ we have that

$$\|u(t)\|_\theta \ \le\ C_0(\|u_0\|_\theta,\ T,\ \theta)+C_1(T,\ \theta)\int_s^t(t-\tau)^{-\theta}\|u(\tau)\|_\theta\,d\tau\,.$$

Therefore, Lemma 2.1 implies that the value $\|u(t)\|_\theta$ is bounded on $[s,\ s+T)$ which is impossible (see Exercise 2.1). Thus, the solution exists for any half-interval $[s,\ s+T)$. For the proof of estimate (2.15) we note that, as above, inequalities (2.3) and (1.15) for the function $w(t)=u_1(t)-u_2(t)$ give us that

$$\|w(t)\|_\theta \ \le\ \|w(s)\|_\theta+\int_s^t\left[\left(\frac{\theta}{t-\tau}\right)^\theta+\lambda_1^\theta\right]M\|w(\tau)\|_\theta\,d\tau\,.$$

If we apply Lemma 2.1, we find that

$$\|w(t)\|_\theta\le C(\theta,\ \lambda_1,\ M)\|w(s)\|\quad\text{for}\quad s\le t\le s+1\,.$$

Therefore, the estimate

$$\|w(t)\|_\theta \ \le\ C^{n+1}\|w(s)\|_\theta \ \le\ C\exp\{(t-s)\ln C\}\|w(s)\|_\theta$$

holds for $t=s+n+\sigma$, where n is natural and $0\le\sigma\le 1$. Thus, **Theorem 2.3 is proved**.

Exercise *2.7* Using estimate (1.14), prove that if the hypotheses of Theorem 2.3 hold, then the inequality

$$\|Q_N(u_1(t)-u_2(t))\|_\theta \ \le$$

$$\le\ \left\{e^{-\lambda_{N+1}(t-s)}+\frac{a_3}{\lambda_{N+1}^{1-\theta}}\ e^{a_2(t-s)}\right\}\|u_1(s)-u_2(s)\|_\theta \qquad (2.18)$$

is valid for any two solutions $u_1(t)$ and $u_2(t)$. Here $Q_N=I-P_N$ and P_N is the orthoprojector onto the span of $\{e_1,\ \dots\ e_N\}$, the number a_2 is the same as in (2.15) and a_3 depends on λ_1, θ, and M.

Let us consider one more case in which we can guarantee the global solvability of problem (2.1). Assume that condition (2.2) holds for $\theta=1/2$ and

$$B(u) = -B_0(u) + B_1(u,\ t)\,, \tag{2.19}$$

where $B_1(u,\ t)$ satisfies the global Lipschitz condition (2.14) with $\theta = 1/2$ and $B_0(u)$ is a potential operator on the space $V = \mathcal{F}_{1/2}$. This means that there exists a Frechét differentiable functional $F(u)$ on V such that $B_0(u) = F'(u)$, i.e.

$$\lim_{\|v\|_{1/2} \to 0} \frac{1}{\|v\|_{1/2}} \left| F(u+v) - F(u) - (B_0(u),\ v) \right| = 0\,.$$

Theorem 2.4.

Let (2.2) ***be valid with*** $\theta = 1/2$ ***and let decomposition*** (2.19) ***take place. Assume that the functional*** $F(u)$ ***is bounded below on*** $V = \mathcal{F}_{1/2}$. ***Then problem*** (2.1) ***has a unique mild solution*** $u(t) \in C([s,\ s+T];\ D(A^{1/2}))$ ***on an arbitrary segment*** $[s,\ s+T]$.

Proof.

Let $u_m(t)$ be an approximate solution to problem (2.1) on a segment $[s,\ s+T]$, where T does not depend on m (see Exercise 2.5):

$$\frac{d}{dt} u_m(t) + A u_m(t) = P_m B(u_m(t),\ t)\,, \qquad u_m(s) = P_m u_0\,. \tag{2.20}$$

Multiplying (2.20) by $\dot{u}_m(t) = \frac{d}{dt} u_m(t)$ scalarwise in the space H, we find that

$$\|\dot{u}_m\|^2 + \frac{d}{dt}\left\{ \frac{1}{2}\|A^{1/2} u_m\|^2 + F(u_m) \right\} = (B_1(u_m,\ t),\ \dot{u}_m) \le$$

$$\le \frac{1}{2}\|\dot{u}_m\|^2 + \|B_1(0,\ t)\|^2 + M^2 \|A^{1/2} u_m\|^2\,.$$

Since $F(u)$ is bounded below, we obtain that

$$\frac{d}{dt} W(u_m(t)) \le a\, W(u_m(t)) + b + \|B_1(0,\ t)\|^2$$

with constants a and b independent of m, where

$$W(u) = \frac{1}{2}\|A^{1/2} u\|^2 + F(u)\,.$$

Therefore, Gronwall's lemma gives us that

$$W(u_m(t)) \le \left(W(u_m(s)) + \frac{b}{a} \right) e^{a(t-s)} + 2 \int_s^t e^{a(t-\tau)} \|B_1(0,\ \tau)\|^2 d\tau$$

for all t in the segment $[s,\ s+T]$ of the existence of approximate solutions. Firstly, this estimate enables us to prove the global existence of approximate solutions (cf. Exercise 2.1). Secondly, by virtue of the continuity of the functional W on $V = \mathcal{F}_{1/2}$ Theorem 2.2 enables us to pass to the limit $m \to \infty$ on an arbitrary segment $[s,\ s+T]$ and prove the global solvability of limit problem (2.1). **Theorem 2.4 is proved.**

Exercise *2.8* Let $u(t)$ be a mild solution to problem (2.1) such that $\|u(t)\|_\theta \le C_T$ for $s \le t \le s+T$. Use Lemma 2.1 to prove that

$$\|u(t)\|_\alpha \le C_T t^{\theta-\alpha}\|u_0\|_\theta, \qquad t \in [s,\ s+T] \tag{2.21}$$

for all $\theta \le \alpha < 1$, where $C_T = C_T(\alpha,\ \theta)$ is a positive constant.

Exercise *2.9* Let $u(t)$ and $v(t)$ be solutions to problem (2.1) with the initial conditions $u_0,\ v_0 \in \mathcal{F}_\theta$ and such that $\|u(t)\|_\theta + \|v(t)\|_\theta \le C_T$ for t lying in a segment $[s,\ s+T]$. Then

$$\|u(t)-v(t)\|_\alpha \le C_T(\theta,\ \alpha)t^{\theta-\alpha}\|u_0-v_0\|_\theta,$$
$$t \in [s,\ s+T], \qquad \theta \le \alpha < 1.$$

Thus, if $B(u,\ t) \equiv B(u)$ and the hypotheses of Theorem 2.3 or 2.4 hold, then equation (2.1) generates a dynamical system $(\mathcal{F}_\theta,\ S_t)$ with the evolutionary operator S_t which is defined by the equality $S_t u_0 = u(t)$, where $u(t)$ is the solution to problem (2.1). The semigroup property of S_t follows from the assertion on the uniqueness of solution.

Exercise *2.10* Show that Theorem 2.4 holds even if we replace the assumption of semiboundedness of $F(u)$ by the condition $F(u) \ge -\alpha\|A^{1/2}u\|^2 - \beta$ for some $\alpha < 1/2$ and $\beta > 0$.

§ 3 Examples

Here we consider several examples of an application of theorems of Section 2. Our presentation is brief here and is organized in several cycles of exercises. More detailed considerations as well as other examples can be found in the books by Henry, Babin and Vishik, and Temam from the list of references to Chapter 2 (see also Sections 6 and 7 of this chapter).

We first remind some definitions and notations. Let Ω be a domain in $\mathbb{R}^d$ $(d \ge 1)$. The Sobolev space $H^m(\Omega)$ of the order m $(m=0,\ 1,\ 2,\ \dots)$ is defined by the formula

$$H^m(\Omega) = \{f \in L^2(\Omega):\ D^j f \in L^2(\Omega),\ |j| \le m\},$$

where $j = (j_1,\ j_2,\ \dots,\ j_d)$, $j_k = 0,\ 1,\ 2,\ \dots$, $|j| = j_1 + \dots + j_d$,

$$D^j f = \partial_{x_1}^{j_1}\partial_{x_2}^{j_2}\dots\partial_{x_d}^{j_d} f(x), \qquad x = (x_1,\ \dots,\ x_d).$$

The space $H^m(\Omega)$ is a separable Hilbert space with the inner product

$$(u, v)_m = \sum_{|j| \le m} \int_\Omega D^j u(x)\, D^j v(x) \mathrm{d}x \ .$$

Below we also use the space $H_0^m(\Omega)$ which is constructed as the closure in $H^m(\Omega)$ of the set $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support. For more detailed information we refer the reader to the handbooks on the theory of Sobolev spaces.

Example *3.1* (nonlinear heat equation)

$$\begin{cases} \partial_t u = \nu \Delta u + f(t, x, u, \nabla u), & x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 . \end{cases} \tag{3.1}$$

Here Ω is a bounded domain in $\mathbb{R}^d$, Δ is the Laplace operator, and ν is a positive constant. Assume that $f(t, x, u, p)$ is a continuous function of its variables which satisfies the Lipschitz condition

$$|f(t, x, u, p) - f(t, x, u, q)| \le K\left(|u_1 - u_2|^2 + \sum_{j=1}^d |p_j - q_j|^2\right)^{1/2} \tag{3.2}$$

with an absolute constant K. It is clear that the operator $B(u, t)$ defined by the formula

$$B(u, t)(x) = f(t, x, u(x), \nabla u(x)),$$

can be estimated as follows:

$$\|B(u_1, t) - B(u_2, t)\| \le K\left(\|u_1 - u_2\|^2 + \sum_j \|\partial_{x_j}(u_1 - u_2)\|^2\right)^{1/2}. \tag{3.3}$$

Here and below $\|\cdot\|$ is the norm in the space $H = L^2(\Omega)$. It is well-known that the operator $A = -\Delta$ with the Dirichlet boundary condition on $\partial\Omega$ is a positive operator with discrete spectrum. Its domain is $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, $D(A^{1/2}) = H_0^1(\Omega)$. We also note that

$$\|A^{1/2} u\|^2 = (Au, u) = \sum_{j=1}^d \int_\Omega \left(\frac{\partial u}{\partial x_j}\right)^2 \mathrm{d}x \ , \qquad u \in D(A^{1/2}) = H_0^1(\Omega).$$

Therefore, equation (3.3) for $B(u, t)$ gives us the estimate

$$\|B(u_1, t) - B(u_2, t)\| \le K\left(\frac{1}{\lambda_1} + 1\right)^{1/2} \|A^{1/2} u\|,$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. Therefore, we can apply Theorem 2.3 with $\theta = 1/2$ to problem (3.1). This theorem guarantees the existence and uniqueness of a mild solution to problem (3.1) in the space $C(\mathbb{R}_+, H_0^1(\Omega))$.

— Exercise *3.1* Assume that $f(t, x, u, p) \equiv f(t, x, u)$ is a continuous function of its arguments satisfying the global Lipschitz condition with respect to the variable u. Prove the global theorem on the existence of mild solutions to problem (3.1) in the space $H = L^2(\Omega)$.

— Example *3.2*

Let us consider problem (3.1) in the case of one spatial variable:

$$\begin{cases} \partial_t u = \nu \partial_x^2 u - g(x, u) + f(t, x, u, \partial_x u), & t > 0,\ x \in (0, 1) \\ u|_{x=0} = u|_{x=1} = 0, \quad u|_{t=0} = u_0 . \end{cases} \tag{3.4}$$

Assume that $g(x, u)$ is a continuously differentiable function with respect to the variable u and $|g'_u(x, u)| \le h(u)$, where $h(u)$ is a function bounded on every compact set of the real axis. We also assume that the function $f(t, x, u, p)$ is continuous and possesses property (3.2). For any element $u \in D(A^{1/2}) = H_0^1(0, 1)$ the following estimates hold:

$$\max_{[0,1]} |u(x)| \le \|u'\|_{L^2(0,1)} \quad \text{and} \quad \|u\|_{L^2(0,1)} \le \|u'\|_{L^2(0,1)},$$

where $u' = \partial_x u$. Therefore, it is easy to find that the inequality

$$\|B(u_1, t) - B(u_2, t)\| \le M(\rho) \|A^{1/2}(u_1 - u_2)\| \tag{3.5}$$

is valid for

$$B(u, t) = -g(x, u(x)) + f(t, x, u(x), u'(x)),$$

provided $\|A^{1/2} u_j\| \equiv \|u'_j\| \le \rho$. Here $\|\cdot\|$ is the norm in the space $H = L^2(0, 1)$ and $M(\rho) = \sup\{h(\xi) : |\xi| < \rho\} + K\sqrt{2}$. Equation (3.5) and Theorem 2.1 guarantee the local solvability of problem (3.4) in the space $H_0^1(\Omega)$. Moreover, if the function

$$\mathcal{F}(x, y) = \int_0^y g(x, \xi) d\xi$$

is bounded below, then we can use Theorem 2.4 to obtain the assertion on the existence of mild solutions to problem (3.4) on an arbitrary segment $[0, T]$.

It should be noted that the reasoning in Example 3.2 is also valid for several spatial variables. However, in order to ensure the fulfilment of the estimate of the form (3.5) one should impose additional conditions on the growth of the function $h(u)$. For example, we can require that the equation

$$h(u) \le C_1 + C_2 |u|^p$$

be fulfilled, where $p \le 2/(d-2)$ if $d > 2$ and p is an arbitrary number if $d = 2$. In this case the inequality of the form (3.5) follows from the Hölder inequality and the continuity of the embedding of the space $H^1(\Omega)$ into $L^q(\Omega)$, where $q = 2d/(d-2)$ if $d = \dim\Omega > 2$ and q is an arbitrary number if $d = 2$, $q \ge 1$.

The results shown in Examples 3.1 and 3.2 also hold for the systems of parabolic equations. For example, a system of reaction-diffusion equations

$$\begin{cases} \partial_t u = \nu\Delta u + f(t,\ u,\ \nabla u), \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x), \end{cases} \tag{3.6}$$

can be considered in a smooth bounded domain $\Omega \subset \mathbb{R}^d$. Here $u = (u_1,\ u_2,\ \ldots,\ u_m)$ and $f(t,\ u,\ \xi)$ is a continuous function from $\mathbb{R}_+ \times \mathbb{R}^{(m+1)d}$ into $\mathbb{R}^m$ such that

$$|f(t,\ u,\ \xi) - f(t,\ v,\ \eta)| \le M(|u-v| + |\xi-\eta|), \tag{3.7}$$

where M is a constant, $u,\ v \in \mathbb{R}^m$, $\xi,\ \eta \in \mathbb{R}^{md}$ and n is an outer normal to $\partial\Omega$.

Exercise *3.2* Prove the global theorem on the existence and uniqueness of mild solutions to problem (3.6) in $\mathcal{F}_{1/2} = [H^1(\Omega)]^m \equiv H^1(\Omega) \times \ldots \times H^1(\Omega)$.

Example *3.3* (nonlocal Burgers equation).

$$\begin{cases} u_t - \nu u_{xx} + (\omega,\ u)u_x = f(t), \quad 0 < x < l,\ t > 0 \\ u|_{x=0} = u|_{x=l} = 0, \quad u|_{t=0} = u_0. \end{cases} \tag{3.8}$$

Here $f(t)$ is a continuous function with the values in $L^2(0,\ l)$,

$$\omega \in L^2(0,\ l), \qquad (\omega,\ u) = \int_0^l \omega(x)u(x,\ t)\,\mathrm{d}x$$

and ν is a positive parameter. Exercises 3.3–3.6 below answer the question on the solvability of problem (3.8).

Exercise *3.3* Prove the local existence of mild solutions to problem (3.8) in the space $\mathcal{F}_{1/2} = H_0^1(0,\ l)$. *Hint*:

$$A = -\nu\partial_{xx}^2, \quad D(A) = H^2(0,\ l) \cap H_0^1(0,\ l),$$

$$B(u) = -(\omega,\ u)u_x + f(t).$$

Exercise *3.4* Consider the Galerkin approximate solutions to problem (3.8)

$$u_m(t) \equiv u_m(t,\ x) = \sqrt{\frac{2}{l}} \cdot \sum_{k=1}^m g_k(t) \sin\frac{\pi k}{l}x.$$

Write out the system of ordinary differential equations to determine $\{g_k(t)\}$. Prove that this system is locally solvable.

— Exercise *3.5* Prove that the equations

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \nu\|\partial_x u_m(t)\|^2 = (f(t),\ u_m(t)) \tag{3.9}$$

and

$$\frac{d}{dt}\|\partial_x u_m(t)\|^2 + \nu\|\partial^2_{xx} u_m(t)\|^2 \le$$
$$\le \frac{2}{\nu}\left(|(\omega,\ u)|^2\|\partial_x u_m(t)\|^2 + \|f(t)\|^2\right) \tag{3.10}$$

are valid for any interval of the existence of the approximate solution $u_m(t)$. Here $\|\cdot\|$ is the norm in $L^2(0,\ l)$.

— Exercise *3.6* Use equations (3.9) and (3.10) to prove the global existence of the Galerkin approximate solutions to problem (3.8) and to obtain the uniform estimate of the form

$$\|\partial_x u_m(t)\| \le C(\|\partial_x u_0\|,\ T), \qquad t \in [0,\ T] \tag{3.11}$$

for any $T > 0$.

Thus, Theorem 2.2 guarantees the global existence and uniqueness of weak solutions to problem (3.8) in $\mathcal{F}_{1/2} = H^1_0(0,\ l)$.

— Example *3.4* (Cahn-Hilliard equation).

$$\begin{cases} u_t + \nu\partial_x^4 u - \partial_x^2(u^3 + au^2 + bu) = 0, \quad x \in (0,\ l), \quad t > 0, \\ \partial_x u|_{x=0} = \partial_x^3 u|_{x=0} = 0, \quad \partial_x u|_{x=l} = \partial_x^3 u|_{x=l} = 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{3.12}$$

where $\nu > 0$, a, and $b \in \mathbb{R}$ are constants. The result of the cycle of Exercises 3.7–3.10 is a theorem on the existence and uniqueness of solutions to problem (3.12).

— Exercise *3.7* Prove that the estimate

$$\|\partial_x^2(u\cdot v)\|^2 \le C(\|u\|^2 + \|\partial_x^2 u\|^2)(\|v\|^2 + \|\partial_x^2 v\|^2)$$

is valid for any two functions $u(x)$ and $v(x)$ smooth on $[0,\ l]$. Use this estimate to ascertain that problem (3.12) is locally uniquely solvable in the space

$$V = \left\{u \in H^2(0,\ l) : u_x\big|_{x=0} = u_x\big|_{x=l} = 0\right\}$$

(*Hint*: $\nu\partial_x^4 + 1 \mapsto A$, $V = D(A^{1/2})$).

Let us consider the Galerkin approximate solution $u_m(t)$ to problem (3.12):

$$u_m(t) \equiv u_m(t,\, x) = \frac{1}{\sqrt{l}}\, g_0(t) + \sqrt{\frac{2}{l}} \cdot \sum_{k=1}^{m} g_k(t) \cos\frac{\pi k}{l} x\,.$$

— Exercise *3.8* Prove that the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} W(u_m(t)) + \int_0^l \Big(\partial_x[\nu\, \partial_x^2 u_m(t,\, x) - p(u_m(t,\, x))]\Big)^2 \mathrm{d}x = 0 \quad (3.13)$$

holds for any interval of the existence of approximate solutions $\{u_m(t)\}$. Here $p(u) = u^3 + a u^2 + b$ and

$$W(u) = \int_0^l \Big(\frac{\nu}{2} u_x^2 + \frac{1}{4} u^4 + \frac{a}{3} u^3 + \frac{b}{2} u^2\Big) \mathrm{d}x\,. \quad (3.14)$$

In particular, equation (3.13) implies that approximate solutions exist for any segment $[0,\, T]$. For $u_0 \in V$ they can be estimated as follows:

$$\|\partial_x u_m(t)\| + \max_{x \in [0,\, l]} |u_m(t,\, x)| \;\le\; C_T\,, \qquad t \in [0,\, T]\,, \quad (3.15)$$

where the number C_T does not depend on m.

— Exercise *3.9* Using (3.15) show that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\|\partial_x^2 u_m(t)\big\|^2 + \big\|\partial_x^4 u_m(t)\big\|^2 \;\le\; C_T\big(1 + \big\|\partial_x^2 u_m(t)\big\|^2\big)\,, \quad t \in (0,\, T]$$

holds for any interval $(0, T)$ and for any approximate solution $u_m(t)$ to problem (3.12). (*Hint*: first prove that $\|u'\|^2_{L^4(0,\, l)} \le$ $\le C \max_{[0,\, l]} |u(x)| \cdot \|u'\|_{L^2(0,\, l)}$).

— Exercise *3.10* Using Theorem 2.2 and the result of the previous exercise, prove the global theorem on the existence and uniqueness of weak solutions to the Cahn-Hilliard equation (3.12) in the space V (see Exercise 3.7).

— Example *3.5* (abstract form of two-dimensional system of Navier-Stokes equations)

In a separable Hilbert space H we consider the evolutionary equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu A u + b(u,\, u) = f(t)\,, \qquad u\big|_{t=0} = u_0\,, \quad (3.16)$$

where A is a positive operator with discrete spectrum, ν is a positive parameter and $b(u, v)$ is a bilinear mapping from $D(A^{1/2}) \times D(A^{1/2})$ into $\mathcal{F}_{-1/2} = D(A^{-1/2})$ possessing the property

$$(b(u, v), v) = 0 \quad \text{for all} \quad u, v \in D(A^{1/2}) \tag{3.17}$$

and such that for all $u, v \in D(A)$ the estimates

$$\|b(u, v)\| \le C_\delta \|A^{1/2-\delta} u\| \cdot \|A^{1/2+\delta} v\|, \qquad 0 < \delta < \frac{1}{2} \tag{3.18}$$

and

$$\|A^\beta b(u, v)\| \le C_{\delta, \beta} \|A^{(1/2)+\delta} u\| \cdot \|A^{1/2+\beta} v\|, \qquad 0 \le \beta \le \frac{1}{2}, \;\; 0 < \delta \le \frac{1}{2} \tag{3.19}$$

hold. We also assume that $f(t)$ is a continuous function with the values in H.

Exercise *3.11* Prove that Theorem 2.1 on the local solvability with $\theta = 1/2 + \beta$ is applicable to problem (3.16), where β is a number from the interval $(0, 1/2)$.

Let $\{e_k\}$ be the basis of eigenvectors of the operator A, let $0 < \lambda_1 \le \lambda_2 \le \ldots$ be the corresponding eigenvalues and let P_m be the orthoprojector onto the span of $\{e_1, \ldots, e_m\}$. We consider the Galerkin approximations of problem (3.16):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} u_m(t) + \nu A u_m(t) + P_m b(u_m(t), u_m(t)) = P_m f(t), \\ u_m(0) = P_m u_0 . \end{cases} \tag{3.20}$$

Exercise *3.12* Show that the estimates

$$\|u_m(t)\|^2 + \nu \int_0^t \|A^{1/2} u_m(\tau)\|^2 \mathrm{d}\tau \le \|u_0\|^2 + \frac{1}{\lambda_1 \nu} \int_0^t \|f(\tau)\|^2 \mathrm{d}\tau \tag{3.21}$$

and

$$\|u_m(t)\|^2 \le \|u_m(0)\|^2 e^{-\lambda_1 \nu t} + \frac{1}{\lambda_1} \left(\frac{F}{\nu}\right)^2 (1 - e^{-\lambda_1 \nu t}) \tag{3.22}$$

are valid for an arbitrary interval of the existence of solutions to problem (3.20). Here $F = \sup_{t \ge 0} \|A^{-1/2} f(t)\|$. Using these estimates, prove the global solvability of problem (3.20).

Exercise *3.13* Show that the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \|A^{1/2} u_m(t)\|^2 + \nu \|A u_m(t)\|^2 \le \frac{2}{\nu} \left(\|b(u_m(t), u_m(t))\|^2 + \|f(t)\|^2\right) \tag{3.23}$$

holds for a solution to problem (3.20).

Using the interpolation inequality (see Exercise 1.13) and estimate (3.18), it is easy to find that

$$\|b(u,\ u)\|^2 \le C\|u\|\cdot\|A^{1/2}u\|^2\|Au\| \le \left(\frac{C}{\nu}\right)^2\|u\|^2\|A^{1/2}u\|^4 + \frac{\nu^2}{4}\|Au\|^2 .$$

Therefore, equation (3.23) implies that

$$\frac{d}{dt}\left\|A^{1/2}u_m(t)\right\|^2 \le \sigma_m(t)\left\|A^{1/2}u_m(t)\right\|^2 + \frac{2}{\nu}\|f(t)\|^2 , \tag{3.24}$$

where $\sigma_m(t) = (2C^2/\nu^3)\|u_m(t)\|^2\cdot\left\|A^{1/2}u_m(t)\right\|^2$. Hence, Gronwall's inequality gives us that

$$\left\|A^{1/2}u_m(t)\right\|^2 \le \left\|A^{1/2}u_0\right\|^2\exp\left\{\int_0^t\sigma_m(\tau)d\tau\right\} + \frac{2}{\nu}\int_0^t\|f(\tau)\|^2\exp\left\{\int_0^t\sigma_m(\xi)d\xi\right\}d\tau .$$

It follows from equation (3.21) that the value $\int_0^t\sigma_m(\tau)d\tau$ is uniformly bounded with respect to m on an arbitrary segment $[0,\ T]$. Consequently, the uniform estimates

$$\left\|A^{1/2}u_m(t)\right\| \le C_T, \qquad t\in[0,\ T], \quad u_0\in D(A^{1/2}) \tag{3.25}$$

are valid for any $T > 0$ and $m = 1,\ 2,\ \ldots$.

Exercise *3.14* Using equations (3.23) and (3.25), prove that if $u_0 \in D(A^{1/2})$, then

$$\frac{d}{dt}\left\|A^{1/2}u_m(t)\right\|^2 + \frac{\nu}{2}\left\|Au_m(t)\right\|^2 \le C_T, \quad t\in[0,\ T]$$

for any $T > 0$.

Exercise *3.15* Let $0 < \beta < \frac{1}{2}$ and let $\sup_{[0,\ T]}\|A^\beta f(t)\|^2 \le C_T$. Prove that

$$\frac{d}{dt}\left\|A^{1/2+\beta}u_m(t)\right\|^2 + \nu\left\|A^{1+\beta}u_m(t)\right\|^2 \le$$

$$\le C_1\left\|A^\beta b(u_m(t),\ u_m(t))\right\|^2 + C_T .$$

Exercise *3.16* Use the results of Exercises 3.14 and 3.15 and inequality (3.19) to prove the global existence of mild solutions to problem (3.16) in the space $D(A^{1/2+\beta})$, provided that $u_0\in D(A^{1/2+\beta})$ and $f(t)$ is a continuous function with the values in $D(A^\beta)$. Here β is an arbitrary number from the interval $(0,\ 1/2)$.

§ 4 Existence Conditions and Properties of Global Attractor

In this section we study the asymptotic properties of the dynamical system generated by the autonomous equation

$$\frac{du}{dt} + Au = B(u), \qquad u|_{t=0} = u_0, \tag{4.1}$$

where, as before, A is a positive operator with discrete spectrum and $B(u)$ is a nonlinear mapping from $\mathcal{F}_\theta$ into H such that

$$\|B(u_1) - B(u_2)\| \le M(\rho)\|A^\theta(u_1 - u_2)\| \tag{4.2}$$

for all $u_1, u_2 \in \mathcal{F}_\theta = D(A^\theta)$ possessing the property $\|A^\theta u_j\| \le \rho$, $0 \le \theta < 1$. We assume that problem (4.1) has a unique mild (in $\mathcal{F}_\theta$) solution on $\mathbb{R}_+$ for any $u_0 \in \mathcal{F}_\theta$. The theorems that guarantee the fulfilment of this requirement and also some examples are given in Sections 2 and 3 of this chapter. It should be also noted that in this section we use some results of Chapter 1 for the proof of main assertions. Further triple numeration is used in the references to the assertions and formulae of Chapter 1 (first digit is the chapter number).

Let $(\mathcal{F}_\theta, S_t)$ be a dynamical system with the evolutionary operator S_t defined by the formula $S_t u_0 = u(t)$, where $u(t)$ is a mild solution to problem (4.1). As shown in Chapter 1, for the system $(\mathcal{F}_\theta, S_t)$ to possess a compact global attractor, it should be dissipative. It turns out that the condition of dissipativity is not only necessary, but also sufficient in the class of systems considered.

Lemma 4.1.

Let $(\mathcal{F}_\theta, S_t)$ be dissipative and let $B_0 = \{u:\ \|A^\theta u\| \le R_0\}$ be its absorbing ball. Then the set $B_\alpha = \{u:\ \|A^\alpha u\| \le R_\alpha\}$ is absorbing for all $\alpha \in (\theta, 1)$, where

$$R_\alpha = (\alpha - \theta)^{\alpha - \theta} R_0 + \frac{\alpha^\alpha}{1-\alpha} \sup\{\|B(u)\|:\ u \in \mathcal{F}_\theta,\ \|A^\theta u\| \le R_0\}\ . \tag{4.3}$$

Proof.

Using equation (2.3) and estimate (1.17), we have

$$\|A^\alpha u(t+1)\| \le (\alpha - \theta)^{\alpha - \theta}\|A^\theta u(t)\| + \int_t^{t+1} \left(\frac{\alpha}{t+1-\tau}\right)^\alpha \|B(u(t))\| dt\ ,$$

where $u(t) = S_t u_0$. Let B be a bounded set in $\mathcal{F}_\theta$. Then $\|S_t y\|_\theta \le R_0$ for all $t \ge t_B$ and $y \in B$. Therefore, the estimate

$$\|B(u(t))\| \le \sup\{\|B(u)\|:\ u \in B_0\}, \quad t \ge t_B$$

holds for $u(t) = S_t y$. Hence, $\|A^\alpha u(t+1)\| \le R_\alpha$ for all $t \ge t_B$. This implies the assertion of the lemma.

Theorem 4.1.

Assume that the dynamical system $(\mathscr{F}_\theta,\ S_t)$ ***generated by problem*** (4.1) ***is dissipative. Then it is compact and possesses a connected compact global attractor*** $\mathscr{A}$. ***This attractor is a bounded set in*** $\mathscr{F}_\alpha$ ***for*** $\theta \le \alpha < 1$ ***and has a finite fractal dimension.***

Proof.

By virtue of Theorem 1.1 the space $\mathscr{F}_\alpha$ is compactly embedded into $\mathscr{F}_\theta$ for $\alpha > \theta$. Therefore, Lemma 4.1 implies that the dynamical system $(\mathscr{F}_\theta,\ S_t)$ is compact. Hence, Theorem 1.5.1 gives us that $(\mathscr{F}_\theta,\ S_t)$ possesses a connected compact global attractor $\mathscr{A}$. Evidently, $\|A^\alpha u\| \le R_\alpha$ for all $u \in \mathscr{A}$, where R_α is defined by equality (4.3). Thus, we should only establish the finite dimensionality of the attractor. Let us apply the method used in the proof of Theorem 1.9.1 and based on Theorem 1.8.1. Let $S_t u_1$ and $S_t u_2$ be semitrajectories of the dynamical system $(\mathscr{F}_\theta,\ S_t)$ such that $\|S_t u_j\|_\theta \le R$ for all $t \ge 0$, $j = 1,\ 2$. Then equations (2.3) and (1.17) give us that

$$\|S_t u_1 - S_t u_2\|_\theta \ \le\ \|u_1 - u_2\|_\theta + M(R)\theta^\theta \int\limits_0^t (t-\tau)^{-\theta} \|S_t u_1 - S_t u_2\|_\theta \mathrm{d}t\ .$$

Using Lemma 2.1, we find that for all $u_j \in \mathscr{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$ the following estimate holds:

$$\|S_t u_1 - S_t u_2\|_\theta \le C\|u_1 - u_2\|_\theta \quad \text{for} \quad 0 \le t \le 1\,,$$

where the constant C depends on θ and R. Therefore,

$$\|S_t u_1 - S_t u_2\|_\theta \ \le\ C\|S_\tau u_1 - S_\tau u_2\|_\theta \quad \text{for} \quad 0 \le \tau \le t \le \tau + 1$$

for the considered u_1 and u_2. Consequently,

$$\|S_t u_1 - S_t u_2\|_\theta \ \le\ C\|S_{[t]} u_1 - S_{[t]} u_2\|_\theta \ \le\ C^{[t]+1}\|u_1 - u_2\|_\theta\,,$$

where $[t]$ is an integer part of the number t. Thus, the estimate of the form

$$\|S_t u_1 - S_t u_2\|_\theta \le C e^{at}\|u_1 - u_2\|_\theta \tag{4.4}$$

is valid, where the constants C and a depend on θ and R. Similarly, Lemma 1.1 gives

$$\|Q_N(S_t u_1 - S_t u_2)\|_\theta \ \le\ e^{-\lambda_{N+1} t}\|u_1 - u_2\|_\theta +$$

$$+ M(R)\int\limits_0^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(t-\tau)} \|S_\tau u_1 - S_\tau u_2\|_\theta \mathrm{d}\tau \tag{4.5}$$

for all $u_1,\ u_2 \in \mathscr{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$, where $Q_N = I - P_N$ and P_N is the orthoprojector onto the first N eigenvectors of the operator A. If we substitute equation (4.4) in the right-hand side of inequality (4.5), then we obtain that

$$\|Q_N(S_t u_1 - S_t u_2)\|_\theta \le (e^{-\lambda_{N+1} t} + CM(R)\, e^{at} J_N(t))\|u_1 - u_2\|_\theta ,$$

where

$$J_N(t) = \int_0^t \Big[\Big(\frac{\theta}{t-\tau}\Big)^\theta + \lambda_{N+1}^\theta\Big] e^{-\lambda_{N+1}(t-\tau)} d\tau .$$

After the change of variable $\xi = \lambda_{N+1}(t-\tau)$ it is easy to find that $J_N(t) \le$ $\le C_0 \lambda_{N+1}^{-1+\theta}$. Therefore,

$$\|Q_N(S_t u_1 - S_t u_2)\|_\theta \le \left(e^{-\lambda_{N+1} t} + \frac{C(R,\ \theta)}{\lambda_{N+1}^{1-\theta}}\, e^{at}\right)\|u_1 - u_2\|_\theta \tag{4.6}$$

for all $u_1,\ u_2 \in \mathcal{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$. Hence, there exist $t_0 > 0$ and N such that

$$\|Q_N(S_{t_0} u_1 - S_{t_0} u_2)\|_\theta \le \delta \|u_1 - u_2\|_\theta , \qquad 0 < \delta < 1 ,$$

for all $u_1,\ u_2 \in \mathcal{F}_\theta$ such that $\|S_t u_j\|_\theta \le R$. However, the attractor $\mathcal{A}$ lies in the absorbing ball B_0. Therefore, this estimate and inequality (4.4) mean that the hypotheses of Theorem 1.8.1 hold for the mapping $V = S_{t_0}$. Hence, the attractor $\mathcal{A}$ has a finite fractal dimension. It can be estimated with the help of the parameters in inequalities (4.4) and (4.6). Thus, **Theorem 4.1 is proved**.

Equations (4.4) and (4.6) which are valid for any $R > 0$ enable us to prove the existence of a fractal exponential attractor of the dynamical system $(\mathcal{F}_\theta,\ S_t)$ in the same way as in Section 9 of Chapter 1.

Theorem 4.2.

Assume that the dynamical system $(\mathcal{F}_\theta,\ S_t)$ ***generated by problem*** (4.1) ***is dissipative. Then it possesses a fractal exponential attractor (inertial set).***

Proof.

It is sufficient to verify that the hypotheses of Theorem 1.9.2 (see (1.9.12)–(1.9.14)) hold for $(\mathcal{F}_\theta,\ S_t)$. Let us show that

$$K = \bigcup_{t \ge t_0} S_t B_\alpha , \qquad B_\alpha = \{u:\ \|A^\alpha u\| \le R_\alpha\}$$

can be taken for the compact K in (1.9.12)–(1.9.14). Here α is a number from the interval $(\theta,\ 1)$ and R_α is defined by formula (4.3). We choose the parameter $t_0 = t_0(B_\alpha)$ such that $S_t K \subset B_\alpha$ for $t \ge t_0$. Since K is a bounded invariant set, equation (4.4) is valid for any $u_1,\ u_2 \in K$ with some constants C and a. It is also easy to verify that K is a compact. Indeed, let $\{k_n\} \subset K$. Then $k_n = S_{t_n} y_n$, therewith we can assume that $k_n \to w$, $y_n \to y \in B_\alpha$ and either $t_n \to t_* < \infty$ or $t_n \to \infty$. In the first case with the help of (4.4) we have

$$\left\|S_{t_n} y_n - S_{t_*} y\right\|_\theta \le C e^{a t_n} \|y_n - y\|_\theta + \left\|S_{t_n} y - S_{t_*} y\right\|_\theta .$$

Therefore, $k_n = S_{t_n} y_n \to S_{t_*} y \in K$. In the second case

$$w = \lim_{n \to \infty} S_{t_n} y_n \in \omega(B_\alpha) \subset S_{t_0}(B_\alpha) \subset K .$$

Here $\omega(B_\alpha)$ is the omega-limit set for the semitrajectories emanating from B_α. Thus, K is a compact invariant absorbing set. In particular this means that condition (1.9.12) is fulfilled, therewith we can take any number for $\gamma > 0$. Conditions (1.9.13) and (1.9.14) follow from equations (4.4) and (4.6). Consequently, it is sufficient to apply Theorem 1.9.2 **to conclude the proof of Theorem 4.2**.

Thus, the dissipativity of the dynamical system $(\mathcal{F}_\theta, S_t)$ generated by problem (4.1) guarantees the existence of a finite-dimensional global attractor and an inertial set. Under some additional conditions concerning $B(u)$ the requirement of dissipativity can be slightly weakened. We give the following definition. Let $\alpha \le \theta$. The dynamical system $(\mathcal{F}_\theta, S_t)$ is said to be $\mathcal{F}_\alpha$***-dissipative*** if there exists $R^*_\alpha > 0$ such that for any set B bounded in $\mathcal{F}_\alpha$ there exists $t_0 = t_0(B)$ such that

$$\left\|A^\alpha S_t y\right\| \equiv \left\|S_t y\right\|_\alpha \le R^*_\alpha \quad \text{for all} \quad y \in B \cap \mathcal{F}_\theta \quad \text{and} \quad t \ge t_0 .$$

Lemma 4.2.

Assume that $B(u)$ satisfies the global Lipschitz condition

$$\left\|B(u_1) - B(u_2)\right\| \le M \left\|A^\theta (u_1 - u_2)\right\| . \tag{4.7}$$

Let the dynamical system $(\mathcal{F}_\theta, S_t)$ generated by mild solutions to problem (4.1) be $\mathcal{F}_\alpha$-dissipative for some $\alpha \in (\theta - 1, \theta]$. Then $(\mathcal{F}_\theta, S_t)$ is a compact dynamical system, i.e. it possesses an absorbing set which is compact in $\mathcal{F}_\theta$.

Proof.

By virtue of Lemma 4.1 it is sufficient to verify that the system $(\mathcal{F}_\theta, S_t)$ is dissipative (i.e. $\mathcal{F}_\theta$-dissipative). If we use expression (2.3) and equation (1.17), then we obtain

$$\|A^\theta u(t+s)\| \le \left(\frac{\theta - \alpha}{s}\right)^{\theta - \alpha} \|A^\alpha u(t)\| + \int_t^{t+s} \left(\frac{\theta}{t+s-\tau}\right)^\theta \|B(u(\tau))\| \mathrm{d}\tau$$

for positive t and s. Here $u(t) = S_t u_0$. Since $\|B(u)\| \le \|B(0)\| + M\|A^\theta u\|$, we have the estimate

$$\|A^\theta u(t+s)\| \le \left(\frac{1}{s}\right)^{\theta - \alpha} \left\{(\theta - \alpha)^{\theta - \alpha} \|A^\alpha u(t)\| + \|B(0)\| \frac{\theta^\theta}{1-\theta}\right\} +$$

$$+M\int_0^s \left(\frac{\theta}{s-\tau}\right)^{\theta} \|A^{\theta}u(t+\tau)\| d\tau$$

for $0 \le s \le 1$. Hence, we can apply Lemma 2.1 to obtain

$$\|A^{\theta}u(t+s)\| \le C(\theta,\ \alpha,\ M)(1+\|A^{\alpha}u(t)\|)\, s^{\alpha-\theta}, \quad 0 \le s \le 1 .$$

Therefore, if $\|S_t u_0\|_{\alpha} \le R_{\alpha}^{*}$ for $t \ge t_0(B)$ and $u_0 \in B \cap \mathcal{F}_{\theta}$, then the latter inequality gives us that

$$\|S_t u_0\|_{\theta} \le C(\theta,\ \alpha,\ M)(1+R_{\alpha}^{*}) \quad \text{for} \quad t \ge 1 + t_0(B),$$

i.e. $(\mathcal{F}_{\theta},\ S_t)$ is a dissipative system. Lemma 4.2 is proved.

Exercise *4.1* Show that the assertion of Lemma 4.2 holds if instead of (4.7) we suppose that $B(u) = B_1(u) + B_2(u)$, where $B_1(u)$ possesses property (4.7) and $B_2(u)$ is such that

$$\sup\{B_2(u):\ \|A^{\alpha}u\| \le R_{\alpha}^{*}\} < \infty .$$

The following assertion contains a sufficient condition of dissipativity of the dynamical system generated by problem (4.1).

Theorem 4.3.

Assume that condition (4.2) ***is fulfilled with*** $\theta = 1/2$ ***and*** $B(u) = -F'(u)$ ***is a potential operator from*** $\mathcal{F}_{1/2}$ ***into*** H ***(the prime stands for the Frechét derivative). Let***

$$F(u) \ge -\alpha, \qquad (F'(u),\ u) - \beta F(u) \ge -\gamma\|A^{1/2}u\|^2 - \delta \tag{4.8}$$

for all $u \in \mathcal{F}_{1/2}$, ***where*** $\alpha,\ \beta,\ \gamma,$ ***and*** δ ***are real parameters, therewith*** $\beta > 0$ ***and*** $\gamma < 1$. ***Then the dynamical system is dissipative in*** $\mathcal{F}_{1/2}$.

Proof.

In view of Theorem 2.4 conditions (4.8) guarantee the existence of the evolutionary operator S_t. Let us verify the dissipativity. As in the proof of Theorem 2.4 we consider the Galerkin approximations $\{u_m(t)\}$. It is evident that

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2 + (F'(u_m),\ u_m) = 0$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\|A^{1/2}u_m(t)\|^2 + F(u_m(t))\right) + \|\dot{u}_m(t)\|^2 = 0 .$$

If we add these equations and use (4.8), then we obtain that

$$\frac{d}{dt}\left\{\frac{1}{2}\|u_m\|^2 + \frac{1}{2}\|A^{1/2}u_m\|^2 + F(u_m)\right\} + (1-\gamma)\|A^{1/2}u_m\|^2 + \beta F(u_m) \le \delta .$$

Let

$$V(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|A^{1/2}u\|^2 + F(u) + \alpha\,.$$

Therefore, it is clear that

$$\frac{d}{dt}V(u_m(t)) + \omega V(u_m(t)) \;\le\; C$$

with some positive constants ω and C that do not depend on m. This (cf. Exercise 1.4.1) easily implies the dissipativity of the system $(\mathcal{F}_{1/2},\ S_t)$, moreover,

$$\frac{1}{2}\left\|A^{1/2}S_t u\right\|^2 \;\le\; V(u_0)\,e^{-\omega t} + \frac{C}{\omega}(1 - e^{-\omega t})\,. \tag{4.9}$$

Theorem 4.3 is proved.

Exercise *4.2* Show that the assertion of Theorem 4.3 holds if (4.2) is fulfilled with $\theta = 1/2$ and

$$B(u) = -F'(u) + B_0(u)\,,$$

where $F(u)$ possesses properties (4.8) and $B_0(u)$ satisfies the estimate $\|B_0(u)\| \le C_\varepsilon + \varepsilon\|A^{1/2}u\|$ for $\varepsilon > 0$ small enough.

Let us look at the examples of Section 3 again. We assume that the function

$$f(t,\ x,\ u,\ p) \equiv f(x,\ u,\ p) \tag{4.10}$$

in Example 3.1 possesses property (3.2) and

$$\int_\Omega f(x,\ u(x),\ \nabla u(x))u(x)dx \;\le\; (\lambda_1 - \delta)\int_\Omega u^2(x)dx + C \tag{4.11}$$

for all $u \in H_0^1(\Omega)$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition on $\partial\Omega$. Here δ and C are positive constants.

Exercise *4.3* Using the Galerkin approximations of problem (3.1) and Lemma 4.2, prove that the dynamical system generated by equation (3.1) is dissipative in $H_0^1(\Omega)$ under conditions (3.2), (4.10), and (4.11).

Therefore, if conditions (3.2), (4.10) and (4.11) are fulfilled, then the dynamical system $(H_0^1(\Omega),\ S_t)$ generated by a mild (in $H_0^1(\Omega)$) solution to problem (3.1) possesses both a finite-dimensional global attractor and an inertial set.

Exercise *4.4* Prove that equation (4.11) holds if

$$f(x,\ u,\ \nabla u) = f_0(x,\ u) + \sum_{j=1}^{d} a_j \frac{\partial u}{\partial x_j}\,,$$

where a_j are real constants and the function $f_0(x,\ u)$ possesses the property

$$f(x, u)u \le (\lambda_1 - \delta)u^2 + C, \quad u \in R$$

for any $x \in \Omega$.

— Exercise *4.5* Consider the dynamical system generated by problem (3.4). In addition to the hypotheses of Example 3.2 we assume that $f(t, x, u, \partial_x u) \equiv 0$ and the function $g(x, u)$ possesses the properties

$$\int_0^y g(x, \xi)\mathrm{d}\xi \ge -\alpha, \qquad y g(x, y) - \beta \int_0^y g(x, \xi)\mathrm{d}\xi \ge -\gamma$$

for some positive α, β, and γ. Then the dynamical system generated by (3.4) is dissipative in $H_0^1(0, 1)$.

— Exercise *4.6* Find the analogue of the result of Exercise 4.5 for the system of reaction-diffusion equations (3.6).

— Exercise *4.7* Using equations (3.9) and (3.10) prove that the dynamical system generated by the nonlocal Burgers equation (3.8) with $f(t) \equiv \equiv f \in L^2(0, l)$ is dissipative in $H_0^1(\Omega)$.

Let us consider a dynamical system (V, S_t) generated by the Cahn-Hilliard equation (3.12). We remind that

$$V = \left\{ u \in H^2(0, l): \; u_x\big|_{x=0} = u_x\big|_{x=l} = 0 \right\}.$$

— Exercise *4.8* Let the function $W(u)$ be defined by equality (3.14). Show that for any positive R and α the set

$$X_{\alpha, R} = \left\{ u \in V: \; W(u) \le R, \; \left| \int_0^l u(x)\mathrm{d}x \right| \le \alpha \right\} \tag{4.12}$$

is a closed invariant subset in V for the dynamical system (V, S_t) generated by problem (3.12).

— Exercise *4.9* Prove that the dynamical system $(X_{\alpha, R}, S_t)$ generated by the Cahn-Hilliard equation on the set $X_{\alpha, R}$ defined by (4.12) is dissipative (*Hint*: cf. Exercise 3.9).

In conclusion of this section let us establish the dissipativity of the dynamical system $(D(A^{1/2+\beta}), S_t)$ generated by the abstract form of the two-dimensional Navier-Stokes system (see Example 3.5) under the assumption that $f(t) \equiv f \in D(A^\beta)$. We consider the dynamical system $(P_m H, S_t^m)$ generated by the Galerkin approximations (see (3.20)) of problem (3.16).

— Exercise *4.10* Using (3.24) prove that

$$t\|A^{1/2}S_t^m u_0\|^2 \le \left(c_0 + \int_0^1 \|A^{1/2}S_\tau^m u_0\|^2 d\tau\right) \times$$

$$\times \exp\left\{c_1 \int_0^1 \left(\|S_\tau^m u_0\|^2 \|A^{1/2}S_\tau^m u_0\|\right)^2 d\tau\right\} \tag{4.13}$$

for all $0 < t \le 1$, where c_0 and c_1 are constants independent of m.

— Exercise *4.11* With the help of (3.21), (3.22) and (4.13) verify the property of dissipativity of the system $(D(A^{1/2+\beta}), S_t)$ in the space $D(A^{1/2})$. Deduce its dissipativity (*Hint*: see Exercise 3.14–3.16).

Thus, the dynamical system generated by the two-dimensional Navier-Stokes equations possesses both a finite-dimensional compact global attractor and an inertial set.

§ 5 *Systems with Lyapunov Function*

In this section we consider problem (4.1) on the assumption that condition (4.2) holds with $\theta = 1/2$ and $B(u)$ is a potential operator, i.e. there exists a functional $F(u)$ on $\mathcal{F}_{1/2} = D(A^{1/2})$ such that its Frechét derivative $F'(u)$ possesses the property

$$B(u) = -F'(u), \qquad u \in D(A^{1/2}). \tag{5.1}$$

Below we also assume that the conditions

$$F(u) \ge -\alpha, \quad (F'(u), u) - \beta F(u) \ge -\gamma\|A^{1/2}u\|^2 - \delta \tag{5.2}$$

are fulfilled for all $u \in D(A^{1/2})$, where $\alpha, \delta \in \mathbb{R}$, $\beta > 0$, and $\gamma < 1$. On the one hand, these conditions ensure the existence and uniqueness of mild (in $D(A^{1/2})$) solutions to problem (4.1) (see Theorem 2.4). On the other hand, they guarantee the existence of a finite-dimensional global attractor $\mathcal{A}$ for the dynamical system $(\mathcal{F}_{1/2}, S_t)$ generated by problem (4.1) (see Theorem 4.3). Conditions (5.1) and (5.2) enable us to obtain additional information on the structure of attractor.

Theorem 5.1.

Assume that conditions (5.1), (5.2), *and* (4.2) *hold with* $\theta = 1/2$. *Then the global attractor* $\mathcal{A}$ *of the dynamical system* $(\mathcal{F}_{1/2}, S_t)$ *generated by*

problem (4.1) ***is a bounded set in*** $\mathcal{F}_1 = D(A)$ ***and it coincides with the unstable manifold emanating from the set of fixed points of the system, i.e.***

$$\mathcal{A} = M_+(\mathcal{N}), \tag{5.3}$$

where $\mathcal{N} = \{z \in D(A) :\ Az = B(z)\}$ ***(for the definition of*** $M_+(\mathcal{N})$ ***see Section*** 6 ***of Chapter*** 1***).***

The proof of the theorem is based on the following lemmata.

Lemma 5.1.

Assume that a semitrajectory $u(t) = S_t u_0$ *possesses the property* $\|A^\alpha u(t)\| \le R_\alpha$ *for all* $t \ge 0$*, where* $1/2 < \alpha < 1$*. Then*

$$\|A^{1/2}(u(t) - u(s))\| \le C(R_\alpha)|t-s|^{\alpha - 1/2} \tag{5.4}$$

for all $t,\ s \ge 0$.

Proof.

For the sake of definiteness we assume that $t > s \ge 0$. Equation (2.3) implies that

$$u(t) - u(s) = (e^{-(t-s)A} - I)\, u(s) + \int_s^t e^{-(t-\tau)A} B(u(\tau))\mathrm{d}\tau .$$

Since

$$e^{-(t-s)A} - I = -\int_s^t A\, e^{-(t-\tau)A}\mathrm{d}\tau ,$$

equation (1.17) gives us that

$$\|A^{1/2}(u(t) - u(s))\| \le c_0 \int_s^t (t-\tau)^{\alpha - 3/2}\mathrm{d}\tau \|A^\alpha u(s)\| +$$

$$+ c_1 \int_s^t (t-\tau)^{-1/2}\mathrm{d}\tau \ \max_{\tau \ge 0}\|B(u(\tau))\| .$$

This implies estimate (5.4).

Lemma 5.2.

There exists $R_1 > 0$ *such that the set* $B_1 = \{u :\ \|Au\| \le R_1\}$ *is absorbing for the dynamical system* $(\mathcal{F}_{1/2},\ S_t)$.

Proof.

By virtue of Theorem 4.3 the system $(\mathcal{F}_{1/2},\ S_t)$ is dissipative. Therefore, it follows from Lemma 4.1 that $B_\alpha = \{u :\ \|A^\alpha u\| \le R_\alpha\}$ is an absorbing set,where

R_α is defined by (4.3), $1/2 < \alpha < 1$. Thus, to prove the lemma it is sufficient to consider semitrajectories $u(t)$ possessing the property $\|A^\alpha u(t)\| \le R_\alpha$, $t \ge 0$. Let us present the solution $u(t)$ in the form

$$u(t) = e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\tau)A}\Big[B(u(\tau)) - B(u(t))\Big]d\tau +$$

$$+ A^{-1}(1 - e^{-(t-s)A})B(u(t)) .$$

Using Lemma 5.1 we find that

$$\|Au(t)\| \le C_0(t-s)^{-1/2}R_{1/2} + C_1(R_\alpha)\int_s^t (t-\tau)^{-\frac{3}{2}+\alpha}d\tau + C_2(R_{1/2}) .$$

This implies that

$$\|Au(t+1)\| \le C(R_\alpha), \qquad t \ge 0 ,$$

provided that $\|A^\alpha u(t)\| \le R_\alpha$ for $t \ge 0$. Therefore, the assertion of Lemma 5.2 follows from Lemma 4.1.

Proof of Theorem 5.1.

The boundedness of the attractor $\mathcal{A}$ in $D(A)$ follows from Lemma 5.2. Let us prove (5.3). We consider the Galerkin approximation $u_m(t)$ of solutions to problem (4.1):

$$\frac{du_m(t)}{dt} + Au_m(t) = P_m B(u_m(t)), \qquad u_m(0) = P_m u_0 .$$

Here P_m is the orthoprojector onto the span of $\{e_1, e_2, \dots, e_m\}$. Let

$$V(u) = \frac{1}{2}(Au, u) + F(u), \quad u \in \mathcal{F}_{1/2} = D(A^{1/2}) . \tag{5.5}$$

It is clear that

$$\frac{d}{dt}V(u_m(t)) = (Au_m(t) + F'(u_m(t)), \dot{u}_m(t)) = -\|P_m[Au_m(t) - B(u_m(t))]\|^2 .$$

This implies that

$$V(u_m(t)) - V(u_m(s)) = -\int_s^t \|P_m[Au_m(\tau) - B(u_m(\tau))]\|^2 d\tau \le$$

$$\le -\int_s^t \|P_N(Au_m(\tau) - B(u_m(\tau)))\|^2 d\tau$$

for $t \ge s$ and for any $N \le m$, where P_N is the orthoprojector onto the span of $\{e_1, \dots, e_N\}$. With the help of Theorem 2.2 and due to the continuity of the functional $V(u)$ we can pass to the limit $m \to \infty$ in the latter equation. As a result, we obtain the estimate

$$V(u(t)) + \int_s^t \|P_N(Au(\tau) - B(u(\tau)))\|^2 d\tau \le V(u(s)), \quad t \ge s, \tag{5.6}$$

for any solution $u(t)$ to problem (4.1) and for any $N \ge 1$. If the semitrajectory $u(t) = S_t u_0$ lies in the attractor $\mathscr{A}$, then we can pass to the limit $N \to \infty$ in (5.6) and obtain the equation

$$V(u(t)) + \int_s^t \|Au(\tau) - B(u(\tau))\|^2 d\tau \le V(u(s)) \tag{5.7}$$

for any $t \ge s \ge 0$ and $u(t) \in \mathscr{A}$. Equation (5.7) implies that the functional $V(u)$ defined by equality (5.5) is the Lyapunov function of the dynamical system $(\mathscr{F}_{1/2}, S_t)$ on the attractor $\mathscr{A}$. Therefore, Theorem 1.6.1 implies equation (5.3). **Theorem 5.1 is proved**.

— Exercise *5.1* Using (5.6) show that any solution $u(t)$ to problem (4.1) with $u_0 \in \mathscr{F}_{1/2}$ possesses the property

$$\int_0^t \|Au(\tau)\|^2 d\tau < \infty, \quad t > 0.$$

Prove the validity of inequality (5.7) for any solution $u(t)$ to problem (4.1).

— Exercise *5.2* Using the results of Exercises 5.1 and 1.6.5 show that if the hypotheses of Theorem 5.1 hold, then a global minimal attractor $\mathscr{A}_{\min}$ of the system $(\mathscr{F}_{1/2}, S_t)$ has the form

$$\mathscr{A}_{\min} = \{w \in D(A):\ Aw - B(w) = 0\}.$$

— Exercise *5.3* Prove that the assertions of Theorems 4.3 and 5.1 hold if we consider the equation

$$\frac{du}{dt} + Au = B(u) + h, \quad u\big|_{t=0} = u_0 \tag{5.8}$$

instead of problem (4.1). Here h is an arbitrary element from H and $B(u)$ possesses properties (5.1), (5.2) and (4.2) with $\theta = 1/2$ (*Hint*: $V_h(u) = V(u) - (h, u)$).

— Exercise *5.4* Let S_t be an evolutionary operator of problem (5.8). Show that for any $R > 0$ there exist numbers $a_R \ge 0$ and $b_R > 0$ such that

$$\left\|A^{1/2}(S_t u_1 - S_t u_2)\right\| \le b_R e^{a_R t} \left\|A^{1/2}(u_1 - u_2)\right\|,$$

provided $\left\|A^{1/2} u_j\right\| \le R$, $j = 1, 2$ (*Hint*: $V_h(S_t u_j) \le V_h(u_j) \le C_R$).

Theorem 5.1 and the reasonings of Section 6 of Chapter 1 reduce the question on the structure of global attractor to the problem of studying the properties of stationary points of the dynamical system under consideration. Under some additional conditions on the operator $B(u)$ it can be proved in general that the number of fixed points is finite and all of them are hyperbolic. This enables us to use the results of Section 6 of Chapter 1 to specify the attractor structure. For some reasons (they will be clear later) it is convenient to deal with the fixed points of the dynamical system generated by problem (5.8).

Thus we consider the equation

$$L(u) \equiv Au - B(u) = h\,, \qquad u \in D(A)\,, \tag{5.9}$$

where as before A is a positive operator with discrete spectrum, $B(u)$ is a nonlinear mapping possessing properties (5.1), (5.2) and (4.2) with $\theta = 1/2$, and h is an element of H.

Lemma 5.3.

For any $h \in H$ problem (5.9) is solvable. If M is a bounded set in H, then the set $L^{-1}(M)$ of solutions to equation (5.9) is bounded in $D(A)$ for $h \in M$. If M is compact in H, then $L^{-1}(M)$ is compact in $D(A)$.

Proof.

Let us consider a continuous functional

$$W(u) = \frac{1}{2}(Au,\ u) + F(u) - (h,\ u) \tag{5.10}$$

on $\mathcal{F}_{1/2} = D(A^{1/2})$. Since $F(u) \ge -\alpha$ for all $u \in \mathcal{F}_{1/2}$, the functional $W(u)$ possesses the property

$$W(u) \ \ge\ \frac{1}{2}\|A^{1/2}u\|^2 - \alpha - \|A^{-1/2}h\|\,\|A^{1/2}u\| \ \ge$$

$$\ge\ \frac{1}{4}\|A^{1/2}u\|^2 - \alpha - \|A^{-1/2}h\|^2\ . \tag{5.11}$$

In particular, this means that $W(u)$ is bounded below. Let us consider the functional $W(u)$ on the subspace $P_m\mathcal{F}_{1/2}$ (P_m is the orthoprojector onto the span of elements $e_1,\ e_2,\ \dots,\ e_m$, as before). By virtue of (5.11) there exists a minimum point u_m of this functional in $P_m\mathcal{F}_{1/2}$ which obviously satisfies the equation

$$Au_m - P_m B(u_m) \ =\ P_m h\,. \tag{5.12}$$

Equation (5.11) also implies that

$$\frac{1}{4}\|A^{1/2}u_m\|^2 \ \le\ W(u_m) + \alpha + \|A^{-1/2}h\|^2 \ \le$$

$$\le\ \inf\{W(u):\ u \in P_m H\} + \alpha + \|A^{-1/2}h\|^2\ .$$

Hence,

$$\left\|A^{1/2}u_m\right\| \le F(0)+\alpha+\|A^{-1/2}h\|^2 \le C(1+\|A^{-1/2}h\|^2)$$

with a constant C independent of m. Thus, equations (5.12) and (4.2) give us that

$$\|Au_m\| \le \|B(u_m)+h\| \le \|B(0)\|+\|h\|+M(\|A^{1/2}u_m\|)\|A^{1/2}u_m\| .$$

Therefore, if $\|h\| \le R$, then $\|Au_m\| \le C(R)$. This estimate enables us to extract a weakly convergent (in $D(A)$) subsequence $\{u_{m_k}\}$ and to pass to the limit as $k \to \infty$ in (5.12) with the help of Theorem 1.1. Thus, the solvability of equation (5.9) is proved. It is obvious that every limit (in $D(A)$) point u of the sequence $\{u_m\}$ possesses the property

$$\|Au\| \le C_R \quad \text{if} \quad \|h\| \le R . \tag{5.13}$$

This means that the complete preimage $L^{-1}(M)$ of any bounded set M in H is bounded in $D(A)$. Now we prove that the mapping L is ***proper***, i.e. the preimage $L^{-1}(M)$ is compact for a compact M. Let $\{u_n\}$ be a sequence from $L^{-1}(M)$. Then the sequence $\{L(u_n)\}$ lies in the compact M and therefore there exist an element $h \in M$ and a subsequence $\{n_k\}$ such that $\|L(u_{n_k})-h\| \to 0$ as $k \to \infty$. By virtue of (5.13) we can also assume that $\{A u_{n_k}\}$ is a weakly convergent sequence in $D(A)$. If we use the equation

$$\|Au_{n_k}-B(u_{n_k})-h\| \to 0 , \qquad k \to \infty ,$$

Theorem 1.1, and property (4.2) with $\theta = 1/2$, then we can easily prove that the sequence $\{u_{n_k}\}$ strongly converges in $D(A)$ to a solution u to the equation $L(u) = h$. Lemma 5.3 is proved.

Lemma 5.4.

In addition to (5.1), (5.2), *and* (4.2) *with* $\theta = 1/2$ *we assume that the mapping* $B(\cdot)$ *is Frechét differentiable, i.e. for any* $u \in \mathcal{F}_{1/2}$ *there exists a linear bounded operator* $B'(u)$ *from* $\mathcal{F}_{1/2}$ *into* H *such that*

$$\|B(u+v)-B(u)-B'(u)v\| = o(\|A^{1/2}v\|) \tag{5.14}$$

for every $v \in \mathcal{F}_{1/2}$, *such that* $\|A^{1/2}v\| \le 1$. *Then the operator* $A-B'(u)$ *is a Fredholm operator for any* $u \in \mathcal{F}_{1/2} = D(A^{1/2})$.

We remind that a densely defined closed linear operator G in H is said to be ***Fredholm*** (of index zero) if

(a) its image is closed; and
(b) $\dim \mathcal{K}er\, G = \dim \mathcal{K}er\, G^* < \infty$.

Proof of Lemma 5.4.

It is clear that the operator $G \equiv A-B'(u)$ has the structure

$$G = A - C A^{1/2} = A^{1/2}(I - A^{-1/2} C) A^{1/2} ,$$

where C is a bounded operator in H $(C = B'(u) A^{-1/2})$. By virtue of Theorem 1.1 the operator $K = A^{-1/2} C$ is compact. This implies the closedness of the image of G. Moreover, it is obvious that

$$\dim \mathcal{K}er\, G = \dim \mathcal{K}er\, (I - K)$$

and

$$\dim \mathcal{K}er\, G^* = \dim \mathcal{K}er\, (I - K^*) .$$

Therefore, the Fredholm alternative for the compact operator gives us that

$$\dim \mathcal{K}er\, G^* = \dim \mathcal{K}er\, G < \infty .$$

Let us introduce the notion of a ***regular value*** of the operator L seen as an element $h \in H$ possessing the property that for every $u \in L^{-1} h \equiv \{v:\ L(v) = h\}$ the operator $L'(u) = A - B'(u)$ is invertible on H. Lemmata 5.3 and 5.4 enable us to use the Sard-Smale theorem (for the statement and the proof see, e.g., the book by A. V. Babin and M. I. Vishik [1]) and state that the set $\mathcal{R}$ of regular values of the mapping $L(u) = Au - B(u)$ is an open everywhere dense set in H. The following assertion is valid for regular values of the operator L.

Lemma 5.5.

Let h be a regular value of the operator L. Then the set of solutions to equation (5.9) *is finite.*

Proof.

By virtue of Lemma 5.3 the set $\mathcal{N} = \{v:\ L(v) = h\}$ is compact. Since $h \in \mathcal{R}$, the operator $L'(u) = A - B'(u)$ is invertible on H for $u \in \mathcal{N}$. It is also evident that $L'(u)$ has a domain $D(A)$. Therefore, by virtue of the uniform boundedness principle $A L'(u)^{-1}$ is a bounded operator for $u \in \mathcal{N}$. Hence, it follows from (5.14) that

$$\|A(v - w)\| \le \|AL'(v)^{-1}\| \cdot \|L'(v)(v - w)\| =$$

$$= \|AL'(v)^{-1}\| \|B(v) - B(w) - B'(v)(v - w)\| = o(\|A^{1/2}(v - w)\|)$$

for any v and w in $\mathcal{N}$. This implies that for every $v \in \mathcal{N}$ there exists a vicinity that does not contain other points of the set $\mathcal{N}$. Therefore, the compact set $\mathcal{N}$ has no condensation points. Hence, $\mathcal{N}$ consists of a finite number of elements. Lemma 5.5 is proved.

In order to prove the hyperbolicity (for the definition see Section 6 of Chapter 1) of fixed points we should first consider linearization of problem (5.8) at these points. Assume that the hypotheses of Lemma 5.4 hold and $v_0 \in D(A)$ is a stationary solution. We consider the problem

$$\frac{du}{dt} + (A - B'(v_0))u = 0\,, \qquad u\big|_{t=0} = u_0\,. \tag{5.15}$$

Its solution can be regarded as a continuous function in $\mathcal{F}_{1/2} = D(A^{1/2})$ which satisfies the equation

$$u(t) = e^{-tA}\,u_0 + \int_0^t e^{-(t-\tau)A} B'(v_0)u(\tau)d\tau\,.$$

If $u_0 \in D(A^{1/2})$, then we can apply Theorem 2.3 on the existence and uniqueness of solution. Let T_t stand for the evolutionary operator of problem (5.15).

— Exercise *5.5* Prove that T_t is a compact operator in every space $\mathcal{F}_\alpha$, $0 \le \alpha < 1$, $t > 0$.

— Exercise *5.6* Prove that for any $\rho > 0$ and $t > 0$ the set of points of the spectrum of the operator T_t that are lying outside the disk $\{\lambda : |\lambda| \le \rho\}$ is finite and the corresponding eigensubspace is finite-dimensional.

— Exercise *5.7* Assume that $B'(v_0)$ is a symmetric operator in H. Prove that the spectrum of the operator T_t is real.

The next assertion contains the conditions wherein the evolutionary operator S_t belongs to the class $C^{1+\alpha}$, $\alpha > 0$, on the set of stationary points.

Theorem 5.2.

Assume that conditions (5.1), (5.2), ***and*** (4.2) ***are fulfilled with*** $\theta = 1/2$. ***Let*** $B(u)$ ***possess a Frechét derivative in*** $\mathcal{F}_{1/2}$ ***such that for any*** $R > 0$

$$\|B(u+v) - B(u) - B'(u)v\| \le C\|A^{1/2}v\|^{1+\alpha}, \qquad \alpha > 0, \tag{5.16}$$

provided that $\|A^{1/2}v\| \le R$, ***where the constant*** $C = C(u, R)$ ***depends on*** u ***and*** R ***only. Then the evolutionary operator*** S_t ***of problem*** (5.8) ***has a Frechét derivative at every stationary point*** v_0. ***Moreover,*** $([S_t(v_0)]', u) = T_t u$, ***where*** T_t ***is the evolutionary operator of linear problem*** (5.15).

Proof.

It is evident that

$$S_t[v_0 + u] - v_0 - T_t u = \int_0^t e^{-(t-\tau)A} \Big\{ B(S_\tau[v_0+u]) - B(v_0) - B'(v_0)T_\tau u \Big\} d\tau\,.$$

Therefore, using (1.17) and (5.16) we have

$$\|A^{1/2}\psi(t)\| \le \int_0^t (2e[t-\tau])^{-1/2}\{C\|A^{1/2}(S_\tau[v_0+u]-v_0)\|^{1+\alpha}+$$

$$+\|B'(v_0)\|\cdot\|A^{1/2}\psi(\tau)\|\}\,d\tau\ ,$$

where $\psi(t) = S_t[v_0+u]-v_0-T_t u$. Using the result of Exercise 5.4 we obtain that

$$\|A^{1/2}(S_t[v_0+u]-v_0)\| \le b_R e^{a_R t}\|A^{1/2}u\| \quad \text{if} \quad \|A^{1/2}u\| \le R\,.$$

Thus,

$$\|A^{1/2}\psi(t)\| \le C_0\sqrt{t}\,e^{(1+\alpha)a_R t}\|A^{1/2}u\|^{1+\alpha}+C_1\int_0^t (t-\tau)^{-1/2}\|A^{1/2}\psi(\tau)\|\,d\tau\,.$$

Consequently, Lemma 2.1 gives us the estimate

$$\|A^{1/2}(S_t[v_0+u]-v_0-T_t u)\| \le C_T\|A^{1/2}u\|^{1+\alpha}, \qquad t\in[0,\ T]\,.$$

This implies the assertion of Theorem 5.2.

Exercise *5.8* Assume that the constant C in (5.16) depends on R only, provided that $\|A^{1/2}u\| \le R$ and $\|A^{1/2}v\| \le R$. Prove that $S_t \in C^{1+\alpha}$ for any $0 < \alpha < 1$.

The reasoning above leads to the following result on the properties of the set of fixed points of problem (5.8).

Theorem 5.3.

Assume that conditions (5.1)***,*** (5.2) ***and*** (4.2) ***are fulfilled with*** $\theta = 1/2$ ***and the operator*** $B(u)$ ***possesses a Frechét derivative in*** $\mathcal{F}_{1/2}$ ***such that equation*** (5.16) ***holds with the constant*** $C = C(R)$ ***depending only on*** R ***for*** $\|A^{1/2}u\| \le R$ ***and*** $\|A^{1/2}v\| \le R$***. Then there exists an open dense (in*** H***) set*** $\mathcal{R}$ ***such that for*** $h \in \mathcal{R}$ ***the set of fixed points of the system*** $(\mathcal{F}_{1/2},\ S_t)$ ***generated by problem*** (5.8) ***is finite. If in addition we assume that*** $B'(z)$ ***is a symmetric operator for*** $z \in D(A)$***, then fixed points are hyperbolic.***

In particular, this theorem means that if $h \in \mathcal{R}$, then the global attractor of the dynamical system generated by equation (5.8) possesses the properties given in Exercises 1.6.9–1.6.12. Moreover, it is possible to apply Theorem 1.6.3 as well as the other results related to finiteness and hyperbolicity of the set of fixed points (see, e.g., the book by A. V. Babin and M. I. Vishik [1]).

Example *5.1*

Let us consider a dynamical system generated by the nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} + g(x,\ u(x)) = h(x), \quad 0 < x < 1, \quad t > 0, \\ u\big|_{x=0} = u\big|_{x=1} = 0, \quad u\big|_{t=0} = u_0(x), \end{cases} \tag{5.17}$$

in $H_0^1(0,\ 1)$. Assume that $g(x,\ u)$ is twice continuously differentiable with respect to its variables and the conditions

$$\int_0^y g(x,\ \xi)\,d\xi \ge -\alpha, \qquad y\,g(x,\ y) - \beta \int_0^y g(x,\ \xi)\,d\xi \ge -\gamma$$

are fulfilled with some positive constants $a,\ b$, and γ.

Exercise *5.9* Prove that the dynamical system generated by equation (5.17) possesses a global attractor $\mathcal{A} = M_+(\mathcal{N})$, where $\mathcal{N}$ is the set of stationary solutions to problem (5.17).

Exercise *5.10* Prove that there exists a dense open set $\mathcal{R}$ in $L^2(0,\ 1)$ such that for every $h(x) \in \mathcal{R}$ the set $\mathcal{N}$ of fixed points of the dynamical system generated by problem (5.17) is finite and all the points are hyperbolic.

It should be noted that if a property of a dynamical system holds for the parameters from an open and dense set in the corresponding space, then it is frequently said that this property is a ***generic property***.

However, it should be kept in mind that the generic property is not the one that holds almost always. For example one can build an open and dense set $\mathcal{R}$ in $[0,\ 1]$, the Lebesgue measure of which is arbitrarily small $(\le \varepsilon)$. To do that we should take

$$\mathcal{R} = \bigcup_{k=1}^{\infty} \left\{ x \in (0,\ 1):\ |x - r_k| < \varepsilon \cdot 2^{-k-2} \right\},$$

where $\{r_k\}$ is a sequence of all the rational numbers of the segment $[0,\ 1]$. Therefore, it should be remembered that generic properties are quite frequently encountered and stay stable during small perturbations of the properties of a system.

§ 6 *Explicitly Solvable Model of Nonlinear Diffusion*

In this section we study the asymptotic properties of solutions to the following nonlinear diffusion equation

$$\begin{cases} u_t - \nu u_{xx} + \left(\varkappa \int_0^1 |u(x,\ t)|^2 \mathrm{d}x - \Gamma\right) u + \rho u_x = 0, \quad 0 < x < 1,\ t > 0, \\ u\big|_{x=0} = u\big|_{x=1} = 0, \quad u\big|_{t=0} = u_0(x), \end{cases} \tag{6.1}$$

where $\nu > 0$, $\varkappa > 0$, Γ and ρ are parameters. The main feature of this problem is that the asymptotic behaviour of its solutions can be completely described with the help of elementary functions. We do not know whether problem (6.1) is related to any real physical process.

Exercise *6.1* Show that Theorem 2.4 which guarantees the global existence and uniqueness of mild solutions is applicable to problem (6.1) in the Sobolev space $H_0^1(0,\ 1)$.

Exercise *6.2* Write out the system of ordinary differential equations for the functions $\{g_k(t)\}$ that determine the Galerkin approximations

$$u_m(t) = \sqrt{2} \sum_{k=1}^{m} g_k(t) \sin \pi k x \tag{6.2}$$

of the order m of a solution to problem (6.1).

Exercise *6.3* Using the properties of the functions $u_m(t)$ defined by equation (6.2) prove that the mild solution $u(t,\ x)$ possesses the properties

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t (\nu\|\partial_x u(\tau)\|^2 + \varkappa\|u(\tau)\|^4 - \Gamma\|u(\tau)\|^2) \mathrm{d}\tau = \frac{1}{2}\|u_0\|^2 \tag{6.3}$$

and

$$\|u(t)\|^2 \le \|u_0\|^2 e^{-2\nu\pi^2 t} + \frac{\Gamma^2}{4\nu\varkappa\pi^2}(1 - e^{-2\nu\pi^2 t}). \tag{6.4}$$

Here and below $\|\cdot\|$ is a norm in $L^2(0,\ 1)$.

Exercise *6.4* Using equations (6.3) and (6.4) prove that the dynamical system generated by problem (6.1) in $H_0^1(0,\ 1)$ is dissipative.

Therefore, by virtue of Theorem 4.1 the dynamical system $(H_0^1(0,\ 1),\ S_t)$ generated by equation (6.1) possesses a finite-dimensional global attractor.

Let $u(t) = u(x,\ t) \in C(\mathbb{R}_+;\ H_0^1(0,\ 1))$ be a mild solution to problem (6.1) with the initial condition $u_0(x) \in H_0^1(0,\ 1)$. Then the function $u(x,\ t)$ can be considered as a mild solution to the linear problem

$$\begin{cases} u_t - \nu u_{xx} + b(t)u + \rho u_x = 0\,, \quad 0 < x < 1, \quad t > 0\ , \\ u|_{x=0} = u|_{x=1} = 0, \quad u|_{t=0} = u_0(x)\ , \end{cases} \tag{6.5}$$

where $b(t)$ is a scalar continuous function defined by the formula

$$b(t) = \varkappa \int_0^1 |u(x,\ t)|^2 \mathrm{d}x - \Gamma\,.$$

We consider the function

$$v(x,\ t) = u(x,\ t)\exp\left\{\int_0^t b(\tau)\,\mathrm{d}\tau + \frac{\rho^2}{4\nu}t - \frac{\rho}{2\nu}x\right\}. \tag{6.6}$$

Then it is easy to check that $v(x,\ t)$ is a mild solution to the heat equation

$$\begin{cases} v_t = \nu v_{xx}, \quad x \in (0,\ 1),\ t > 0\ , \\ v|_{x=0} = v|_{x=1} = 0, \quad v|_{t=0} = u_0(x)\exp\left\{-\dfrac{\rho}{2\nu}x\right\}. \end{cases} \tag{6.7}$$

The following assertion shows that the asymptotic properties of equation (6.7) completely determine the dynamics of the system generated by problem (6.1).

Lemma 6.1.

Every mild (in $H_0^1(0,\ 1)$) solution $u(x,\ t)$ to problem (6.1) *can be rewritten in the form*

$$u(x,\ t) = \frac{w(x,\ t)}{\left\{1 + 2\varkappa\displaystyle\int_0^t \|w(\tau)\|^2\,\mathrm{d}\tau\right\}^{1/2}}\,, \tag{6.8}$$

where $w(x,\ t)$ has the form

$$w(x,\ t) = v(x,\ t)\,\exp\left\{\left(\Gamma - \frac{\rho^2}{4\nu}\right)t + \frac{\rho}{2\nu}x\right\} \tag{6.9}$$

and $v(x,\ t)$ is the solution to problem (6.7).

Proof.

Let $w(x, t)$ have the form (6.9). Then we obtain from (6.6) that

$$w(x, t) = u(x, t) \exp\left\{ \varkappa \int_0^t \|u(\tau)\|^2 d\tau \right\}. \qquad (6.10)$$

Therefore,

$$\|w(x, t)\|^2 = \|u(x, t)\|^2 \exp\left\{ 2\varkappa \int_0^t \|u(\tau)\|^2 d\tau \right\} = \frac{1}{2\varkappa} \frac{d}{dt} \exp\left\{ 2\varkappa \int_0^t \|u(\tau)\|^2 d\tau \right\}.$$

Hence,

$$\exp\left\{ 2\varkappa \int_0^t \|u(\tau)\|^2 d\tau \right\} = 1 + 2\varkappa \int_0^t \|w(\tau)\|^2 d\tau .$$

This and equation (6.10) imply (6.8). Lemma 6.1 is proved.

Now let us find the fixed points of problem (6.1). They satisfy the equation

$$-\nu u_{xx} + (\varkappa \|u\|^2 - \Gamma) u + \rho u_x = 0, \quad u|_{x=0} = u|_{x=1} = 0 .$$

Therefore, $u(x) = w(x) \exp\left\{ \frac{\rho}{2\nu} x \right\}$, where $w(x)$ is the solution to the problem

$$-\nu w_{xx} + \left(\varkappa \|u\|^2 - \Gamma + \frac{\rho^2}{4\nu} \right) w = 0, \quad w|_{x=0} = w|_{x=1} = 0 .$$

However, this problem has a nontrivial solution $w(x)$ if and only if

$$w(x) = C \sin \pi n x \quad \text{and} \quad \varkappa \|u\|^2 - \Gamma + \frac{\rho^2}{4\nu} + \nu(\pi n)^2 = 0 ,$$

where n is a natural number. Since $u = w \exp\left\{ \frac{\rho}{2\nu} x \right\}$, we obtain the equation

$$\varkappa C^2 \int_0^1 e^{\frac{\rho}{\nu} x} \sin^2 \pi n x \, dx + \nu(\pi n)^2 - \Gamma + \frac{\rho^2}{4\nu} = 0$$

which can be used to find the constant C. After the integration we have

$$\varkappa C^2 \frac{\nu}{\rho} (e^{\rho/\nu} - 1) \frac{2 n^2 \nu^2 \pi^2}{\rho^2 + 4 n^2 \nu^2 \pi^2} = \Gamma - \frac{\rho^2}{4\nu} - \nu(\pi n)^2 .$$

The constant C can be found only when the parameter n possesses the property $\Gamma - \rho^2/(4\nu) - \nu(\pi n)^2 > 0$. Thus, we have the following assertion.

Lemma 6.2.

Let $\gamma = \gamma(\Gamma, \nu, \rho) \equiv \Gamma - \frac{\rho^2}{4\nu}$. *If* $\gamma \le \nu\pi^2$, *then problem* (6.1) *has a unique fixed point* $\overline{u}_0(x) \equiv 0$. *If* $(\pi n)^2 < \gamma \le \pi^2(n+1)^2$ *for some* $n \ge 1$, *then the fixed points of problem* (6.1) *are*

$$\overline{u}_0(x) \equiv 0, \qquad \overline{u}_{\pm k}(x) = \pm\mu_k e^{\frac{\rho}{2\nu}x} \sin\pi k x, \qquad k = 1, 2, \dots, n, \tag{6.11}$$

where

$$\mu_k \equiv \mu_k(\Gamma, \rho, \nu) = \frac{1}{4\nu^2\pi k}\sqrt{\frac{2\rho\delta_k(\Gamma, \rho, \nu)}{\varkappa\left(\exp\left(\frac{\rho}{\nu}\right) - 1\right)}}, \tag{6.12}$$

$$\delta_k(\Gamma, \rho, \nu) = [4\Gamma\nu - \rho^2 - 4(\pi n\nu)^2] \cdot [\rho^2 + 4(\pi n\nu)^2].$$

Exercise *6.5* Show that every subspace

$$H_N = \mathrm{Lin}\left\{e^{\frac{\rho}{2\nu}x} \sin\pi k x : \quad k = 1, 2, \dots, N\right\} \tag{6.13}$$

is positively invariant for the dynamical system $(H_0^1(0, 1), S_t)$ generated by problem (6.1).

Theorem 6.1.

Let $\gamma \equiv \Gamma - \frac{\rho^2}{4\nu} < \nu\pi^2$. ***Then for any mild solution*** $u(t)$ ***to problem*** (6.1) ***the estimate***

$$\|\partial_x u(t)\| \le C(\rho, \nu) e^{-(\nu\pi^2 - \gamma)t} \|\partial_x u_0\|, \qquad t \ge 0, \tag{6.14}$$

is valid. If $\gamma \ge \nu\pi^2$ ***and*** $\nu\pi^2 N^2 > \gamma$, ***then the subspace*** H_{N-1} ***defined by formula*** (6.13) ***is exponentially attracting:***

$$\mathrm{dist}_{H_0^1(0,1)}(S_t u_0, H_{N-1}) \le C(\rho, \nu) e^{-(\nu\pi^2 N^2 - \gamma)t} \|\partial_x u_0\|. \tag{6.15}$$

Here S_t ***is the evolutionary operator in*** $H_0^1(0, 1)$ ***corresponding to*** (6.1).

Proof.

Let $w(x, t)$ be of the form (6.9). Then

$$w_\rho(t) \equiv e^{-\frac{\rho}{2\nu}x} w(t) = \sum_{k=1}^{\infty} e^{-(\nu(\pi k)^2 - \gamma)t} C_{k,\rho} e_k(x), \tag{6.16}$$

where $e_k(x) = \sqrt{2}\sin\pi k x$ and

$$C_{k,\rho} = \int_0^1 e^{-\frac{\rho}{2\nu}x} u_0(x) e_k(x) \mathrm{d}x.$$

Assume that $\gamma < \nu\pi^2$. Then it is obvious that

$$\|\partial_x w_\rho(t)\|^2 = \sum_{k=1}^{\infty} (\pi k)^2 C_{k,\rho}^2 \exp\{-2(\nu(\pi k)^2 - \gamma)t\} \le$$
$$\le \exp\{-2(\nu\pi^2 - \gamma)t\}\left\|\partial_x\left(e^{-\frac{\rho}{2\nu}x} u_0\right)\right\|^2. \tag{6.17}$$

However, equation (6.8) implies that

$$\left\|\partial_x\left(e^{-\frac{\rho}{2\nu}x} u(t)\right)\right\| \le \|\partial_x w_\rho(t)\|.$$

Therefore, estimate (6.14) follows from (6.17) and from the obvious equality $\|u\| \le 1/\pi\|\partial_x u\|$. When $\gamma \ge \nu\pi^2$ and $N > (1/\pi)\sqrt{\gamma/\nu}$ the function $w(t)$ in (6.9) can be rewritten in the form

$$w(t) = h_N(t) + e^{\frac{\rho}{2\nu}x} w_{\rho,N}(t), \tag{6.18}$$

where

$$h_N(t) = \sqrt{2}\, e^{\frac{\rho}{2\nu}x} \sum_{k=1}^{N-1} e^{-(\nu(\pi k)^2 - \gamma)t}\, C_{k,\rho} \sin \pi k x \in H_{N-1}$$

and $w_{\rho,N}(t)$ can be estimated (cf. (6.17)) as follows:

$$\|\partial_x w_{\rho,N}(t)\|^2 \le \exp\{-2(\nu(\pi N)^2 - \gamma)t\}\left\|\partial_x\left(e^{-\frac{\rho}{2\nu}x} u_0\right)\right\|^2. \tag{6.19}$$

Thus, it is clear that

$$\operatorname{dist}_{H_0^1(0,\,1)}(S_t u_0, H_{N-1}) \le C(\rho, \nu)\|\partial_x w_{\rho,N}(t)\|.$$

Consequently, estimate (6.15) is valid. **Theorem 6.1 is proved**.

In particular, Theorem 6.1 means that if $\gamma \equiv \Gamma - \rho^2/(4\nu) < \nu\pi^2$, then the global attractor of problem (6.1) consists of a single zero element, whereas if $\gamma \ge \nu\pi^2$, the attractor lies in an exponentially attracting invariant subspace H_{N_0}, where $N_0 = [(1/\pi)\sqrt{\gamma/\nu}]$ and $[\cdot]$ is a sign of the integer part of a number.

The following assertion shows that the global minimal attractor of problem (6.1) consists of fixed points of the system. It also provides a description of the corresponding basins of attraction.

Theorem 6.2.

Let $\gamma \equiv \Gamma - \dfrac{\rho^2}{4\nu} > \nu\pi^2$. *Assume that the number* $\pi^{-1}\sqrt{\gamma/\nu}$ *is not integer. Suppose that* N_0 *is the greatest integer such that* $\nu(\pi N_0)^2 < \gamma$. *Let*

$$C_j(u_0) = \sqrt{2}\int_0^1 e^{-\frac{\rho}{2\nu}x}\, u_0(x)\sin\pi jx\,\mathrm{d}x$$

and let $u(t)$ ***be a mild solution to problem*** (6.1).

(a) ***If*** $C_j(u_0) = 0$ ***for all*** $j = 1, 2, \ldots, N_0$, ***then***

$$\|\partial_x u(t)\| \le C(\nu, \rho)\exp\{-(\nu\pi^2(N_0+1)^2 - \gamma)t\}\|\partial_x u_0\|, \qquad t \ge 0. \quad (6.20)$$

(b) ***If*** $C_j(u_0) \ne 0$ ***for some*** j ***between*** 1 ***and*** N_0, ***then there exist positive numbers*** $C = C(\nu, \rho, \gamma;\, u_0)$ ***and*** $\beta = \beta(\gamma, \nu)$ ***such that***

$$\|\partial_x(u(t) - \bar{u}_{\sigma k})\| \le Ce^{-\beta t}, \qquad t \ge 0, \quad (6.21)$$

where $\bar{u}_{\pm k}(x)$ ***is defined by formula*** (6.11), $\sigma = \operatorname{sign} C_k(u_0)$, ***and*** k ***is the smallest index between*** 1 ***and*** N_0 ***such that*** $C_k(u_0) \ne 0$.

Proof.

In order to prove assertion (a) it is sufficient to note that the value $h_N(t)$ is identically equal to zero in decomposition (6.18) when $N = N_0 + 1$. Therefore, (6.20) follows from (6.19).

Now we prove assertion (b). In this case equation (6.18) can be rewritten in the form

$$w(t) = g_k(t) + h_k(t) + w_{\rho,\, N_0+1}(t), \quad (6.22)$$

where

$$g_k(t) = \sqrt{2}\, e^{\frac{\rho}{2\nu}x}\, e^{-[\nu(\pi k)^2 - \gamma]t}\, C_k(u_0)\sin\pi kx,$$

$$h_k(t) = \sqrt{2}\, e^{\frac{\rho}{2\nu}x}\sum_{j=k+1}^{N_0} e^{-[\nu(\pi j)^2 - \gamma]t}\, C_{j,\,\rho}\sin\pi jx,$$

and $h_k(t) \equiv 0$ if $k = N_0$. Moreover, the estimate

$$\|\partial_x w_{\rho,\, N_0+1}(t)\| \le C(\rho, \nu)e^{-(\nu\pi^2(N_0+1)^2 - \gamma)t}\|\partial_x u_0\| \quad (6.23)$$

is valid for $w_{\rho,\, N_0+1}(t)$. It is also evident that

$$\|\partial_x h_k(t)\| \le C(\rho, \nu)\, e^{(\gamma - \nu\pi^2(k+1)^2)t}\|\partial_x u_0\|. \quad (6.24)$$

Since

$$\left|\|w(t)\|^2 - \|g_k(t)\|^2\right| \le (\|w(t)\| + \|g_k(t)\|)\,\left|\|w(t)\| - \|g_k(t)\|\right| \le$$

$$\le (2\|g_k(t)\| + \|h_k(t)\| + \|w_{\rho,\, N_0+1}(t)\|)\,(\|h_k(t)\| + \|w_{\rho,\, N_0+1}(t)\|),$$

using (6.23) and (6.24) we obtain

$$\left|\|w(t)\|^2 - \|g_k(t)\|^2\right| \le C\exp\{2[\gamma - \nu\pi^2(k^2 + (k+1)^2)]t\}\|\partial_x u_0\|^2.$$

Integration gives

$$2\varkappa\int_0^t \|g_k(\tau)\|^2 d\tau = 4\varkappa[C_k(u_0)]^2\int_0^1 e^{\frac{\rho}{\nu}x}\sin^2\pi kx\,dx\int_0^t e^{2(\gamma-\nu(\pi k)^2)\tau}d\tau =$$

$$= 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2(e^{2(\gamma-\nu(\pi k)^2)t}-1)\ ,$$

where μ_k is defined by formula (6.12). Hence,

$$1+2\varkappa\int_0^t \|w(\tau)\|^2 d\tau = 1+2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 e^{2(\gamma-\nu(\pi k)^2)t}+a_k(t)\,, \qquad (6.25)$$

where

$$|a_k(t)| \le C\Big[1+\exp\Big\{2\Big(\gamma-\nu\pi^2\frac{k^2+(k+1)^2}{2}\Big)t\Big\}\Big]\|\partial_x u_0\|^2, \qquad k \le N_0-1\,.$$

Let

$$v_k(t) = \frac{g_k(t)}{\Big(1+2\varkappa\int_0^t \|w(\tau)\|^2 d\tau\Big)^{1/2}}\,.$$

We consider the case when $1 \le k \le N_0-1$. Equations (6.23)–(6.25) imply that

$$\|\partial_x(u(t)-v_k(t))\| \le \|\partial_x w_{\rho,\,N_0+1}(t)\| + \frac{\|\partial_x h_k(t)\|}{\Big(1+2\varkappa\int_0^t \|w(\tau)\|^2 d\tau\Big)^{1/2}} \le$$

$$\le C(\rho,\,\nu)\|\partial_x u_0\|\Big\{e^{-(\nu\pi^2(N_0+1)^2-\gamma)t}+$$

$$+\frac{\exp\{(\gamma-\nu\pi^2(k+1)^2)t\}}{\Big(1+2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 e^{2(\gamma-\nu(\pi k)^2)t}+a_k(t)\Big)^{1/2}}\Big\}\,.$$

Therefore,

$$\|\partial_x(u(t)-v_k(t))\| \le$$

$$\le C(\rho,\,\nu,\,u_0)\Big\{e^{-(\nu\pi^2(N_0+1)^2-\gamma)t}+e^{-\nu\pi^2((k+1)^2-k^2)t}\,G(t)^{-1/2}\Big\}\ ,$$

where

$$G(t) = (1 + a_k(t))\, e^{-2(\gamma - \nu\pi^2 k^2)t} + 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 .$$

It follows from (6.25) that $G(t) > 0$ for all $t \geq 0$ and $G(0) = 1$. Moreover, the above-mentioned estimate for $a_k(t)$ enables us to state that

$$\lim_{t\to\infty} G(t) = 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 > 0 .$$

This implies that there exists a constant $G_{\min} = G_{\min}(\gamma, k, \nu, u_0) > 0$ such that $G(t) \geq G_{\min}$ for all $t \geq 0$. Consequently,

$$\|\partial_x(u(t) - v_k(t))\| \leq C(\rho, \nu, \gamma; u_0)\, e^{-\alpha t}, \tag{6.26}$$

where $\alpha = \alpha(k, \nu, \gamma) = \min\{\nu\pi^2(N_0+1)^2 - \gamma,\ \nu\pi^2(2k+1)\}$. Now we consider the value $v_k(t)$. Evidently,

$$v_k(t) = \varphi_k(t, u_0)\,\bar{u}_{\sigma k}(x),$$

where

$$\varphi_k(t, u_0) = \frac{\sqrt{2}\,|C_k(u_0)| \exp\{(\gamma - \nu\pi^2 k^2)t\}}{\mu_k\Big(1 + 2\Big(\frac{C_k(u_0)}{\mu_k}\Big)^2 \exp\{2(\gamma - \nu\pi^2 k^2)t\} + a_k(t)\Big)^{1/2}} .$$

Simple calculations give us that

$$|\varphi_k(t, u_0) - 1| \leq C(\rho, \nu, \gamma, u_0)\, e^{-\bar{\beta} t}$$

with the constant $\bar{\beta} = \nu\pi^2(2k+1)$. This and equation (6.26) imply (6.21), provided $k \leq N_0 - 1$. We offer the reader to analyse the case when $k = N_0$ on his/her own. **Theorem 6.2 is proved**.

Theorem 6.2 enables us to obtain a complete description of the basins of attraction of each fixed point of the dynamical system $(H_0^1(0, 1), S_t)$ generated by problem (6.1).

Corollary 6.1.

Let $\gamma \equiv \Gamma - \rho^2/(4\nu) > \nu\pi^2$. Assume that the number $\pi^{-1}\sqrt{\gamma/\nu}$ is not integer. Let N_0 be the greatest integer possessing the property $\nu(\pi N_0)^2 < \gamma$. We denote

$$C_j(u) = \sqrt{2}\int_0^1 e^{-\frac{\rho}{2\nu}x}\, u(x) \sin \pi j x \,\mathrm{d}x$$

and define the sets

$$\mathcal{D}_k = \{u \in H_0^1(0, 1):\ C_j(u) = 0,\ j = 1, \dots, k-1,\ C_k(u) > 0\},$$

$$\mathcal{D}_{-k} = \{u \in H_0^1(0, 1):\ C_j(u) = 0,\ j = 1, \dots, k-1,\ C_k(u) < 0\}$$

for $k = 1, 2, \ldots, N_0$. *We also assume that*

$$\mathcal{D}_0 = \{u \in H_0^1(0, 1):\ C_j(u) = 0,\ j = 1, 2, \ldots, N_0\}\ .$$

Then for any $l = 0, \pm 1, \pm 2, \ldots, \pm N_0$ *we have that*

$$\lim_{t \to \infty} \|S_t v - \bar{u}_l\|_{H_0^1(0,\ 1)} = 0 \Leftrightarrow v \in \mathcal{D}_l\ ,$$

where $\{\bar{u}_l\}$ *are the fixed points of problem* (6.1) *which are defined by equalities* (6.11).

The next assertion gives us a complete description of the global attractor of problem (6.1).

Theorem 6.3.

Assume that the hypotheses of Theorem 6.2 ***hold and*** N_0 ***is the same as in Theorem*** 6.2. ***Then the global attractor*** $\mathcal{A}$ ***of the dynamical system*** $(H_0^1(0, 1), S_t)$ ***generated by equation*** (6.1) ***is the closure of the set***

$$\mathrm{A} = \left\{ v(x) = \frac{\sqrt{2} \sum_{k=1}^{N_0} \xi_k\, e^{\frac{\rho}{2\nu}x} \sin \pi k x}{\left(1 + 2\varkappa \sum_{k,\ j=1}^{N_0} \xi_k \xi_j \dfrac{\alpha_{kj}}{\nu_k + \nu_j}\right)^{1/2}}\ :\ \xi_j \in \mathbb{R} \right\}, \tag{6.27}$$

where $\nu_k = \gamma - \nu(\pi k)^2$, $k = 1, 2, \ldots, N_0$, ***and***

$$\alpha_{kj} = 2 \int_0^1 e^{\frac{\rho}{\nu}x} \sin \pi k x \cdot \sin \pi j x\, \mathrm{d}x, \qquad k, j = 1, 2, \ldots, N_0.$$

Every complete trajectory $\{u(t):\ t \in \mathbb{R}\}$ ***which lies in the attractor and does not coincide with any of the fixed points*** $\bar{u}_k$, $k = 0, \pm 1, \ldots, \pm N_0$, ***has the form***

$$u(t) = \frac{\sqrt{2} \sum_{k=1}^{N_0} \xi_k\, e^{\nu_k t}\, e^{\frac{\rho}{2\nu}x} \sin \pi k x}{\left(1 + 2\varkappa \sum_{k,\ j=1}^{N_0} \xi_k \xi_j \dfrac{\alpha_{kj}}{\nu_k + \nu_j}\, e^{(\nu_k + \nu_j)t}\right)^{1/2}}\ , \tag{6.28}$$

where ξ_k ***are real numbers,*** $k = 1, 2, \ldots, N_0$, $t \in \mathbb{R}$.

— Exercise *6.6* Show that for all $\xi \in \mathbb{R}^{N_0}$ the function

$$a(t, \xi) = 2\varkappa \sum_{k,\ j=1}^{N_0} \xi_k \xi_j \frac{\alpha_{kj}}{\nu_k + \nu_j}\, e^{(\nu_k + \nu_j)t}, \qquad t \ge 0 \tag{6.29}$$

is nonnegative and it is monotonely nondecreasing with respect to t (Hint: $a_t'(t, \xi) \ge 0$ and $a(t, \xi) \to 0$ as $t \to -\infty$).

Proof of Theorem 6.3.

Let h belong to the set A given by formula (6.27). Then by virtue of Lemma 6.1 we have

$$S_t h = \frac{w(x,\ t)}{\left(1 + 2\varkappa \int_0^t \|w(\tau)\|^2 \mathrm{d}\tau\right)^{1/2}}, \tag{6.30}$$

where

$$w(t) = \sqrt{\frac{2}{1 + a(0,\ \xi)}}\ e^{\frac{\rho}{2\nu}x} \sum_{k=1}^{N_0} \xi_k e^{\nu_k t} \sin \pi k x$$

and the value $a(t,\ \xi)$ is defined according to (6.29). Simple calculations show that

$$2\varkappa \int_0^t \|w(\tau)\|^2 \mathrm{d}\tau = \frac{a(t,\ \xi) - a(0,\ \xi)}{1 + a(0,\ \xi)}.$$

Therefore, it is easy to see that $S_t h = u(t)$ for $t \ge 0$, where $u(t)$ has the form (6.28). It follows that $S_t \mathrm{A} = \mathrm{A}$ and that a complete trajectory $u(t)$ lying in A has the form (6.28). In particular, this means that $\mathrm{A} \subset \mathscr{A}$. To prove that $\overline{\mathrm{A}} = \mathscr{A}$ it is sufficient to verify using Theorem 6.1 and the reduction principle (see Theorem 1.7.4) that for any element $h \in H_{N_0}$ there exists a semitrajectory $v(t) \subset \mathrm{A}$ such that

$$\lim_{t \to \infty} \|\partial_x (S_t h - v(t))\| = 0$$

uniformly with respect to h from any bounded set in H_{N_0}. To do this, it should be kept in mind that for

$$h = \sqrt{2} \sum_{k=1}^{N_0} \xi_k\, e^{\frac{\rho}{2\nu}x} \sin \pi k x \ \in\ H_{N_0} \tag{6.31}$$

$S_t h$ has the form (6.30) with

$$w(t) = \sqrt{2}\ e^{\frac{\rho}{2\nu}x} \sum_{k=1}^{N_0} \xi_k e^{\nu_k t} \sin \pi k x\,.$$

Therefore, it is easy to find that

$$S_t h = \frac{w(t)}{(1 + a(t,\ \xi) - a(0,\ \xi))^{1/2}},$$

where $a(t,\ \xi)$ is given by formula (6.29). Hence, if we choose

$$v(t) = \frac{w(t)}{(1 + a(t,\ \xi))^{1/2}} \ \in\ \mathrm{A}\,,$$

we find that

$$S_t h - v(t) = \psi(t, h) S_t h, \tag{6.32}$$

where

$$\psi(t, h) = 1 - \left(1 - \frac{a(0, \xi)}{1 + a(t, \xi)}\right)^{1/2}.$$

Using the obvious inequality

$$1 - (1-x)^{1/2} \le x, \quad 0 \le x \le 1,$$

we obtain that

$$\psi(t, h) \le \frac{a(0, \xi)}{1 + a(t, \xi)},$$

provided that h has the form (6.31). It is evident that

$$a(t, \xi) \ge a(0, \xi) + c_0 \int_0^t e^{2\nu_{N_0}\tau} d\tau \cdot \sum_{k=1}^{N_0} \xi_k^2 .$$

Therefore,

$$\psi(t, h) \le C\left(\int_0^t e^{2\nu_{N_0}\tau} d\tau\right)^{-1}.$$

Consequently, the dissipativity property of S_t in $H_0^1(0, 1)$ and equations (6.32) give us that

$$\|\partial_x(S_t h - v(t))\| \le C\left(\int_0^t e^{2\nu_{N_0}\tau} d\tau\right)^{-1}, \quad t \ge t_B$$

for all $h \in B \subset H_{N_0}$, where B is an arbitrary bounded set. Thus, $\overline{\mathrm{A}} = \mathscr{A}$ and therefore **Theorem 6.3 is proved**.

— Exercise *6.7* Show that the set A coincides with the unstable manifold $M_+(0)$ emanating from zero, provided that the hypotheses of Theorem 6.3 hold.

— Exercise *6.8* Show that the set $\mathrm{A} = M_+(0)$ from Theorem 6.3 can be described as follows:

$$\mathrm{A} = \left\{ v = \sqrt{2} \sum_{k=1}^{N_0} \eta_k e^{\frac{\rho}{2\nu}x} \sin \pi k x : \right.$$

$$\left. \left(2\varkappa \sum_{k,\, j=1}^{N_0} \eta_k \eta_j \frac{\alpha_{kj}}{\nu_k + \nu_j} < 1, \quad \eta \in \mathbb{R}^{N_0} \right) \right\} .$$

Therewith, the global attractor $\mathcal{A}$ has the form

$$\mathcal{A} = \left\{ v = \sqrt{2} \sum_{k=1}^{N_0} \eta_k \, e^{\frac{\rho}{2\nu}x} \sin \pi k x : \; 2\varkappa \sum_{k,\,j=1}^{N_0} \eta_k \eta_j \frac{\alpha_{kj}}{\nu_k + \nu_j} \le 1 \right\}.$$

— Exercise *6.9* Prove that the boundary

$$\partial \mathrm{A} = \left\{ v = \sqrt{2} \sum_{k=1}^{N_0} \eta_k \, e^{\frac{\rho}{2\nu}x} \sin \pi k x : \right.$$

$$\left. \left(2\varkappa \sum_{k,\,j=1}^{N_0} \eta_k \eta_j \frac{\alpha_{kj}}{\nu_k + \nu_j} = 1 , \quad \eta \in \mathbb{R}^{N_0} \right) \right\}$$

of the set A is a strictly invariant set.

— Exercise *6.10* Show that any trajectory γ lying in $\partial \mathrm{A}$ has the form $\{u(t),$ $t \in \mathbb{R}\}$, where

$$u(t) = \frac{\sum_{k=1}^{N_0} \eta_k \, e^{\nu_k t} \, e^{\frac{\rho}{2\nu}x} \sin \pi k x}{\sqrt{\varkappa} \left(\sum_{k,\,j=1}^{N_0} \eta_k \eta_j \frac{\alpha_{kj}}{\nu_k + \nu_j} \, e^{(\nu_k + \nu_j)t} \right)^{1/2}}, \qquad \eta \in \mathbb{R}^{N_0}.$$

— Exercise *6.11* Using the result of Exercise 6.10 find the unstable manifold $M_+(\bar{u}_k)$ emanating from the fixed point $\bar{u}_k$, $k = \pm 1, \pm 2, \ldots, \pm N_0$.

— Exercise *6.12* Find out for which pairs of fixed points $\{\bar{u}_k, \bar{u}_j\}$, $k, j =$ $= \pm 1, \pm 2, \ldots, \pm N_0$, there exists a heteroclinic trajectory connecting them, i.e. a complete trajectory $\{u_{kj}(t) : \; t \in \mathbb{R}\}$ such that

$$\bar{u}_k = \lim_{t \to -\infty} u_{kj}(t), \qquad \bar{u}_j = \lim_{t \to +\infty} u_{kj}(t).$$

— Exercise *6.13* Display graphically the global attractor $\mathcal{A}$ on the plane generated by the vectors $e_1 = e^{[\rho/(2\nu)]x} \sin \pi x$ and $e_2 = e^{[\rho/(2\nu)]x} \times$ $\times \sin 2\pi x$ for $N_0 = 2$.

Exercise 6.14 Study the structure of the global attractor of the dynamical system generated by the equation

$$\begin{cases} u_t - \nu u_{xx} + \left(\varkappa \int\limits_0^1 (|u(x,t)|^2 + |v(x,t)|^2)\,dx - \Gamma \right) u - \alpha v = 0, \quad 0 < x < 1, \ t > 0, \\ v_t - \nu v_{xx} + \left(\varkappa \int\limits_0^1 (|u(x,t)|^2 + |v(x,t)|^2)\,dx - \Gamma \right) v + \alpha u = 0, \quad 0 < x < 1, \ t > 0, \\ u|_{x=0} = u|_{x=1} = v|_{x=0} = v|_{x=1} = 0 \end{cases}$$

in $H_0^1(0, 1) \times H_0^1(0, 1)$, where ν, $\varkappa$, Γ, α are positive parameters.

§ 7 Simplified Model of Appearance of Turbulence in Fluid

In 1948 German mathematician E. Hopf suggested (see the references in [3]) to consider the following system of equations in order to illustrate one of the possible scenarios of the turbulence appearance in fluids:

$$\begin{cases} u_t = \mu u_{xx} - v * v - w * w - u * 1, & (7.1) \\ v_t = \mu v_{xx} + v * u + v * a + w * b, & (7.2) \\ w_t = \mu w_{xx} + w * u - v * b + w * a, & (7.3) \end{cases}$$

where the unknown functions u, v, and w are even and 2π-periodic with respect to x and

$$(f * g)(x) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(x-y)\, g(y)\, dy .$$

Here $a(x)$ and $b(x)$ are even 2π-periodic functions and μ is a positive constant. We also set the initial conditions

$$u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x), \quad w|_{t=0} = w_0(x). \tag{7.4}$$

As in the previous section the asymptotic behaviour of solutions to problem (7.1)–(7.4) can be explicitly described.

Let us introduce the necessary functional spaces. Let

$$H = \{ f \in L^2_{loc}(\mathbb{R}) : \ f(x) = f(-x) = f(x + 2\pi) \}.$$

Evidently H is a separable Hilbert space with the inner product and the norm defined by the formulae:

$$(f,\ g) = \int_0^{2\pi} f(x)\, g(x)\, \mathrm{d}x\,, \qquad \|f\|^2 = (f,\ f)$$

There is a natural orthonormal basis

$$\left\{\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos x,\ \frac{1}{\sqrt{\pi}}\cos 2x,\ \dots\right\}$$

in this space. The coefficients $C_n(f)$ of decomposition of the function $f \in H$ with respect to this basis have the form

$$C_0(f) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x)\, \mathrm{d}x\,, \qquad C_n(f) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos nx\, \mathrm{d}x\,.$$

— Exercise *7.1* Let $f,\ g \in H$. Then $f * g \in H$ and

$$\|f * g\| \le \frac{1}{\sqrt{2\pi}} \|f\| \cdot \|g\|\,. \tag{7.5}$$

The Fourier coefficients C_n of the functions $f * g$, f, and g obey the equations

$$C_0(f * g) = \frac{1}{\sqrt{2\pi}} C_0(f)\, C_0(g)\,, \qquad C_n(f * g) = \frac{1}{2\sqrt{\pi}} C_n(f)\, C_n(g)\,. \tag{7.6}$$

— Exercise *7.2* Let p_m be the orthoprojector onto the span of elements $\{\cos kx,\ k = 0,\ 1,\ \dots,\ m\}$ in H. Show that

$$p_m(f * g) = (p_m f) * g = f * (p_m g)\,. \tag{7.7}$$

Let us consider the Hilbert space $\mathbb{H} = H^3 \equiv H \times H \times H$ with the norm $\|(u;\ v;\ w)\|_{\mathbb{H}} = (\|u\|^2 + \|v\|^2 + \|w\|^2)^{1/2}$ as the phase space of problem (7.1)–(7.4). We define an operator A by the formula

$$A(u,\ v,\ w) = (u - \mu u_{xx};\ v - \mu v_{xx};\ w - \mu w_{xx})\,, \quad (u;\ v;\ w) \in D(A)$$

on the domain

$$D(A) = [H^2_{\mathrm{loc}}(\mathbb{R})]^3 \cap \mathbb{H}\,,$$

where $H^2_{\mathrm{loc}}(\mathbb{R})$ is the second order Sobolev space.

— Exercise *7.3* Prove that A is a positive operator with discrete spectrum. Its eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ have the form:

$$\lambda_{3k} = \lambda_{3k+1} = \lambda_{3k+2} = 1 + \mu k^2\,, \qquad k = 0,\ 1,\ 2,\ \dots\,,$$

while the corresponding eigenelements are defined by the formulae

$$\left.\begin{aligned} e_0 = \frac{1}{\sqrt{2\pi}}(1;\ 0;\ 0), \quad e_1 = \frac{1}{\sqrt{2\pi}}(0;\ 1;\ 0), \quad e_2 = \frac{1}{\sqrt{2\pi}}(0;\ 0;\ 1), \\ e_{3k} = \frac{1}{\sqrt{\pi}}(\cos kx;\ 0;\ 0), \quad e_{3k+1} = \frac{1}{\sqrt{\pi}}(0;\ \cos kx;\ 0), \\ e_{3k+2} = \frac{1}{\sqrt{\pi}}(0;\ 0;\ \cos kx), \end{aligned}\right\} \tag{7.8}$$

where $k = 1,\ 2,\ \dots$

Let

$$\begin{aligned} b_1(u,\ v,\ w) &= -v*v - w*w - u*1, \\ b_2(u,\ v,\ w) &= v*u + v*a + w*b, \\ b_3(u,\ v,\ w) &= w*u - v*b + w*a. \end{aligned}$$

Equation (7.5) implies that $b_j(u,\ v,\ w) \in H$, provided a, b, u, v, and w are the elements of the space H, $j = 1,\ 2,\ 3$. Therefore, the formula

$$B(u,\ v,\ w) = (b_1(u,\ v,\ w) + u;\ b_2(u,\ v,\ w) + v;\ b_3(u,\ v,\ w) + w)$$

gives a continuous mapping of the space $\mathbb{H}$ into itself.

Exercise *7.4* Prove that

$$\|B(y_1) - B(y_2)\|_{\mathbb{H}} \le C\Big(1 + \|a\| + \|b\| + \|y_1\|_{\mathbb{H}} + \|y_2\|_{\mathbb{H}}\Big)\|y_1 - y_2\|_{\mathbb{H}},$$

where $y_j = (u_j;\ v_j;\ w_j) \in \mathbb{H}$, $j = 1,\ 2$.

Thus, if $a,\ b \in H$, then problem (7.1)–(7.4) can be rewritten in the form

$$\frac{dy}{dt} + Ay = B(y), \qquad y|_{t=0} = y_0,$$

where A and B satisfy the hypotheses of Theorem 2.1. Therefore, the Cauchy problem (7.1)–(7.4) has a unique mild solution $y(t) = (u(t),\ v(t),\ w(t))$ in the space $\mathbb{H}$ on a segment $[0,\ T]$, provided that $a,\ b \in H$. In order to prove the global existence theorem we consider the Galerkin approximations of problem (7.1)–(7.4). The Galerkin approximate solution $y_m(t)$ of the order $3m$ with respect to basis (7.8) can be presented in the form

$$y_m(t) = \Big(u^{(m)}(t);\ v^{(m)}(t);\ w^{(m)}(t)\Big) = \sum_{k=0}^{m-1}\Big(u_k(t);\ v_k(t);\ w_k(t)\Big)\cos kx, \tag{7.9}$$

where $u_k(t)$, $v_k(t)$, and $w_k(t)$ are scalar functions. By virtue of equations (7.7) it is easy to check that the functions $u^{(m)}(t)$, $v^{(m)}(t)$, and $w^{(m)}(t)$ satisfy equations (7.1)–(7.3) and the initial conditions

$$u^{(m)}(0) = p_m u_0, \qquad v^{(m)}(0) = p_m v_0, \qquad w^{(m)}(0) = p_m w_0. \tag{7.10}$$

Thus, approximate solutions exist, locally at least. However, if we use (7.1)–(7.3) we can easily find that

$$\frac{1}{2}\cdot\frac{d}{dt}\left\{\|u^{(m)}(t)\|^2+\|v^{(m)}(t)\|^2+\|w^{(m)}(t)\|^2\right\} +$$

$$+\ \mu\left\{\left\|u_x^{(m)}\right\|^2+\left\|v_x^{(m)}\right\|^2+\left\|w_x^{(m)}\right\|^2\right\} =$$

$$= -(u^{(m)}*1,\ u^{(m)}) + (v^{(m)}*a,\ v^{(m)}) + (w^{(m)}*a,\ w^{(m)})\ .$$

Therefore, inequality (7.5) leads to the relation

$$\frac{d}{dt}\|y_m(t)\|_{\mathbb{H}}^2 \le C(1+\|a\|)\|y_m(t)\|_{\mathbb{H}}^2\ .$$

This implies the global existence of approximate solutions $y_m(t)$ (see Exercise 2.1). Therefore, Theorem 2.2 guarantees the existence of a mild solution to problem (7.1)–(7.4) in the space $\mathbb{H}=H^3$ on the time interval of any length T. Moreover, the mild solution $y(t)=(u(t);\ v(t);\ w(t))$ possesses the property

$$\max_{[0,\,T]}\|y(t)-y_m(t)\| \to 0\,,\qquad m\to\infty$$

for any segment $[0,\,T]$. Approximate solution $y_m(t)$ has the structure (7.9).

— Exercise *7.5* Show that the scalar functions $\{u_k(t);\ v_k(t);\ w_k(t)\}$ involved in (7.9) are solutions to the system of equations:

$$\begin{cases} \dot{u}_k+\mu k^2 u_k = -v_k^2-w_k^2-u_0\delta_{k0}\,, & (7.11)\\ \dot{v}_k+\mu k^2 v_k = v_k u_k+v_k a_k+w_k b_k\,, & (7.12)\\ \dot{w}_k+\mu k^2 w_k = w_k u_k-v_k b_k+w_k a_k\,. & (7.13)\end{cases}$$

Here $k=1,\ 2,\ \ldots\,,\ \delta_{k0}=0$ for $k\neq 0$, $\delta_{00}=1$, the numbers $a_k=$ $=C_k(a)$ and $b_k=C_k(b)$ are the Fourier coefficients of the functions $a(x)$ and $b(x)$.

Thus, equations (7.1)–(7.3) generate a dynamical system $(\mathbb{H},\ S_t)$ with the evolutionary operator S_t defined by the formula

$$S_t\,y_0 = (u(t);\ v(t);\ w(t))\,,$$

where $(u(t);\ v(t);\ w(t))$ is a mild solution (in $\mathbb{H}$) to the Cauchy problem (7.1)–(7.4), $y_0=(u_0,\ v_0,\ w_0)$. An interesting property of this system is given in the following exercise.

— Exercise *7.6* Let $\mathscr{L}_k$ be the span of elements $\{e_{3k},\ e_{3k+1},\ e_{3k+2}\}$, where $k=0,\ 1,\ 2,\ \ldots$ and $\{e_n\}$ are defined by equations (7.8). Then the subspace $\mathscr{L}_k$ of the phase space $\mathbb{H}$ is positively invariant with respect to S_t $(S_t\,\mathscr{L}_k\subset\mathscr{L}_k)$.

Therefore, the phase space $\mathbb{H}$ of the dynamical system $(\mathbb{H}, S_t)$ falls into the orthogonal sum

$$\mathbb{H} = \sum_{k=0}^{\infty} \oplus \mathscr{L}_k$$

of invariant subspaces. Evidently, the dynamics of the system $(\mathbb{H}, S_t)$ in the subspace $\mathscr{L}_k$ is completely determined by the system of three ordinary differential equations (7.11)–(7.13).

Lemma 7.1.

Assume that $a, b \in H$ *and*

$$C_0 = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} a(x)\mathrm{d}x < 0 . \tag{7.14}$$

Then the dynamical system $(\mathbb{H}, S_t)$ *is dissipative.*

Proof.

Let us fix N and then consider the initial conditions $y_0 = (u_0, v_0, w_0)$ from the subspace

$$\mathbb{H}_N = \sum_{k=0}^{N} \oplus \mathscr{L}_k = \sum_{k=0}^{N} \oplus \operatorname{Lin}\{e_{3k}, e_{3k+1}, e_{3k+2}\},$$

where $\{e_n\}$ are defined by equations (7.8). It is clear that $\mathbb{H}_N$ is positively invariant and the trajectory

$$y(t) = (u(t); v(t); w(t)) = \sum_{k=0}^{N} (u_k(t); v_k(t); w_k(t)) \cos kx$$

of the system is a function satisfying (7.1)–(7.4) in the classical sense. Let p_m be the orthoprojector in H onto the span of elements $\{\cos kx : n = 0, 1, \ldots, m\}$. We introduce a new variable $\tilde{u}(t) = u(t) + \alpha^m$ instead of the function $u(t)$. Here $\alpha^m = (p_m - p_0)a$. Equations (7.1)–(7.3) can be rewritten in the form

$$\begin{cases} \tilde{u}_t - \mu \tilde{u}_{xx} = -v*v - w*w + (-\tilde{u} + \alpha^m)*1 - \mu \alpha^m_{xx} , & (7.15) \\ v_t - \mu v_{xx} = v*\tilde{u} + v*(q_m a) + w*b + v*(p_0 a) , & (7.16) \\ w_t - \mu w_{xx} = w*\tilde{u} - v*b + w*(q_m a) + w*(p_0 a) , & (7.17) \end{cases}$$

where $q_m = 1 - p_m$. The properties of the convolution operation (see Exercises 7.1 and 7.2) enable us to show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{u}\|^2+\|v\|^2+\|w\|^2) + \mu(\|\tilde{u}_x\|^2+\|v_x\|^2+\|w_x\|^2) =$$
$$= ((-\tilde{u}+\alpha^m)*1,\ \tilde{u}) + \mu(\alpha_x^m,\ \tilde{u}_x) + (v*(q_m a),\ v) +$$
$$+ (w*(q_m a),\ w) + (v*(p_0 a),\ v) + (w*(p_0 a),\ w)\ . \tag{7.18}$$

It is clear that

$$((-\tilde{u}+\alpha^m)*1,\ \tilde{u}) = -[C_0(\tilde{u})]^2, \qquad (v*(p_0 a),\ v) = \frac{1}{\sqrt{2\pi}}C_0(a)[C_0(v)]^2,$$

where $C_0(f)$ is the zeroth Fourier coefficient of the function $f(x) \in H$. Moreover, the estimate

$$\|q_m v\| \le \|v_x\|, \qquad m \ge 1,$$

holds. We choose $m \ge 1$ such that $(1/\sqrt{2\pi})\|q_m a\| \le \mu/2$. Then equation (7.18) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}\|^2+\|v\|^2+\|w\|^2\right) + \mu\left(\|\tilde{u}_x\|^2+\|v_x\|^2+\|w_x\|^2\right) +$$
$$+ 2|C_0(\tilde{u})|^2 + \sqrt{\frac{2}{\pi}}|C_0(a)|\left(|C_0(v)|^2+|C_0(w)|^2\right) \le \mu\|\alpha_x^m\|^2.$$

If we use the inequality

$$\|f\|^2 \le |C_0(f)|^2 + \|f_x\|^2,$$

then we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\tilde{u}\|^2+\|v\|^2+\|w\|^2) + \nu(\|u\|^2+\|v\|^2+\|w\|^2) \le \mu\|\alpha_x^m\|^2,$$

where $\nu = \min(\mu,\ 2,\ \sqrt{2/\pi}|C_0(a)|)$. Consequently, the estimate

$$\|\tilde{y}(t)\|_{\mathbb{H}}^2 \le \|\tilde{y}(0)\|_{\mathbb{H}}^2 e^{-\nu t} + \frac{\mu}{\nu}(1-e^{-\nu t})\|\alpha_x^m\|^2 \tag{7.19}$$

is valid for $\tilde{y}(t) = (\tilde{u}(t),\ v(t),\ w(t))$, provided $\|q_m a\| \le \mu\sqrt{\pi/2}$ and $\tilde{y}_0 = y_0 + (\alpha^m,\ 0,\ 0)$ where $y_0 \in \mathbb{H}_N$. By passing to the limit we can extend inequality (7.19) over all the elements $y_0 \in \mathbb{H}$. Thus, the system $(\mathbb{H},\ S_t)$ possesses an absorbing set

$$\mathcal{B}_m = \{(u,\ v,\ w):\ \|u+(p_m-p_0)a\|^2+\|v\|^2+\|w\|^2 \le R_m^2\}, \tag{7.20}$$

where m is such that $\|a - p_m a\| \le \mu\sqrt{\pi/2}$ and

$$R_m^2 = \mu\|[(p_m-p_0)a]_x\|^2\left(\min\left(\mu,\ 2,\ \sqrt{\frac{2}{\pi}}|C_0(a)|\right)\right)^{-1} + 1\,.$$

Exercise *7.7* Show that the ball $\mathcal{B}_m$ defined by equality (7.20) is positively invariant.

Exercise 7.8 Consider the restriction of the dynamical system $(\mathbb{H}, S_t)$ to the subspace

$$\widetilde{\mathbb{H}} = \left\{ h(x) = (u(x);\ v(x);\ w(x)) \in \mathbb{H},\quad \int_0^{2\pi} h(x)\,dx = 0 \right\} = \sum_{n \ge 1} \mathscr{L}_n .$$

Show that $(\widetilde{\mathbb{H}}, S_t)$ is dissipative not depending on the validity of condition (7.14).

Lemma 7.2.

Assume that the hypotheses of Lemma 7.1 *hold and let p_m be the orthoprojector onto the span of elements $\{\cos kx:\ k = 0, 1, \dots, m\}$ in H, $q_m = 1 - p_m$. Then the estimate*

$$\|y_m(t)\|^2_{\mathbb{H}} \le \|y_0\|^2_{\mathbb{H}} \exp\left\{ -2\left((m+1)^2 \mu - \frac{\|q_m a\|}{\sqrt{2\pi}} \right) t \right\} \tag{7.21}$$

holds for all m such that $\|q_m a\| < \mu\sqrt{2\pi}(m+1)^2$. Here $y_m(t) = (q_m u(t);\ q_m v(t);\ q_m w(t))$ and $(u(t);\ v(t);\ w(t))$ is the mild solution to problem (7.1)–(7.4) *with the initial condition $y_0 = (u_0;\ v_0;\ w_0)$.*

Proof.

As in the proof of Lemma 7.1 we assume that $y_0 = (u_0;\ v_0;\ w_0) \in \mathbb{H}_N$ for some N. If we apply the projector q_m to equalities (7.15)–(7.17), then we get the equations

$$\begin{cases} u^m_t - \mu u^m_{xx} = -v^m * v^m - w^m * w^m , \\ v^m_t - \mu v^m_{xx} = v^m * u^m + w^m * b + v^m * q_m a , \\ w^m_t - \mu w^m_{xx} = w^m * u^m - v^m * b + w^m * q_m a , \end{cases}$$

where $u^m = q_m u$, $v^m = q_m v$ and $w^m = q_m w$. Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u^m\|^2 + \|v^m\|^2 + \|w^m\|^2 \right) + \mu\left(\|u^m_x\|^2 + \|v^m_x\|^2 + \|w^m_x\|^2 \right) \le$$

$$\le \frac{1}{\sqrt{2\pi}} \|q_m a\| \left(\|v^m\|^2 + \|w^m\|^2 \right) \tag{7.22}$$

as in the proof of the previous lemma. It is easy to check that $\|(q_m h)_x\|^2 \ge (m+1)^2 \|h\|^2$ for every $h \in p_N H$. Therefore, inequality (7.22) implies that

$$\frac{1}{2}\frac{d}{dt} \|y_m(t)\|^2_{\mathbb{H}} + \left(\mu (m+1)^2 - \frac{1}{\sqrt{2\pi}} \|q_m a\| \right) \|y_m(t)\|^2_{\mathbb{H}} \le 0 .$$

Hence, equation (7.21) is valid. Lemma 7.2 is proved.

Lemmata 7.1 and 7.2 enable us to prove the following assertion on the existence of the global attractor.

Theorem 7.1.

Let $a, b \in H$ ***and let condition*** (7.14) ***hold. Then the dynamical system*** $(\mathbb{H}, S_t)$ ***generated by the mild solutions to problem*** (7.1)–(7.4) ***possesses a global attractor*** $\mathcal{A}_\mu$. ***This attractor is a compact connected set. It lies in the finite-dimensional subspace***

$$\mathbb{H}_N = \sum_{k=0}^{N} \oplus \operatorname{Lin}\{e_{3k}, e_{3k+1}, e_{3k+1}\},$$

where the vectors $\{e_n\}$ ***are defined by equalities*** (7.8) ***and the parameter*** N ***is defined as the smallest number possessing the property*** $\|q_N a\| < \mu\sqrt{2\pi}(N+1)^2$. ***Here*** q_N ***is the orthoprojector onto the subspace generated by the elements*** $\{\cos nx : n \ge N+1\}$ ***in*** H.

To prove the theorem it is sufficient to note that the dynamical system is compact (see Lemma 4.1). Therefore, we can use Theorem 1.5.1. In particular, it should be noted that belonging of the attractor $\mathcal{A}_\mu$ to the subspace $\mathcal{H}_N$ means that $\dim_f \mathcal{A}_\mu \le 3(N+1)$, where N is an arbitrary number possessing the property $\|q_N a\| < \mu\sqrt{2\pi}(N+1)^2$. Below we describe the structure of the attractor and evaluate its dimension exactly.

According to Lemma 7.2 the subspace $\mathbb{H}_N$ is a uniformly exponentially attracting and positively invariant set. Therefore, by virtue of Theorem 1.7.4 it is sufficient to study the structure of the global attractor of the finite-dimensional dynamical system $(\mathbb{H}_N, S_t)$. To do that it is sufficient to study the qualitative behaviour of the trajectory in each invariant subspace $\mathcal{L}_k$, $0 \le k \le N$ (see Exercise 7.6). This behaviour is completely described by equations (7.11)–(7.13) which get transformed into system (1.6.4)–(1.6.6) studied before if we take $\mu = \mu k^2 + \delta_{k0}$, $\nu = \mu k^2 - a_k$, and $\beta = b_k$. Therefore, the results contained in Section 6 of Chapter 1 lead us to the following conclusion.

Theorem 7.2.

Let the hypotheses of Theorem 7.1 ***hold. Then the global minimal attractor*** $\mathcal{A}_{\min}^{(\mu)}$ ***of the dynamical system*** $(\mathbb{H}, S_t)$ ***generated by mild solutions to problem*** (7.1)–(7.4) ***has the form*** $\mathcal{A}_{\min}^{(\mu)} = \{0\} \cup \Sigma_\mu$, ***where***

$$\Sigma_\mu = \bigcup_{k \in J_\mu(a)} \bigcup_{0 \le \varphi < 2\pi} \left\{ (u_k;\ r_k\cos\varphi;\ r_k\sin\varphi)\frac{\cos kx}{\sqrt{\pi}} \right\}.$$

Here $u_k = \mu k^2 - a_k$, $r_k = k(a_k\mu - \mu^2 k^2)^{1/2}$, ***the values*** $a_k = C_k(a)$ ***are the***

Fourier coefficients of the function $a(x)$, ***and the number*** k ***ranges over the set of indices*** $J_\mu(a)$ ***such that*** $0 < \mu k^2 < a_k$. ***Topologically*** Σ_μ ***is a torus (i.e. a cross product of circumferences) of the dimension*** $\text{Card}\, J_\mu(a)$. ***The global attractor*** $\mathcal{A}_\mu$ ***of the system*** $(\mathbb{H}, S_t)$ ***can be obtained from*** $\mathcal{A}_{\min}^{(\mu)}$ ***by attaching the unstable manifold*** $M_+(0)$ ***emanating from the zero element of the space*** $\mathbb{H}$. ***Moreover,*** $\dim \mathcal{A}_\mu = 2\, \text{Card}\, J_\mu(a)$.

It should be noted that appearance of a limit invariant torus of high dimension possesssing the structure described in Theorem 7.2 is usually assosiated with the Landau-Hopf picture of turbulence appearance in fluids. Assume that the parameter μ gradually decreases. Then for some fixed choice of the function $a(x)$ the following picture is sequentially observed. If μ is large enough, then there exists only one attracting fixed point in the system. While μ decreases and passes some critical value μ_1, this fixed point loses its stability and an attracting limit cycle arises in the system. A subsequent decrease of μ leads to the appearance of a two-dimensional torus. It exists for some interval of values of μ: $\mu_3 < \mu < \mu_2 \ (< \mu_1)$. Then tori of higher dimensions arise sequentially. Therefore, the character of asymptotic behaviour of typical trajectories becomes more complicated as μ decreases. According to the Landau-Hopf scenario, movement along an infinite-dimensional torus corresponds to the turbulence.

§ 8 On Retarded Semilinear Parabolic Equations

In this section we show how the above-mentioned ideas can be used in the study of the asymptotic properties of dynamical systems generated by the retarded perturbations of problem (2.1). It should be noted that systems corresponding to ordinary retarded differential equations are quite well-studied (see, e. g., the book by J. Hale [4]). However, there are only occasional journal publications on the retarded partial differential equations. The exposition in this section is quite brief. We give the reader an opportunity to restore the missing details independently.

As before, let A be a positive operator with discrete spectrum in a separable Hilbert space H and let $C(a, b, \mathcal{F}_\theta)$ be the space of strongly continuous functions on the segment $[a, b]$ with the values in $\mathcal{F}_\theta = D(A^\theta)$, $\theta \ge 0$. Further we also use the notation $C_\theta = C(-r, 0; \mathcal{F}_\theta)$, where $r > 0$ is a fixed number (with the meaning of the delay time). It is clear that C_θ is a Banach space with the norm

$$|v|_{C_\theta} \equiv \max\left\{\|A^\theta v(\sigma)\| : \ \sigma \in [-r;\ 0]\right\} .$$

Let B be a (nonlinear) mapping of the space C_θ into H possessing the property

$$\|B(v_1) - B(v_2)\| \le M(R)|v_1 - v_2|_{C_\theta}, \qquad 0 \le \theta < 1, \tag{8.1}$$

for any $v_1, v_2 \in C_\theta$ such that $|v_j|_{C_\theta} \le R$, where $R > 0$ is an arbitrary number and $M(R)$ is a nondecreasing function. In the space H we consider a differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = B(u_t), \qquad t \ge t_0, \tag{8.2}$$

where u_t denotes the element from C_θ determined with the help of the function $u(t)$ by the equality

$$u_t(\sigma) = u(t + \sigma), \qquad \sigma \in [-r, 0].$$

We equip equation (8.2) with the initial condition

$$u_{t_0}(\sigma) = u(t_0 + \sigma) = v(\sigma), \qquad \sigma \in [-r, 0], \tag{8.3}$$

where v is an element from C_θ.

The simplest example of problem (8.2) and (8.3) is the Cauchy problem for the nonlinear retarded diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f_1(u(t, x)) - f_2(u(t-r, x)), \quad x \in \Omega, \ t > t_0, \\ u|_{\partial\Omega} = 0, \quad u(\sigma)|_{\sigma \in [t_0 - r, t_0]} = v(\sigma). \end{cases} \tag{8.4}$$

Here r is a positive parameter, $f_1(u)$ and $f_2(u)$ are the given scalar functions. As in the non-retarded case (see Section 2), we give the following definition.

A function $u(t) \in C(t_0 - r, t_0 + T; \mathcal{F}_\theta)$ is called a ***mild (in*** $\mathcal{F}_\theta$***) solution*** to problem (8.2) and (8.3) on the half-interval $[t_0, t_0 + T)$ if (8.3) holds and $u(t)$ satisfies the integral equation

$$u(t) = e^{-(t - t_0)A} v(0) + \int_{t_0}^{t} e^{-(t - t_0)A} B(u_\tau) \mathrm{d}\tau. \tag{8.5}$$

The following analogue of Theorem 2.1 on the local solvability of problem (8.2) and (8.3) holds.

Theorem 8.1.

Assume that (8.1) ***holds. Then for any initial condition*** $v \in C_\theta$ ***there exists*** $T > 0$ ***such that problem*** (8.2) ***and*** (8.3) ***has a unique mild solution on the half-interval*** $[t_0, t_0 + T)$.

Proof.

As in Section 2, we use the fixed point method. For the sake of simplicity we consider the case $t_0 = 0$ (for arbitrary $t_0 \in \mathbb{R}$ the reasoning is similar). In the space $C_\theta(0, T) \equiv C(0, T; \mathcal{F}_\theta)$ we consider a ball

$$\mathcal{B}_\rho = \{w \in C_\theta(0, T): \ |w - \hat{v}|_{C_\theta(0, T)} \le \rho\},$$

where $\hat{v}(t) = \exp\{-At\}v(0)$ and $v(\sigma) \in C_\theta$ is the initial condition for problem (8.2) and (8.3). We also use the notation

$$|w|_{C_\theta(0, T)} \equiv \max\{\|A^\theta w(t)\| : \ t \in [0, T]\}.$$

We consider the mapping K from $C_\theta(0, T)$ into itself defined by the formula

$$[Kw](t) = \hat{v}(t) + \int_0^t e^{-(t-\tau)A} B(w_\tau) d\tau .$$

Here we assume that $w(\sigma) = v(\sigma)$ for $\sigma \in [-r, 0)$. Using the estimate (see Exercise 1.23)

$$\|A^\theta e^{-sA}\| \le \left(\frac{\theta}{es}\right)^\theta, \qquad s > 0, \tag{8.6}$$

(for $\theta = 0$ we suppose $\theta^\theta = 1$) we have that

$$\|A^\theta (Kw_1(t) - Kw_2(t))\| \le \int_0^t \left(\frac{\theta}{e(t-\tau)}\right)^\theta \|B(w_{1,\tau}) - B(w_{2,\tau})\| d\tau . \tag{8.7}$$

If $w \in \mathcal{B}_\rho$, then

$$\max_{[0, T]} \|A^\theta w(t)\| \le \rho + \|A^\theta v(0)\| .$$

This easily implies that

$$|w_\tau|_{C_\theta} \le \rho + |v|_{C_\theta}, \qquad \tau \in [0, T],$$

where, as above, $w_\tau \in C_\theta$ is defined by the formula

$$w_\tau(\sigma) = w(\tau + \sigma), \qquad \sigma \in [-r, 0].$$

Hence, estimate (8.1) for $w_j \in \mathcal{B}_\rho$ gives us that

$$\|B(w_{1,\tau}) - B(w_{2,\tau})\| \le M(\rho + |v|_{C_\theta}) |w_{1,\tau} - w_{2,\tau}|_{C_\theta} .$$

Since $w_1(\sigma) = w_2(\sigma)$ for $\sigma \in [-r, 0)$, the last estimate can be rewritten in the form

$$\|B(w_{1,\tau}) - B(w_{2,\tau})\| \le M(\rho + |v|_{C_\theta}) |w_1 - w_2|_{C_\theta(0, T)}$$

for $\tau \in [0, T]$. Therefore, (8.7) implies that

$$|Kw_1 - Kw_2|_{C_\theta(0, T)} \le \frac{1}{1-\theta}\left(\frac{\theta}{e}\right)^\theta T^{1-\theta} M(\rho + |v|_{C_\theta}) |w_1 - w_2|_{C_\theta(0, T)},$$

if $w_j \in \mathcal{B}_\rho$, $j = 1, 2$. Similarly, we have that

$$|Kw - \hat{v}|_{C_\theta(0, T)} \le \frac{1}{1-\theta}\left(\frac{\theta}{e}\right)^\theta T^{1-\theta} \left\{\|B(0)\| + M(\rho + |v|_{C_\theta})(\rho + |v|_{C_\theta})\right\}$$

for $w \in \mathcal{B}_\rho$. These two inequalities enable us to choose $T = T(\theta, \rho, |v|_{C_\theta}) > 0$ such that K is a contractive mapping of $\mathcal{B}_\rho$ into itself. Consequently, there exists a unique fixed point $w(t) \in C_\theta(0, T)$ of the mapping K. The structure of the operator K implies that $w(+0) = [Kw](+0) = v(0)$. Therefore, the function

$$u(t) = \begin{cases} w(t), & t \in [0, T], \\ v(t), & t \in [-r, 0], \end{cases}$$

lies in $C(-r, T; \mathcal{F}_\theta)$ and is a mild solution to problem (8.2), (8.3) on the segment $[0, T]$. Thus, **Theorem 8.1 is proved**.

In many aspects the theory of retarded equations of the type (8.2) is similar to the corresponding reasonings related to the problem without delay (see (2.1)). The exercises below partially confirm that.

— Exercise *8.1* Prove the assertions similar to the ones in Exercises 2.1–2.5 and in Theorem 2.2.

— Exercise *8.2* Assume that the constant $M(R)$ in (8.1) does not depend on R. Prove that problem (8.2) and (8.3) has a unique mild solution on $[t_0, \infty)$, provided $v(\sigma) \in C_\theta$. Moreover, for any pair of solutions $u_1(t)$ and $u_2(t)$ the estimate

$$\|u_1(t) - u_2(t)\|_\theta \le a_1 e^{a_2(t-t_0)} |v_1 - v_2|_{C_\theta} \tag{8.8}$$

is valid, where $v_j(\sigma)$ is the initial condition for $u_j(t)$ (see (8.3)).

For the sake of simplicity from now on we restrict ourselves to the case when the mapping B has the form

$$B(v) = B_0(v(0)) + B_1(v), \qquad v = v(\sigma) \in C_{1/2}, \tag{8.9}$$

where $B_0(\cdot)$ is a continuous mapping from $\mathcal{F}_{1/2} \equiv D(A^{1/2})$ into H, $B_1(\cdot)$ continuously maps $C_{1/2}$ into H and possesses the property $B_1(0) = 0$. We also assume that $B_0(\cdot)$ is a potential operator, i.e. there exists a continuously Frechét differentiable function $F(u)$ on $\mathcal{F}_{1/2}$ such that $B_0(u) = -F'(u)$. We require that

$$F(u) \ge -\alpha, \qquad (F'(u), u) - \beta F(u) \ge \gamma \|u\|^2 - \delta \tag{8.10}$$

for all $u \in \mathcal{F}_{1/2}$, where α, β, γ, and δ are real parameters, β and γ are positive (cf. Section 2). As to the retarded term $B_1(v)$, we consider the uniform estimate

$$\|B_1(v_1) - B_2(v_2)\| \le M \int_{-r}^{0} \|A^{1/2}(v_1(\sigma) - v_2(\sigma))\| d\sigma \tag{8.11}$$

to be valid. Here M is an absolute constant and $v_j(\sigma) \in C \equiv C_{1/2}$.

— Exercise *8.3* Assume that conditions (8.9)–(8.11) hold. Then problem (8.2) and (8.3) has a unique mild solution (in $\mathcal{F}_{1/2}$) on any segment $[t_0,\ t_0+T]$ for every initial condition v from $C = C_{1/2}$.

Therefore, we can define an evolutionary operator S_t acting in the space $C \equiv C_{1/2}$ by the formula

$$(S_t v)(\sigma) = u_t(\sigma) \equiv u(t+\sigma), \qquad \sigma \in [-r,\ 0], \tag{8.12}$$

where $u(t)$ is a mild solution to problem (8.2) and (8.3).

— Exercise *8.4* Prove that the operator S_t given by formula (8.12) satisfies the semigroup property: $S_t \circ S_\tau = S_{t+\tau}$, $S_0 = I$, $t,\ \tau \ge 0$ and the pair $(C_{1/2},\ S_t)$ is a dynamical system.

Theorem 8.2.

Let conditions (8.9)–(8.11) ***and*** (8.1) ***with*** $\theta = 1/2$ ***hold. Assume that the parameters in*** (8.10) ***and*** (8.11) ***satisfy the equation***

$$\frac{r^2}{2\gamma}(2+\gamma)M^2 \exp\left\{r \cdot \min(2,\ \gamma,\ \beta)\right\} \le \min(2,\ \gamma,\ \beta)\,.$$

Then the dynamical system $(C,\ S_t)$ ***generated by equality*** (8.12) ***is a dissipative compact system.***

Proof.

We reason in the same way as in the proof of Theorem 4.3. Using the Galerkin approximations it is easy to find that a solution to problem (8.2) and (8.3) satisfies the equalities

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|A^{1/2}u(t)\|^2 + (F'(u(t)),\ u(t)) = (B_1(u_t),\ u(t))$$

and

$$\frac{1}{2}\frac{d}{dt}(\|A^{1/2}u(t)\|^2 + 2F(u(t))) + \|\dot{u}(t)\|^2 = (B_1(u_t),\ \dot{u}(t))\ .$$

If we add these two equations and use (8.10), then we get that

$$\frac{d}{dt}V(u(t)) + \|A^{1/2}u(t)\|^2 + \gamma\|u(t)\|^2 + \|\dot{u}(t)\|^2 + \beta F(u(t)) \le$$
$$\le \delta + \|B_1(u_t)\|(\|u(t)\| + \|\dot{u}(t)\|)\,, \tag{8.13}$$

where

$$V(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|A^{1/2}u\|^2 + F(u) + \alpha\ . \tag{8.14}$$

Using (8.11) it is easy to find that there exists a constant $D_0 > 0$ such that

$$\frac{d}{dt}V(u(t)) + \omega_1 V(u(t)) \le D_0 + \frac{1}{2}\omega_2 \int_{t-r}^{t} \|A^{1/2}u(\tau)\|^2 d\tau ,$$

where

$$\omega_1 = \min(2, \gamma, \beta), \qquad \omega_2 = \frac{r}{2}\Big(1+\frac{2}{\gamma}\Big)M^2 .$$

Consequently, the inequality

$$\dot{\psi}(t) + \omega_1 \psi(t) \le D_0 + \omega_2 \int_{t-r}^{t} \psi(\tau) d\tau$$

is valid for $\psi(t) = V(u(t))$. Therefore, we have

$$\dot{\varphi}(t) \le D_0 e^{\omega_1 t} + \omega_2 e^{\omega_1 r} \int_{t-r}^{t} \varphi(\tau) d\tau$$

for the function

$$\varphi(t) = e^{\omega_1 t}\psi(t) = e^{\omega_1 t} V(u(t)) .$$

If we integrate this inequality from 0 to t, then we obtain

$$\varphi(t) \le \varphi(0) + \frac{D_0}{\omega_1}(e^{\omega_1 t} - 1) + \omega_2 e^{\omega_1 r} r \int_{-r}^{t} \varphi(\tau) d\tau .$$

Therefore, Gronwall's lemma gives us that

$$V(u(t)) \le (V(u(0)) + C_0 |v|^2_{C_{1/2}}) e^{-\omega_3 t} + C_1 ,$$

provided that

$$\omega_2 e^{\omega_1 r} r < \omega_1 .$$

Here C_1 and C_2 are positive numbers, $\omega_3 = \omega_1 - \omega_2 e^{\omega_1 r} r > 0$. This implies the dissipativity of the dynamical system (C, S_t). In order to prove its compactness we note that the reasoning similar to the one in the proof of Lemma 4.1 enables us to prove the existence of the absorbing set $\mathcal{B}_\theta$ which is a bounded subset of the space $C_\theta = C(-r, 0; D(A^\theta))$ for $1/2 < \theta < 1$. Using the equality (cf. (8.5))

$$u(t) = e^{-(t-s)A} u(s) + \int_{s}^{t} e^{-(t-\tau)A} B(u_\tau) d\tau$$

for $t \ge s$ large enough, we can show that the equation

$$\|A^{1/2}(u(t) - u(s))\| \le C|t-s|^\beta$$

holds for the solution $u(t)$ in the absorbing set $\mathcal{B}_\theta$. Here the constant C depends on $\mathcal{B}_\theta$ and the parameters of the problem only, $\beta > 0$. This circumstance enables us to prove the existence of a compact absorbing set for the dynamical system $(C,\ S_t)$. **Theorem 8.2 is proved**.

Theorem 8.2 and the results of Chapter 1 enable us to prove the following assertion on the attractor of problem (8.2) and (8.3).

Theorem 8.3.

Assume that the hypotheses of Theorem 8.2 *hold. Then the dynamical system* $(C,\ S_t)$ *possesses a compact connected global attractor* $\mathscr{A}$ *which is a bounded set in the space* $C_\theta = C(-r,\ 0;\ \mathscr{F}_\theta)$ *for each* $\theta < 1$.

It should be noted that the finite dimensionality of this attractor can be proved in some cases.

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Chapter 3

Inertial Manifolds

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If an infinite-dimensional dynamical system possesses a global attractor of finite dimension (see the definitions in Chapter 1), then there is, at least theoretically, a possibility to reduce the study of its asymptotic regimes to the investigation of properties of a finite-dimensional system. However, as the structure of attractor cannot be described in details for the most interesting cases, the constructive investigation of this finite-dimensional system cannot be carried out. In this respect some ideas related to the method of integral manifolds and to the reduction principle are very useful. They have led to appearance and intensive use of the concept of inertial manifold of an infinite-dimensional dynamical system (see [1]–[8] and the references therein). This manifold is a finite-dimensional invariant surface, it contains a global attractor and attracts trajectories exponentially fast. Moreover, there is a possibility to reduce the study of limit regimes of the original infinite-dimensional system to solving of a similar problem for a class of ordinary differential equations.

In this chapter we present one of the approaches to the construction of inertial manifolds (IM) for an evolutionary equation of the type:

$$\frac{du}{dt} + Au = B(u, t), \qquad u\big|_{t=0} = u_0. \tag{0.1}$$

Here $u(t)$ is a function of the real variable t with the values in a separable Hilbert space $\mathbf{H}$. We pay the main attention to the case when A is a positive linear operator with discrete spectrum and $B(u, t)$ is a nonlinear mapping of $\mathbf{H}$ subordinated to A in some sense. The approach used here for the construction of inertial manifolds is based on a variant of the Lyapunov-Perron method presented in the paper [2]. Other approaches can be found in [1], [3]–[7], [9], and [10]. However, it should be noted that all the methods for the construction of IM known at present time require a quite strong condition on the spectrum of the operator A: the difference $\lambda_{N+1} - \lambda_N$ of two neighbouring eigenvalues of the operator A should grow sufficiently fast as $N \to \infty$.

§ 1 Basic Equation and Concept of Inertial Manifold

In a separable Hilbert space H we consider a Cauchy problem of the type

$$\frac{du}{dt} + Au = B(u, t), \qquad t > s, \quad u\big|_{t=s} = u_0, \quad s \in \mathbb{R}, \tag{1.1}$$

where A is a positive operator with discrete spectrum (for the definition see Section 1 of Chapter 2) and $B(\cdot, \cdot)$ is a nonlinear continuous mapping from $D(A^\theta) \times \mathbb{R}$

into H, $0 \le \theta < 1$, possessing the properties

$$\|B(u,\ t)\| \le M(1 + \|A^{\theta}u\|) \tag{1.2}$$

and

$$\|B(u_1,\ t) - B(u_2,\ t)\| \le M\|A^{\theta}(u_1 - u_2)\| \tag{1.3}$$

for all u, u_1, and u_2 from the domain $\mathcal{F}_{\theta} = D(A^{\theta})$ of the operator A^{θ}. Here M is a positive constant independent of t and $\|\cdot\|$ is the norm in the space H. Further it is assumed that $\{e_k\}$ is the orthonormal basis in H consisting of the eigenfunctions of the operator A:

$$A e_k = \lambda_k e_k, \qquad 0 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{k \to \infty} \lambda_k = \infty .$$

Theorem 2.3 of Chapter 2 implies that for any initial condition $u_0 \in \mathcal{F}_{\theta}$ problem (1.1) has a unique mild (in $\mathcal{F}_{\theta}$) solution $u(t)$ on every half-interval $[s,\ s+T)$, i.e. there exists a unique function $u(t) \in C(s,\ s+T;\ \mathcal{F}_{\theta})$ which satisfies the integral equation

$$u(t) = e^{-(t-s)A}u_0 + \int_s^t e^{-(t-\tau)A}B(u(\tau),\ \tau)\,d\tau \tag{1.4}$$

for all $t \in [s,\ s+T)$. This solution possesses the property (see (2.6) in Chapter 2)

$$\|A^{\beta}(u(t+\sigma) - u(t))\| \le C\sigma^{\theta-\beta}, \qquad 0 \le \beta \le \theta$$

for $0 < \sigma < 1$ and $t > s$. Moreover, for any pair of mild solutions $u_1(t)$ and $u_2(t)$ to problem (1.1) the following inequalities hold (see (2.2.15)):

$$\|A^{\theta}u(t)\| \le a_1 e^{a_2(t-s)}\|A^{\theta}u(s)\|, \qquad t \ge s \tag{1.5}$$

and (cf. (2.2.18))

$$\|Q_N A^{\theta}u(t)\| \le \left\{e^{-\lambda_{N+1}(t-s)} + M(1+k)a_1\,\lambda_{N+1}^{-1+\theta}\,e^{a_2(t-s)}\right\}\|A^{\theta}u(s)\|, \tag{1.6}$$

where $u(t) = u_1(t) - u_2(t)$, a_1 and a_2 are positive numbers depending on θ, λ_1, and M only. Hereinafter $Q_N = I - P_N$, where P_N is the orthoprojector onto the first N eigenvectors of the operator A. Moreover, we use the notation

$$k = \theta^{\theta}\int_0^{\infty}\xi^{-\theta}e^{-\xi}\,d\xi \quad \text{for } \theta > 0 \quad \text{and} \quad k = 0 \quad \text{for } \theta = 0 . \tag{1.7}$$

Further we will also use the following so-called dichotomy estimates proved in Lemma 1.1 of Chapter 2:

$$\|A^{\theta}e^{-tA}P_N\| \le \lambda_N^{\theta}\,e^{\lambda_N|t|}, \qquad t \in \mathbb{R};$$

$$\|e^{-tA}Q_N\| \le e^{-\lambda_{N+1}t}, \qquad t \ge 0; \tag{1.8}$$

$$\left\|A^{\theta} e^{-tA} Q_N\right\| \le \left[(\theta/t)^{\theta} + \lambda_{N+1}^{\theta}\right] e^{-\lambda_{N+1} t}, \qquad t > 0, \quad \theta > 0.$$

The ***inertial manifold*** (IM) of problem (1.1) is a collection of surfaces $\{\mathbf{M}_t,\ t \in \mathbb{R}\}$ in H of the form

$$\mathbf{M}_t = \{p + \Phi(p,\ t) :\ p \in P_N H,\ \ \Phi(p,\ t) \in (1 - P_N)\mathcal{F}_{\theta}\},$$

where $\Phi(p,\ t)$ is a mapping from $P_N H \times \mathbb{R}$ into $(1 - P_N)\mathcal{F}_{\theta}$ satisfying the Lipschitz condition

$$\left\|A^{\theta}(\Phi(p_1,\ t) - \Phi(p_2,\ t))\right\| \le C\left\|A^{\theta}(p_1 - p_2)\right\| \tag{1.9}$$

with the constant C independent of p_j and t. We also require the fulfillment of the invariance condition (if $u_0 \in \mathbf{M}_s$, then the solution $u(t)$ to problem (1.1) possesses the property $u(t) \in \mathbf{M}_t$, $t \ge s$) and the condition of the uniform exponential attraction of bounded sets: there exists $\gamma > 0$ such that for any bounded set $B \subset H$ there exist numbers C_B and $t_B > s$ such that

$$\sup\left\{\operatorname{dist}_{\mathcal{F}_{\theta}}(u(t,\ u_0),\ \mathbf{M}_t) :\ u_0 \in B\right\} \le C_B\, e^{-\gamma(t - t_B)}$$

for all $t \ge t_B$. Here $u(t,\ u_0)$ is a mild solution to problem (1.1).

From the point of view of applications the existence of an inertial manifold (IM) means that a regular separation of fast (in the subspace $(I - P_N)H$) and slow (in the subspace $P_N H$) motions is possible. Moreover, the subspace of slow motions turns out to be finite-dimensional. It should be noted in advance that such separation is not unique. However, if the global attractor exists, then every IM contains it.

When constructing IM we usually use the methods developed in the theory of integral manifolds for central and central-unstable cases (see [11], [12]).

If the inertial manifold exists, then it continuously depends on t, i.e.

$$\lim_{t \to s}\left\|A^{\theta}(\Phi(p,\ s) - \Phi(p,\ t))\right\| = 0$$

for any $p \in P_N H$ and $s \in \mathbb{R}$. Indeed, let $u(t)$ be the solution to problem (1.1) with $u_0 = p + \Phi(p,\ s)$, $p \in P_N H$. Then $u(t) \in \mathbf{M}_t$ for $t \ge s$ and hence

$$u(t) = P_N u(t) + \Phi(P_N u(t),\ t).$$

Therefore,

$$\Phi(p,\ t) - \Phi(p,\ s) = \left[\Phi(p,\ t) - \Phi(P_N u(t),\ t)\right] + $$
$$+ \left[u(t) - u_0\right] + \left[p - P_N u(t)\right].$$

Consequently, Lipschitz condition (1.9) leads to the estimate

$$\left\|A^{\theta}(\Phi(p,\ s) - \Phi(p,\ t))\right\| \le C\left\|A^{\theta}(u(t) - u_0)\right\|.$$

Since $u(t) \in C(s,\ +\infty,\ D(A^{\theta}))$, this estimate gives us the required continuity property of $\Phi(p,\ t)$.

Exercise *1.1* Prove that the estimate

$$\|A^{\beta}(\Phi(p,\ t+\sigma)-\Phi(p,\ t))\| \le C_{\beta}(p,\ N)\sigma^{\theta-\beta}$$

holds for $\Phi(p,\ t)$ when $0\le\sigma\le 1$, $0\le\beta\le\theta$, $t\in\mathbb{R}$.

The notion of the inertial manifold is closely related to the notion of the ***inertial form***. If we rewrite the solution $u(t)$ in the form $u(t)=p(t)+q(t)$, where $p(t)=P_N u(t)$, $q(t)=Q_N u(t)$, and $Q_N=I-P_N$, then equation (1.1) can be rewritten as a system of two equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}p(t)+Ap(t) = P_N B(p(t)+q(t)) , \\ \dfrac{\mathrm{d}}{\mathrm{d}t}q(t)+Aq(t) = Q_N B(p(t)+q(t)) , \\ p\big|_{t=s}=p_0\equiv P_N u_0, \quad q\big|_{t=s}=q_0\equiv Q_N u_0 . \end{cases}$$

By virtue of the invariance property of IM the condition $(p_0,\ q_0)\in\mathbf{M}_s$ implies that $(p(t),\ q(t))\in\mathbf{M}_t$, i.e. the equality $q_0=\Phi(p_0,\ s)$ implies that $q(t)=\Phi(p(t),\ t)$. Therefore, if we know the function $\Phi(p,\ t)$ that gives IM, then the solution $u(t)$ lying in $\mathbf{M}_t$ can be found in two stages: at first we solve the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t)+Ap(t) = P_N B(p(t)+\Phi(p(t),\ t)), \qquad p\big|_{t=s}=p_0, \tag{1.10}$$

and then we take $u(t)=p(t)+\Phi(p(t),\ t)$. Thus, the qualitative behaviour of solutions $u(t)$ lying in IM is completely determined by the properties of differential equation (1.10) in the finite-dimensional space $P_N H$. Equation (1.10) is said to be the inertial form (IF) of problem (1.1). In the autonomous case ($B(u,\ t)\equiv B(u)$) one can use the attraction property for IM and the reduction principle (see Theorem 7.4 of Chapter 1) in order to state that the finite-dimensional IF completely determines the asymptotic behaviour of the dynamical system generated by problem (1.1).

Exercise *1.2* Let $\Phi(p,\ t)$ give the inertial manifold for problem (1.1). Show that IF (1.10) is uniquely solvable on the whole real axis, i.e. there exists a unique function $p(t)\in C(-\infty,\ \infty;\ P_N H)$ such that equation (1.10) holds.

Exercise *1.3* Let $p(t)$ be a solution to IF (1.10) defined for all $t\in\mathbb{R}$. Prove that $u(t)=p(t)+\Phi(p(t),\ t)$ is a mild solution to problem (1.1) defined on the whole time axis and such that $u\big|_{t=s}=p_0+\Phi(p_0,\ t)$.

Exercise *1.4* Use the results of Exercises 1.2 and 1.3 to show that if IM $\{\mathbf{M}_t\}$ exists, then it is strictly invariant, i.e. for any $u\in\mathbf{M}_t$ and $s<t$ there exists $u_0\in\mathbf{M}_t$ such that $u=u(t)$ is a solution to problem (1.1).

In the sections to follow the construction of IM is based on a version of the Lyapunov-Perron method presented in the paper by Chow-Lu [2]. This method is based on the following simple fact.

Lemma 1.1.

Let $f(t)$ be a continuous function on $\mathbb{R}$ with the values in H such that

$$\|Q_N f(t)\| \le C, \qquad t \in \mathbb{R}.$$

Then for the mild solution $u(t)$ (on the whole axis) to equation

$$\frac{d}{dt} u + A u = f(t) \tag{1.11}$$

to be bounded in the subspace $Q_N \mathcal{F}_\theta$ it is necessary and sufficient that

$$u(t) = e^{-(t-s)A} p + \int_s^t e^{-(t-\tau)A} P_N f(\tau)\, d\tau + \int_{-\infty}^t e^{-(t-\tau)A} Q_N f(\tau)\, d\tau \tag{1.12}$$

for $t \in \mathbb{R}$, where p is an element from $P_N H$ and s is an arbitrary real number.

We note that the solution to problem (1.11) on the whole axis is a function $u(t) \in C(\mathbb{R}, H)$ satisfying the equation

$$u(t) = e^{-(t-s)A} u(s) + \int_s^t e^{-(t-\tau)A} f(\tau)\, d\tau$$

for any $s \in \mathbb{R}$.

Proof.

It is easy to prove (do it yourself) that equation (1.12) gives a mild solution to (1.11) with the required property of boundedness. Vice versa, let $u(t)$ be a solution to equation (1.11) such that $\|Q_N u(t)\|_\theta$ is bounded. Then the function $q(t) = Q_N u(t)$ is a bounded solution to equation

$$\frac{d}{dt} q(t) + A q(t) = Q_N f(t).$$

Consequently, Lemma 2.1.2 implies that

$$q(t) = \int_{-\infty}^t e^{-(t-\tau)A} Q_N f(\tau)\, d\tau.$$

Therefore, in order to prove (1.12) it is sufficient to use the constant variation formula for a solution to the finite-dimensional equation

$$\frac{dp}{dt} + Ap = P_N f(t), \qquad p(t) = P_N u(t).$$

Thus, Lemma 1.1 is proved.

Lemma 1.1 enables us to obtain an equation to determine the function $\Phi(p, t)$. Indeed, let us assume that $B(u, t)$ is bounded and there exists $\mathbf{M}_t$ with the function $\Phi(p, t)$ possessing the property $\|A^\theta \Phi(p, t)\| \le C$ for all $p \in P_N H$ and $t \in \mathbb{R}$. Then the solution to problem (1.1) lying in $\mathbf{M}_t$ has the form

$$u(t) = p(t) + \Phi(p(t), t).$$

It is bounded in the subspace $Q_N H$ and therefore it satisfies the equation of the form

$$u(t) = e^{-(t-s)A} p + \\ + \int_s^t e^{-(t-\tau)A} P_N B(u(\tau), \tau) d\tau + \int_{-\infty}^t e^{-(t-\tau)A} Q_N B(u(\tau), \tau) d\tau \,, \quad (t \in \mathbb{R}). \quad (1.13)$$

Moreover,

$$\Phi(p, s) = Q_N u(s) = \int_{-\infty}^s e^{-(s-\tau)A} Q_N B(u(\tau), \tau) d\tau \,. \quad (1.14)$$

Actually it is this fact that forms the core of the Lyapunov-Perron method. It is proved below that under some conditions (i) integral equation (1.13) is uniquely solvable for any $p \in P_N H$ and (ii) the function $\Phi(p, s)$ defined by equality (1.14) gives IM.

In the construction of IM with the help of the Lyapunov-Perron method an important role is also played by the results given in the following exercises.

— Exercise *1.5* Assume that $\sup\{e^{-\gamma(s-t)}\|f(t)\|:\ t < s\} < \infty$, where γ is any number from the interval $(\lambda_N, \lambda_{N+1})$ and $s \in \mathbb{R}$. Let $u(t)$ be a mild solution (on the whole axis) to equation (1.11). Show that $u(t)$ possesses the property

$$\sup_{t<s}\{e^{-\gamma(s-t)}\|A^\theta u(t)\|\} < \infty$$

if and only if equation (1.12) holds for $t < s$.
Hint: consider the new unknown function

$$w(t) = e^{\gamma(t-s)} u(t)$$

instead of $u(t)$.

— Exercise *1.6* Assume that $f(t)$ is a continuous function on the semiaxis $[s, +\infty)$ with the values in H such that for some γ from the interval $(\lambda_N, \lambda_{N+1})$ the equation

$$\sup\{e^{-\gamma(s-t)}\|f(t)\|:\ t \in [s, +\infty)\} < \infty$$

holds. Prove that for a mild solution $u(t)$ to equation (1.11) on the semiaxis $[s, +\infty)$ to possess the property

$$\sup\{e^{-\gamma(s-t)}\|A^\theta u(t)\|:\ t\in[s,+\infty)\}<\infty$$

it is necessary and sufficient that

$$u(t) = e^{-(t-s)A}q + \int_s^t e^{-(t-\tau)A}Q_N f(\tau)\,\mathrm{d}\tau - \\ - \int_t^{+\infty} e^{-(t-\tau)A}P_N f(\tau)\,\mathrm{d}\tau \ , \tag{1.15}$$

where $t \ge s$ and q is an element of $Q_N D(A^\theta)$. *Hint*: see the hint to Exercise 1.5.

§ 2 Integral Equation for Determination of Inertial Manifold

In this section we study the solvability and the properties of solutions to a class of integral equations which contains equation (1.13) as a limit case. Broader treatment of the equation of the type (1.13) is useful in connection with some problems of the approximation theory for IM.

For $s \in \mathbb{R}$ and $0 < L \le \infty$ we define the space $C_s \equiv C_{\gamma,\,\theta}(s-L,\, s)$ as the set of continuous functions $v(t)$ on the segment $[s-L,\, s]$ with the values in $D(A^\theta)$ and such that

$$|v|_s \equiv \sup_{t\in[s-L,\,s]}\{e^{-\gamma(s-t)}\|A^\theta u(t)\|\} < \infty .$$

Here γ is a positive number. In this space we consider the integral equation

$$v(t) = \mathrm{B}_p^{s,\,L}[v](t), \qquad s-L \le t \le s , \tag{2.1}$$

where

$$\mathrm{B}_p^{s,L}[v](t) = e^{-(t-s)A}p - \int_t^s e^{-(t-\tau)A}PB(v(\tau),\,\tau)\,\mathrm{d}\tau \ + \\ + \int_{s-L}^t e^{-(t-\tau)A}QB(v(\tau),\,\tau)\,\mathrm{d}\tau \ .$$

Hereinafter the index N of the projectors P_N and Q_N is omitted, i.e. P is the orthoprojector onto $\mathrm{Lin}\{e_1,\,\dots,\,e_N\}$ and $Q = 1-P$. It should be noted that the most significant case for the construction of IM is when $L=\infty$.

Chapter 3

Lemma 2.1.

Let at least one of two conditions be fulfilled:

$$0 < L < \infty \quad \text{and} \quad M\left(\frac{\theta^\theta}{1-\theta} L^{1-\theta} + L\lambda_{N+1}^\theta\right) \le q < 1 \tag{2.2}$$

or $0 < L \le \infty$ *and*

$$\lambda_{N+1} - \lambda_N \ge \frac{2M}{q}((1+k)\lambda_{N+1}^\theta + \lambda_N^\theta), \quad 0 < q < 1, \tag{2.3}$$

where k *is defined by equation* (1.7). *Then for any fixed* $s \in \mathbb{R}$ *there exists a unique function* $v_s(t;\, p) \in C_s$ *satisfying equation* (2.1) *for all* $t \in [s-L,\, s]$, *where* γ *is an arbitrary number from the segment* $[\lambda_N,\, \lambda_{N+1}]$ *in the case of* (2.2) *and* $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$ *in the case of* (2.3). *Moreover,*

$$|v(\,\cdot\,;\; p_1) - v(\,\cdot\,;\; p_1)|_s \le (1-q)^{-1}\|A^\theta(p_1 - p_2)\| \tag{2.4}$$

and

$$|v_s|_s \le (1-q)^{-1}\{D_1 + \|A^\theta p\|\}, \tag{2.5}$$

where

$$D_1 = M(1+k)\lambda_{N+1}^{-1+\theta} + M\lambda_N^{-1+\theta}. \tag{2.6}$$

Proof.

Let us apply the fixed point method to equation (2.1). Using (1.8) it is easy to check (similar estimates are given in Chapter 2) that

$$\left\|A^\theta(\mathrm{B}_{p_1}^{s,\,L}(v_1)(t) - \mathrm{B}_{p_2}^{s,\,L}(v_2)(t))\right\| \le$$

$$\le e^{\lambda_N(s-t)}\|A^\theta(p_1 - p_2)\| + \int_t^s \lambda_N^\theta e^{\lambda_N(\tau - t)} M\|v_1(\tau) - v_2(\tau)\|_\theta \, d\tau +$$

$$+ \int_{s-L}^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(t-\tau)} M\|v_1(\tau) - v_2(\tau)\|_\theta \, d\tau \le$$

$$\le e^{\lambda_N(s-t)}\|A^\theta(p_1 - p_2)\| + (q_1(s,\, t) + q_2(s,\, t))\, e^{\gamma(s-t)}|v_1 - v_2|_s \,,$$

where

$$q_1(s,\, t) = M \int_{s-L}^t \left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-(\lambda_{N+1} - \gamma)(t-\tau)} \, d\tau \tag{2.7}$$

and

$$q_2(s, t) = M \int_t^s \lambda_N^\theta e^{(\lambda_N - \gamma)(\tau - t)} d\tau . \qquad (2.8)$$

Therefore, if the estimate

$$q_1(s, t) + q_2(s, t) \le q , \qquad s - L \le t \le s \qquad (2.9)$$

holds, then

$$\left| \mathbf{B}_{p_1}^{s, L}[v_1] - \mathbf{B}_{p_2}^{s, L}[v_2] \right|_s \le \left\| A^\theta(p_1 - p_2) \right\| + q \left| v_1 - v_2 \right|_s . \qquad (2.10)$$

Let us estimate the values $q_1(s, t)$ and $q_2(s, t)$. Assume that (2.2) is fulfilled. Then it is evident that

$$q_1(s, t) \le M\theta^\theta \int_{s-L}^t (t-\tau)^{-\theta} d\tau + M\lambda_{N+1}^\theta (t - s + L) =$$

$$= M\frac{\theta^\theta}{1-\theta}(t - s + L)^{1-\theta} + M\lambda_{N+1}^\theta (t - s + L)$$

and

$$q_2(s, t) \le M\lambda_N^\theta (s - t) \le M\lambda_{N+1}^\theta (s - t)$$

for $\lambda_N \le \gamma \le \lambda_{N+1}$. Therefore,

$$q_1(s, t) + q_2(s, t) \le M\left(\frac{\theta^\theta}{1-\theta}(t - s + L)^{1-\theta} + \lambda_{N+1}^\theta L\right).$$

Consequently, equation (2.2) implies (2.9). Now let the spectral condition (2.3) be fulfilled. Then

$$q_1(s, t) \le \int_{-\infty}^t \frac{M\theta^\theta}{(t-\tau)^\theta} e^{-(\lambda_{N+1} - \gamma)(t - \tau)} d\tau + \frac{M\lambda_{N+1}^\theta}{\lambda_{N+1} - \gamma}$$

for all $\gamma < \lambda_{N+1}$. We change the variable in integration $\xi = (\lambda_{N+1} - \gamma)(t - \tau)$ and find that

$$q_1(s, t) \le \frac{Mk}{(\lambda_{N+1} - \gamma)^{1-\theta}} + \frac{M\lambda_{N+1}^\theta}{\lambda_{N+1} - \gamma} ,$$

where the constant k is defined by (1.7). It is also evident that

$$q_2(s, t) \le \frac{M\lambda_N^\theta}{\gamma - \lambda_N}$$

provided that $\gamma > \lambda_N$. Equation (2.3) implies that $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$ lies in the interval $(\lambda_N, \lambda_{N+1})$. If we choose the parameter γ in such way, then we get

$$q_1(s,\ t)+q_2(s,\ t)\ \le\ \frac{M(1+k)\lambda_{N+1}^{\theta}}{\lambda_{N+1}-\lambda_N-\frac{2M}{q}\lambda_N^{\theta}}+\frac{q}{2}.$$

Hence, equation (2.3) implies (2.9). Therefore, estimate (2.10) is valid, provided that the hypotheses of the lemma hold. Moreover, similar reasoning enables us to show that

$$\left|\mathrm{B}_p^{s,\,L}[v]\right|_s\ \le\ D_1+\|A^{\theta}p\|+q|v|_s\,, \tag{2.11}$$

where D_1 is defined by formula (2.6). In particular, estimates (2.10) and (2.11) mean that when s, L, and p are fixed, the operator $\mathrm{B}_p^{s,\,L}$ maps C_s into itself and is contractive. Therefore, there exists a unique fixed point $v_s(t,\ p)$. Evidently it possesses properties (2.4) and (2.5). Lemma 2.1 is proved.

Lemma 2.1 enables us to define a collection of manifolds $\{\mathrm{M}_s^L\}$ by the formula

$$\mathrm{M}_s^L=\{p+\Phi^L(p,\ s):\ p\in PH\},$$

where

$$\Phi^L(p,\ s)=\int\limits_{s-L}^{s}e^{-(s-\tau)A}QB(v(\tau),\ \tau)\,\mathrm{d}\tau\ \equiv\ Qv(s;\ p). \tag{2.12}$$

Here $v(t)=v(t;\ p)$ is the solution to integral equation (2.1). Some properties of the manifolds $\{\mathrm{M}_s^L\}$ and the function $\Phi^L(p,\ s)$ are given in the following assertion.

Theorem 2.1.

Assume that at least one of two conditions (2.2) ***and*** (2.3) ***is satisfied. Then the mapping*** $\Phi^L(\cdot,\ s)$ ***from*** PH ***into*** QH ***possesses the properties***

a) $$\|A^{\theta}\Phi^L(p,\ s)\|\ \le\ D_2+q(1-q)^{-1}\{D_1+\|A^{\theta}p\|\} \tag{2.13}$$

for any $p\in PH$, ***hereinafter*** D_1 ***is defined by formula*** (2.6) ***and***

$$D_2=M(1+k)\lambda_{N+1}^{-1+\theta}\ ; \tag{2.14}$$

b) ***the manifold*** M_s^L ***is a Lipschitzian surface and***

$$\left\|A^{\theta}\Phi^L(p_1,\ s)-\Phi^L(p_2,\ s)\right\|\ \le\ \frac{q}{1-q}\left\|A^{\theta}(p_1-p_2)\right\| \tag{2.15}$$

for all $p_1,\ p_2\in PH$ ***and*** $s\in\mathbb{R}$;

c) ***if*** $u(t)\equiv u(t,\ s;\ p+\Phi_s^L(p))$ ***is the solution to problem*** (1.1) ***with the initial data*** $u_0=p+\Phi^L(p,\ s)$, $p\in PH$, ***then*** $Qu(t)=\Phi^L(Pu(t),\ t)$ ***for*** $L=\infty$. ***In case of*** $L<\infty$ ***the inequality***

$$\|A^{\theta}(Qu(t)-\Phi^L(Pu(t),\ t))\|\ \le$$

$$\le\ D_2(1-q)^{-1}e^{-\gamma L}+q(1-q)^{-2}e^{-\gamma(t-s)}\{D_1+\|A^{\theta}p\|\} \tag{2.16}$$

holds for all $s \le t \le s+L$***, where*** γ ***is an arbitrary number from the segment*** $[\lambda_N, \lambda_{N+1}]$ ***if*** (2.2) ***is fulfilled and*** $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$ ***when*** (2.3) ***is fulfilled;***

d) ***if*** $B(u, t) \equiv B(u)$ ***does not depend on*** t***, then*** $\Phi^L(p, s) \equiv \Phi^L(p)$***, i.e.*** $\Phi^L(p, t)$ ***is independent of*** t.

Proof.

Equations (2.12) and (1.8) imply that

$$\|A^\theta \Phi^L(p, s)\| \le M \int_{s-L}^{s} \left[\left(\frac{\theta}{s-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(s-\tau)} (1 + \|A^\theta v(\tau)\|)\,\mathrm{d}\tau \le$$

$$\le M \int_{s-L}^{s} \left[\left(\frac{\theta}{s-\tau}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1}(s-\tau)}\,\mathrm{d}\tau + q_1(s, s)|v|_s .$$

By virtue of (2.9) we have that $q_1(s, s) < q$. Therefore, when we change the variable in integration $\xi = \lambda_{N+1}(s-\tau)$ with the help of equation (2.5) we obtain (2.13). Similarly, using (2.4) and (1.8) one can prove property (2.15).

Let us prove assertion (c). We fix $t_0 \in [s, s+L]$ and assume that $w(t)$ is a function on the segment $[s, s+L]$ such that $w(t) = u(t)$ for $t \in [s, t_0]$ and $w(t) = v_s(t)$ for $t \in [s-L, s]$. Here $v_s(t)$ is the solution to integral equation (2.1). Using equations (1.4) and (2.1) we obtain that

$$w(t) = e^{-(t-s)A}(p + \Phi^L(p, s)) + \int_s^t e^{-(t-\tau)A} B(w(\tau), \tau)\,\mathrm{d}\tau =$$

$$= e^{-(t-s)A} p + \int_s^t e^{-(t-\tau)A} PB(w(\tau), \tau)\,\mathrm{d}\tau + \int_{s-L}^t e^{-(t-\tau)A} QB(w(\tau), \tau)\,\mathrm{d}\tau \quad (2.17)$$

for $s \le t \le t_0$. Evidently, equation (2.17) also remains true for $t \in [s-L, s]$. Equation (1.4) gives us that

$$p = e^{-(s-t_0)A} p(t_0) + \int_{t_0}^{s} e^{-(s-\tau)A} PB(w(\tau), \tau)\,\mathrm{d}\tau .$$

Therefore, the substitution in (2.17) gives us that

$$w(t) = \mathrm{B}_{p(t_0)}^{t_0, L}[w](t) + b_L(t_0, s; t) \quad (2.18)$$

for all $t \in [t_0-L, t_0]$, where $p(t) = Pu(t)$ and

$$b_L(t_0, s; t) = \int_{s-L}^{t_0-L} e^{-(t-\tau)A} QB(v_s(\tau), \tau)\,\mathrm{d}\tau . \quad (2.19)$$

In particular, if $L = \infty$ equation (2.18) turns into equation (2.1) with $s = t_0$ and $p = p(t_0)$. Therefore, equation (2.12) implies the invariance property $Qu(t_0) = \Phi^\infty(Pu(t_0),\ t_0)$. Let us estimate the value (2.19). If we reason in the same way as in the proof of Lemma 2.1, then we obtain that

$$\|A^\theta b_L(t_0,\ s;\ t)\| \le e^{-(t-t_0+L)\lambda_{N+1}}\left\{q_1^*(s,\ t_0-L) + q_1(s,\ t_0-L)\, e^{\gamma(s-t_0+L)}|v_s|_s\right\},$$

where $q_1(s,\ t)$ is defined by formula (2.7) and

$$q_1^*(s,\ t) = M\int\limits_{s-L}^{t}\left(\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right)e^{-\lambda_{N+1}(t-\tau)}d\tau. \tag{2.20}$$

Therefore, simple calculations give us that

$$\|A^\theta b_L(t_0,\ s;\ t)\| \le e^{-(t-t_0+L)\lambda_{N+1}}\left\{D_2 + e^{\gamma(s-t_0+L)}q|v_s|_s\right\}, \tag{2.21}$$

where D_2 is defined by formula (2.14). Let $v_{t_0}(t)$ be the solution to integral equation (2.1) for $s = t_0$ and $p = Pu(t_0)$. Then using (2.12), (2.18), and (2.1) we find that

$$Qu(t_0) - \Phi^L(Pu(t_0),\ t_0) = Q(w(t_0) - v_{t_0}(t_0)). \tag{2.22}$$

However, for all $t \in [t_0-L,\ t_0]$ we have that

$$w(t) - v_{t_0}(t) = \mathrm{B}_{p(t_0)}^{t_0,\ L}[w](t) - \mathrm{B}_{p(t_0)}^{t_0,\ L}[v_{t_0}](t) + b_L(t_0,\ s;\ t).$$

Therefore, the contractibility property of the operator $\mathrm{B}_p^{t_0,\ L}$ gives us that

$$(1-q)|w - v_{t_0}|_{t_0} \le \sup_{t\in[t_0-L,\ t_0]}\left\{e^{-\gamma(t_0-t)}\|A^\theta b_L(t_0,\ s;\ t)\|\right\}.$$

Hence, it follows from (2.21) and (2.22) that

$$\|A^\theta(Qu(t_0)) - \Phi^L(Pu(t_0),\ t_0)\| \le \|A^\theta(w(t_0) - v_{t_0}(t_0))\| \le$$

$$\le |w - v_{t_0}|_{t_0} \le (1-q)^{-1}\left\{e^{-\gamma L}D_2 + q\, e^{-\gamma(t_0-s)}|v_s|_s\right\}.$$

This and equation (2.5) imply (2.16). Therefore, assertion (c) is proved.

In order to prove assertion (d) it should be kept in mind that if $\mathcal{B}(u,\ t) \equiv \mathcal{B}(u)$, then the structure of the operator $\mathrm{B}_p^{s,\ L}$ enables us to state that

$$\mathrm{B}_p^{s,\ L}[v](t-h) = \mathrm{B}_p^{s+h,\ L}[v_h](t)$$

for $s+h-L \le t \le s+h$, where $v_h(t) = v(t-h)$. Therefore, if $v(t) \in C_{\gamma,\ \theta}(s-L,\ s)$ is a solution to integral equation (2.1), then the function

$$v_h(t) \equiv v(t-h) \in C_{\gamma,\ \theta}(s+h-L,\ s+h)$$

is its solution when $s+h$ is written instead of s. Consequently, equation (2.12) gives us that

$$\Phi^L(p,\ s+h) = Qv_h(s+h) = Qv(s) = \Phi^L(p,\ s)\ .$$

Thus, **Theorem 2.1 is proved**.

— Exercise *2.1* Show that if $\|B(u,\ t)\| \le M$, then inequalities (2.13) and (2.16) can be replaced by the relations

$$\|A^\theta \Phi^L(p,\ s)\| \le D_2\,, \tag{2.23}$$

$$\|A^\theta(Qu(t) - \Phi^L(Pu((t),\ t)))\| \le D_2(1-q)^{-1}e^{-\gamma L}\,, \tag{2.24}$$

where D_2 is defined by formula (2.14).

§ 3 Existence and Properties of Inertial Manifolds

In particular, assertion (c) of Theorem 2.1 shows that if the spectral gap condition

$$\lambda_{N+1} - \lambda_N \ge \frac{2M}{q}((1+k)\lambda_{N+1}^\theta + \lambda_N^\theta)\,, \qquad 0 < q < 1\,, \tag{3.1}$$

is fulfilled, then the collection of surfaces

$$\mathbf{M}_s = \{p + \Phi(p,\ s):\ p \in PH\}\,, \qquad s \in \mathbb{R}\,, \tag{3.2}$$

is invariant, i.e.

$$U(t,\ s)\mathbf{M}_s \subset \mathbf{M}_t\,, \qquad -\infty < s \le t < \infty\,. \tag{3.3}$$

Here $\Phi(p,\ s) = \Phi^\infty(p,\ s)$ is defined by formula (2.12) for $L = \infty$ and $U(t,\ s)$ is the evolutionary operator corresponding to problem (1.1). It is defined by the formula $U(t,\ s)u_0 = u(t)$, where $u(t)$ is a mild solution to problem (1.1).

In this section we show that collection (3.2) possesses the property of exponential uniform attraction. Hence, $\{\mathbf{M}_t\}$ is an inertial manifold for problem (1.1). Moreover, Theorem 3.1 below states that $\{\mathbf{M}_t\}$ is an ***exponentially asymptotically complete*** IM, i.e. for any solution $u(t) = U(t,\ s)u_0$ there exists a solution $\tilde{u}(t) = U(t,\ s)\tilde{u}_0$ lying in the manifold $(\tilde{u}(t) \in \mathbf{M}_t,\ t \ge s)$ such that

$$\|A^\theta(u(t) - \tilde{u}(t))\| \le Ce^{-\eta(t-s)}\,, \qquad \eta > 0\,, \quad t > s\,.$$

In this case the solution $\tilde{u}(t)$ is said to be an ***induced trajectory*** for $u(t)$ on the manifold $\mathbf{M}_t$. In particular, the existence of induced trajectories means that the solution to original infinite-dimensional problem (1.1) can be naturally associated with the solution to the system of ordinary differential equations (1.10).

Theorem 3.1.

Assume that spectral gap condition (3.1) ***is valid for some*** $q < 2 - \sqrt{2}$. ***Then the manifold*** $\{\mathbf{M}_s,\ s \in \mathbb{R}\}$ ***given by formula*** (3.2) ***is inertial for problem*** (1.1). ***Moreover, for any solution*** $u(t) = U(t,\ s)u_0$ ***there exists an induced trajectory*** $u^*(t) = U(t,\ s)u_0^*$ ***such that*** $u^*(t) \in \mathbf{M}_t$ ***for*** $t \ge s$ ***and***

$$\|A^\theta(u(t) - u^*(t))\| \ \le\ \frac{2(1-q)}{(2-q)^2 - 2}\, e^{-\gamma(t-s)} \|A^\theta(Qu_0 - \Phi(P_0u,\ s))\|\ , \qquad (3.4)$$

where $\gamma = \lambda_N + \frac{2M}{q}\lambda_N^\theta$ ***and*** $t \ge s$.

Proof.

Obviously it is sufficient just to prove the existence of an induced trajectory $u^*(t) \in \mathbf{M}_t$ possessing property (3.4). Let $u(t)$ be a mild solution to problem (1.1), $u(t) = U(t,\ s)u_0$. We construct the induced trajectory $u^*(t) = U(t,\ s)u_0^*$ for $u(t)$ in the form $u^*(t) = u(t) + w(t)$, where $w(t)$ lies in the space $C_s^+ \equiv C_{s,\ \gamma}(s,\ +\infty,\ D(A^\theta))$ of continuous functions on the semiaxis $[s,\ +\infty)$ such that

$$|w|_{s,\ +} \equiv \sup_{t \ge s} \{e^{\gamma(t-s)}\|A^\theta w(t)\|\} < \infty\,, \qquad (3.5)$$

where $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$. We introduce the notation

$$F(w,\ t) = B(u(t) + w,\ t) - B(u(t)) \qquad (3.6)$$

and consider the integral equation (cf. (1.15))

$$w(t) \ =\ \mathbf{B}_s^+[w](t) \equiv e^{-(t-s)A}q(w) + \int_s^t e^{-(t-\tau)A}QF(w(\tau),\ \tau)\,d\tau\ -$$

$$-\int_t^{+\infty} e^{-(t-\tau)A}PF(w(\tau),\ \tau)\,d\tau\,, \qquad t \in [s,\ +\infty)\ , \qquad (3.7)$$

in the space C_s^+. Here the value $q(w) \in QD(A^\theta)$ is chosen from the condition

$$u^*(s) = u(s) + w(s) \in \mathbf{M}_s\,,$$

i.e. such that

$$Qu_0 + Qw(s) = \Phi(Pu_0 + Pw(s),\ s)\,.$$

Therefore, by virtue of (3.7) we have

$$q(w) \ =\ -Qu_0 + \Phi\left(Pu_0 - \int_s^{+\infty} e^{-(s-\tau)A}PF(w(\tau),\ \tau)\,d\tau,\ s\right). \qquad (3.8)$$

Thus, in order to prove inequality (3.4) it is sufficient to prove the solvability of integral equation (3.7) in the space C_s^+ and to obtain the estimate of the solution. The preparatory steps for doing this are hidden in the following exercises.

Exercise 3.1 Assume that $F(w,\ t)$ has the form (3.6). Show that for any

$$w(t),\ \overline{w}(t) \in C_s^+ = C_{s,\ \gamma}(s,\ +\infty;\ D(A^\theta))$$

and for $t \ge s$ the following inequalities hold:

$$\|F(w(t),\ t)\| \le e^{-\gamma(t-s)} M |w|_{s,\ +}\ , \tag{3.9}$$

$$\|F(w(t),\ t) - F(\overline{w}(t),\ t)\| \le e^{-\gamma(t-s)} M |w - \overline{w}|_{s,\ +}\ . \tag{3.10}$$

Exercise 3.2 Using (1.8) prove that the equations

$$\left(\int\limits_t^{+\infty} \|A^\theta e^{-(t-\tau)A} P\| e^{-\gamma(\tau - s)} \mathrm{d}\tau \right) \le \frac{\lambda_N^\theta}{\gamma - \lambda_N} \cdot e^{-\gamma(t-s)}, \tag{3.11}$$

$$\left(\int\limits_s^t \|A^\theta e^{-(t-\tau)A} Q\|\, e^{-\gamma(\tau - s)} \mathrm{d}\tau \right) \le$$

$$\le \frac{k(\lambda_{N+1} - \gamma)^\theta + \lambda_{N+1}^\theta}{\lambda_{N+1} - \gamma} \cdot e^{-\gamma(t-s)} \tag{3.12}$$

hold for $\lambda_N < \gamma < \lambda_{N+1}$ and $t \ge s$. Here k is defined by formula (1.7).

Lemma 3.1.

Assume that spectral gap condition (3.1) *holds with* $q < 2 - \sqrt{2}$. *Then* B_s^+ *is a continuous contractive mapping of the space* C_s^+ *into itself. The unique fixed point* w *of this mapping satisfies the estimate*

$$|w|_{s,\ +} \le \frac{2(1-q)}{(2-q)^2 - 2} \left\| A^\theta (Q u_0 - \Phi(P u_0,\ s)) \right\|. \tag{3.13}$$

Proof.

If we use (3.7), then we find that

$$\left\| A^\theta \mathrm{B}_s^+[w](t) \right\| \le e^{-(t-s)\lambda_{N+1}} \|A^\theta q(w)\| +$$

$$+ \int\limits_s^t \|A^\theta e^{-(t-\tau)A} Q\| \|F(w(\tau),\ \tau)\| \mathrm{d}\tau + \int\limits_t^{+\infty} \|A^\theta e^{-(t-\tau)A} P\| \|F(w(\tau),\ \tau)\| \mathrm{d}\tau$$

for $t > s$. Therefore, (3.9), (3.11), and (3.12) give us that

$$\left\| A^\theta \mathrm{B}_s^+[w](t) \right\| \le e^{-(t-s)\lambda_{N+1}} \|A^\theta q(w)\| +$$

$$+ \left\{ \frac{\lambda_N^\theta}{\gamma - \lambda_N} + \frac{(1+k)\lambda_{N+1}^\theta}{\lambda_{N+1} - \gamma} \right\} M e^{-\gamma(t-s)} |w|_{s,\ +}\ .$$

Since $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$, spectral gap condition (3.1) implies that

$$\left\|A^\theta \mathbf{B}_s^+[w](t)\right\| \le e^{-(t-s)\lambda_{N+1}}\|A^\theta q(w)\| + q e^{-\gamma(t-s)}|w|_{s,+} . \tag{3.14}$$

Similarly with the help of (3.10)–(3.12) we have that

$$\left\|A^\theta(\mathbf{B}_s^+[w](t) - \mathbf{B}_s^+[\overline{w}](t))\right\| \le$$
$$\le e^{-(t-s)\lambda_{N+1}}\|A^\theta(q(w) - q(\overline{w}))\| + q e^{-\gamma(t-s)}|w - \overline{w}|_{s,+} \tag{3.15}$$

for any $w,\ \overline{w} \in C_s^+$. From equations (3.8), (3.9), and (2.15) we obtain that

$$\left\|A^\theta(q(w) + Qu_0 - \Phi(Pu_0,\ s))\right\| \le \frac{qM}{1-q}\int_s^{+\infty}\|A^\theta e^{-(s-\tau)A}P\|\, e^{-\gamma(\tau-s)}\mathrm{d}\tau \cdot |w|_{s,+} .$$

Therefore, (3.11) implies that

$$\|A^\theta q(w)\| \le \left\|A^\theta(Qu_0 - \Phi(Pu_0,\ s))\right\| + \frac{q^2}{2(1-q)}|w|_{s,+} .$$

Similarly we have that

$$\|A^\theta(q(w) - q(\overline{w}))\| \le \frac{q^2}{2(1-q)}|w - \overline{w}|_{s,+} . \tag{3.17}$$

It follows from (3.14)–(3.17) that

$$\left|\mathbf{B}_s^+[w]\right|_{s,+} \le \left\|A^\theta(Qu_0 - \Phi(Pu_0,\ s))\right\| + \frac{q}{2}\cdot\frac{2-q}{1-q}|w|_{s,+} , \tag{3.18}$$

$$\left|\mathbf{B}_s^+[w] - \mathbf{B}_s^+[\overline{w}]\right|_{s,+} \le \frac{q}{2}\cdot\frac{2-q}{1-q}|w - \overline{w}|_{s,+} .$$

Therefore, if $q < 2 - \sqrt{2}$, then the operator $\mathbf{B}_s^+$ is continuous and contractive in C_s^+. Estimate (3.13) of its fixed point follows from (3.18). Lemma 3.1 is proved.

In order to complete the proof of Theorem 3.1 we must prove that the function

$$u^*(t) = u(t) + w(t)$$

is a mild solution to problem (1.1) lying in $\{\mathbf{M}_t,\ t \ge s\}$ (here $w(t)$ is a solution to integral equation (3.7)). We can do that by using the result of Exercise 1.2, the invariance of the collection $\{\mathbf{M}_t\}$, and the fact that equality (3.8) is equivalent to the equation $u^*(s) \in \mathbf{M}_s$. **Theorem 3.1 is completely proved**.

Exercise *3.3* Show that if the hypotheses of Theorem 3.1 hold, then the induced trajectory $u^*(t)$ is uniquely defined in the following sense: if there exists a trajectory $u^{**}(t)$ such that $u^{**}(t) \in \mathbf{M}_t$ for $t \ge s$ and

$$\|A^\theta(u(t) - u^{**}(t))\| \le Ce^{-\gamma(t-s)}$$

with $\gamma \ge \lambda_N + \dfrac{2M}{q}\lambda_N^\theta$, then $u^{**}(t) \equiv u^*(t)$ for $t \ge s$.

The construction presented in the proof of Theorem 3.1 shows that in order to build the induced trajectory for a solution $u(t)$ with the exponential order of decrease γ given, it is necessary to have the information on the behaviour of the solution $u(t)$ for ***all*** values $t \ge s$. In this connection the following simple fact on the exponential closeness of the solution $u(t)$ to its projection $Pu(t) + \Phi(Pu(t),\ t)$ onto the manifold appears to be useful sometimes.

— Exercise *3.4* Show that if the hypotheses of Theorem 3.1 hold, then the estimate

$$\|A^\theta(Qu(t) - \Phi(Pu(t),\ t))\| \le$$
$$\le \frac{2}{(2-q)^2 - 2} e^{-\gamma(t-s)} \|A^\theta(Pu_0 - \Phi(Pu_0,\ t))\|$$

is valid for any solution $u(t)$ to problem (1.1). Here $\gamma = \lambda_N + (2M/q)\lambda_N^\theta$ and $t \ge s$ (*Hint*: add the value $\Phi((Pu^*(t),\ t) - Qu^*(t)) = 0$ to the expression under the norm sign in the left-hand side. Here $u^*(t)$ is the induced trajectory for $u(t)$).

It is evident that the inertial manifold $\{\mathrm{M}_t\}$ consists of the solutions $u(t)$ to problem (1.1) which are defined for all real t (see Exercises 1.3 and 1.4). These solutions can be characterized as follows.

Theorem 3.2.

Assume that spectral gap condition (3.1) ***holds with*** $q < 2 - \sqrt{2}$ ***and*** $\{\mathrm{M}_t\}$ ***is the inertial manifold for problem*** (1.1) ***constructed in Theorem*** 3.1. ***Then for a solution*** $u(t)$ ***to problem*** (1.1) ***defined for all*** $t \in \mathbb{R}$ ***to lie in the inertial manifold*** $(u(t) \in \mathrm{M}_t)$***, it is necessary and sufficient that***

$$|u|_s \equiv \sup\{e^{-\gamma(s-t)}\|A^\theta u(t)\| : -\infty < t \le s\} < \infty \tag{3.19}$$

for each $s \in \mathbb{R}$***, where*** $\gamma = \lambda_N + \frac{2M}{q}\lambda_N^\theta$.

Proof.

If $u(t) \in \mathrm{M}_t$, then $u(t) = Pu(t) + \Phi(Pu(t),\ t)$. Therefore, equation (2.13) implies that

$$\|A^\theta u(t)\| \le D_2 + \frac{qD_1}{1-q} + \frac{1}{1-q}\|A^\theta Pu(t)\|. \tag{3.20}$$

The function $p(t) = Pu(t)$ satisfies the equation

$$p(t) = e^{-(t-s)A}p(s) + \int_s^t e^{-(t-\tau)A} PB(u(\tau),\ \tau)\,d\tau$$

for all real t and s. Therefore, we have that

$$\|A^\theta p(t)\| \le e^{(s-t)\lambda_N}\|A^\theta p(s)\| + M\lambda_N^\theta \int_t^s e^{(\tau-t)\lambda_N}(1+\|A^\theta u(\tau)\|)\,d\tau$$

for $t \le s$. With the help of (3.20) we find that

$$\|A^\theta p(t)\| \le C(s, N, q)e^{(s-t)\lambda_N} + \frac{M\lambda_N^\theta}{1-q}\int_t^s e^{(\tau-t)\lambda_N}\|A^\theta p(\tau)\|\,d\tau$$

for $t \le s$, where

$$C(s, N, q) = \|A^\theta p(s)\| + M\Big(1 + D_2 + \frac{qD_1}{1-q}\Big)\frac{1}{\lambda_N^{1-\theta}}.$$

Hence, the inequality

$$\varphi(t) \le C(s, N, q) + \frac{M\lambda_N^\theta}{1-q}\int_t^s \varphi(\tau)\,d\tau$$

holds for the function $\varphi(t) = \|A^\theta p(t)\|\, e^{(t-s)\lambda_N}$ and $t \le s$. If we introduce the function $\psi(t) = \int_t^s \varphi(\tau)\,d\tau$, then the last inequlity can be rewritten in the form

$$\psi'(t) + \frac{M\lambda_N^\theta}{1-q}\psi(t) \ge -C(s, N, q), \qquad t \le s,$$

or

$$\frac{d}{dt}\left[\psi(t)\exp\left\{\frac{M\lambda_N^\theta}{1-q}\,t\right\}\right] \ge -C(s, N, q)\exp\left\{\frac{M\lambda_N^\theta}{1-q}\,t\right\}, \qquad t \le s.$$

After the integration over the segment $[t, s]$ and a simple transformation it is easy to obtain the estimate

$$\|A^\theta p(t)\| \le C(s, N, q)\exp\left\{\Big(\lambda_N + \frac{M\lambda_N^\theta}{1-q}\Big)(s-t)\right\}. \tag{3.21}$$

Obviously for $q < 2-\sqrt{2}$ we have that

$$\lambda_N + \frac{M\lambda_N^\theta}{1-q} < \gamma = \lambda_N + \frac{2M}{q}\lambda_N^\theta.$$

Therefore, equations (3.21) and (3.20) imply (3.19).

Vice versa, we assume that equation (3.19) holds for the solution $u(t)$. Then

$$\|B(u(t))\| \le e^{\gamma(s-t)}M(1+|u|_s), \qquad t \le s. \tag{3.22}$$

It is evident that $q(t) = e^{-\gamma(s-t)}Qu(t)$ is a bounded (on $(-\infty, s]$) solution to the equation

$$\frac{dq}{dt} + (A-\gamma)q = F(t),$$

where $F(t) = \exp\{-\gamma(s-t)\}\, QB(u(t))$. By virtue of (3.22) the function $F(t)$ is bounded in QH. It is also clear that $A_\gamma = A-\gamma$ is a positive operator with discrete spectrum in QH. Therefore, Lemma 1.1 is applicable. It gives

$$Qu(t) = \int_{-\infty}^{t} e^{-(t-\tau)A}\, QB(u(\tau))\, d\tau .$$

Using the equation for $Pu(t)$ it is now easy to find that

$$u(t) = \mathrm{B}_p^{s,\,\infty}[u](t), \qquad t \le s ,$$

where $p = Pu(s)$ and $\mathrm{B}_p^{s,\,\infty}[u]$ is the integral operator similar to the one in (2.1). Hence, we have that $Qu(s) = \Phi(Pu(s),\, s)$ accoring to definition (2.12) of the function $\Phi(p,\, s) = \Phi^\infty(p,\, s)$. Thus, **Theorem 3.2 is proved**.

The following assertion shows that IM $\mathrm{M}_s \equiv \mathrm{M}_s^\infty$ can be approximated by the manifolds $\{\mathrm{M}_s^L\}$, $L < \infty$, with the exponential accuracy (see (2.12)).

Theorem 3.3.

Assume that spectral gap condition (3.1) ***is fulfilled with*** $q < 1$. ***We also assume that the function*** $\Phi^L(p,\, s)$ ***is defined by equality*** (2.12) ***for*** $0 < L \le \infty$. ***Then the estimate***

$$\left\| A^\theta(\Phi^{L_1}(p,\, s) - \Phi^{L_2}(p,\, s))\right\| \le$$

$$\le D_2(1-q)^{-1}\, e^{-\gamma_N L_{\min}} + \frac{1+q}{2(1-q)^2}\{D_1 + \|A^\theta p\|\}\, e^{-\delta_N L_{\min}} , \tag{3.23}$$

is valid with $L_{\min} = \min(L_1,\, L_2)$, $0 < L_1,\ L_2 \le \infty$; ***the constants*** D_1 ***and*** D_2 ***are defined by equations*** (2.6) ***and*** (2.14);

$$\gamma_N = \lambda_N + \frac{2M}{q}\lambda_N^\theta , \qquad \delta_N = \frac{2M(1-q)}{q(1+q)}\lambda_N^\theta .$$

Proof.

Let $0 < L_1 < L_2 < \infty$. Definition (2.12) implies that

$$\Phi^{L_1}(p,\, s) - \Phi^{L_2}(p,\, s) = Q(v_1(s) - v_2(s)), \tag{3.24}$$

where $v_j(t)$ is a solution to integral equation (2.1) with $L = L_j$, $j = 1,\ 2$. The operator $\mathrm{B}_p^{s,\, L_2}$ acting in $C_{\gamma,\,\theta}(s-L_2,\, s)$ (see (2.1)) can be represented in the form

$$B_p^{s,\, L_2}[v](t) = B_p^{s,\, L_1}[v](t) + b(v;\ t,\ s), \qquad t \in [s-L_1,\ s],$$

where

$$b(v;\ t,\ s) = \int_{s-L_2}^{s-L_1} e^{-(t-\tau)A}\, QB(v(\tau),\ \tau)\, d\tau$$

and $v(t)$ is an arbitrary element in $C_{\gamma,\,\theta}(s-L_2,\ s)$. Therefore, if $v_j(t)$ is a solution to problem (2.1) with $L = L_j$, then

$$v_1(t)-v_2(t) = \mathrm{B}_p^{s,\,L_1}[v_1] - \mathrm{B}_p^{s,\,L_1}[v_2] - b(v_2;\ t,\ s) \tag{3.25}$$

for all $s-L_1 \le t \le s$. Let us estimate the value $b(v_2;\ t,\ s)$. As before it is easy to verify that

$$\left\|A^{\theta} b(v_2;\ t,\ s)\right\| \ \le\ e^{-(t-s+L_1)\lambda_{N+1}}\left\{r_1(s-L_1)+r_2(s-L_1)\left|v_2\right|_{s,\,*}\right\}$$

for all $t \in [s-L_1,\ s]$, where

$$r_1(t) = M\int_{-\infty}^{t}\|A^{\theta}e^{-(t-\tau)A}Q\|\,d\tau\,,$$

$$r_2(t) = M\int_{-\infty}^{t}\|A^{\theta}e^{-(t-\tau)A}Q\|\ e^{\gamma_*(s-\tau)}\,d\tau\ ,$$

and the norm $\left|v_2\right|_{s,\,*}$ is defined using the constants $q^* = \dfrac{1+q}{2}$ and $\gamma_* = \lambda_N + \dfrac{2M}{q_*}\lambda_N^{\theta}$ by the formula

$$\left|v_2\right|_{s,\,*} = \sup\left\{e^{-\gamma_*(s-t)}\left\|A^{\theta}v_2(t)\right\|:\ \ t \in [s-L_2,\ s]\right\}.$$

Evidently, spectral gap condition (3.1) implies the same equation with the parameter q^* instead of q. Therefore, simple calculations based on (1.8) give us that

$$r_1(t) \le D_2 \quad \text{and} \quad r_2 \le e^{-\gamma_*(s-t)}\frac{q^*}{2}\,,$$

where D_2 is defined by formula (2.14). Using Lemma 2.1 under condition (2.3) with q^* instead of q we obtain that

$$\left|v_2\right|_{s,\,*} \ \le\ (1-q^*)^{-1}\{D_1+\|A^{\theta}p\|\}\,,$$

where D_1 is given by formula (2.6). Therefore, finaly we have that

$$\left\|A^{\theta} b(v_2;\ t,\ s)\right\| \ \le\ e^{-(t-s+L_1)\lambda_{N+1}}\left\{D_2+\frac{q^*}{2(1-q^*)}\,e^{\gamma_* L_1}(D_1+\|A^{\theta}p\|)\right\}$$

for all $t \in [s-L_1,\ s]$. Consequently,

$$\sup\left\{e^{\gamma(t-s)}\left\|A^{\theta}b(v_2;\ t,\ s)\right\|:\ t \in [s-L_1,\ s]\right\} \ \le$$

$$\le\ e^{-\gamma L_1}\left\{D_2+\frac{q^*}{2(1-q^*)}\,e^{\gamma_* L_1}(D_1+\|A^{\theta}p\|)\right\}.$$

Therefore, since $B_p^{s,\,L_1}$ is a contractive operator in $C_{\gamma,\,\theta}(s-L_1,\ s)$, equation (3.25) gives us that

$$(1-q)|v_1-v_2|_{C_{\gamma,\,\theta}(s-L_1,\ s)} \le$$

$$\le\ e^{-\gamma L_1}D_2+\frac{1}{2}\cdot\frac{1+q}{1-q}\ e^{-(\gamma-\gamma_*)L_1}\Big(D_1+\|A^\theta p\|\Big)\,.$$

Here we also use the equality $q^*=\frac{1}{2}(1+q)$. Hence, estimate (3.23) follows from (3.24). **Theorem 3.3 is proved**.

— Exercise *3.5* Show that in the case when $\|B(u,\ t)\|\le M$ equation (3.23) can be replaced by the inequality

$$\Big\|A^\theta(\Phi^{L_1}(p,\ s)-\Phi^{L_2}(p,\ s))\Big\|\ \le\ D_2(1-q)^{-1}e^{-\gamma_N L_{\min}}\,.$$

— Exercise *3.6* Assume that the hypotheses of Theorem 3.1 hold. Then the estimate

$$\|A^\theta(Qu(t)-\Phi^L(Pu(t),\ t))\|\ \le$$

$$\le\ C\Big(1+\|A^\theta u_0\|\Big)e^{-\gamma_N(t-s)}+C_R\,e^{-\alpha\lambda_N^\theta L}$$

holds for $t\ge t_*$ and for any solution $u(t)$ to problem (1.1) possessing the dissipativity property: $\|A^\theta u(t)\|\le R$ for $t\ge t_*\ge s$ and for some R and t_*. Here $\gamma_N=\lambda_N+\frac{2M}{q}\lambda_N^\theta$ and the constant $\alpha>0$ does not depend on N.

Therefore, if the hypotheses of Theorem 3.1 hold, then a bounded solution to problem (1.1) gets into the exponentially small (with respect to λ_N^θ and L) vicinity of the manifold $\{\mathbf{M}_s^L:\ -\infty<s<\infty\}$ at an exponential velocity.

According to (2.12) in order to build an approximation $\{\mathbf{M}_s^L\}$ of the inertial manifold $\{\mathbf{M}_s\}$ we should solve integral equation (2.1) for L large enough. This equation has the same structure both for $L<\infty$ and for $L=\infty$. Therefore, it is impossible to use the surfaces $\{\mathbf{M}_s^L\}$ directly for the effective approximation of $\{\mathbf{M}_s\}$. However, by virtue of contractiveness of the operator $B_p^{s,\,\infty}$ in the space $C_s^-=$ $=C_{\gamma,\,\theta}(-\infty,\ s)$, its fixed point $v_s(t)$ which determines $\mathbf{M}_s$ can be found with the help of iterations. This fact enables us to construct the collection $\{\mathbf{M}_{n,\ s}\}$ of approximations for $\{\mathbf{M}_s\}$ as follows. Let $v_0=v_{0,\ s}(t;\ p)$ be an element of C_s^-. We take

$$v_n\equiv v_{n,\ s}(t,\ p)=B_p^{s,\,\infty}[v_{n-1}](t)\,,\qquad n=1,\ 2,\ \dots\,,$$

and define the surfaces $\{\mathbf{M}_{n,\ s}\}$ by the formula

$$\mathbf{M}_{n,\ s}=\{p+\Phi_n(p,\ s):\ p\in PH\}\,,$$

where $\Phi_n(p,\ s)=Qv_{n,\,s}(p,\ s)\,,\quad n=1,\ 2,\ \dots$

— Exercise *3.7* Let $v_0 \equiv p$ and let $B(u, t) \equiv B(u)$. Show that

$$\Phi_0(p, s) \equiv 0 \quad \text{and} \quad \Phi_1(p, s) = A^{-1} Q B(p).$$

— Exercise *3.8* Assume that spectral gap condition (3.1) is fulfilled. Show that

$$\left\| A^\theta(\Phi_n(p, s) - \Phi(p, s)) \right\| \le q^n \left(|v_0|_s + (1-q)^{-1} \left[D_1 + \| A^\theta p \| \right] \right),$$

where D_1 is defined by formula (2.6) and $\Phi(p, s)$ is the function that determines the inertial manifold.

— Exercise *3.9* Prove the assertion for $\Phi_n(p, s)$ similar to the one in Exercise 3.5.

Theorems represented above can also be used in the case when the original system is dissipative and estimates (1.2) and (1.3) are not assumed to be uniform with respect to $u \in D(A^\theta)$. The dissipativity property enables us to restrict ourselves to the consideration of the trajectories lying in a vicinity of the absorbing set when we study the asymptotic behaviour of solutions to problem (0.1). In this case it is convenient to modify the original problem. Assume that the mapping $B(u, t)$ is continuous with respect to its arguments and possesses the properties

$$\|B(u, t)\| \le C_\rho, \qquad \|B(u_1, t) - B(u_2, t)\| \le C_\rho \|A^\theta(u_1 - u_2)\| \tag{3.26}$$

for any $\rho > 0$ and for all u, u_1, and u_2 lying in the ball $B_\rho = \{v : \|A^\theta v\| \le \rho\}$. Let $\chi(s)$ be an infinitely differentiable function on $R_+ = [0, \infty)$ such that

$$\chi(s) = 1, \quad 0 \le s \le 1; \qquad \chi(s) = 0, \; s \ge 2;$$
$$0 \le \chi(s) \le 1, \quad |\chi'(s)| \le 2, \quad s \in R_+.$$

We define the mapping $B_R(u, t)$ by assuming that

$$B_R(u, t) = \chi(R^{-1}\|A^\theta u\|) B(u, t), \quad u \in D(A^\theta). \tag{3.27}$$

— Exercise *3.10* Show that the mapping $B_R(u, t)$ possesses the properties

$$\left\| A^\theta B_R(u, t) \right\| \le M,$$

$$\|B_R(u_1, t) - B_R(u_2, t)\| \le M \left\| A^\theta(u_1 - u_2) \right\|, \tag{3.28}$$

where $M = C_{2R}(1 + 2/R)$ and C_ρ is a constant from (3.26).

Let us now assume that $B(u, t)$ satisfies condition (3.26) and the problem

$$\frac{du}{dt} + Au = B(u, t), \qquad u|_{t=0} = u_0, \tag{3.29}$$

has a unique mild solution on any segment $[s, s+T]$ and possesses the following dissipativity property: there exists $R_0 > 0$ such that for any $R > 0$ the relation

$$\left\|A^{\theta}u(t,\ s;\ u_0)\right\| \le R_0 \quad \text{for all} \quad t-s \ge t_0(R) \tag{3.30}$$

holds, provided that $\left\|A^{\theta}u_0\right\| \le R$. Here $u(t,\ s;\ u_0)$ is the solution to problem (3.29).

— Exercise *3.11* Show that the asymptotic behaviour of solutions to problem (3.29) completely coincides with the asymptotic behaviour of solutions to the problem

$$\frac{du}{dt} + Au = B_{2R_0}(u,\ t), \qquad u\big|_{t=s} = u_0, \tag{3.31}$$

where B_{2R_0} is defined by formula (3.27) and R_0 is the constant from equation (3.30).

— Exercise *3.12* Assume that for a solution to problem (3.29) the invariance property of the absorbing ball is fulfilled: if $\left\|A^{\theta}u_0\right\| \le R_0$, then $\left\|A^{\theta}u(t,\ s;\ u_0)\right\| \le R$ for all $t \le s$. Let $\mathbf{M}_t$ be the invariant manifold of problem (3.31). Then the set $\mathbf{M}_t^{R_0} = \mathbf{M}_t \cap \{u: \|A^{\theta}u\| \le R_0\}$ is invariant for problem (3.29): if $u_0 \in \mathbf{M}_s^{R_0}$, then $u(t,\ s;\ u_0) \in \mathbf{M}_s^{R_0}$, $t \ge s$.

Thus, if the appropriate spectral gap condition for problem (3.29) is fulfilled, then there exists a finite-dimensional surface which is a locally invariant exponentially attracting set.

In conclusion of this section we note that the version of the Lyapunov-Perron method represented here can also be used for the construction (see [13]) of inertial manifolds for retarded semilinear parabolic equations similar to the ones considered in Section 8 of Chapter 2. In this case both the smallness of retardation and the fulfilment of the spectral gap condition of the form (3.1) are required.

§ 4 Continuous Dependence of Inertial Manifold on Problem Parameters

Let us consider the Cauchy problem

$$\frac{du}{dt} + Au = B^*(u,\ t), \qquad u\big|_{t=s} = u_0, \quad s \in \mathbb{R} \tag{4.1}$$

in the space H together with problem (1.1). Assume that $B^*(u,\ t)$ is a nonlinear mapping from $D(A^{\theta}) \times \mathbb{R}$ into H possessing properties (1.2) and (1.3) with the same constant M as in problem (1.1). If spectral gap condition (3.1) is fulfilled, then problem (4.1) (as well as (1.1)) possesses an invariant manifold

$$\mathbf{M}_s^* = \{p + \Phi^*(p,\ s):\ p \in PH\}, \qquad s \in \mathbb{R}. \tag{4.2}$$

The aim of this section is to obtain an estimate for the distance between the manifolds $\mathbf{M}_s$ and $\mathbf{M}_s^*$. The main result is the following assertion.

Theorem 4.1.

Assume that conditions (1.2), (1.3)***, and*** (3.1) ***are fulfilled both for problems*** (1.1) ***and*** (4.1). ***We also assume that***

$$\|B(v,\ t) - B^*(v,\ t)\| \ \le\ \rho_1 + \rho_2 \|A^\theta v\| \tag{4.3}$$

for all $v \in D(A^\theta)$ ***and*** $t \in \mathbb{R}$***, where*** ρ_1 ***and*** ρ_2 ***are positive numbers. Then the equation***

$$\sup_{s\ \in R} \|A^\theta(\Phi(p,\ s) - \Phi^*(p,\ s))\| \ \le\ C_1(q,\ \theta)\frac{\rho_1 + \rho_2}{\lambda_N^{1-\theta}} + C_2(q,\ \theta,\ M)\,\rho_2 \|A^\theta p\|$$

is valid for the functions $\Phi(p,\ s)$ ***and*** $\Phi^*(p,\ s)$ ***which give the invariant manifolds for problems*** (1.1) ***and*** (4.1) ***respectively. Here the numbers*** $C_1(q,\ \theta)$ ***and*** $C_2(q,\ \theta,\ M)$ ***do not depend on*** N ***and*** ρ_j.

Proof.

Equation (2.12) with $L = \infty$ implies that

$$\|A^\theta(\Phi(p,\ s) - \Phi^*(p,\ s))\| \ \le \int\limits_{-\infty}^{s} \|A^\theta e^{-(s-\tau)A} Q\|\,\|B(v(\tau),\ \tau) - B^*(v^*(\tau),\ \tau)\|\,d\tau\ ,$$

where $v(\tau)$ and $v^*(\tau)$ are solutions to the integral equations of the type (2.1) corresponding to problems (1.1) and (4.1) respectively. Equations (1.3) and (4.3) give us that

$$\|B(v(\tau),\ \tau) - B^*(v^*(\tau),\ \tau)\| \ \le\ M\|A^\theta(v(\tau) - v^*(\tau))\| + \Big(\rho_1 + \rho_2\|A^\theta v(\tau)\|\Big) \ \le$$

$$\le\ e^{\gamma(s-\tau)}(M|v - v^*|_s + \rho_2 |v|_s) + \rho_1 \tag{4.4}$$

for $\tau \le s$, where

$$|w|_s = \operatorname*{ess\ sup}_{t \le s} \left\{e^{-\gamma(s-t)}\|A^\Theta w(t)\|\right\} \tag{4.5}$$

and $\gamma = \lambda_N + \frac{2M}{q}\lambda_N^\theta$ as before. Hence, after simple calculations as in Section 2 we find that

$$\|A^\theta(\Phi(p,\ s) - \Phi^*(p,\ s))\| \ \le\ \frac{q}{2}\Big(|v - v^*|_s + \frac{\rho_2}{M}|v|_s\Big) + \rho_1\frac{1+k}{\lambda_{N+1}^{1-\theta}}\,. \tag{4.6}$$

Let us estimate the value $|v - v^*|_s$. Since v and v^* are fixed points of the corresponding operator $\mathbf{B}_p^{s,\ \infty}$, we have that

$$\|A^{\theta}(v(t)-v^*(t))\| \le \int_t^s \|A^{\theta} e^{-(t-\tau)A} P\| \|B(v(\tau),\tau)-B^*(v^*(\tau),\tau)\| d\tau +$$

$$+\int_{-\infty}^s \|A^{\theta} e^{-(t-\tau)A} Q\| \|B(v(\tau),\tau)-B^*(v^*(\tau),\tau)\| d\tau .$$

Therefore, by using spectral gap condition (3.1) and estimate (4.4) as above it is easy to find that

$$|v-v^*|_s \le q|v-v^*|_s + \frac{q\rho_2}{M}\cdot|v|_s + \rho_1\left(\frac{1}{\lambda_N^{1-\theta}} + \frac{1+k}{\lambda_{N+1}^{1-\theta}}\right).$$

Consequently,

$$|v-v^*|_s \le \frac{q}{1-q}\cdot\frac{\rho_2}{M}\cdot|v|_s + \frac{\rho_1}{1-q}\cdot\frac{2+k}{\lambda_N^{1-\theta}}.$$

Therefore, equation (4.6) implies that

$$\|A^{\theta}(\Phi(p,s)-\Phi^*(p,s))\| \le \frac{q}{1-q}\cdot\frac{\rho_2}{2M}\cdot|v|_s + \rho_1\cdot\frac{2-q}{2-2q}\cdot\frac{2+k}{\lambda_N^{1-\theta}}.$$

Hence, estimate (2.5) gives us the inequality

$$\|A^{\theta}(\Phi(p,s)-\Phi^*(p,s))\| \le \frac{\rho_1+\rho_2}{(1-q)^2}\cdot\frac{2+k}{\lambda_N^{1-\theta}} + \frac{q}{2(1-q)^2}\cdot\frac{\rho_2}{M}\|A^{\theta}p\|.$$

This **implies the assertion of Theorem 4.1**.

Let us now consider the Galerkin approximations $u_m(t)$ of problem (1.1). We remind (see Chapter 2) that the Galerkin approximation of the order m is defined as a function $u_m(t)$ with the values in $P_m H$, this function being a solution to the problem

$$\frac{du_m}{dt} + Au_m = P_m B(u_{m_0}), \qquad u_m\big|_{t=s} = u_{0m} . \tag{4.7}$$

Here P_m is the orthoprojector onto the span of elements $\{e_1, \dots, e_m\}$ in H.

Exercise *4.1* Assume that spectral gap condition (3.1) holds and $m \ge N+1$. Show that problem (4.7) possesses an invariant manifold of the form

$$\mathbf{M}_s^{(m)} = \{p + \Phi^{(m)}(p,s):\ p \in PH\}$$

in $P_m H$, where the function $\Phi^{(m)}(p,s):\ PH \to (P_m - P)H$ is defined by equation similar to (2.12).

The following assertion holds.

Theorem 4.2.

Assume that spectral gap condition (3.1) ***holds. Let*** $\Phi(p, s)$ ***and*** $\Phi^{(m)}(p, s)$ ***be the functions defined by the formulae of the type*** (2.12) ***and let these functions give invariant manifolds for problems*** (1.1) ***and*** (4.7) ***for*** $m \ge N+1$ ***respectively. Then the estimate***

$$\|A^\theta(\Phi(p, s) - \Phi^{(m)}(p, s))\| \le \frac{C(q, M, \theta)}{\lambda_{m+1}^{1-\theta}} \left\{ 1 + \frac{D_1 + \|A^\theta p\|}{1 - \dfrac{\lambda_{N+1}}{\lambda_{m+1}}} \right\} \tag{4.8}$$

is valid, where the constant D_1 ***is defined by formula*** (2.6).

Proof.

It is evident that

$$(\Phi(p, s) - \Phi^{(m)}(p, s)) = Q\big[v(s; p) - v^{(m)}(s; p)\big], \tag{4.9}$$

where $v(t, p)$ and $v^{(m)}(t, p)$ are solutions to the integral equations

$$v(t) = \mathrm{B}_p^{s,\infty}[v](t), \qquad -\infty < t \le s,$$

and

$$v^{(m)}(t) = P_m \mathrm{B}_p^{s,\infty}[v^{(m)}](t), \qquad -\infty < t \le s.$$

Here $\mathrm{B}_p^{s,\infty}$ is defined as in (2.1). Since

$$v(t) - v^{(m)}(t) = (I - P_m)v(t) + P_m\big[\mathrm{B}_p^{s,\infty}[v](t) - \mathrm{B}_p^{s,\infty}[v^{(m)}](t)\big],$$

we have

$$\|A^\theta(v(t) - v^{(m)}(t))\| = \big\|A^\theta(1 - P_m)v(t)\big\| + \big\|A^\theta[\mathrm{B}_p^{s,\infty}[v](t) - \mathrm{B}_p^{s,\infty}[v^{(m)}](t)]\big\|.$$

The contractiveness property of the operator $\mathrm{B}_p^{s,\infty}$ leads to the equation

$$\|A^\theta(v(t) - v^{(m)}(t))\| = \big\|A^\theta(1 - P_m)v(t)\big\| + q \cdot |v - v^{(m)}|_s \, e^{\gamma(s-t)}.$$

In particular, this implies that

$$|v - v^{(m)}|_s \equiv \sup_{t<s} e^{-\gamma(s-t)} \|A^\theta(v(t) - v^{(m)}(t))\| \le (1-q)^{-1} |(1 - P_m)v|_s .$$

Hence, with the help of (4.9) we find that

$$\|A^\theta(\Phi(p, s) - \Phi^{(m)}(p, s))\| \le \|A^\theta(v(s) - v^{(m)}(s))\| \le |v - v^{(m)}|_s \le \tag{4.10}$$

$$\le (1-q)^{-1} |(1 - P_m)v|_s .$$

Let us estimate the value $|(1-P_m)v|_s$. It is clear that

$$(1-P_m)v(t) = \int_{-\infty}^{t} e^{-(t-\tau)A}(1-P_m)B(v(\tau))\,d\tau\,.$$

Therefore, Lemma 2.1.1 (see also (1.8)) gives us that

$$\|A^\theta(1-P_m)v(t)\| \le M\int_{-\infty}^{t}\left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right]e^{-(t-\tau)\lambda_{m+1}}\,d\tau\, +$$

$$+M\int_{-\infty}^{t}\left[\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{m+1}^\theta\right]e^{-\lambda_{m+1}(t-\tau)}\;e^{(t-\tau)\gamma}\,d\tau\;|v|_s\cdot e^{\gamma(s-t)}\;,$$

where

$$\gamma \equiv \lambda_N + \frac{2M}{q}\lambda_N^\theta < \lambda_{N+1} < \lambda_{m+1}$$

as above. Simple calculations analogous to the ones in Lemma 2.1 imply that

$$\|A^\theta(1-P_m)v(t)\| \le \frac{M(1+k)}{\lambda_{m+1}^{1-\theta}} + \frac{M(1+k)\lambda_{m+1}^\theta}{\lambda_{m+1}-\gamma}\;e^{\gamma(s-t)}\,|v|_s\,,$$

where the constant k has the form (1.7). Consequently, using (2.5) we obtain

$$|(1-P_m)v|_s \le \frac{M(1+k)}{\lambda_{m+1}^{1-\theta}}\left(1+\left(1-\frac{\lambda_{N+1}}{\lambda_{m+1}}\right)^{-1}|v|_s\right) \le$$

$$\le \frac{M(1+k)}{\lambda_{m+1}^{1-\theta}}\left(1+\left(1-\frac{\lambda_{N+1}}{\lambda_{m+1}}\right)^{-1}(1-q)^{-1}(D_1+\|A^\theta p\|)\right).$$

This and (4.10) imply estimate (4.8). **Theorem 4.2 is proved**.

Exercise *4.2* In addition assume that the hypotheses of Theorem 4.2 hold and $\|B(u,\,t)\| \le M$. Show that in this case estimate (4.8) has the form

$$\|A^\theta(\Phi(p,\,s)-\Phi^{(m)}(p,\,s))\| \le C(q;\,M,\,\theta)\lambda_{m+1}^{-1+\theta}\,.$$

§ 5 *Examples and Discussion*

Example *5.1*

Let us consider the nonlinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} + f(x,\ u,\ t), \quad 0 < x < l,\ t > 0\ , & (5.1) \\ u\big|_{x=0} = u\big|_{x=l} = 0\ , & (5.2) \\ u\big|_{t=0} = u_0(x)\ . & (5.3) \end{cases}$$

Assume that ν is a positive parameter and $f(x,\ u,\ t)$ is a continuous function of its variables which possesses the properties

$$\left|f(x,\ u_1,\ t) - f(x,\ u_2,\ t)\right| \le M\left|u_1 - u_2\right|, \quad |f(x,\ 0,\ t)| \le \frac{M}{\sqrt{l}}\ .$$

Problem (5.1)–(5.3) generates a dynamical system in $L^2(0,\ l)$ (see Section 3 of Chapter 2). Therewith

$$A = -\nu \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad D(A) = H_0^1(0,\ L) \cap H^2(0,\ l),$$

where $H^s(0,\ l)$ is the Sobolev space of the order s. The mapping $B(\cdot\ ,\ t)$ given by the formula $u(x) \to f(x,\ u(x),\ t)$ satisfies conditions (1.2) and (1.3) with $\theta = 0$. In this case spectral gap condition (2.3) has the form

$$\nu \frac{\pi^2}{l^2}((N+1)^2 - N^2) \ \ge\ \frac{4M}{q}\ .$$

Thus, problem (5.1)–(5.3) possesses an inertial manifold of the dimension N, provided that

$$N > -\frac{1}{2} + \frac{2M}{\nu q} \frac{l^2}{\pi^2} \tag{5.4}$$

for some $q < 2 - \sqrt{2}$.

Exercise *5.1* Find the conditions under which the inertial manifold of problem (5.1)–(5.3) is one-dimensional. What is the structure of the corresponding inertial form?

Exercise *5.2* Consider problem (5.1) and (5.3) with the Neumann boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=l} = 0 \tag{5.5}$$

Show that problem (5.1), (5.3), and (5.5) has an inertial manifold of the dimension $N+1$, provided condition (5.4) holds for some

$N \geq 0$. (*Hint*: $A = -\nu(\mathrm{d}^2/\mathrm{d}x^2) + \varepsilon$ with condition (5.5), $B(u, t) = -\varepsilon u + f(x, u, t)$, where $\varepsilon > 0$ is small enough).

Exercise *5.3* Find the conditions on the parameters of problem (5.1), (5.3), and (5.5) under which there exists a one-dimensional inertial manifold. Show that if $f(x, u, t) \equiv f(u, t)$, then the corresponding inertial form is of the type

$$\dot{p}(t) = f(p(t), t), \qquad p|_{t=0} = p_0 .$$

Example *5.2*

Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} + f\left(x, u, \dfrac{\partial u}{\partial x}, t\right), \quad 0 < x < l, \ t > 0, \\ u|_{x=0} = u|_{x=l} = 0, \quad u|_{t=0} = u_0(x). \end{cases} \tag{5.6}$$

Here $\nu > 0$ and $f(x, u, \xi, t)$ is a continuous function of its variables such that

$$|f(x, u_1, \xi, t) - f(x, u_2, \xi, t)| \leq L_1|u_1 - u_2| + L_2|\xi_1 - \xi_2| \tag{5.7}$$

for all $x \in (0, l)$, $t \geq 0$ and

$$\int_0^l [f(x, 0, 0, t)]^2 \,\mathrm{d}x \leq L_3^2 ,$$

where L_j are nonnegative numbers. As in Example 5.1 we assume that

$$A = -\nu \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad D(A) = H_0^1(0, l) \cap H^2(0, l), \quad B(u, t) = f\left(x, u, \frac{\partial u}{\partial x}, t\right).$$

It is evident that

$$\|B(u_1, t) - B(u_2, t)\| \leq L_1\|u_1 - u_2\| + L_2\left\|\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}\right\|.$$

Here $\|\cdot\|$ is the norm in $L^2(0, l)$. By using the obvious inequality

$$\left\|\frac{\partial u}{\partial x}\right\|^2 \geq \left(\frac{\pi}{l}\right)^2 \|u\|^2, \qquad u \in H_0^1(0, l),$$

we find that

$$\|B(u_1, t) - B(u_2, t)\| \leq \frac{1}{\sqrt{\nu}}\left(L_1 \frac{l}{\pi} + L_2\right)\|A^{1/2}(u_1 - u_2)\|.$$

Hence, conditions (1.2) and (1.3) are fulfilled with

$$\theta = \frac{1}{2}, \qquad M = \max\left\{L_3; \ \frac{1}{\sqrt{\nu}}\left(\frac{l}{\pi}L_1 + L_2\right)\right\}.$$

Therewith spectral gap condition (2.3) acquires the form

$$\sqrt{\nu}\,\frac{\pi}{l}(2N+1) \;\geq\; \frac{2M}{q}(2N+1+k(N+1)),$$

where

$$k = \frac{1}{\sqrt{2}}\int_0^x \xi^{-1/2}e^{-\xi}\,d\xi = \sqrt{\frac{\pi}{2}}\,.$$

Thus, the equation

$$1+\sqrt{\frac{\pi}{2}}\cdot\frac{N+1}{2N+1} \;\leq\; \frac{\pi q\sqrt{\nu}}{2lM}$$

or

$$2+\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\pi}{2}}\cdot\frac{1}{2N+1} \;\leq\; \frac{\pi q\sqrt{\nu}}{lM}$$

must be valid for some $0 < q < 2-\sqrt{2}$. We can ensure the fulfilment of this condition only in the case when

$$2+\sqrt{\frac{\pi}{2}} \;<\; \frac{\pi q_0\sqrt{\nu}}{lM}\,, \qquad q_0 = 2-\sqrt{2}\,,$$

i.e. if

$$M \equiv \max\left\{L_3;\; \frac{1}{\sqrt{\nu}}\left(\frac{l}{\pi}L_1+L_2\right)\right\} \;<\; 2\pi\frac{\sqrt{\nu}}{l}\cdot\frac{\sqrt{2}-1}{2\sqrt{2}+\sqrt{\pi}}\,. \tag{5.9}$$

Thus, in order to apply the above-presented theorems to the construction of the inertial manifold for problem (5.6) one should pose some additional conditions (see (5.7) and (5.9)) on the nonlinear term $f(x,\,u,\,\partial u/\partial x,\,t)$ or require that the diffusion coefficient ν be large enough.

— Exercise *5.4* Assume that $f(x,\,u,\,\xi,\,t) = \varepsilon\bar{f}(x,\,u,\,\xi,\,t)$ in (5.6), where the function $\bar{f}$ possesses properties (5.7) and (5.8) with arbitrary $L_j \geq 0$. Show that problem (5.6) has an inertial manifold for any $0 < \varepsilon < \varepsilon_0$, where

$$\varepsilon_0 = 2\pi\frac{\sqrt{\nu}}{l}\cdot\frac{\sqrt{2}-1}{2\sqrt{2}+\sqrt{\pi}}\cdot\left[\max\left\{L_3;\;\frac{1}{\nu}\left(\frac{l}{\sqrt{\pi}}L_1+L_2\right)\right\}\right]^{-1}.$$

Characterize the dependence of the dimension of inertial manifold on ε.

— Exercise *5.5* Study the question on the existence of an inertial manifold for problem (5.6) in which the Dirichlet boundary condition is replaced by the Neumann boundary condition (5.5).

It should be noted that

$$\lambda_n = C_d n^{2/d}(1 + o(1)), \quad n \to \infty, \quad d = \dim \Omega,$$

where λ_n are the eigenvalues of the linear part of the equation of the type

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(x, u, \nabla u, t), \quad x \in \Omega, \quad t > 0,$$

in a multidimensional bounded domain Ω. Therefore, we can not expect that Theorem 3.1 is directly applicable in this case. In this connection we point out the paper [3] in which the existence of IM for the nonlinear heat equation is proved in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \le 3)$ that satisfies the so-called "principle of spatial averaging" (the class of these domains contains two- and three-dimensional cubes).

It is evident that the most severe constraint that essentially restricts an application of Theorem 3.1 is spectral gap condition (3.1). In some cases it is possible to weaken or modify it a little. In this connection we mention papers [6] and [7] in which spectral gap condition (3.1) is given with the parameters $q = 2$ and $k = 0$ for $0 \le \theta < 1$. Besides it is not necessary to assume that the spectrum of the operator A is discrete. It is sufficient just to require that the selfadjoint operator A possess a gap in the positive part of the spectrum such that for its edges the spectral condition holds. We can also assume the operator A to be sectorial rather than selfadjoint (for example, see [6]).

Unfortunately, we cannot get rid of the spectral conditions in the construction of the inertial manifold. One of the approaches to overcome this difficulty runs as follows: let us consider the regularization of problem (0.1) of the form

$$\frac{du}{dt} + Au + \varepsilon A^m u = B(u, t), \quad u\big|_{t=0} = u_0. \tag{5.10}$$

Here $\varepsilon > 0$ and the number $m > 0$ is chosen such that the operator $\tilde{A} = A + \varepsilon A^m$ possesses spectral gap condition (3.1). Therewith IM for problem (5.10) should be naturally called an approximate IM for system (0.1). Other approaches to the construction of the approximate IM are presented below.

It should also be noted that in spite of the arising difficulties the number of equations of mathematical physics for which it is possible to prove the existence of IM is large enough. Among these equations we can name the Cahn-Hillard equations in the domain $\Omega = (0, L)^d$, $d = \dim \Omega \le 2$, the Ginzburg-Landau equations ($\Omega = (0, L)^d$, $d \le 2$), the Kuramoto-Sivashinsky equation, some equations of the theory of oscillations ($d = 1$), a number of reaction-diffusion equations, the Swift-Hohenberg equation, and a non-local version of the Burgers equation. The corresponding references and an extended list of equations can be found in survey [8].

In conclusion of this section we give one more interesting application of the theorem on the existence of an inertial manifold.

Example 5.3

Let us consider the system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(u, \nabla u), \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial \Omega} = 0, \tag{5.11}$$

in a bounded domain $\Omega \subset R^d$. Here $u = (u_1, \ldots, u_m)$ and the function $f(u, \xi)$ satisfies the global Lipschitz condition:

$$|f(u, \xi) - f(v, \eta)| \le L\{|u - v|^2 + |\xi - \eta|^2\}^{1/2}, \tag{5.12}$$

where $u, v \in \mathbb{R}^m$, $\xi, \eta \in \mathbb{R}^{md}$, and $L > 0$. We also assume that $|f(0, 0)| \le L$. Problem (5.11) can be rewritten in the form (0.1) in the space $H = [L^2(\Omega)]^m$ if we suppose

$$Au = -\nu \Delta u + u, \qquad B(u) = u + f(u, \nabla u).$$

It is clear that the operator A is positive in its natural domain and it has a discrete spectrum. Equation (5.12) implies that the relation

$$\|B(u) - B(v)\| \le L\left\{\|u - v\|^2 + \|\nabla(u - v)\|^2\right\}^{1/2} + \|u - v\| \le$$

$$\le \left(1 + L\max\left\{1; \frac{1}{\sqrt{\nu}}\right\}\right)\left\{\|u - v\|^2 + \nu\|\nabla(u - v)\|^2\right\}^{1/2}$$

is valid for $B(u)$. Thus,

$$\|B(u) - B(v)\| \le M\|A^{1/2}(u - v)\|,$$

where

$$M = 1 + L\max\left\{1; \frac{1}{\sqrt{\nu}}\right\}.$$

Therefore, problem (5.11) generates an evolutionary semigroup S_t (see Chapter 2) in the space $D(A^{1/2})$. An important property of S_t is the following: the subspace $\mathbf{L}$ which consists of constant vectors is invariant with respect to this semigroup. The dimension of this subspace is equal to m. The action of the semigroup in this subspace is generated by a system of ordinary differential equations

$$\frac{du}{dt} = f(u, 0), \qquad u(t) \in \mathbf{L}. \tag{5.13}$$

Exercise 5.6 Assume that equation (5.12) holds for $\xi = \eta = 0$. Show that equation (5.13) is uniquely solvable on the whole time axis for any initial condition and the equation

$$\sup_{t \le s}\left\{e^{-L(s-t)}|u(s)|\right\} < \infty \tag{5.14}$$

holds for any $s \in \mathbb{R}$.

The subspace $\mathbf{L}$ consists of the eigenvectors of the operator A corresponding to the eigenvalue $\lambda_1 = 1$. The next eigenvalue has the form $\lambda_2 = \nu\mu_1 + 1$, where μ_1 is the first nonzero eigenvalue of the Laplace operator with the Neumann boundary condition on $\partial\Omega$. Therefore, spectral gap equation (3.1) can be rewritten in the form

$$\nu\mu_1 \ \geq\ \frac{2}{q}\left(1 + L\max\left\{1;\ \frac{1}{\sqrt{\nu}}\right\}\right)\left(\left(1+\sqrt{\frac{\pi}{2}}\right)\sqrt{\nu\mu_1 + 1} + 1\right) \tag{5.15}$$

for $N = m$ and $\theta = 1/2$, where $0 < q < 2 - \sqrt{2}$. It is clear that there exists $\nu_0 > 0$ such that equation (5.15) holds for all $\nu \geq \nu_0$. Therefore, we can apply Theorem 3.1 to find that if ν is large enough, then there exists IM of the type

$$\mathbf{M} = \left\{p + \Phi(p):\ p \in \mathbf{L},\ \Phi:\ \mathbf{L} \to H \ominus \mathbf{L}\right\}.$$

The invariance of the subspace $\mathbf{L}$ and estimate (5.14) enable us to use Theorem 3.2 and to state that $\mathbf{L} \subset \mathbf{M}$. This easily implies that $\Phi(p) \equiv 0$, i.e. $\mathbf{M} = \mathbf{L}$. Thus, Theorem 3.1 gives us that for any solution $u(t)$ to problem (5.11) there exists a solution $\tilde{u}(t)$ to the system of ordinary differential equations (5.13) such that

$$\|u(t) - \tilde{u}(t)\|_1 \ \leq\ Ce^{-\gamma t}, \qquad t \geq 0,$$

where the constant $\gamma > 0$ does not depend on $u(t)$ and $\|\cdot\|_1$ is the Sobolev norm of the first order.

Exercise 5.7 Consider the problem

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} + f(x,\ u),\ \ 0 < x < \pi;\quad u\big|_{x=0} = u\big|_{x=\pi} = 0, \tag{5.16}$$

where the function $f(x,\ u)$ has the form

$$f(x,\ u) = g_1(u_1,\ u_2)\sin x + g_2(u_1,\ u_2)\sin 2x.$$

Here

$$u_j = \frac{2}{\pi}\int_0^{\pi} u(x)\sin jx\, dx, \quad j = 1,\ 2,$$

and $g_j(u_1,\ u_2)$ are continuous functions such that

$$|g_j(u_1,\ u_2) - g_j(v_1,\ v_2)| \ \leq$$

$$\leq\ L_j\left(|u_1 - v_1|^2 + |u_2 - v_2|^2\right)^{1/2}; \qquad g_j(0,\ 0) = 0.$$

Show that if

$$\nu \ >\ \frac{2}{5(\sqrt{2}-1)}\sqrt{\pi(L_1^2 + L_2^2)},$$

then the dynamical system generated by problem (5.16) has the two-dimensional (flat) inertial manifold

$$M = \{p_1 \sin x + p_2 \sin 2x:\ p_1,\ p_2 \in \mathbb{R}\}$$

and the corresponding inertial form is:

$$\dot p_1 + \nu p_1 = g_1(p_1,\ p_2)\,, \qquad \dot p_2 + 4\nu p_2 = g_2(p_1,\ p_2)\,.$$

— Exercise *5.8* Study the question on the existence of an inertial manifold for the Hopf model of turbulence appearance (see Section 7 of Chapter 2).

§ 6 *Approximate Inertial Manifolds for Semilinear Parabolic Equations*

Even in the cases when the existence of IM can be proved, the question concerning the effective use of the inertial form

$$\partial_t p + Ap = PB(p + \Phi(p,\ t),\ t) \tag{6.1}$$

is not simple. The fact is that it is not practically possible to find a more or less explicit solution to the integral equation for $\Phi(p,\ t)$ even in the finite-dimensional case. In this connection we face the problem of approximate or asymptotic construction of an invariant (inertial) manifold. Various aspects of this problem related to finite-dimensional systems are presented in the book by Ya. Baris and O. Lykova [14].

For infinite-dimensional systems the problem of construction of an approximate IM can be interpreted as a problem of reduction, i.e. as a problem of constructive description of finite-dimensional projectors P and functions $\Phi(\,\cdot\,,\ t):\ PH \to \to (1-P)H$ such that an equation of form (6.1) "inherits" (of course, this needs to be specified) all the peculiarities of the long-time behaviour of the original system (0.1). It is clear that the manifolds arising in this case have to be close in some sense to the real IM (in fact, the dynamics on IM reproduces all the essential features of the qualitative behaviour of the original system). Under such a formulation a problem of construction of IM acquires secondary importance, so one can directly construct a sequence of approximate IMs. Usually (see the references in survey [8]) the problem of the construction of an approximate IM can be formulated as follows: find a surface of the form

$$\mathrm{M}_t = \{p + \Phi(p,\ t):\ p \in PH\}\,, \tag{6.2}$$

which attracts all the trajectories of the system in its small vicinity. The character of closeness is determined by the parameter λ_{N+1}^{-1} related to the decomposition

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} + Ap = PB(p+q,\ t)\ , \\ \dfrac{\mathrm{d}q}{\mathrm{d}t} + Aq = (1-P)B(p+q,\ t)\ . \end{cases} \tag{6.3}$$

We obtain the trivial approximate IM $\mathbf{M}_t^{(0)}$ if we put $\Phi(p,\, t) = \Phi_0(p,\, t) \equiv 0$ in (6.2). In this case $\mathbf{M}_t^{(0)}$ is a finite-dimensional subspace in $\mathscr{H}$ whereas inertial form (6.1) turns into the standard Galerkin approximation of problem (0.1) corresponding to this subspace. One can find the simplest non-trivial approximation $\mathbf{M}_t^{(1)}$ using formula (6.2) and assuming that

$$\Phi(p,\ t) = \Phi_1(p,\ t) \equiv A^{-1}(1-P)B(p,\ t)\,. \tag{6.4}$$

The consideration of system (0.1) on $\mathbf{M}_t^{(1)}$ leads to the second equation of equations (6.3) being replaced by the equality $Aq = (1-P)B(p,\ t)$. The results of the computer simulation (see the references in survey [8]) show that the use of just the first approximation to IM has a number of advantages in comparison with the traditional Galerkin method (some peculiarities of the qualitative behaviour of the system can be observed for a smaller number of modes).

There exist several methods of the construction of an approximate IM. We present the approach based on Lemma 2.1 which enables us to construct an approximate IM of the exponential order, i.e. the surfaces in the phase space H such that their exponentially small (with respect to the parameter λ_{N+1}) vicinities uniformly attract all the trajectories of the system. For the first time this approach was used in paper [15] for a class of stochastic equations in the Hilbert space. Here we give its deterministic version.

Let us consider the integral equation (see(2.1))

$$v(t) = \mathbf{B}_p^{s,\,L}[v](t)\,, \qquad s-L \le t \le s$$

and assume that $L = \rho\lambda_{N+1}^{-\theta}$, where the parameter ρ possesses the property

$$q \equiv M\left(\frac{\theta^\theta}{1-\theta}\,\lambda_1^{-(1-\theta)}\,\rho^{1-\theta} + \rho\right) < 1\,. \tag{6.5}$$

In this case equations (2.2) hold. Hence, Lemma 2.1 enables us to construct a collection of manifolds $\{\mathbf{M}_s^L\}$ for $L = \rho\lambda_{N+1}^{-\theta}$ with the help of the formula

$$\mathbf{M}_s^L = \{p + \Phi^L(p,\ s):\ p \in PH\}, \tag{6.6}$$

where

$$\Phi^L(p,\ s) = \int_{s-L}^{s} e^{-(s-\tau)A}\,QB(v(\tau),\ \tau)\,\mathrm{d}\tau \equiv Qv(s,\ p)\ . \tag{6.7}$$

Here $v(t) = v(t,\ p)$ is a solution to integral equation (2.1) and $L = \rho\lambda_{N+1}^{-\theta}$.

Exercise *6.1* Show that both the function $\Phi^L(p, s)$ and the surface $\mathbf{M}_s^L$ do not depend on s in the autonomous case $(B(u, t) \equiv B(t))$.

The following assertion is valid.

Theorem 6.1.

There exist positive numbers $\rho_1 = \rho_1(M, \theta, \lambda_1)$ ***and*** $\Lambda = \Lambda(M, \theta, \lambda_1)$ ***such that if***

$$\lambda_{N+1}^{1-\theta} \ge \Lambda\rho^{-1}, \qquad L = \rho\lambda_{N+1}^{-\theta}, \quad 0 < \rho \le \rho_1, \tag{6.8}$$

then the mappings $\Phi^L(\cdot, s): PH \to QH$ ***defined by equation*** (6.7) ***possess the property***

$$\|A^\theta(Qu(t) - \Phi^L(Pu(t), t))\| \le$$

$$\le C_R^{(1)} \exp\left\{-\frac{\sigma_0}{\rho}\lambda_{N+1}^\theta(t - t_*)\right\} + C_R^{(2)} \exp\left\{-\frac{\rho}{2}\lambda_{N+1}^{1-\theta}\right\} \tag{6.9}$$

for all $t \ge t_* + L/2$. ***Here*** $\sigma_0 > 0$ ***is an absolute constant and*** $u(t)$ ***is a mild solution to problem*** (1.1) ***such that***

$$\|A^\theta u(t)\| \le R \quad \textit{for} \quad t \in [t_*, +\infty). \tag{6.10}$$

If $\|B(u, t)\| \le M$, ***then estimate*** (6.9) ***can be rewritten as follows:***

$$\|A^\theta(Qu(t) - \Phi^L(Pu(t), t))\| \le$$

$$\le C_R \exp\left\{-\frac{\sigma_0}{\rho}\lambda_{N+1}^\theta(t - t_*)\right\} + D_2 \exp\{-\rho\lambda_{N+1}^{1-\theta}\}, \tag{6.11}$$

where D_2 ***is defined by equality*** (2.14).

Proof.
Let

$$\hat{u}(t) = U(t, s; Pu(s) + \Phi^L(Pu(s), s)), \qquad t_* \le s \le t,$$

where $U(t, s; v)$ is a mild solution to problem (1.1) with the initial condition $v \in D(A^\theta)$ at the moment s. Therewith $u(t) = U(t, 0; u_0)$. It is evident that

$$Qu(t) - \Phi^L(Pu(t), t) = Q(u(t) - \hat{u}(t)) +$$

$$+ \left[Q\hat{u}(t) - \Phi^L(P\hat{u}(t), t)\right] + \left[\Phi^L(P\hat{u}(t), t) - \Phi^L(Pu(t), t)\right]. \tag{6.12}$$

Let us estimate each term in this decomposition. Equation (1.6) implies that

$$\|A^\theta Q(u(t) - \hat{u}(t))\| \le \overline{\alpha}_N(t - s)\|A^\theta(Qu(s) - \Phi^L(Pu(s), s))\|, \tag{6.13}$$

where

$$\overline{\alpha}_N(\tau) = e^{-\lambda_{N+1}\tau} + M(1+k)a_1\,\lambda_{N+1}^{-1+\theta}\,e^{a_2\tau}.$$

Using (2.16) we find that

$$\|A^\theta(Q\hat{u}(s) - \Phi^L(P\hat{u}(s),\ s))\| \le \beta_N(L,\ t-s) \tag{6.14}$$

where

$$\beta_N(L,\ \tau) = D_2(1-q)^{-1}e^{-\gamma L} + q(1-q)^{-2}(R+D_1)\,e^{-\gamma\tau},$$

moreover, the second term in $\beta_N(L,\ \tau)$ can be omitted if $\|B(u,\ t)\| \le M$ (see Exercise 2.1). At last equations (2.15) and (1.5) imply that

$$\begin{aligned}&\|A^\theta(\Phi^L(P\hat{u}(t),\ t)) - \Phi^L(Pu(t),\ t)\| \le \\ &\le a_1\frac{q}{1-q}\,e^{a_2(t-s)}\|A^\theta(Qu(s) - \Phi^L(Pu(s),\ s))\| .\end{aligned} \tag{6.15}$$

Thus, equations (6.12)–(6.15) give us the inequality

$$d(t) \le \alpha_N(t-s)d(s) + \beta_N(L,\ t-s) \tag{6.16}$$

for $t \ge s \ge t_*$, where

$$d(t) = \|A^\theta(Qu(t) - \Phi^L(Pu(t),\ t))\|$$

and

$$\begin{aligned}\alpha_N(\tau) &= \overline{\alpha}_N(\tau) + a_1\frac{q}{1-q}\,e^{a_2\tau} = \\ &= e^{-\lambda_{N+1}\tau} + a_1\left[M(1+k)\,\lambda_{N+1}^{-1+\theta} + q(1-q)^{-1}\right]e^{a_2\tau}.\end{aligned}$$

It follows from (6.16) that under the condition $s + L/2 \le t \le s+L$ the equation

$$d(t) \le \alpha_{N,\,L}\,d(s) + \beta_N(L,\ L/2) \tag{6.17}$$

holds with

$$\alpha_{N,\,L} = e^{-\lambda_{N+1}\frac{L}{2}} + a_1\left[M(1+k)\,\lambda_{N+1}^{-1+\theta} + \frac{q}{1-q}\right]e^{a_2L}.$$

It is clear that $\alpha_{N,\,L} \le 1/2$ if

$$\lambda_{N+1}L > 4\ln 2\,, \qquad \lambda_{N+1}^{-1+\theta} > 16\,a_1M(1+k)$$

and

$$a_2L < \ln 2\,, \qquad q \le (1+16\,a_1)^{-1}. \tag{6.18}$$

Let $\rho_1 = \rho_1(M,\ \theta,\ \lambda_1)$ be such that equation (6.18) holds for $L = \rho\lambda_{N+1}^{-\theta}$ and for the parameter q of the form (6.5) with $0 < \rho \le \rho_1$. Then equation (6.8) with $\Lambda = 4(1+4a_1M(1+k)\rho_1)$ implies that $\alpha_{N,\,L} \le 1/2$. Let $t_n = t_* + (1/2)nL$. Then it follows from (6.17) that

$$d(t_{n+1}) \le \frac{1}{2} d(t_n) + \beta_N(L,\ L/2), \qquad n = 0,\ 1,\ 2,\ \ldots$$

After iterations we find that

$$d(t_n) \le 2^{-n} d(t_0) + 2\,\beta_N\Big(L,\ \frac{L}{2}\Big), \qquad n = 0,\ 1,\ 2,\ \ldots \tag{6.19}$$

Equation (6.17) also gives us that

$$d(t) \le \frac{1}{2} d(t_n) + \beta_N\Big(L,\ \frac{L}{2}\Big), \qquad t_n + \frac{L}{2} \le t \le t_n + L\,.$$

Therefore, it follows from (6.19) that

$$d(t) \le 2\exp\left\{-\frac{2}{L}(t-t_*)\ln 2\right\} d(t_*) + 2\,\beta_N\Big(L,\ \frac{L}{2}\Big)$$

for all $t \ge t_* + L/2$. This implies (6.9) and (6.11) if we take $\gamma = \lambda_{N+1}$ in the equation for $\beta_N(L,\ L/2)$. Thus, **Theorem 6.1 is proved**.

In particular, it should be noted that relations (6.9) and (6.11) also mean that a solution to problem (0.1) possessing the property (6.10) reaches the layer of the thickness $\varepsilon_N = c_1 \exp\{-c_2\, \lambda_{N+1}^{1-\theta}\}$ adjacent to the surface $\{\mathbf{M}_t^L\}$ given by equation (6.6) for t large enough. Moreover, it is clear that if problem (0.1) is autonomous $(B(u,\ t) \equiv B(u))$ and if it possesses a global attractor, then the attractor lies in this layer. In the autonomous case $\mathbf{M}^L$ does not depend on t (see Exercise 6.1). These observations give us some information about the position of the attractor in the phase space. Sometimes they enable us to establish the so-called localization theorems for the global attractor.

Exercise *6.2* Let $\|B(u,\ t)\| \le M$. Use equations (1.4) and (1.8) to show that

$$\|A^\theta u(t)\| \le e^{-\lambda_1(t-s)} \|A^\theta u_0\| + R_0\,,$$

where $R_0 = M(1+k)\,\lambda_1^{-1+\theta}$.

In particular, the result of this exercise means that assumption (6.10) holds for any $R > R_0$ and for t_* large enough under the condition $\|B(u,\ t)\| \le M$. In the general case equation (6.10) is a variant of the dissipativity property.

Exercise *6.3* Let $v_0 = v_{0,\,s}^L(t,\ s)$ be a function from $C_{\gamma,\,\theta}(s-L,\ s)$. Assume that

$$v_n^L \equiv v_{n,\,s}^L(t,\ s) = \mathbf{B}_p^{s,\,L}[v_{n-1}](t), \qquad n = 1,\ 2,\ \ldots$$

and

$$\Phi_n^L(p,\ s) = Q v_{n,\,s}^L(s,\ p), \qquad n = 0,\ 1,\ 2,\ \ldots$$

Show that the assertions of Theorem 6.1 remain true for the function $\Phi_n^L(p,\ s)$ if we add the term

$$q^n\left(|v_0|_s + c_0(D_1+R)\right)$$

to the right-hand sides of equations (6.9) and (6.11). Here q is defined by equality (6.5) and $|v_0|_s$ is the norm of the function v_0 in the space $C_{\gamma,\,\theta}(s-L,\, s)$.

Therefore, the function $\Phi_n^L(p,\, s)$ generates a collection of approximate inertial manifolds of the exponential (with respect to λ_{N+1}) order for n large enough.

Example *6.1*

Let us consider the nonlinear heat equation in a bounded domain $\Omega \subset \mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u,\, \nabla u), \quad x \in \Omega,\ t > s\ , \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=s} = u_0(x)\ . \end{cases} \tag{6.20}$$

Assume that the function $f(w,\, \xi)$ possesses the properties

$$\left|f(u_1,\, \xi_1) - f(u_2,\, \xi_2)\right| \le C_1\left(|u_1 - u_2| + |\xi_1 - \xi_2|\right), \qquad |f(u,\, \xi)| \le C_2\,.$$

We use Theorem 6.1 and the asymptotic formula

$$\lambda_N \sim c_0 N^{2/d}, \qquad N \to \infty\,,$$

for the eigenvalues of the operator $-\Delta$ in $\Omega \subset \mathbb{R}^d$ to obtain that in the Sobolev space $H_0^1(\Omega)$ for any N there exists a finite-dimensional Lipschitzian surface $\mathbf{M}_N$ of the dimension N such that

$$\mathrm{dist}_{H_0^1(\Omega)}(u(t),\, \mathbf{M}_N) \le C_1 \exp\{-\sigma_1 N^{1/d}(t-t_*)\} + C_2 \exp\{-\sigma_2 N^{1/d}\}$$

for $t \ge t_*$ and for any mild (in $H_0^1(\Omega)$) solution $u(t)$ to problem (6.20). Here t_* is large enough, C_j and σ_j are positive constants.

Exercise *6.4* Consider the abstract form of the two-dimensional system of the Navier-Stokes equations

$$\frac{du}{dt} + \nu A u + b(u,\, u) = f(t), \qquad u\big|_{t=0} = u_0 \tag{6.21}$$

(see Example 3.5 and Exercises 4.10 and 4.11 of Chapter 2). Assume that $\|A^{1/2} f(t)\| < C$ for $t \ge 0$. Use the dissipativity property for (6.21) and the formula

$$c_0 k \le \frac{\lambda_k}{\lambda_1} \le c_1 k$$

for the eigenvalues of the operator A to show that there exists a collection of functions $\{\Phi(p,\, t) :\, t \ge 1\}$ from $PD(A)$ into $(1-P)D(A)$ possessing the properties

a) $\|A\Phi(p,\, t)\| \le c_1 N^{-1/2}$;

$\|A(\Phi(p_1,\, t) - \Phi(p_2,\, t))\| \le c_2 \|A(p_1 - p_2)\|$

for any $p,\ p_1,\ p_2 \in PD(A)$;

b) for any solution $u(t) \in D(A)$ to problem (6.21) there exists $t^* > 1$ such that

$$\|AQ(u(t) - \Phi(Pu(t),\, t))\| \le$$

$$\le c_3 \exp\{-\sigma_0 N^{1/2}(t - t_*)\} + c_4 \exp\{-\sigma_1 N^{1/2}\}.$$

Here P is the orthoprojector onto the first N eigenelements of the operator A.

— Exercise *6.5* Use Theorem 6.1 to construct approximate inertial manifolds for (a) the nonlocal Burgers equation, (b) the Cahn-Hilliard equation, and (c) the system of reaction-diffusion equations (see Sections 3 and 4 of Chapter 2).

In conclusion of the section we note (see [8], [9]) that in the autonomous case the approximate IM can also be built using the equation

$$\langle \Phi'(p);\ -Ap + PB(p + \Phi(p))\rangle + A\Phi(p) = QB(p + \Phi(p)). \tag{6.22}$$

Here $p \in PH$, $Q = I - P$, $\Phi'(p)$ is the Frechét derivative and $\langle \Phi'(p),\, w\rangle$ is its value at the point p on the element w. At least formally, equation (6.22) can be obtained if we substitute the pair $\{p(t);\ \Phi(p(t))\}$ into equation (6.3). The second of equations (6.3) implicitly contains a small parameter λ_{N+1}^{-1}. Therefore, using (6.22) we can suggest an iteration process of calculation of the sequence $\{\Phi_m\}$ giving the approximate IM:

$$A\Phi_k(p) = QB(p + \Phi_{\nu_1(k)}(p)) +$$

$$+ \langle \Phi'_{\nu_2(k)}(p)\ ;\ Ap - PB(p + \Phi_{\nu_3(k)}(p))\rangle\ , \qquad k \ge 1\,, \tag{6.23}$$

where the integers $\nu_i(k)$ are such that

$$0 \le \nu_i(k) \le k - 1\,, \qquad \lim_{k \to \infty} \nu_i(k) = \infty\,, \qquad i = 1,\ 2,\ 3\,.$$

One should also choose the zeroth approximation and concretely define the form of the values $\nu_i(k)$ (for example, we can take $\Phi_0(v) \equiv 0$ and $\nu_i(k) = k - 1$, $i = 1,\ 2,\ 3$). When constructing a sequence of approximate IMs one has to solve only a linear stationary problem on each step. From the point of view of concrete calculations this gives certain advantages in comparison with the construction used in Theorem 6.1. However, these manifolds have the power order of approximation only (for detailed discussion of this construction and for proofs see [9]).

— Exercise *6.6* Prove that the mapping $\Phi_1(v)$ has the form (6.4) under the condition $\Phi_0(v) \equiv 0$. Write down the equation for $\Phi_2(v)$ when $\nu_1(2) = 1$, $\nu_2(2) = \nu_3(2) = 0$.

§ 7 Inertial Manifold for Second Order in Time Equations

The approach to the construction of IM given in Sections 2–4 is essentially based on the fact that the system has form (0.1) with a selfadjoint positive operator A. However, there exists a wide class of problems which cannot be reduced to this form. From the point of view of applications the important representatives of this class are second order in time systems arising in the theory of nonlinear oscillations:

$$\begin{cases} \dfrac{d^2u}{dt^2} + 2\varepsilon\dfrac{du}{dt} + Au = B(u,\ t), \quad t > s,\ \ \varepsilon > 0, \\ u\big|_{t=s} = u_0, \quad \dfrac{du}{dt}\Big|_{t=s} = u_1 . \end{cases} \tag{7.1}$$

In this section we study the existence of IM for problem (7.1). We assume that A is a selfadjoint positive operator with discrete spectrum (μ_k and e_k are the corresponding eigenvalues and eigenelements) and the mapping $B(u,\ t)$ possesses the properties of the type (1.2) and (1.3) for $0 \le \theta \le 1/2$, i.e. $B(u,\ t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H such that

$$\|B(0,\ t)\| \le M_0,$$
$$\|B(u_1,\ t) - B(u_2,\ t)\| \le M_1\|A^\theta(u_1 - u_2)\|, \tag{7.2}$$

where $0 \le \theta \le 1/2$ and $u_1,\ u_2 \in D(\mathcal{A}^\theta) \equiv \mathcal{F}_\theta$.

The simplest example of a system of the form (7.1) is the following nonlinear wave equation with dissipation:

$$\begin{cases} \dfrac{\partial^2u}{\partial t^2} + 2\varepsilon\dfrac{\partial u}{\partial t} - \dfrac{\partial^2u}{\partial x^2} + f\left(x,\ t;\ u,\ \dfrac{\partial u}{\partial x}\right) = 0, \quad 0 < x < L,\ \ t > s, \\ u\big|_{x=0} = u\big|_{x=L} = 0, \\ u\big|_{t=s} = u_0(x), \quad \dfrac{\partial u}{\partial t}\Big|_{t=s} = u_1(x) . \end{cases} \tag{7.3}$$

Let $\mathcal{H} = D(A^{1/2}) \times H$. It is clear that $\mathcal{H}$ is a separable Hilbert space with the inner product

$$(U,\ V) = (Au_0,\ v_0) + (u_1,\ v_1), \tag{7.4}$$

where $U = (u_0;\ u_1)$ and $V = (v_0;\ v_1)$ are elements of $\mathcal{H}$. In the space $\mathcal{H}$ problem (7.1) can be rewritten as a system of the first order:

$$\frac{d}{dt}U(t) + \mathbf{A}U(t) = \mathbf{B}(U(t),\ t), \quad t > s; \qquad U\big|_{t=s} = U_0. \tag{7.5}$$

Here

$$U(t) = \left(u(t);\ \frac{\partial u(t)}{\partial t}\right), \qquad U_0 = (u_0;\ u_1) \in \mathcal{H}.$$

The linear operator $\mathbf{A}$ and the mapping $\mathbf{B}(U,\ t)$ are defined by the equations:

$$\mathbf{A}U = (-u_1;\ Au_0 + 2\varepsilon u_1), \qquad D(\mathbf{A}) = D(A) \times D(A^{1/2}), \tag{7.6}$$

$$\mathbf{B}(U,\ t) = (0;\ B(u_0,\ t)), \quad U = (u_0;\ u_1).$$

— Exercise *7.1* Prove that the eigenvalues and eigenvectors of the operator $\mathbf{A}$ have the form:

$$\lambda_n^{\pm} = \varepsilon \pm \sqrt{\varepsilon^2 - \mu_n}, \quad f_n^{\pm} = (e_n;\ -\lambda_n^{\pm} e_n), \quad n = 1,\ 2,\ \ldots, \tag{7.7}$$

where μ_n and e_n are the eigenvalues and eigenvectors of A.

— Exercise *7.2* Display graphically the spectrum of the operator A on the complex plane.

These exercises show that although problem (7.1) can be represented in the form (7.5) which is formally identical to (0.1) we cannot use Theorem 3.1 here. Nevertheless, after a small modification the reasoning of Sections 2–4 enables us to prove the existence of IM for problem (7.1). Such a modification based on an idea from [16] is given below.

First of all we prove the solvability of problem (7.1). Let us first consider the linear problem

$$\begin{cases} \dfrac{d^2u}{dt^2} + 2\varepsilon\dfrac{du}{dt} + Au = h(t), \quad t > s\ , \\ u\big|_{t=s} = u_0, \quad \dfrac{du}{dt}\Big|_{t=s} = u_1\ . \end{cases} \tag{7.8}$$

These equations can also be rewritten in the form (cf. (7.5))

$$\frac{d}{dt}U(t) + \mathbf{A}U(t) = H(t), \qquad U\big|_{t=s} = U_0, \tag{7.9}$$

where $U(t) = (u(t);\ \dot{u}(t))$ and $H(t) = (0;\ h(t))$. We define a ***mild solution*** to problem (7.8) (or (7.9)) on the segment $[s,\ s+T]$ as a function $u(t)$ from the class

$$\mathbf{L}_{s,\,T} \equiv C(s,\ s+T;\ \mathcal{F}_{1/2}) \cap C^1(s,\ s+T;\ H) \cap C^2(s,\ s+T;\ \mathcal{F}_{-1/2})$$

which satisfies equations (7.8). Here $\mathcal{F}_\theta = D(A^\theta)$ as before (see Chapter 2). One can prove the existence and uniqueness of mild solutions to (7.8) using the Galerkin method, for example. The ***approximate Galerkin solution*** of the order m is defined as a function

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k$$

satisfying the equations

$$\begin{cases} (\ddot{u}_m(t),\ e_j) + 2\varepsilon(\dot{u}_m(t),\ e_j) + (Au_m(t),\ e_j) = (h(t),\ e_j), \quad t > s\ , \\ (u_m(s),\ e_j) = (u_0,\ e_j), \quad (\dot{u}_m(s),\ e_j) = (u_1,\ e_j) \end{cases} \tag{7.10}$$

for $j = 1,\ 2,\ \dots,\ m$. Moreover, we assume that $g_j(t) \in C^1(s,\ s+T)$ and $\dot{g}_j(t)$ is absolutely continuous. Hereinafter we use the notation $\dot{v}(t) = \mathrm{d}v/\mathrm{d}t$. Evidently equations (7.10) can be rewritten in the form

$$\begin{cases} \ddot{u}_m(t) + 2\,\varepsilon\,\dot{u}_m(t) + A\,u_m(t) = p_m h(t)\ , \\ u_m\big|_{t=s} = p_m u_0\,, \quad \dot{u}_m\big|_{t=s} = p_m u_1\ , \end{cases} \tag{7.11}$$

where p_m is the orthoprojector onto $\mathrm{Lin}\{e_1,\ \dots\ e_m\}$ in H.

In the exercises given below it is assumed that

$$h(t) \in L^\infty(\mathbb{R},\ H)\,, \qquad u_0 \in D(A^{1/2})\,, \qquad u_1 \in H\,. \tag{7.12}$$

— **Exercise** *7.3* Show that problem (7.10) is uniquely solvable on any segment $[s,\ s+T]$ and $u_m(t) \in \mathbf{L}_{s,\ T}$.

— **Exercise** *7.4* Show that the energy equality

$$\frac{1}{2}\Big(\|\dot{u}_m(t)\|^2 + \|A^{1/2}u_m(t)\|^2\Big) + 2\,\varepsilon\int_s^t \|\dot{u}_m(\tau)\|^2\,\mathrm{d}\tau =$$

$$= \frac{1}{2}\Big(\|p_m u_1\|^2 + \|A^{1/2}p_m u_0\|^2\Big) + \int_s^t (h(\tau),\ \dot{u}_m(\tau))\,\mathrm{d}\tau \tag{7.13}$$

holds for any solution to problem (7.10).

— **Exercise** *7.5* Using (7.11) and (7.13) prove the a priori estimate

$$\|A^{-1/2}\,\ddot{u}_m(t)\|^2 + \|\dot{u}_m(t)\|^2 + \|A^{1/2}u_m(t)\| \ \le\ C(T,\ u_0,\ u_1)$$

for the approximate Galerkin solution $u_m(t)$ to problem (7.8).

— **Exercise** *7.6* Using the linearity of problem (7.11) show that for every two approximate solutions $u_m(t)$ and $u_{m'}(t)$ the estimate

$$\|A^{-1/2}(\ddot{u}_m(t) - \ddot{u}_{m'}(t))\|^2 +$$

$$+ \|\dot{u}_m(t) - \dot{u}_{m'}(t)\| + \|A^{1/2}(u_m(t) - u_{m'}(t))\|^2 \ \le$$

$$\leq C_T \left\|(p_m - p_{m'}) u_1\right\|^2 +$$

$$+ \left\|A^{1/2}(p_m - p_{m'}) u_0\right\|^2 + \operatorname*{ess\ sup}_{\tau \in [s,\ s+T]} \left\|(p_m - p_{m'}) h(\tau)\right\|^2$$

holds for all $t \in [s,\ s+T]$, where $T > 0$ is an arbitrary number.

Exercise 7.7 Using the results of Exercises 7.5 and 7.6 show that we can pass to the limit $n \to \infty$ in equations (7.11) and prove the existence and uniqueness of mild solutions to problem (7.8) on every segment $[s,\ s+T]$ under the condition (7.12).

Exercise 7.8 For a mild solution $u(t)$ to problem (7.8) prove the energy equation:

$$\frac{1}{2}\left(\|\dot{u}(t)\|^2 + \|A^{1/2} u(t)\|^2\right) + 2\varepsilon \int_s^t \|\dot{u}(\tau)\|^2 d\tau =$$

$$= \frac{1}{2}\left(\|u_1\|^2 + \left\|A^{1/2} u_0\right\|^2\right) + \int_s^t (h(\tau),\ \dot{u}(\tau)) d\tau\ . \tag{7.14}$$

In particular, the exercises above show that for $h(t) \equiv 0$ problem (7.8) generates a linear evolutionary semigroup $e^{-t\mathrm{A}}$ in the space $\mathcal{H} = D(A^{1/2}) \times H$ by the formula

$$e^{-t\mathrm{A}}(u_0;\ u_1) = (u(t);\ \dot{u}(t)), \tag{7.15}$$

where $u(t)$ is a mild solution to problem (7.8) for $h(t) \equiv 0$. Equation (7.14) implies that the semigroup $e^{-t\mathrm{A}}$ is contractive for $\varepsilon \geq 0$.

Exercise 7.9 Assume that conditions (7.12) are fulfilled. Show that the mild solution to problem (7.8) can be presented in the form

$$(\dot{u}(t);\ u(t)) \ = \ e^{-(t-s)\mathrm{A}}(u_0;\ u_1) + \int_s^t e^{-(t-s)\mathrm{A}}(0;\ h(\tau)) d\tau\ , \tag{7.16}$$

where the semigroup $e^{-t\mathrm{A}}$ is defined by equation (7.15).

Let us now consider nonlinear problem (7.1) and define its ***mild solution*** as a function $U(t) \equiv (u(t);\ \dot{u}(t)) \in C(s,\ s+T;\ \mathcal{H})$ satisfying the integral equation

$$U(t) = e^{-(t-s)\mathrm{A}} U_0 + \int_s^t e^{-(t-s)\mathrm{A}} \mathrm{B}(U(\tau),\ \tau) d\tau \tag{7.17}$$

on $[s,\ s+T]$. Here $\mathrm{B}(U(t),\ t) = (0;\ B(u(t),\ t))$ and $U_0 = (u_0;\ u_1)$.

— Exercise *7.10* Show that the estimates

$$\|\mathbf{B}(U,\ t)\|_{\mathcal{H}} \le M(1+\|U\|_{\mathcal{H}}),$$

$$\|\mathbf{B}(U_1,\ t)-\mathbf{B}(U_2,\ t)\|_{\mathcal{H}} \le M\|U_1-U_2\|_{\mathcal{H}}$$

hold in the space $\mathcal{H}=D(A^{1/2})\times H$. Here M is a positive constant.

— Exercise *7.11* Follow the reasoning used in the proof of Theorems 2.1 and 2.3 of Chapter 2 to prove the existence and uniqueness of a mild solution to problem (7.1) on any segment $[s,\ s+T]$.

Thus, in the space $\mathcal{H}$ there exists a continuous evolutionary family of operators $S(t,\ s)$ possessing the properties

$$S(t,\ t)=I, \qquad S(t,\ \tau)\circ S(\tau,\ s)=S(t,\ s),$$

and

$$S(t,\ s)U_0 = (u(t);\ \dot{u}(t)),$$

where $u(t)$ is a mild solution to problem (7.1) with the initial condition $U_0 = (u_0;\ u_1)$.

Let condition $\varepsilon^2>\mu_{N+1}$ hold for some integer N. We consider the decomposition of the space $\mathcal{H}$ into the orthogonal sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 ,$$

where

$$\mathcal{H}_1 = \mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k):\ \ k=1,\ 2,\ \dots N\}$$

and $\mathcal{H}_2$ is defined as the closure of the set

$$\mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k):\ k\ge N+1\}.$$

— Exercise *7.12* Show that the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant with respect to the operator $\mathbf{A}$. Find the spectrum of the restrictions of the operator $\mathbf{A}$ to each of these spaces.

Let us introduce the following inner products in the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ (the purpose of this introduction will become apparent further):

$$\begin{aligned}\langle U,\ V\rangle_1 &= \varepsilon^2(u_0,\ v_0)-(Au_0,\ v_0)+(\varepsilon u_0+u_1,\ \varepsilon v_0+v_1)\ ,\\ \langle U,\ V\rangle_2 &= (Au_0,\ v_0)+(\varepsilon^2-2\mu_{N+1})(u_0,\ v_0)+(\varepsilon u_0+u_1,\ \varepsilon v_0+v_1)\ .\end{aligned} \tag{7.18}$$

Here $U=(u_0;\ u_1)$ and $V=(v_0;\ v_1)$ are elements from the corresponding subspace $\mathcal{H}_i$. Using (7.18) we define a new inner product and a norm in $\mathcal{H}$ by the equalities:

$$\langle U,\ V\rangle = \langle U_1,\ V_1\rangle_1+\langle U_2,\ V_2\rangle_2, \qquad |U| = \langle U,\ U\rangle^{1/2},$$

where $U=U_1+U_2$ and $V=V_1+V_2$ are decompositions of the elements U and V into the orthogonal terms $V_i,\ U_i\in\mathcal{H}_i,\ \ i=1,\ 2$.

Chapter 3

Lemma 7.1.

The estimates

$$|U|_1 \geq \frac{1}{\mu_N^{\theta}} \sqrt{\varepsilon^2 - \mu_N}\,\|A^{\theta} u_0\|, \qquad U = (u_0;\ u_1) \in \mathcal{H}_1; \tag{7.19}$$

$$|U|_2 \geq \frac{1}{\mu_{N+1}^{\theta}}\, \delta_{N,\,\varepsilon} \|A^{\theta} u_0\|, \qquad U = (u_0;\ u_1) \in \mathcal{H}_2 \tag{7.20}$$

hold for $0 \leq \theta \leq 1/2$. *Here*

$$\delta_{N,\,\varepsilon} = \sqrt{\mu_{N+1}}\ \min\left(1,\ \sqrt{\frac{\varepsilon^2 - \mu_{N+1}}{\mu_{N+1}}}\right). \tag{7.21}$$

Proof.

Let $U = (u_0;\ u_1) \in \mathcal{H}_1$. It is evident that in this case $\|A^{\beta} u_0\| \leq \mu_N^{\beta} \|u_0\|$ for any $\beta > 0$. Therefore,

$$|U|_1^2 \ \geq\ \varepsilon^2 \|u_0\|^2 - \|A^{1/2} u_0\|^2 \ \geq\ \mu_N^{-2\theta} (\varepsilon^2 - \mu_N) \|A^{\theta} u_0\|^2,$$

i.e. equation (7.19) holds. Let $U \in \mathcal{H}_2$. Then using the inequality

$$\|A^{\beta} u_0\| \geq \mu_{N+1}^{\beta} \|u_0\|, \qquad \beta > 0, \qquad u_0 \in \overline{\mathrm{Lin}\{e_k :\ k \geq N+1\}} \tag{7.22}$$

for $0 < \delta \leq 1$ we find that

$$|U|_2^2 \ \geq\ \delta^2 \|A^{1/2} u_0\|^2 + (\varepsilon^2 - (1+\delta^2)\,\mu_{N+1}) \|u_0\|^2.$$

If we take $\delta = \delta_{N,\,\varepsilon}\, \mu_{N+1}^{-1/2}$ and use (7.22), then we obtain estimate (7.20). The lemma is proved.

In particular, this lemma implies the estimate

$$\|A^{\theta} u_0\| \leq \mu_{N+1}^{\theta}\, \delta_{N,\,\varepsilon}^{-1} |U| \tag{7.23}$$

for any $U = (u_0;\ u_1) \in \mathcal{H}$, where $0 \leq \theta \leq 1/2$ and $\delta_{N,\,\varepsilon}$ has the form (7.21).

— Exercise *7.13* Prove the equivalence of the norm $|\cdot|$ and the norm generated by the inner product (7.4).

— Exercise *7.14* Show that we can take $\delta_{N,\varepsilon} = \sqrt{\varepsilon^2 - \mu_{N+1}}$ for $\theta = 0$ in (7.20) and (7.23).

— Exercise *7.15* Prove that the eigenvectors $\{f_k^{\pm}\}$ of the operator $\mathbf{A}$ (see (7.7)) possess the following orthogonal properties:

$$\langle f_n^+, f_k^+ \rangle = \langle f_n^-, f_k^- \rangle = \langle f_n^+, f_k^- \rangle = 0, \qquad k \neq n,$$
$$\langle f_k^+, f_k^- \rangle = 0, \qquad 1 \leq k \leq N. \tag{7.24}$$

Note that the last of these equations is one of the reasons of introducing a new inner product.

Let $P_{\mathscr{H}_i}$ be the orthoprojector onto the subspace $\mathscr{H}_i$ in $\mathscr{H}$, $i = 1,\ 2$.

Lemma 7.2.

The equality

$$\left|e^{-\mathrm{A}t}P_{\mathscr{H}_2}\right| = e^{-\lambda_{N+1}^{-}t}, \qquad t \ge 0, \tag{7.25}$$

is valid. Here $|\cdot|$ *is the operator norm which is induced by the corresponding vector norm.*

Proof.

Let $U \in \mathscr{H}_2$. We consider the function $\psi(t) = |e^{-\mathrm{A}t}U|^2$. Since $\mathscr{H}_2$ is invariant with respect to $e^{-\mathrm{A}t}$, the equation

$$\psi(t) = (Au(t),\ u(t)) + (\varepsilon^2 - 2\,\mu_{N+1})(u(t),\ u(t)) + \|\dot{u} + \varepsilon u\|^2$$

holds, where $u(t)$ is a solution to problem (7.8) for $h(t) \equiv 0$. After simple calculations we obtain that

$$\frac{d\psi}{dt} + 2\,\varepsilon\,\psi \;=\; 4(\varepsilon^2 - \mu_{N+1})(\dot{u} + \varepsilon u,\ u)\,.$$

It is evident that

$$2\sqrt{\varepsilon^2 - \mu_{N+1}}\,(\dot{u} + \varepsilon u,\ u) \;\le\; (\varepsilon^2 - \mu_{N+1})\|u\|^2 + \|\dot{u} + \varepsilon u\|^2 \;\le\; \psi(t)\,.$$

Therefore,

$$\frac{d\psi}{dt} + 2\,\varepsilon\,\psi \;\le\; 2\sqrt{\varepsilon^2 - \mu_{N+1}}\;\psi\,.$$

Consequently,

$$\psi(t) \le e^{-\lambda_{N+1}^{-}t}\,\psi(0), \qquad t > 0\,. \tag{7.26}$$

If we now notice that

$$\exp\{-\mathrm{A}t\}\,f_{N+1}^{-} \;=\; e^{-\lambda_{N+1}^{-}t}\,f_{N+1}^{-}\;,$$

then equation (7.26) implies (7.25). Thus, Lemma 7.2 is proved.

Let us consider the subspaces

$$\mathscr{H}_1^{\pm} = \mathrm{Lin}\{f_k^{\pm}\ :\ k \le N\}\,.$$

Equation (7.24) gives us that the subspaces are orthogonal to each other and therefore $\mathscr{H}_1 = \mathscr{H}_1^{+} \oplus \mathscr{H}_1^{-}$. Using (7.24) it is easy to prove (do it yourself) that

$$\left|e^{At}P_{\mathcal{H}_1^-}\right| \le e^{\lambda_N^-|t|}, \qquad t \in \mathbb{R}, \tag{7.27}$$

$$\left|e^{-At}P_{\mathcal{H}_1^+}\right| \le e^{-\lambda_N^+|t|}, \qquad t > 0. \tag{7.28}$$

We use the following pair of orthogonal (with respect to the inner product $\langle\cdot, \cdot\rangle$) projectors in the space $\mathcal{H}$

$$P = P_{\mathcal{H}_1^-}, \qquad Q = I - P = P_{\mathcal{H}_1^+} + P_{\mathcal{H}_2}$$

to construct the inertial manifold of problem (7.1) (or (7.5)). Lemma 7.2 and equations (7.27) and (7.28) imply the dichotomy equations

$$\left|e^{At}P\right| \le e^{\lambda_N^-|t|}, \quad t \in \mathbb{R}; \qquad \left|e^{-At}Q\right| \le e^{-\lambda_{N+1}^-|t|}, \quad t > 0. \tag{7.29}$$

We remind that $\lambda_N^- = \varepsilon - \sqrt{\varepsilon^2 - \mu_k}$ and $\varepsilon^2 > \mu_{N+1}$.

The assertion below plays an important role in the estimates to follow.

Lemma 7.3.

Let $\mathrm{B}(U, t) = (0; B(u_0, t))$, *where* $U = (u_0; u_1) \in \mathcal{H}$ *and* $B(u_0)$ *possesses properties* (7.2). *Then*

$$\left|\mathrm{B}(U, t)\right| \le M_0 + K_N|U|, \qquad U \in \mathcal{H},$$

$$\left|\mathrm{B}(U_1, t) - \mathrm{B}(U_2, t)\right| \le K_N\left|U_1 - U_2\right|, \qquad U_1, U_2 \in \mathcal{H}, \tag{7.30}$$

where

$$K_N = M_1\, \mu_{N+1}^{\theta - 1/2} \max\left(1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}}\right). \tag{7.31}$$

The proof of this lemma follows from the structure of the mapping $\mathrm{B}(U, t)$ and from estimates (7.2) and (7.23).

— Exercise *7.16* Show that one can take $K_N = M_1(\varepsilon^2 - \mu_{N+1})^{-1/2}$ for $\theta = 0$ in (7.30) (*Hint*: see Exercise 7.14).

Let us now consider the integral equation (cf. (2.1) for $L = \infty$)

$$V(t) = \mathfrak{B}_p^s[V](t) \equiv$$

$$\equiv e^{-(t-s)A}p - \int_t^s e^{-(t-\tau)A}P\mathrm{B}(V(\tau), \tau)\,d\tau + \int_{-\infty}^t e^{-(t-\tau)A}Q\mathrm{B}(V(\tau), \tau)\,d\tau \tag{7.32}$$

in the space C_s of continuous vector-functions $U(t)$ on $(-\infty,\ s]$ with the values in $\mathcal{H}$ such that the norm

$$\| U \| \equiv \sup_{t < s} e^{-\gamma(s-t)} |U(t)| < \infty , \qquad \gamma = \frac{1}{2}(\lambda_{N+1}^{-} + \lambda_N^{-}),$$

is finite. Here $p \in P\mathcal{H}$ and $t \in (-\infty,\ s)$.

— Exercise *7.17* Show that the right-hand side of equation (7.32) is a continuous function of the variable t with the values in $\mathcal{H}$.

Lemma 7.4.

The operator $\mathfrak{B}_p^s$ maps the space C_s into itself and possesses the properties

$$\| \mathfrak{B}_p^s[V] \| \le |p| + M_0 \left(\frac{1}{\lambda_N^{-}} + \frac{1}{\lambda_{N+1}^{-}} \right) + \frac{4K_N}{\lambda_{N+1}^{-} - \lambda_N^{-}} \| V \| \tag{7.33}$$

and

$$\| \mathfrak{B}_p^s[V_1] - \mathfrak{B}_p^s[V_2] \| \le \frac{4K_N}{\lambda_{N+1}^{-} - \lambda_N^{-}} \| V_1 - V_2 \| . \tag{7.34}$$

Proof.

Let us prove (7.34). Evidently, equations (7.29) and (7.30) imply that

$$\left| \mathfrak{B}_p^s[V_1(t)] - \mathfrak{B}_p^s[V_2(t)] \right| \le K_N \int_t^s e^{\lambda_N^{-}(\tau - t)} |V_1(\tau) - V_2(\tau)| \, d\tau +$$

$$+ K_N \int_{-\infty}^t e^{-\lambda_{N+1}^{-}(t-\tau)} |V_1(\tau) - V_2(\tau)| \, d\tau .$$

Since

$$|V_1(\tau) - V_2(\tau)| \le e^{\gamma(s-\tau)} \| V_1 - V_2 \| ,$$

it is evident that

$$\left| \mathfrak{B}_p^s[V_1](t) - \mathfrak{B}_p^s[V_2](t) \right| \le q e^{\gamma(s-t)} \| V_1 - V_2 \|$$

with

$$q = K_N \left\{ \int_t^s e^{(\lambda_N^{-} - \gamma)(\tau - t)} d\tau + \int_{-\infty}^t e^{-(\lambda_{N+1}^{-} - \gamma)(t-\tau)} d\tau \right\} .$$

Simple calculations show that $q \le 4K_N(\lambda_{N+1}^{-} - \lambda_N^{-})^{-1}$. Consequently, equation (7.34) holds. Equation (7.33) can be proved similarly. Lemma 7.4 is proved.

Thus, if for some $q < 1$ the condition

$$\lambda_{N+1}^{-} - \lambda_{N}^{-} \geq \frac{4K_N}{q} \tag{7.35}$$

holds, then equation (7.32) is uniquely solvable in C_s and its solution V can be estimated as follows:

$$\|V\| \leq (1-q)^{-1}\left(|p| + M_0\left(\frac{1}{\lambda_N^-} + \frac{1}{\lambda_{N+1}^-}\right)\right). \tag{7.36}$$

Therefore, we can define a collection of manifolds $\{\mathrm{M}_s\}$ in the space $\mathcal{H}$ by the formula

$$\mathrm{M}_s = \{p + \Phi(p,\ s) \colon p \in P\mathcal{H}\}, \tag{7.37}$$

where

$$\Phi(p,\ s) = \int_{-\infty}^{s} e^{-(t-\tau)\mathrm{A}} Q\mathrm{B}(V(\tau),\ \tau)\,d\tau\,. \tag{7.38}$$

Here $V(\tau)$ is a solution to integral equation (7.32). The main result of this section is the following assertion.

Theorem 7.1.

Assume that

$$\varepsilon^2 > \mu_{N+1} \quad \textit{and} \quad \lambda_{N+1}^{-} - \lambda_{N}^{-} \geq \frac{4K_N}{q} \tag{7.39}$$

for some $0 < q < 1$, ***where*** $\lambda_k^- = \varepsilon - \sqrt{\varepsilon^2 - \mu_k}$ ***and*** K_N ***is defined by formula*** (7.31). ***Then the function*** $\Phi(p,\ s)$ ***given by equality*** (7.38) ***satisfies the Lipschitz condition***

$$|\Phi(p_1,\ s) - \Phi(p_2,\ s)| \leq \frac{q}{2(1-q)}|p_1 - p_2| \tag{7.40}$$

and the manifold M_s ***is invariant with respect to the evolutionary operator*** $S(t,\ \tau)$ ***generated by the formula***

$$S(t,\ \tau)U_0 = (u(t);\ \dot{u}(t)), \qquad t \geq s,$$

in $\mathcal{H}$, ***where*** $u(t)$ ***is a solution to problem*** (7.1) ***with the initial condition*** $U_0 = (u_0;\ u_1)$. ***Moreover, if*** $0 < q < 2 - \sqrt{2}$, ***then there exist initial conditions*** $U_0^* = (u_0^*;\ u_1^*) \in \mathrm{M}_s$ ***such that***

$$|S(t,\ s)U_0 - S(t,\ s)U_0^*| \leq C_q\, e^{-\gamma(t-s)}|QU_0 - \Phi(PU_0,\ s)|$$

for $t \geq s$, ***where*** $\gamma = \frac{1}{2}(\lambda_N^- + \lambda_{N+1}^-)$.

The proof of the theorem is based on Lemma 7.4 and estimates (7.29) and (7.30). It almost entirely repeats the corresponding reasonings in Sections 2 and 3. We give the reader an oppotunity to recover the details of the reasonings as an exercise.

Let us analyse condition (7.39). Equation (7.31) implies that (7.39) holds if

$$\varepsilon^2 \ge 2\mu_{N+1}, \qquad \frac{\mu_{N+1}-\mu_N}{2\sqrt{\varepsilon^2-\mu_N}} \ge \frac{4}{q} M_1\, \mu_{N+1}^{\theta-1/2}. \tag{7.41}$$

However, if we assume that

$$\mu_{N+1}-\mu_N \ge \frac{8\sqrt{2}}{q} M_1\, \mu_{N+1}^{\theta}, \tag{7.42}$$

then for condition (7.41) to be fulfilled it is sufficient to require that

$$2\mu_{N+1} \le \varepsilon^2 \le 2\mu_{N+1}+\mu_N. \tag{7.43}$$

Thus, if for some N conditions (7.42) and (7.43) hold, then the assertions of Theorem 7.1 are valid for system (7.1). This enables us to formulate the assertion on the existence of IM as follows.

Theorem 7.2.

Assume that the eigenvalues μ_N ***of the operator*** A ***possess the properties***

$$\inf_N \frac{\mu_N}{\mu_{N+1}} > 0 \quad \textit{and} \quad \mu_{N(k)+1} = c_0 k^{\rho}(1+o(1)), \quad \rho>0, \quad k\to\infty, \tag{7.44}$$

for some sequence $\{N(k)\}$ ***which tends to infinity and satisfies the estimate***

$$\mu_{N(k)+1}-\mu_{N(k)} \ge \frac{8\sqrt{2}}{q} M_1\, \mu_{N(k)+1}^{\theta}, \qquad 0<q<2-\sqrt{2}\,.$$

Then there exists $\varepsilon_0>0$ ***such that the assertions of Theorem*** 7.1 ***hold for all*** $\varepsilon \ge \varepsilon_0$.

Proof.

Equation (7.44) implies that there exists k_0 such that the intervals

$$\left[2\mu_{N(k)+1},\ 2\mu_{N(k)+1}+\mu_{N(k)}\right], \qquad k \ge k_0,$$

cover some semiaxis $[\varepsilon_0, +\infty)$. Indeed, otherwise there would appear a subsequence $\{N(k_j)\}$ such that

$$\mu_{N(k_j)} < 2\left(\mu_{N(k_j+1)+1}-\mu_{N(k_j)+1}\right)$$

But that is impossible due to (7.44). Consequently, for any $\varepsilon \ge \varepsilon_0$ there exists $N=N_\varepsilon$ such that equations (7.42), (7.43) as well as (7.39) hold.

Exercise *7.18* Consider problem (7.3) with the function $f\!\left(x, t, u, \frac{\partial u}{\partial x}\right) = f(x, t, u)$ possessing the property

$$|f(x, t, u_1)-f(x, t, u_2)| \le L|u_1-u_2|.$$

Use Theorem 7.1 to find a domain in the plane of the parameters (ε, L) for which one can guarantee the existence of an inertial manifold.

§ 8 Approximate Inertial Manifolds for Second Order in Time Equations

As seen from the results of Section 7, in order to guarantee the existence of IM for a problem of the type

$$\begin{gathered} \frac{d^2 u}{dt^2} + \gamma \frac{du}{dt} + A u = B(u) , \\ u\big|_{t=0} = u_0 , \quad \frac{du}{dt}\bigg|_{t=0} = u_1 , \end{gathered} \tag{8.1}$$

we have to require that the parameter $\gamma = 2\varepsilon > 0$ be large enough and the spectral gap condition (see (7.41)) be valid for the operator A. Therefore, as in the case with parabolic equations there arises a problem of construction of an approximate inertial manifold without any assumptions on the behaviour of the spectrum of the operator A and the parameter $\gamma > 0$ which characterizes the resistance force.

Unfortunately, the approach presented in Section 6 is not applicable to the equation of the type (8.1) without any additional assumptions on γ. First of all, it is connected with the fact that the regularizing effect which takes place in the case of parabolic equations does not hold for second order equations of the type (8.1) (in the parabolic case the solution at the moment $t > 0$ is smoother than its initial condition).

In this section (see also [17]) we suggest an iteration scheme that enables us to construct an approximate IM as a solution to a class of linear problems. For the sake of simplicity, we restrict ourselves to the case of autonomous equations $(B(u, t) \equiv \equiv B(u))$. The suggested scheme is based on the equation in functional derivatives such that the function giving the original true IM should satisfy it. This approach was developed for the parabolic equation in [9] (see also [8]). Unfortunately, this approach has two defects. First, approximate IMs have the power order (not the exponential one as in Section 6) and, second, we cannot prove the convergence of approximate IMs to the exact one when the latter exists.

Thus, in a separable Hilbert space H we consider a differential equation of the type (8.1) where γ is a positive number, A is a positive selfadjoint operator with discrete spectrum and $B(\cdot)$ is a nonlinear mapping from the domain $D(A^{1/2})$ of the operator $A^{1/2}$ into H such that for some integer $m \ge 2$ the function $B(u)$ lies in C^m as a mapping from $D(A^{1/2})$ into H and for every $\rho > 0$ the following estimates hold:

$$\left\| \langle B^{(k)}(u);\ w_1, \ldots, w_k \rangle \right\| \le C_\rho \prod_{j=1}^{k} \left\| A^{1/2} w_j \right\| , \tag{8.2}$$

$$\left\| \langle B^{(k)}(u) - B^{(k)}(u^*);\ w_1, \ldots, w_k \rangle \right\| \le C_\rho \| A^{1/2}(u-u^*) \| \prod_{j=1}^{k} \left\| A^{1/2} w_j \right\|, \quad (8.3)$$

where $k = 0, 1, \ldots, m$, $\|\cdot\|$ is a norm in the space H, $\|A^{1/2}u\| \le \rho$, $\|A^{1/2}u^*\| \le \rho$, and $w_j \in D(A^{1/2})$. Here $B^{(k)}(u)$ is the Frechét derivative of the order k of $B(u)$ and $\langle B^{(k)}(u);\ w_1, \ldots, w_k\rangle$ is its value on the elements $w_1, \ldots, w_k$.

Let $L_{m,R}$ be a class of solutions to problem (8.1) possessing the following properties of regularity:

I) for $k = 0, 1, \ldots, m-1$ and for all $T > 0$

$$u^{(k)}(t) \in C(0, T; D(A))$$

and

$$u^{(m)}(t) \in C(0, T; D(A^{1/2})), \qquad u^{(m+1)}(t) \in C(0, T; H),$$

where $C(0, T; V)$ is the space of strongly continuous functions on $[0, T]$ with the values in V, hereinafter $u^{(k)}(t) = \partial_t^k u(t)$;

II) for any $u \in L_{m,R}$ the estimate

$$\|u^{(k+1)}(t)\|^2 + \|A^{1/2}u^{(k)}(t)\|^2 + \|Au^{(k-1)}(t)\|^2 \le R^2 \quad (8.4)$$

holds for $k = 1, \ldots, m$ and for $t \ge t^*$, where t^* depends on u_0 and u_1 only.

In fact, the classes $L_{m,R}$ are studied in [18]. This paper contains necessary and sufficient conditions which guarantee that a solution belongs to a class $L_{m,R}$. It should be noted that in [18] the nonlinear wave equation of the type

$$\begin{aligned} &\partial_t^2 u + \gamma \partial_t u - \Delta u + g(u) = f(x), \quad x \in \Omega, \ t > 0, \\ &u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x), \end{aligned} \quad (8.5)$$

serves as the main example. Here $\gamma > 0$, $f(x) \in C^\infty(\overline{\Omega})$ and the conditions set on the function $g(s)$ from $C^\infty(\mathbb{R})$ are such that we can take $g(u) = \sin u$ or $g(u) = u^{2p+1}$, where $p = 0, 1, 2, \ldots$ for $d = \dim \Omega \le 2$ and $p = 0, 1$ for $d = 3$. In this example the classes $L_{m,R}$ are nonempty for all m. Other examples will be given in Chapter 4.

We fix an integer N and assume $P = P_N$ to be the projector in H onto the subspace generated by the first N eigenvectors of the operator A. Let $Q = I - P$. If we apply the projectors P and Q to equation (8.1), then we obtain the following system of two equations for $p(t) = PU(t)$ and $q(t) = Qu(t)$:

$$\begin{aligned} &\partial_t^2 p + \gamma \partial_t p + Ap = PB(p+q), \\ &\partial_t^2 q + \gamma \partial_t q + Aq = QB(p+q). \end{aligned} \quad (8.6)$$

The reasoning below is formal. Its goal is to obtain an iteration scheme for the determination of an approximate IM. We assume that system (8.6) has an invariant manifold of the form

$$\mathbf{M} = \{(p + h(p, \dot{p});\ \dot{p} + l(p, \dot{p})):\ p, \dot{p} \in PH\} \quad (8.7)$$

in the phase space $D(A^{1/2}) \times H$. Here h and l are smooth mappings from $PH \times PH$ into $QD(A)$. If we substitute $q(t) = h(p(t),\ \partial_t p(t))$ and $\partial_t q(t) = l(p(t),\ \partial_t p(t))$ in the second equality of (8.6), then we obtain the following equation:

$$\langle \delta_p l;\ \dot p\rangle + \langle \delta_{\dot p} l;\ -\gamma \dot p - A p + PB(p + h(p,\ \dot p))\rangle +$$
$$+ \gamma l(p,\ \dot p) + A h(p,\ \dot p) \ = \ QB(p + h(p,\ \dot p))\ .$$

The compatibility condition

$$l(p(t),\ \partial_t p(t)) \ = \ \partial_t h(p(t),\ \partial_t p(t))$$

gives us that

$$l(p,\ \dot p) \ = \ \langle \delta_p h;\ \dot p\rangle + \langle \delta_{\dot p} h;\ -\gamma \dot p - A p + PB(p + h(p,\ \dot p))\rangle\ .$$

Hereinafter $\delta_p f$ and $\delta_{\dot p} f$ are the Frechét derivatives of the function $f(p, \dot p)$ with respect to p and $\dot p$; $\langle \delta_p f;\ w\rangle$ and $\langle \delta_{\dot p} f;\ w\rangle$ are values of the corresponding derivatives on an element w.

Using these formal equations, we can suggest the following iteration process to determine the class of functions $\{h_k;\ l_k\}$ giving the sequence of approximate IMs with the help of (8.7):

$$A h_k(p,\ \dot p) = QB(p + h_{k-1}(p,\ \dot p)) - \gamma l_{\nu(k)}(p,\ \dot p) - \langle \delta_p l_{k-1};\ \dot p\rangle -$$
$$- \langle \delta_{\dot p} l_{k-1};\ -\gamma \dot p - A p + PB(p + h_{k-1}(p,\ \dot p))\rangle\ , \tag{8.8}$$

where $k = 1,\ 2,\ 3,\ \ldots$ and the integers $\nu(k)$ should be choosen such that $k - 1 \le$ $\le \nu(k) \le k$. Here $l_k(p,\ \dot p)$ is defined by the formula

$$l_k(p,\ \dot p) = \langle \delta_p h_{k-1};\ \dot p\rangle + \langle \delta_{\dot p} h_{k-1};\ -\gamma \dot p - A p + PB(p + h_{k-1}(p,\ \dot p))\rangle\ , \tag{8.9}$$

where $k = 1,\ 2,\ 3,\ \ldots$. We also assume that

$$h_0(p,\ \dot p) \equiv l_0(p,\ \dot p) \equiv 0\ . \tag{8.10}$$

Exercise *8.1* Find the form of $h_1(p,\ \dot p)$ and $l_1(p,\ \dot p)$ for $\nu(1) = 0$ and for $\nu(1) = 1$.

The following assertion contains information on the smoothness properties of the functions h_n and l_n which will be necessary further.

Theorem 8.1.

Assume that the class of functions $\{h_n;\ l_n\}$ ***is defined according to*** (8.8)–(8.10). ***Then for each*** n ***the functions*** h_n ***and*** l_n ***belong to the class*** C^m ***as mappings from*** $PH \times PH$ ***into*** QH ***and for all integers*** $\alpha,\ \beta \ge 0$ ***such that*** $\alpha + \beta \le m$ ***the estimates***

$$\left\|A\langle D^{\alpha,\,\beta}h_n(p,\,\dot p);\;w_1,\,\dots,\,w_\alpha;\;\dot w_1,\,\dots,\,\dot w_\beta\rangle\right\| \le$$

$$\le C_{\alpha,\,\beta,\,R}\prod_{i=1}^{\alpha}\|Aw_i\|\cdot\prod_{i=1}^{\beta}\|A^{1/2}\dot w_i\|\,, \tag{8.11}$$

$$\left\|A^{1/2}\langle D^{\alpha,\,\beta}l_n(p,\,\dot p);\quad w_1,\,\dots,\,w_\alpha;\quad \dot w_1,\,\dots,\,\dot w_\beta\rangle\right\| \le$$

$$\le C_{\alpha,\,\beta,\,R}\prod_{i=1}^{\alpha}\|Aw_i\|\cdot\prod_{i=1}^{\beta}\|A^{1/2}\dot w_i\| \tag{8.12}$$

are valid for all p ***and*** $\dot p$ ***from*** PH ***such that*** $\|Ap\| \le R$ ***and*** $\|A^{1/2}\dot p\| \le R$. ***Hereinafter*** $D^{\alpha,\,\beta}f$ ***is the mixed Frechét derivative of the function*** f ***of the order*** α ***with respect to*** p ***and of the order*** β ***with respect to*** $\dot p$***; the values*** w_j ***and*** $\dot w_j$ ***are from*** PH. ***Moreover, if*** $\alpha = 0$ ***or*** $\beta = 0$***, then the corresponding products in*** (8.11) ***and*** (8.12) ***should be omitted.***

Proof.

We use induction with respect to n. It follows from (8.10) and (8.2) that estimates (8.11) and (8.12) are valid for $n = 0,\ 1$. Assume that (8.11) and (8.12) hold for all $n \le k-1$. Then the following lemma holds.

Lemma 8.1.

Let $F_\nu(p,\,\dot p) = B(p + h_\nu(p,\,\dot p))$ *and let*

$$F_\nu^{\alpha,\,\beta}(w) = \langle D^{\alpha,\,\beta}F_\nu(p,\,\dot p);\;w_1,\,\dots,\,w_\alpha;\;\dot w_1,\,\dots,\,\dot w_\beta\rangle\,.$$

Then for $\nu \le k-1$ *and for all integers* $\alpha,\ \beta \ge 0$ *such that* $\alpha + \beta \le m$ *the estimate*

$$F_\nu^{\alpha,\,\beta}(w) \;\le\; C\prod_{i=1}^{\alpha}\|Aw_i\|\cdot\prod_{i=1}^{\beta}\|A^{1/2}\dot w_i\| \tag{8.13}$$

holds, where $w_j,\ \ \dot w_j,\ \ p,\ \ \dot p \in PH$ *and* $\|Ap\| \le R,\ \ \|A^{1/2}\dot p\| \le R$.

Proof.

It is evident that $F_\nu^{\alpha,\,\beta}(w)$ is the sum of terms of the type

$$B_\nu^s(y) = \langle B^{(s)}(p + h_\nu(p,\,\dot p));\quad y_1,\,\dots,\,y_s\rangle\,,\qquad s \ge 0\,.$$

Here y_σ is one of the values of the form:

$$y_* = w_\sigma + \langle\delta_p h_\nu;\;\;w_\sigma\rangle\,,$$

$$y_{**} = \langle D^{\sigma,\,\tau}h_\nu;\quad w_1,\,\dots,\,w_\alpha;\quad \dot w_1,\,\dots,\,\dot w_\beta\rangle\,.$$

Equation (8.2) implies that

$$\left\| B_\nu^s(y) \right\| \le C_R \prod_{j=1}^{s} \left\| A^{1/2} y_j \right\| .$$

Therefore, the induction hypothesis gives us (8.13).

Let us prove (8.12). The induction hypothesis implies that it is sufficient to estimate the derivatives of the second term in the right-hand side of (8.9). It has the form

$$\langle \delta_{\dot p} h_{k-1};\ \ D_k(p,\, \dot p) \rangle , \tag{8.14}$$

where

$$D_k(p,\, \dot p) = -\gamma \dot p - A p + PB(p + h_{k-1}(p,\, \dot p)) .$$

The Frechét derivatives of value (8.14)

$$\langle D^{\alpha,\,\beta} \langle \delta_{\dot p} h_{k-1};\ D_k(p,\, \dot p) \rangle;\ w_1,\, \dots,\, w_\alpha;\ \dot w_1,\, \dots,\, \dot w_\beta \rangle$$

are sums of the terms of the type

$$G(\sigma,\, \tau) \equiv \langle D^{\sigma,\, \tau+1} h_{k-1}(p,\, \dot p);\ w_{j_1},\, \dots,\, w_{j_\sigma};\ \dot w_{i_1},\, \dots,\, \dot w_{i_\tau},\ y_{\sigma,\,\tau} \rangle ,$$

where

$$y_{\sigma,\,\tau} = \langle D^{\alpha-\sigma,\ \beta-\tau} D_k(p,\, \dot p);\ w_{\omega_1},\, \dots,\, w_{\omega_{\alpha-\sigma}};\ \dot w_{\rho_1},\, \dots,\, \dot w_{\rho_{\beta-\tau}} \rangle .$$

Here $0 \le \sigma \le \alpha$, $0 \le \tau \le \beta$ and the sets of indices possess the following properties:

$$\{j_1,\, \dots,\, j_\sigma\} \cap \{\omega_1,\, \dots,\, \omega_{\alpha-\sigma}\} = \varnothing ,$$

$$\{j_1,\, \dots,\, j_\sigma\} \cup \{\omega_1,\, \dots,\, \omega_{\alpha-\sigma}\} = \{1,\ 2,\ \dots,\ \alpha\} ;$$

$$\{i_1,\, \dots,\, i_\tau\} \cap \{\rho_1,\, \dots,\, \rho_{\beta-\tau}\} = \varnothing ,$$

$$\{i_1,\, \dots,\, i_\tau\} \cup \{\rho_1,\, \dots,\, \rho_{\beta-\tau}\} = \{1,\ 2,\ \dots,\ \beta\} .$$

The induction hypothesis implies that

$$\| AG(\sigma,\, \tau) \| \le C \prod_{\theta=1}^{\sigma} \left\| A w_{j_\theta} \right\| \cdot \prod_{\theta=1}^{\tau} \left\| A^{1/2} \dot w_{i_\theta} \right\| \cdot \left\| A^{1/2} y_{\sigma,\,\tau} \right\| .$$

Using the induction hypothesis again as well as Lemma 8.1 and the inequality

$$\left\| A^{1/2} P h \right\| \le \lambda_N^{1/2} \| P h \| ,$$

we obtain an estimate of the following form (if $\sigma = \alpha$ or $\tau = \beta$, then the corresponding product should be considered to be equal to 1):

$$\left\| A^{1/2} y_{\sigma,\,\tau} \right\| \le C(1 + \lambda_N^{1/2}) \prod_{\theta=1}^{\alpha-\sigma} \left\| A w_{\omega_\theta} \right\| \cdot \prod_{\theta=1}^{\beta-\tau} \left\| A^{1/2} \dot w_{\rho_\theta} \right\| .$$

Hereinafter λ_k is the k-th eigenvalue of the operator A. Thus, it is possible to state that

$$\|AG(\sigma, \tau)\| \le C(1+\lambda_N^{1/2})\prod_i \|Aw_i\| \cdot \prod_i \|A^{1/2}\dot{w}_i\| . \qquad (8.15)$$

Using the inequality

$$\|Qh\| \le \lambda_{N+1}^{-s}\|A^s Qh\| , \qquad s > 0 , \qquad (8.16)$$

and equation (8.15) it is easy to find that estimates (8.12) are valid for $n = k$. If we use (8.8), (8.12) and follow a similar line of reasoning, we can easily obtain (8.11). **Theorem 8.1 is proved**.

Theorem 8.1 and equation (8.4) imply the following lemma.

Lemma 8.2.

Assume that $u(t)$ is a solution to problem (8.1) *lying in $L_{m,R}$, $m \ge 1$. Let $p(t) = Pu(t)$ and let*

$$q_s(t) = h_s(p(t), \partial_t p(t)) , \qquad \bar{q}_s(t) = l_s(p(t), \partial_t p(t)) . \qquad (8.17)$$

Then the estimates

$$\left\|A^{1/2}\bar{q}_s^{(j)}(t)\right\|^2 + \left\|Aq_s^{(j)}(t)\right\|^2 \le C_{R,m}$$

with $0 \le j \le m-1$ and

$$\left\|\bar{q}_s^{(m)}(t)\right\|^2 + \left\|A^{1/2}q_s^{(m)}(t)\right\|^2 \le C_{R,m}$$

are valid for t large enough.

Proof.

It should be noted that $q_s^{(j)}(t)$ is the sum of terms of the form

$$\langle D^{\alpha,\beta}h_s(p, \partial_t p), p^{(i_1)}(t), \dots, p^{(i_\alpha)}(t); \; p^{(\tau_1+1)}(t), \dots, p^{(\tau_\beta+1)}(t)\rangle ,$$

where α, β, $i_1, \dots, i_\alpha$, $\tau_1, \dots, \tau_\beta$ are nonnegative integers such that

$$1 \le \alpha+\beta \le j , \qquad i_1+\dots+i_\alpha+\tau_1+\dots+\tau_\beta = j .$$

Similar equation also holds for $\bar{q}_s^{(i)}(t)$. Further one should use Theorem 8.1 and the estimates

$$\|p^{(k+1)}(t)\|^2 + \|A^{1/2}p^{(k)}(t)\|^2 + \|Ap^{(n-1)}(t)\|^2 \le R^2 , \qquad t \ge t_* , \quad 1 \le k \le m ,$$

which follow from (8.4).

Let us define the induced trajectories of the system by the formula

$$U_s(t) = (u_s(t); \bar{u}_s(t)) ,$$

where $s = 0, 1, 2, \dots$ and

$$u_s(t) = p(t) + q_s(t) , \qquad \bar{u}_s(t) = \partial_t p(t) + \bar{q}_s(t) . \qquad (8.18)$$

Here $p(t) = Pu(t)$, $u(t)$ is a solution to problem (8.1); $q_s(t)$ and $\overline{q}_s(t)$ are defined with the help of (8.17). Assume that $u(t)$ lies in $L_{m,\,R}$. Then Lemma 8.2 implies that the induced trajectories can be estimated as follows:

$$\left\|A^{1/2}\,\overline{u}_s^{(j)}(t)\right\| + \left\|Au_s^{(j)}(t)\right\| \le C_{R,\,s}\,, \qquad 0 \le j \le m-1\,;$$

$$\left\|\overline{u}_s^{(m)}(t)\right\|^2 + \left\|A^{1/2}u_s^{(m)}(t)\right\|^2 \le C_{R,\,s}$$

for t large enough. Using (8.3), (8.4), and the last estimates, it is easy to prove the following assertion (do it yourself).

Lemma 8.3.

Let

$$E_s(t) = B(p(t)+q(t)) - B(p(t)+q_s(t))\,.$$

Then

$$\left\|E_s^{(j)}(t)\right\| \le C_{R,\,j}\sum_{i=0}^{j}\left\|A^{1/2}(q^{(i)}(t) - q_s^{(i)}(t))\right\|$$

for $j = 0,\ 1,\ \ldots,\ m$ *and for* t *large enough.*

The main result of this section is the following assertion.

Theorem 8.2.

Let $u(t)$ ***be a solution to problem*** (8.1) ***lying in*** $L_{m,\,R}$ ***with*** $m \ge 2$. ***Assume that*** $h_n(p,\,\dot{p})$ ***and*** $l_n(p,\,\dot{p})$ ***are defined by*** (8.8)–(8.10). ***Then the estimates***

$$\left\|A\partial_t^j(u(t) - u_n(t))\right\| \le C_{n,\,R}\,\lambda_{N+1}^{-n/2}\,, \tag{8.19}$$

$$\left\|A^{1/2}\,\partial_t^j(\partial_t u(t) - \overline{u}_n(t))\right\| \le C_{n,\,R}\,\lambda_{N+1}^{-n/2}\,, \tag{8.20}$$

are valid for $n \le m-1$ ***and for*** t ***large enough. Here*** $0 \le j \le m-n-1$, $u_n(t)$ ***and*** $\overline{u}_n(t)$ ***are defined by*** (8.18), ***and*** λ_{N+1} ***is the*** $(N+1)$***-th eigenvalue of the operator*** A.

Proof.

Let us consider the difference between the solution $u(t)$ and the trajectory induced by this solution:

$$\chi_s(t) = u(t) - u_s(t)\,, \qquad \overline{\chi}_s(t) = \partial_t u(t) - \overline{u}_s(t)\,, \qquad s \ge 0\,,$$

where $\overline{u}_s(t)$ and $u_s(t)$ are defined by formula (8.18). Since $\chi_0(t) = q(t)$, equation (8.4) implies that

$$\left\|A^{1/2}\chi_0^{(j+1)}(t)\right\| + \left\|A\chi_0^{(j)}(t)\right\| \le C\,, \qquad j = 0,\ 1,\ 2,\ \ldots,\ m-1\,, \tag{8.21}$$

for t large enough. Equations (8.8)–(8.10) also give us that

$$A\chi_1(t) = -\chi_0''(t) - \gamma\chi_0'(t) + QE_0(t).$$

We use Lemma 8.3 and equation (8.21) to find that

$$\left\|A\chi_1^{(j)}(t)\right\| \le C\lambda_{N+1}^{-1/2}, \quad j = 0, 1, \dots, m-2,$$

for t large enough. Therefore, equation (8.19) holds for $n = 0, 1$ and for t large enough. From equations (8.6), (8.8), and (8.9) it is easy to find that

$$A\chi_k = -\partial_t\overline{\chi}_{k-1} - \gamma\partial_t\overline{\chi}_{\nu(k)-1} - \gamma\langle\delta_{\dot p}h_{\nu(k)-1};\ PE_{\nu(k)-1}\rangle - \\ - \langle\delta_{\dot p}l_{k-1};\ PE_{k-1}\rangle + QE_{k-1}$$

and

$$\overline{\chi}_k = \partial_t\chi_{k-1} + \langle\delta_{\dot p}h_{k-1};\ PE_{k-1}\rangle. \tag{8.22}$$

Lemma 8.4.

The estimates

$$\left\|A\,\partial_t^j\langle\delta_{\dot p}h_\nu;\ PE_\nu\rangle\right\| \le C\lambda_N^{1/2}\sum_{s=0}^{j}\left\|A^{1/2}\chi_\nu^{(s)}(t)\right\| \tag{8.23}$$

and

$$\left\|A^{1/2}\,\partial_t^j\langle\delta_{\dot p}l_\nu;\ PE_\nu\rangle\right\| \le C\lambda_N^{1/2}\sum_{s=0}^{j}\left\|A^{1/2}\chi_\nu^{(s)}(t)\right\| \tag{8.24}$$

are valid for t large enough and for each $\nu \ge 0$, where $j = 0, 1, \dots, m-1$.

Proof.

Let $f_\nu = h_\nu$ or $f_\nu = l_\nu$. It is clear that the value $\partial_t^j\langle\delta_{\dot p}f_\nu;\ PE_\nu\rangle$ is the algebraic sum of terms of the form:

$$\langle D^{\alpha,\,\beta+1}f_\nu;\ p^{\gamma_1}, \dots, p^{\gamma_\alpha};\ p^{\sigma_1}, \dots, p^{\sigma_\beta},\ \partial_t^s PE_\nu\rangle.$$

Therefore, Theorem 8.1 and Lemma 8.3 imply (8.23) and (8.24). Lemma 8.4 is proved.

We use Lemmata 8.3 and 8.4 as well as inequality (8.16) to obtain that

$$\left\|A\chi_k^{(j)}(t)\right\| \le c_{k,j}\,\lambda_{N+1}^{-1}\left\{\sum_{s=0}^{j+2}\left\|A\chi_{k-1}^{(s)}(t)\right\| + \sum_{s=0}^{j+1}\left\|A\chi_{\nu(k)-1}^{(s)}(t)\right\|\right\} + \\ + d_{kj}\,\lambda_{N+1}^{-1/2}\sum_{s=0}^{j}\left\|A\chi_{k-1}^{(s)}(t)\right\|, \tag{8.25}$$

where $j = 0, 1, \dots, m-2$ and the numbers $c_{k,j}$ and $d_{k,j}$ do not depend on N.

If we now assume that (8.19) holds for $n \le k-1$, then equation (8.25) implies (8.19) for $n = k$ and for $k \le m-1$. Using (8.22) and (8.23) we obtain equation (8.20). **Theorem 8.2 is proved**.

Corollary 8.1

Let the manifold $\mathbf{M}_n$ *have the form* (8.7) *with* $h(p, \dot{p}) = h_n(p, \dot{p})$ *and* $l(p, \dot{p}) = l_n(p, \dot{p})$. *We also assume that* $U(t) = (u(t); \dot{u}(t))$, *where* $u(t)$ *is the solution to problem* (0.1) *from the class* $L_{m,R}$. *Then*

$$\operatorname{dist}_{D(A)\times D(A^{1/2})}(U(t), \mathbf{M}_n) \le C_n \lambda_{N+1}^{-n/2}, \qquad n = 0,\ 1,\ 2 \ldots m-1 .$$

Thus, the thickness of the layer that attracts the trajectories in the phase space has the power order with respect to λ_{N+1} unlike the semilinear parabolic equations of Section 6.

Example *8.1*

Let us consider the nonlinear wave equation (8.5). Let $d = \dim\Omega \le 2$. We assume the following (cf. [18]) about the function $g(s)$:

$$\varliminf_{|s|\to\infty} s^{-1}\int_0^s g(\sigma)d\sigma \ge 0 ;$$

there exists $C_1 > 0$ such that

$$\varliminf_{|s|\to\infty} s^{-1}\left(sg(s) - C_1\int_0^s g(\sigma)d\sigma\right) \ge 0 ;$$

for any m there exists $\beta(m) > 0$ such that

$$|g^{(m)}(s)| \le C_2(1 + |s|^{\beta(m)}). \tag{8.26}$$

Under these assumptions the solution $u(t)$ lies in $L_{m,R}$ for $R > 0$ large enough if and only if the initial data satisfy some compatibility conditions [18]. Moreover, the global attractor $\mathcal{A}$ of system (8.5) exists and any trajectory lying in $\mathcal{A}$ possesses properties (8.4) for all $t \in \mathbb{R}$ and $k = 1,\ 2,\ \ldots$, [18]. It is easy to see that Theorem 8.2 is applicable here (the form of A, $B(\cdot)$ and H is evident in this case). In particular, Theorem 8.2 gives us that for a trajectory $U(t) = (u(t);\ \partial_t u(t))$ of problem (8.5) which lies in the global attractor $\mathcal{A}$ the estimate

$$\left\{\left\|A\partial_t^j(u(t) - u_n(t))\right\|^2 + \left\|A^{1/2}\partial_t^j(\partial_t u(t) - \bar{u}_n(t))\right\|^2\right\}^{1/2} \le C_{n,R,j}\,\lambda_{N+1}^{-n/2}$$

holds for all $n = 1,\ 2,\ \ldots$, all $j = 1,\ 2,\ \ldots$, and all $t \in \mathbb{R}$. Here $\bar{u}_n(t)$ and $u_n(t)$ are defined with the help of (8.18). Therewith

$$\sup\{\operatorname{dist}(U, \mathbf{M}_n) : U \in \mathcal{A}\} \le c_n \lambda_{N+1}^{-n/2}, \qquad n = 1, 2, \ldots, \tag{8.27}$$

where $\mathbf{M}_n$ is a manifold of the type (8.7) with $h = h_n(p, \dot{p})$ and $l = l_n(p, \dot{p})$. Here $\operatorname{dist}(U, \mathbf{M}_n)$ is the distance between U and $\mathbf{M}_n$ in the space $D(A) \times D(A^{1/2})$. Equation (8.27) gives us some information on the location of the global attractor in the phase space.

Other examples of usage of the construction given here can be found in papers [17] and [19] (see also Section 9 of Chapter 4).

§ 9 *Idea of Nonlinear Galerkin Method*

Approximate inertial manifolds have proved to be applicable to the computational study of the asymptotic behaviour of infinite-dimensional dissipative dynamical systems (for example, see the discussion and the references in [8]). Their usage leads to the appearance of the so-called nonlinear Galerkin method [20] based on the replacement of the original problem by its approximate inertial form. In this section we discuss the main features of this method using the following example of a second order in time equation of type (8.1):

$$\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} + Au = B(u), \qquad u\big|_{t=0} = u_0, \qquad \frac{du}{dt}\bigg|_{t=0} = u_1. \tag{9.1}$$

If all conditions on A and $B(\cdot)$ given in the previous section are fulfilled, then Theorem 8.2 is valid. It guarantees the existence of a family of mappings $\{h_k;\ l_k\}$ from $PH \times PH$ into QH possessing the properties:

1) there exist constants $M_j \equiv M_j(n, \rho)$ and $L_j \equiv L_j(n, \rho)$, $j = 1, 2$, such that

$$\|Ah_n(p_0, \dot{p}_0)\| \le M_1, \qquad \|A^{1/2} l_n(p_0, \dot{p}_0)\| \le M_2\, \mathrm{e}, \tag{9.2}$$

$$\|A(h_n(p_1, \dot{p}_1) - h_n(p_2, \dot{p}_2))\| \le L_1\Big(\|A(p_1 - p_2)\| + \|A^{1/2}(\dot{p}_1 - \dot{p}_2)\|\Big), \tag{9.3}$$

$$\|A^{1/2}(l_n(p_1, \dot{p}_1) - l_n(p_2, \dot{p}_2))\| \le L_1\Big(\|A(p_1 - p_2)\| + \|A^{1/2}(\dot{p}_1 - \dot{p}_2)\|\Big) \tag{9.4}$$

   for all p_j and $\dot{p}_j$ from PH such that

$$\|Ap_j\|^2 + \|A^{1/2}\dot{p}_j\|^2 \le \rho^2, \qquad j = 0, 1, \quad \rho > 0;$$

2) for any solution $u(t)$ to problem (9.1) which lies in $L_{m,R}$ for $m \ge 2$ the estimate

$$\left\{\left\|A(u(t)-u_n(t))\right\|^2+\left\|A^{1/2}(\partial_t u(t)-\overline{u}_n(t))\right\|^2\right\}^{1/2} \le C_{n,\,R}\,\lambda_{N+1}^{-n/2} \tag{9.5}$$

is valid (see Theorem 8.2) for all $n \le m-1$ and t large enough. Here

$$\begin{aligned} u_n(t) &= p(t)+h_n(p(t),\ \partial_t p(t))\ , \\ \overline{u}_n(t) &= \partial_t p(t)+l_n(p(t),\ \partial_t p(t))\ , \end{aligned} \tag{9.6}$$

λ_{N+1} is the $(N+1)$-th eigenvalue of A, and R is the constant from (8.4).

The family $\{h_k;\ l_k\}$ is defined with the help of a quite simple procedure (see (8.8) and (8.9)) which can be reduced to the process of solving of stationary equations of the type $Av=g$ in the subspace QH. Moreover,

$$h_0(p,\ \dot p) \equiv l_0(p,\ \dot p) \equiv 0\,, \qquad h_1(p,\ \dot p)=A^{-1}QB(p)\,, \qquad l_1(p,\ \dot p)\equiv 0\,. \tag{9.7}$$

In particular, estimates (9.5) and (9.6) mean (see Corollary 8.1) that trajectories $U(t)=(u(t);\ \partial_t u(t))$ of system (9.1) are attracted by a small (for N large enough) vicinity of the manifold

$$M_n=\{(p+h_n(p,\ \dot p);\ \dot p+l_n(p,\ \dot p)):\ \ p,\ \dot p\in PH\}\,. \tag{9.8}$$

The sequence of mappings $\{h_n(p,\ \dot p)\}$ generates a family of approximate inertial forms of problem (9.1):

$$\partial_t^2 p+\gamma\partial_t p+Ap=PB(p+h_n(p,\ \partial_t p))\,. \tag{9.9}$$

A finite-dimensional dynamical system in PH which approximates (in some sense) the original system corresponds to each form. For $n=0$ equation (9.9) transforms into the standard Galerkin approximation of problem (9.1) (due to (9.7)). If $n>0$, then we obtain a class of numerical methods which can be naturally called the non-linear Galerkin methods. However, we cannot use equation (9.9) in the computational study directly. The point is that, first, in the calculation of $h_n(p,\ \dot p)$ we have to solve a linear equation in the infinite-dimensional space QH and, second, we can lose the dissipativity property. Therefore, we need additional regularization. It can be done as follows. Assume that $f_n(p,\ \dot p)$ stands for one of the functions $h_n(p,\ \dot p)$ or $l_n(p,\ \dot p)$. We define the value

$$f_n^*(p,\ \dot p)\equiv f_{N,\,M,\,n}(p,\ \dot p)=\chi\Big(R^{-1}\big(\|Ap\|^2+\|A^{1/2}\dot p\|^2\big)^{1/2}\Big)P_M f_n(p,\ \dot p)\,, \tag{9.10}$$

where $\chi(s)$ is an infinitely differentiable function on $\mathbb{R}_+$ such that a) $0\le\chi(s)\le 1$; b) $\chi(s)=1$ for $0\le s\le 1$; c) $\chi(s)=0$ for $s\ge 2$; R is the radius of dissipativity (see (8.4) for $k=0$) of system (9.1); P_M is the orthoprojector in H onto the subspace generated by the first M eigenvectors of the operator A, $M>N$. We consider the following N-dimensional evolutionary equation in the subspace $P_N H$:

$$\begin{aligned} &\partial_t^2 p^* + \gamma \partial_t p^* + A p^* = P_N B(p^* + h_n^*(p^*, \partial_t p^*)) \ , \\ &p^*\big|_{t=0} = P_N u_0 \, , \quad \partial_t p^*\big|_{t=0} = P_N u_1 \ . \end{aligned} \tag{9.11}$$

Exercise *9.1* Prove that problem (9.11) has a unique solution for $t > 0$ and the corresponding dynamical system is dissipative in $P_N H \times P_N H$.

We call problem (9.11) a nonlinear Galerkin (n, N, M)-approximation of problem (9.1). The following assertion is valid.

Theorem 9.1.

Assume that the mappings $h_n(p, \dot{p})$ ***and*** $l_n(p, \dot{p})$ ***satisfy equations*** (9.2)–(9.5) ***for*** $n \le m-1$ ***and for some*** $m \ge 2$. ***Moreover, we assume that*** (9.5) ***is valid for all*** $t > 0$. ***Let*** h_n^* ***and*** l_n^* ***be defined by*** (9.10) ***with the help of*** h_n ***and*** l_n ***and let***

$$u_n^*(t) = p^*(t) + h_n^*(p^*(t), \ \partial_t p^*(t)),$$

$$\overline{u}_n^*(t) = \partial_t p^*(t) + l_n^*(p^*(t), \ \partial_t p^*(t)),$$

where $p^*(t)$ ***is a solution to problem*** (9.11). ***Then the estimate***

$$\left\{ \left\| A^{1/2}(u(t) - u_n^*(t)) \right\|^2 + \left\| \partial_t u(t) - \overline{u}_n^*(t) \right\|^2 \right\}^{1/2} \le$$

$$\le \ (\alpha_1 \lambda_{N+1}^{-(n+1)/2} + \alpha_2 \lambda_{M+1}^{-1/2}) \exp(\beta t) \tag{9.12}$$

holds, where $u(t)$ ***is a solution to problem*** (9.1) ***which lies in*** $L_{m,R}$ ***for*** $m \ge 2$ ***and possesses property*** (8.4) ***for*** $k = 1$ ***and for all*** $t > 0$. ***Here*** $n \le$ $\le m-1$, α_1, α_2 ***and*** β ***are positive constants independent of*** M ***and*** N, λ_k ***is the*** k***-th eigenvalue of the operator*** A.

Proof.

Let $p(t) = P_N u(t)$. We consider the values

$$u(t) - u_n^*(t) = p(t) - p^*(t) + [Q_N u(t) - h_n(p(t), \ \partial_t p(t))] +$$
$$+ [h_n(p(t), \ \partial_t p(t)) - h_n^*(p^*(t), \ \partial_t p^*(t))]$$

and

$$\partial_t u(t) - \overline{u}_n^*(t) = \partial_t (p(t) - p^*(t)) + [Q_N \partial_t u(t) - l_n(p(t), \ \partial_t p(t))] +$$
$$+ [l_n(p(t), \ \partial_t p(t)) - l_n^*(p^*(t), \ \partial_t p^*(t))] \ .$$

The equalities

$$P_M h_n(p(t), \ \partial_t p(t)) = h_n^*(p(t), \ \partial_t p(t))$$

and

$$P_M l_n(p(t),\ \partial_t p(t)) = l_n^*(p(t),\ \partial_t p(t))$$

are valid for the class of solutions under consideration. Therefore, we use (9.5) to find that

$$\left\|A^{1/2}(u(t)-u_n^*(t))\right\| \le$$
$$\le C_1\Big(\left\|A^{1/2}(p(t)-p^*(t))\right\| + \left\|\partial_t p(t) - \partial_t p^*(t)\right\|\Big) + \frac{C_2}{\lambda_{M+1}^{1/2}} + \frac{C_{n,\,R}}{\lambda_{N+1}^{(n+1)/2}} \qquad (9.13)$$

and

$$\left\|\partial_t u(t) - \partial_t u_n^*(t)\right\| \le$$
$$\le C_3\Big(\left\|A^{1/2}(p(t)-p^*(t))\right\| + \left\|\partial_t p(t) - \partial_t p^*(t)\right\|\Big) + \frac{C_4}{\lambda_{M+1}^{1/2}} + \frac{C_{n,\,R}}{\lambda_{n+1}^{(n+1)/2}}\ . \qquad (9.14)$$

Therefore, we must compare the solution $p^*(t)$ to problem (9.11) with the value $p(t) = P_N u(t)$ which satisfies the equation

$$\partial_t^2 p + \gamma \partial_t p + A p \ = \ Q_N B(p + Q_N u) \qquad (9.15)$$

with the same initial conditions as the function $p^*(t)$. Let $r(t) = p(t) - p^*(t)$. Then it follows from (9.11) and (9.15) that

$$\partial_t^2 r(t) + \gamma \partial_t r(t) + A\, r(t) \ = F(t,\ p^*,\ u)\ ,$$
$$r(0) = 0, \quad \partial_t r(0) = 0\ , \qquad (9.16)$$

where

$$F(t,\ p^*,\ u) = Q_N\big[B(u(t)) - B(u_n^*(t))\big]\,.$$

Due to the dissipativity of problems (9.11) and (9.15) we use (9.13) to obtain

$$\|F(t,\ p^*,\ u)\| \ \le\ C_R\Big(\left\|A^{1/2} r(t)\right\|^2 + \|\dot r(t)\|^2\Big)^{1/2} + C_{n,\,R}\,\lambda_{N+1}^{-(n+1)/2} + C\,\lambda_{M+1}^{-1/2}$$

for the class of solutions under consideration. Therefore, equation (9.16) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\dot r(t)\|^2 + \left\|A^{1/2} r(t)\right\|^2\Big) \le\ \overline{C}_R\Big(\|\dot r(t)\|^2 + \left\|A^{1/2} r(t)\right\|^2\Big) + \overline{C}_{n,\,R}\,\lambda_{N+1}^{-(n+1)} + C\,\lambda_{M+1}^{-1}\,.$$

Hence, Gronwall's lemma gives us that

$$\|\dot r(t)\|^2 + \left\|A^{1/2} r(t)\right\|^2 \ \le\ (C_{n,\,R}\ \lambda_{N+1}^{-(n+1)} + C\,\lambda_{M+1}^{-1})\ e^{\overline{C}_R t}\,.$$

This and equations (9.13) and (9.14) imply estimate (9.12). **Theorem 9.1 is proved**.

If we take $n = 0$ and $N = M$ in Theorem 9.1, then estimate (9.12) changes into the accuracy estimate of the standard Galerkin method of the order N. Therefore, if the parameters N, M, and n are compatible such that $\lambda_{M+1} \le \lambda_{N+1}^{n+1}$, then the error of the corresponding nonlinear Galerkin method has the same order of smallness as in the standard Galerkin method which uses M basis functions. However, if we use the nonlinear method, we have to solve a number of linear algebraic systems of the order $M-N$ and the Cauchy problem for system (9.11) which consists of N equations. In particular, in order to determine the value $h_1(p, \dot{p})$ we must solve the equation

$$A h_1(p, \dot{p}) = (P_M - P_N) Q B(p)$$

for $n = 1$ and choose the numbers N and M such that $\lambda_{M+1} \le \lambda_{N+1}^2$. Moreover, if $\lambda_k \cong c_0 k^\sigma (1 + o(1))$, $\sigma > 0$, as $k \to \infty$, then the values N and M must be compatible such that $M \le c_\sigma N^2$.

We note that Theorem 9.1 as well as the corresponding variant of the nonlinear Galerkin method can be used in the study of the asymptotic properties of solutions to the nonlinear wave equation (8.5) under some conditions on the nonlinear term $g(u)$. Other applications of Theorem 9.1 can also be pointed out.

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Chapter 4

The Problem on Nonlinear Oscillations of a Plate in a Supersonic Gas Flow

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In this chapter we use the ideas and the results of Chapters 1 and 3 to study in details the asymptotic behaviour of a class of problems arising in the nonlinear theory of oscillations of distributed parameter systems. The main object is the following second order in time equation in a separable Hilbert space H:

$$\frac{d^2}{dt^2}u + \gamma \frac{d}{dt}u + A^2 u + M\left(\|A^{1/2}u\|^2\right) A u + L u = p(t), \tag{0.1}$$

$$u\big|_{t=0} = u_0, \quad \frac{du}{dt}\bigg|_{t=0} = u_1, \tag{0.2}$$

where A is a positive operator with discrete spectrum in H, $M(s)$ is a real function (its properties are described below), L is a linear operator in H, $p(t)$ is a given bounded function with the values in H, and γ is a nonnegative parameter. The problem of type (0.1) and (0.2) arises in the study of nonlinear oscillations of a plate in the supersonic flow of gas. For example, in Berger's approach (see [1, 2]), the dynamics of a plate can be described by the following quasilinear partial differential equation:

$$\partial_t^2 u + \gamma \partial_t u + \Delta^2 u + \left(\Gamma - \int_\Omega |\nabla u|^2 \, dx\right) \Delta u + \rho \partial_{x_1} u = p(x, t), \tag{0.3}$$

$$x \in (x_1, x_2) \subset \Omega \subset \mathbb{R}^2, \qquad t > 0$$

with boundary and initial conditions of the form

$$u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0(x), \quad \partial_t u\big|_{t=0} = u_1(x). \tag{0.4}$$

Here Δ is the Laplace operator in the domain Ω; $\gamma > 0$, $\rho \ge 0$, and Γ are constants; and $p(x, t)$, $u_0(x)$, and $u_1(x)$ are given functions. Equations (0.3)–(0.4) describe nonlinear oscillations of a plate occupying the domain Ω on a plane which is located in a supersonic gas flow moving along the x_1-axis. The aerodynamic pressure on the plate is taken into account according to Ilyushin's "piston" theory (see, e. g., [3]) and is described by the term $\rho \partial_{x_1} u$. The parameter ρ is determined by the velocity of the flow. The function $u(x, t)$ measures the plate deflection at the point x and the moment t. The boundary conditions imply that the edges of the plate are hinged. The function $p(x, t)$ describes the transverse load on the plate. The parameter Γ is proportional to the value of compressive force acting in the plane of the plate. The value γ takes into account the environment resistance.

Our choice of problem (0.1) and (0.2) as the base example is conditioned by the following circumstances. First, using this model we can avoid significant technical difficulties to demonstrate the main steps of reasoning required to construct a solution and to prove the existence of a global attractor for a nonlinear evolutionary second order in time partial differential equation. Second, a study of the limit regimes of system (0.3)–(0.4) is of practical interest. The point is that the most important (from the point of view of applications) type of instability which can be found in the system under consideration is the flutter, i.e. autooscillations of a plate subjected to aerodynamical loads. The modern look on the flutter instability of a plate is the following: there arises the Andronov-Hopf bifurcation leading to the appearance of a stable limit cycle in the system. However, there are experimental and numerical data that enable us to conjecture that an increase in flow velocity may result in the complication of the dynamics and appearance of chaotic fluctuations [4]. Therefore, the study of the existence and properties of the attractor of the given problem enables us to better understand the mechanism of appearance of a nonlinear flutter.

§ 1 Spaces

As above (see Chapter 2), we use the scale of spaces $\mathcal{F}_s$ generated by a positive operator A with discrete spectrum acting in a separable Hilbert space H. We remind (see Section 2.1) that the space $\mathcal{F}_s$ is defined by the equation

$$\mathcal{F}_s \equiv D(A^s) = \left\{ v : \sum_{k=1}^{\infty} c_k e_k : \sum_{k=1}^{\infty} c_k^2 \lambda_k^{2s} < \infty \right\},$$

where $\{e_k\}$ is the orthonormal basis of the eigenelements of the operator A in H, $\lambda_1 \le \lambda_2 \le \ldots$ are the corresponding eigenvalues and s is a real parameter (for $s = 0$ we have $\mathcal{F}_s = H$ and for $s < 0$ the space $\mathcal{F}_s$ should be treated as a class of formal series). The norm in $\mathcal{F}_s$ is given by the equality

$$\|v\|_s^2 = \sum_{k=1}^{\infty} c_k \lambda_k^{2s} \quad \text{for} \quad v = \sum_{k=1}^{\infty} c_k e_k .$$

Further we use the notation $L^2(0, T; \mathcal{F}_s)$ for the set of measurable functions on the segment $[0, T]$ with the values in the space $\mathcal{F}_s$ such that the norm

$$\|v\|_{L^2(0,\, T;\, \mathcal{F}_s)} = \left(\int_0^T \|v(t)\|_s^2 \, dt \right)^{1/2}$$

is finite. The notation $L^p(0, T; X)$ has a similar meaning for $1 \le p \le \infty$.

We remind that a function $u(t)$ with the values in a separable Hilbert space H is said to be ***Bochner measurable*** on a segment $[0, T]$ if it is a limit of a sequence of functions

$$u_N(t) = \sum_{k=1}^{N} u_{N,\,k} \, \chi_{N,\,k}(t)$$

for almost all $t \in [0, T]$, where $u_{N,\,k}$ are elements of H and $\chi_{N,\,k}(t)$ are the characteristic functions of the pairwise disjoint Lebesgue measurable sets $A_{N,\,k}$. One can prove (see, e.g., the book by K. Yosida [5]) that for separable Hilbert spaces under consideration a function $u(t)$ is measurable if and only if the scalar function $(u(t), h)_H$ is measurable for every $h \in H$. Furthermore, a function $u(t)$ is said to be ***Bochner integrable*** over $[0, T]$ if

$$\int_0^T \|u(t) - u_N(t)\|_H^2 \, dt \to 0, \qquad N \to \infty,$$

where $\{u_N(t)\}$ is a sequence of simple functions defined above. The integral of the function $u(t)$ over a measurable set $S \subset [0, T]$ is defined by the equation

$$\int_S u(t)\, dt = \lim_{N \to \infty} \int_0^T \chi_S(\tau) u_N(\tau)\, d\tau,$$

where $\chi_S(\tau)$ is the characteristic function of the set S and the integral of a simple function in the right-hand side of the equality is defined in an obvious way.

For the function with the values in Hilbert spaces most facts of the ordinary Lebesgue integration theory remain true.

— Exercise *1.1* Let $u(t)$ be a function on $[0, T]$ with the values in a separable Hilbert space H. If there exists a sequence of measurable functions $u_n(t)$ such that $u_n(t) \to u(t)$ almost everywhere, then $u(t)$ is also measurable.

— Exercise *1.2* Show that a measurable function $u(t)$ with the values in H is integrable if and only if $\|u(t)\| \in L^1(0, T)$. Therewith

$$\left\| \int_B u(t)\, d\tau \right\| \le \int_B \|u(t)\|\, d\tau$$

for any measurable set $B \subset [0, T]$.

— Exercise *1.3* Let a function $u(t)$ be integrable over $[0, T]$ and let B be a measurable set from $[0, T]$. Show that

$$\int_B (u(\tau), h)_H \, d\tau = \left(\int_B u(\tau)\, d\tau, h \right)_H$$

for any $h \in H$.

— Exercise *1.4* Show that the space $L^2(0, T; \mathcal{F}_s)$ can be described as a set of series

$$h(t) = \sum_{k=1}^{\infty} c_k(t) e_k,$$

where $c_k(t)$ are scalar functions that are square-integrable over $[0, T]$ and such that

$$\sum_{k=1}^{\infty} \lambda_k^{2s} \int_0^T [c_k(t)]^2 \, dt < \infty . \tag{1.1}$$

Below we also use the space $C(0, T; \mathcal{F}_s)$ of strongly continuous functions on $[0, T]$ with the values in $\mathcal{F}_s$ and the norm

$$|v|_{C(0, T; \mathcal{F}_s)} = \max_{t \in [0, T]} \|v(t)\|_s .$$

Exercise *1.5* Let $u(t)$ be a function with the values in $\mathcal{F}_s$ integrable over $[0, T]$. Show that the function

$$v(t) = \int_0^t u(\tau)\, d\tau$$

lies in $C(0, T; \mathcal{F}_s)$. Moreover, $v(t)$ is an absolutely continuous function with the values in $\mathcal{F}_s$, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any collection of disjoint segments $[\alpha_k, \beta_k] \subset [0, T]$ the condition $\sum_k (\beta_k - \alpha_k) < \delta$ implies that

$$\sum_k \|v(\beta_k) - v(\alpha_k)\|_s < \varepsilon .$$

Exercise *1.6* Show that for any absolutely continuous function $v(t)$ on $[0, T]$ with the values in $\mathcal{F}_s$ there exists a function $u(t)$ with the values in $\mathcal{F}_s$ such that it is integrable over $[0, T]$ and

$$v(t) = v(0) + \int_0^t u(\tau)\, d\tau , \qquad t \in [0, T] .$$

(*Hint:* use the one-dimensional variant of this assertion).

The space

$$W_T = \left\{ v(t) : \ v(t) \in L^2(0, T; \mathcal{F}_1), \quad \dot{v}(t) \in L^2(0, T; H) \right\} \tag{1.2}$$

with the norm

$$|v|_{W_T} = \left(\|v\|^2_{L^2(0, T; \mathcal{F}_1)} + \|\dot{v}\|^2_{L^2(0, T; H)} \right)^{1/2}$$

plays an important role below. Hereinafter the derivative $\dot{v}(t) = dv/dt$ stands for a function integrable over $[0, T]$ and such that

$$v(t) = h + \int_0^t \dot{v}(\tau)\, d\tau$$

almost everywhere for some $h \in H$ (see Exercises 1.5 and 1.6). Evidently, the space W_T is continuously embedded into $C(0, T; H)$, i.e. every function $u(t)$ from W_T lies in $C(0, T; H)$ and

$$\max_{t \in [0, T]} \|u(t)\| \le C|u|_{W_T},$$

where C is a constant. This fact is strengthened in the series of exercises given below.

— Exercise *1.7* Let p_m be the projector onto the span of the set $\{e_k : k = 1, \ldots, m\}$ and let $v(t) \in W_T$. Show that $p_m v(t)$ is absolutely continuous and possesses the property

$$\frac{d}{dt}(p_m v(t)) = p_m \dot{v}(t) \in L^2(0, T; p_m \mathcal{F}_1).$$

— Exercise *1.8* The equations

$$\|p_m v(t)\|_{1/2}^2 = \|p_m v(s)\|_{1/2}^2 + 2\int_s^t (p_m v(\tau), p_m \dot{v}(\tau))_{1/2}\, d\tau \quad (1.3a)$$

and

$$(t-s)\|p_m v(t)\|_{1/2}^2 =$$

$$= \int_s^t \left(\|p_m v(\tau)\|_{1/2}^2 + 2(\tau - s)(p_m v(\tau), p_m \dot{v}(\tau))_{1/2}\right) d\tau \quad (1.3b)$$

are valid for any $0 \le s \le t \le T$ and $v(t) \in W_T$.

— Exercise *1.9* Use (1.3) to prove that

$$\sup_{t \in [0, T]} \|p_m v(t)\|_{1/2} \le C_T |v|_{W_T} \quad (1.4a)$$

and

$$\sup_{t \in [0, T]} \|(p_m - p_k) v(t)\|_{1/2} \le C_T |(p_m - p_k) v|_{W_T}. \quad (1.4b)$$

— Exercise *1.10* Use (1.4) to prove that W_T is continuously embedded into $C(0, T; \mathcal{F}_{1/2})$ and

$$\max_{t \in [0, T]} \|v(t)\|_{1/2} \le C_T |v|_{W_T}.$$

The following three exercises result in a particular case of Dubinskii's theorem (see Exercise 1.13).

— Exercise *1.11* Let $\{h_k(t)\}_{k=0}^{\infty}$ be an orthonormal basis in $L^2(0, T; \mathbb{R})$ consisting of the trigonometric functions

$$h_0(t) = \frac{1}{\sqrt{T}}, \quad h_{2k-1}(t) = \sqrt{\frac{2}{T}}\sin\frac{2\pi k}{T}x, \quad h_{2k}(t) = \sqrt{\frac{2}{T}}\cos\frac{2\pi k}{T}x,$$

$k = 1, 2, \ldots$. Show that $f(t) \in L^2(0, T; \mathcal{F}_s)$ if and only if

$$f(t) = \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} c_{kj}\, h_k(t) e_j \tag{1.5}$$

and

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty} \lambda_j^{2s} \left|c_{kj}\right|^2 < \infty .$$

— Exercise *1.12* Show that the space W_T can be described as a set of series of the form (1.5) such that

$$\sum_{k,\, j=1}^{\infty} \left(k^2 + \lambda_j^2\right)\left|c_{kj}\right|^2 < \infty .$$

— Exercise *1.13* Use the method of the proof of Theorem 2.1.1 to show that W_T is compactly embedded into the space $L^2(0, T; \mathcal{F}_s)$ for any $s < 1$.

— Exercise *1.14* Show that W_T is compactly embedded into $C(0, T; H)$. *Hint:* use Exercise 1.10 and the reasoning which is usually applied to prove the Arzelà theorem on the compactness of a collection of scalar continuous functions.

§ 2 Auxiliary Linear Problem

In this section we study the properties of a solution to the following linear problem:

$$\begin{cases} \dfrac{d^2u}{dt^2} + \gamma\dfrac{du}{dt} + A^2u + b(t)Au = h(t) , & (2.1) \\[2ex] u\Big|_{t=0} = u_0, \quad \dfrac{du}{dt}\Big|_{t=0} = u_1 . & (2.2) \end{cases}$$

Here A is a positive operator with discrete spectrum. The vectors $h(t)$, u_0, u_1 as well as the scalar function $b(t)$ are given (for the corresponding hypotheses see the assertion of Theorem 2.1).

The main results of this section are the proof of the theorem on the existence and uniqueness of weak solutions to problem (2.1) and (2.2) and the construction of the evolutionary operator for the system when $h(t) \equiv 0$. In fact, the approach we use here is well-known (see, e.g., [6] and [7]).

A ***weak solution*** to problem (2.1) and (2.2) on a segment $[0, T]$ is a function $u(t) \in W_T$ such that $u(0) = u_0$ and the equation

$$-\int_0^T (\dot{u}(t) + \gamma u(t),\ \dot{v}(t))\,dt + \int_0^T (Au(t) + b(t)u(t),\ Av(t))\,dt =$$

$$= (u_1 + \gamma u_0,\ v(0)) + \int_0^T (h(t),\ v(t))\,dt \tag{2.3}$$

holds for any function $v(t) \in W_T$ such that $v(T) = 0$. As above, $\dot{u}$ stands for the derivative of u with respect to t.

Exercise *2.1* Prove that if a weak solution $u(t)$ exists, then it satisfies the equation

$$(\dot{u}(t) + \gamma u(t),\ w) = (u_1 + \gamma u_0,\ w) -$$

$$-\int_0^t (Au(\tau) + b(\tau)u(\tau),\ Aw)\,d\tau + \int_0^t (h(\tau),\ w)\,d\tau \tag{2.4}$$

for every $w \in \mathcal{F}_1$ (*Hint:* take $v(t) = \int_t^T \varphi(\tau)\,d\tau \cdot w$ in (2.3), where $\varphi(t)$ is a scalar function from $C[0, T]$).

Theorem 2.1

Let $u_0 \in \mathcal{F}_1$, $u_1 \in \mathcal{F}_0$, ***and*** $\gamma \geq 0$. ***We also assume that*** $b(t)$ ***is a bounded continuous function on*** $[0, T]$ ***and*** $h(t) \in L^\infty(0, T; \mathcal{F}_0)$, ***where*** T ***is a positive number. Then problem*** (2.1) ***and*** (2.2) ***has a unique weak solution*** $u(t)$ ***on the segment*** $[0, T]$. ***This solution possesses the properties***

$$u(t) \in C(0, T; \mathcal{F}_1), \qquad \dot{u}(t) \in C(0, T; \mathcal{F}_0) \tag{2.5}$$

and satisfies the energy equation

$$\frac{1}{2}\left(\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right) + \gamma\int_0^t \|\dot{u}(\tau)\|^2\,d\tau + \int_0^t b(\tau)(Au(\tau),\ \dot{u}(\tau))\,d\tau =$$

$$= \frac{1}{2}\left(\|u_1\|^2 + \|Au_0\|^2\right) + \int_0^t (h(\tau),\ \dot{u}(\tau))\,d\tau\ . \tag{2.6}$$

Proof.

We use the compactness method to prove this theorem. At first we construct approximate solutions to problem (2.1) and (2.2). The approximate Galerkin solution (to this problem) of the order m with respect to the basis $\{e_k\}$ is considered to be the function

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k \tag{2.7}$$

satisfying the equations

$$(\ddot{u}_m + \gamma \dot{u}_m + A^2 u_m + b(t) A u_m - h(t),\ e_j) = 0\,, \tag{2.8}$$

$$(u_m(0),\ e_j) = (u_0,\ e_j), \quad (\dot{u}_m(0),\ e_j) = (u_1,\ e_j), \quad j = 1,\ 2,\ \ldots,\ m\,. \tag{2.9}$$

Here $g_k(t) \in C^1(0,\ T)$ and $\dot{g}_k(t)$ is absolutely continuous. Due to the orthogonality of the basis $\{e_k\}$ equations (2.8) and (2.9) can be rewritten as a system of ordinary differential equations:

$$\ddot{g}_k + \gamma \dot{g}_k + \lambda_k^2 g_k - b(t)\lambda_k g_k = h_k(t) \equiv (h(t),\ e_k)\,\mathrm{e},$$

$$g_k(0) = (u_0,\ e_k), \quad \dot{g}_k(0) = (u_1,\ e_k), \quad k = 1,\ 2,\ \ldots,\ m\,.$$

Lemma 2.1

Assume that $\gamma > 0$, $a \ge 0$, $b(t)$ is continuous, and $c(t)$ is a measurable bounded function. Then the Cauchy problem

$$\begin{cases} \ddot{g} + \gamma \dot{g} + a(a - b(t))g = c(t), & t \in [0,\ T]\,, \\ g(0) = g_0, \quad \dot{g}(0) = g_1 \end{cases} \tag{2.10}$$

is uniquely solvable on any segment $[0,\ T]$. Its solution possesses the property

$$\dot{g}(t)^2 + a^2 g(t)^2 \le \left(g_1^2 + a^2 g_0^2 + \frac{1}{2\gamma}\int_0^t c(\tau)^2\, d\tau \right) e^{b_0 t}, \quad t \in [0,\ T]\,, \tag{2.11}$$

where $b_0 = \max_t |b(t)|$. Moreover, if $b(t) \in C^1(0,\ T)$, $c(t) \equiv 0$ and for all $t \in [0,\ T]$ the conditions

$$-\frac{1}{2}a + \frac{\gamma^2}{a} \le b(t) \le \frac{1}{2}a\,, \qquad \dot{b}(t) + \gamma\left(\frac{a}{4} - b(t)\right) \ge 0\,, \tag{2.12}$$

hold, then the following estimate is valid:

$$\dot{g}(t)^2 + a^2 g(t)^2 \le 3(g_1^2 + a^2 g_0^2)\exp\left(-\frac{\gamma}{2}t\right)\,. \tag{2.13}$$

Proof.

Problem (2.10) is solvable at least locally, i.e. there exists $\bar{t}$ such that a solution exists on the half-interval $[0,\ \bar{t}\,)$. Let us prove estimate (2.11) for the interval of existence of solution. To do that, we multiply equation (2.10) by $\dot{g}(t)$. As a result, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{g}^2+a^2g^2)+\gamma\dot{g}^2 \;=\; a\,b(t)g\dot{g}+c(t)\dot{g}\,.$$

We integrate this equality and use the equations

$$a|b(t)g\dot{g}| \;\le\; \frac{1}{2}\max_t|b(t)|\left(\dot{g}^2+a^2g^2\right), \qquad c\dot{g} \;\le\; \gamma\dot{g}^2+\frac{1}{4\gamma}c^2\,,$$

to obtain that

$$\dot{g}(t)^2+a^2g(t) \;\le\; g_1^2+a^2g_0^2+\frac{1}{2\gamma}\int_0^t c^2(\tau)\,\mathrm{d}\tau+b_0\int_0^t(\dot{g}(\tau)^2+a^2g(\tau)^2)\,\mathrm{d}\tau\,.$$

This and Gronwall's lemma give us (2.11).

In particular, estimate (2.11) enables us to prove that the solution $g(t)$ can be extended on a segment $[0,\ T]$ of arbitrary length. Indeed, let us assume the contrary. Then there exists a point $\bar{t}$ such that the solution can not be extended through it. Therewith equation (2.11) implies that

$$\dot{g}(t)^2+a^2g(t)^2 \;\le\; C(T;\ g_0,\ g_1)\,, \qquad 0<t<\bar{t}<T\,.$$

Therefore, (2.10) gives us that the derivative $\ddot{g}(t)$ is bounded on $[0,\ \bar{t}\,)$. Hence, the values

$$\dot{g}(t) \;=\; g_0+\int_0^t\ddot{g}(\tau)\,\mathrm{d}\tau\,, \qquad g(t) \;=\; g_0+\int_0^t\dot{g}(\tau)\,\mathrm{d}\tau$$

are continuous up to the point $\bar{t}$. If we now apply the local theorem on existence to system (2.10) with the initial conditions at the point $\bar{t}$ that are equal to $g(\bar{t})$ and $\dot{g}(\bar{t})$, then we obtain that the solution can be extended through $\bar{t}$. This contradiction implies that the solution $g(t)$ exists on an arbitrary segment $[0,\,T]$.

Let us prove estimate (2.13). To do that, we consider the function

$$V(t) \;=\; \frac{1}{2}\left(\dot{g}^2+a(a-b(t))\,g^2\right)+\frac{\gamma}{2}\left(g\dot{g}+\frac{\gamma}{2}g^2\right). \tag{2.14}$$

Using the inequality

$$-\frac{1}{2\gamma}\dot{g}^2-\frac{\gamma}{2}g^2 \;\le\; g\dot{g} \;\le\; \frac{1}{2\gamma}\dot{g}^2+\frac{\gamma}{2}g^2\,,$$

it is easy to find that the equation

$$\frac{1}{4}\left(\dot{g}^2+a^2g^2\right) \;\le\; V(t) \;\le\; \frac{3}{4}\left(\dot{g}^2+a^2g^2\right) \tag{2.15}$$

holds under the condition

$$\frac{1}{2}a - b(t) \geq 0\,, \qquad a\left[\frac{1}{2}a + b(t)\right] - \gamma^2 \geq 0\,.$$

Further we use (2.10) with $c(t) \equiv 0$ to obtain that

$$\frac{dV}{dt} = -\frac{\gamma}{2}\dot{g}^2 - \frac{1}{2}(a\dot{b} + a\gamma(a-b))g^2\,.$$

Consequently, with the help of (2.15) we get

$$\frac{dV}{dt} + \frac{\gamma}{2}V \leq 0$$

under conditions (2.12). This implies that

$$V(t) \leq V(0)\exp\left(-\frac{\gamma}{2}t\right).$$

We use (2.15) to obtain estimate (2.13). Thus, Lemma 2.1 is proved.

Exercise *2.2* Assume that $\gamma \leq 0$ in Lemma 2.1. Show that problem (2.10) is uniquely solvable on any segment $[0, T]$ and the estimate

$$\dot{g}(t) + a^2 g(t)^2 \leq \left(g_1^2 + a^2 g_0^2\right) e^{(b_0 + 2|\gamma| + \delta)t} +$$

$$+ \frac{1}{\delta}\int_0^t c(\tau)^2 e^{(b_0 + 2|\gamma| + \delta)(t-\tau)} d\tau$$

is valid for $g(t)$ and for any $\delta > 0$, where $b_0 = \max_t |b(t)|$.

Lemma 2.1 implies the existence of a sequence of approximate solutions $\{u_m(t)\}$ to problem (2.1) and (2.2) on any segment $[0, T]$.

Exercise *2.3* Show that every approximate solution u_m is a solution to problem (2.1) and (2.2) with $u_0 = u_{0m}$, $u_1 = u_{1m}$, and $h(x, t) = h_m(x, t)$, where

$$u_{im} = p_m u_i = \sum_{k=1}^{m} (u_i, e_k)e_k\,,$$

$$h_m = p_m h = \sum_{k=1}^{m} (h(t), e_k)e_k\,, \tag{2.16}$$

and p_m is the orthoprojector onto the span of elements $\{e_k : k = 1, 2, \ldots, m\}$ in $\mathcal{F}_0 = H$.

Let us prove that the sequence of approximate solutions $\{u_m\}$ is convergent.

At first we note that

$$\Delta_{m,\,l}(t) = \|\dot u_m - \dot u_{m+l}\|^2 + \|A(u_m - u_{m+l})\|^2 = \sum_{k=m+1}^{m+l} (\dot g_k(t)^2 + \lambda_k^2 g_k(t)^2)$$

for every $t \in [0, T]$. Therefore, by virtue of Lemma 2.1 we have that

$$\Delta_{m,\,l}(t) \le e^{b_0 t} \sum_{k=m+1}^{m+l} \left[(u_1, e_k)^2 + \lambda_k^2 (u_0, e_k)^2 + \frac{1}{2\gamma} \int_0^T (h(t), e_k)^2\, \mathrm{d}t \right]$$

for $\gamma > 0$. Moreover, in the case $\gamma = 0$, the result of Exercise 2.2 gives that

$$\Delta_{m,\,l}(t) \le e^{(b_0+1)t} \sum_{k=m+1}^{m+l} \left[(u_1, e_k)^2 + \lambda_k^2 (u_0, e_k)^2 + \int_0^T (h(t), e_k)^2\, \mathrm{d}t \right].$$

These equations imply that the sequences $\{\dot u_m(t)\}$ and $\{A u_m(t)\}$ are the Cauchy sequences in the space $C(0, T; H)$ on any segment $[0, T]$. Consequently, there exists a function $u(t)$ such that

$$\dot u(t) \in C(0, T; H), \qquad u(t) \in C(0, T; \mathcal{F}_1),$$

$$\lim_{m \to \infty} \max_{[0,\,T]} \left(\|\dot u_m(t) - \dot u(t)\|^2 + \|u_m(t) - u(t)\|_1 \right) = 0. \tag{2.17}$$

Equations (2.8) and (2.9) further imply that

$$-\int_0^T \left(\dot u_m(t) + \gamma u_m(t),\ \dot v(t) \right) \mathrm{d}t + \int_0^T \left(A u_m(t) + b(t) u_m(t),\ A v(t) \right) \mathrm{d}t =$$

$$= (u_{1m} + \gamma u_{0m},\ v(0)) + \int_0^T (h_m(t),\ v(t))\, \mathrm{d}t$$

for all functions $v(t)$ from W_T such that $v(T) = 0$. Here u_{im}, and $h_m(t)$, $i = 0, 1$, are defined by (2.16). We use equation (2.17) to pass to the limit in this equation and to prove that the function $u(t)$ satisfies equality (2.3). Moreover, it follows from (2.17) that $u(0) = u_0$. Therefore, the function $u(t)$ is a weak solution to problem (2.1) and (2.2).

In order to prove the uniqueness of weak solutions we consider the function

$$v_s(t) = \begin{cases} -\int_t^s u(\tau)\, \mathrm{d}\tau, & t < s, \\ 0, & s \le t \le T, \end{cases} \tag{2.18}$$

for $s \in [0, T]$. Here $u(t)$ is a weak solution to problem (2.1) and (2.2) for $h = 0$, $u_0 = 0$, and $u_1 = 0$. Evidently $v_s(t) \in W_T$. Therefore,

$$-\int_0^T (\dot{u}(t)+\gamma u(t),\ \dot{v}_s(t))\,dt + \int_0^T (Au(t)+b(t)u(t),\ Av_s(t))\,dt = 0\,.$$

Due to the structure of the function $v_s(t)$ we obtain that

$$\frac{1}{2}\left(\|u(s)\|^2 + \|Av_s(0)\|^2\right) + \gamma \int_0^s \|u(t)\|^2\,dt = J_s(u,\ v)\,, \tag{2.19}$$

where

$$J_s(u,\ v) = \int_0^s (b(t)u(t),\ Av_s(t))\,dt\,.$$

It is evident that $Av_s(t) = Av_s(0) - Av_t(0)$ for $t \le s$. Therefore,

$$|J_s(u,\ v)| \;\le\; b_0\left(\|Av_s(0)\| \int_0^s \|u(t)\|\,dt + \int_0^s \|u(t)\|\cdot\|Av_t(0)\|\,dt\right) \;\le$$

$$\le\; \frac{1}{4}\|Av_s(0)\|^2 + s\,b_0^2 \int_0^s \|u(t)\|^2\,dt + \frac{b_0}{2}\int_0^s \left(\|u(t)\|^2 + \|Av_t(0)\|^2\right)dt\,.$$

If we substitute this estimate into equation (2.19), then it is easy to find that

$$\|u(s)\|^2 + \|Av_s(0)\|^2 \;\le\; C_T \int_0^s \left(\|u(t)\|^2 + \|Av_t(0)\|^2\right)dt\,,$$

where $s \in [0,\ T]$ and C_T is a positive constant depending on the length of the segment $[0,\ T]$. This and Gronwall's lemma imply that $u(t) \equiv 0$.

Let us prove the energy equation. If we multiply equation (2.8) by $\dot{g}_j(t)$ and summarize the result with respect to j, then we find that

$$\frac{1}{2}\frac{d}{dt}\left(\|\dot{u}_m\|^2 + \|Au_m\|^2\right) + \gamma\,\|\dot{u}_m\|^2 + b(t)(Au_m,\ \dot{u}_m) \;=\; (h,\ \dot{u}_m)\,.$$

After integration with respect to t we use (2.17) to pass to the limit and obtain (2.6).

Theorem 2.1 is completely proved.

Exercise *2.4* Prove that the estimate

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \;\le\; \left(\|u_1\|^2 + \|Au_0\|^2 + \frac{1}{2\gamma}\int_0^t \|h(\tau)\|^2\,d\tau\right)e^{b_0 t} \tag{2.20}$$

is valid for a weak solution $u(t)$ to problem (2.1) and (2.2). Here $b_0 = \max\{|b(t)|:\ t \ge 0\}$ and $\gamma > 0$.

— Exercise *2.5* Let $u(t)$ be a weak solution to problem (2.1) and (2.2). Prove that $A^2u(t) \in C(0, T; \mathcal{F}_{-1})$ and

$$\dot{u}(t) + \gamma u(t) = u_1 + \gamma u_0 - \int_0^t (A^2 u(\tau) + b(\tau) A u(\tau) - h(\tau))\, d\tau .$$

Here we treat the equality as an equality of elements in $\mathcal{F}_{-1}$. (*Hint:* use the results of Exercises 2.1 and 2.1.3).

— Exercise *2.6* Let $u(t)$ be a weak solution to problem (2.1) and (2.2) constructed in Theorem 2.1. Then the function $\dot{u}(t)$ is absolutely continuous as a vector-function with the values in $\mathcal{F}_{-1}$ while the derivative $\ddot{u}(t)$ belongs to the space $L^\infty(0, T; \mathcal{F}_{-1})$. Moreover, the function $u(t)$ satisfies equation (2.1) if we treat it as an equality of elements in $\mathcal{F}_{-1}$ for almost all $t \in [0, T]$.

In particular, the result of Exercise 2.6 shows that a weak solution satisfies equation (2.1) in a stronger sense then (2.4).

We also note that the assertions of Theorem 2.1 and Exercises 2.4–2.6 with the corresponding changes remain true if the initial condition is given at any other moment t_0 which is not equal to zero.

Now we consider the case $h(t) \equiv 0$ and construct the evolutionary operator of problem (2.1) and (2.2). To do that, let us consider the family of spaces

$$\mathcal{H}_\sigma = \mathcal{F}_{1+\sigma} \times \mathcal{F}_\sigma , \qquad \sigma \ge 0 .$$

Every space $\mathcal{H}_\sigma$ is a set of pairs $y = (u; v)$ such that $u \in \mathcal{F}_{1+\sigma}$ and $v \in \mathcal{F}_\sigma$. We define the inner product in $\mathcal{H}_\sigma$ by the formula

$$(y_1, y_2)_{\mathcal{H}_\sigma} = (u_1, u_2)_{1+\sigma} + (v_1, v_2)_\sigma .$$

— Exercise *2.7* Prove that $\mathcal{H}_{\sigma_1}$ is compactly embedded into $\mathcal{H}_\sigma$ for $\sigma_1 > \sigma$.

In the space $\mathcal{H}_0$ we define the evolutionary operator $U(t; t_0)$ of problem (2.1) and (2.2) for $h(t) \equiv 0$ by the equation

$$U(t; t_0) y = (u(t); \dot{u}(t)), \tag{2.21}$$

where $u(t)$ is a solution to (2.1) and (2.2) at the moment t with initial conditions that are equal to $y = (u_0; u_1)$ at the moment t_0.

The following assertion plays an important role in the study of asymptotic behaviour of solutions to problem (0.1) and (0.2).

Theorem 2.2

Assume that the function $b(t)$ *is continuously differentiable in* (2.1) *and such that*

$$b_0 = \sup_t |b(t)| < \infty, \qquad b_1 = \sup_t |\dot{b}(t)| < \infty .$$

Then the evolutionary operator $U(t;\ \tau)$ *of problem* (2.1) *and* (2.2) *for* $h(t) \equiv 0$ *is a linear bounded operator in each space* $\mathcal{H}_\sigma$ *for* $\sigma \geq 0$ *and it possesses the properties:*

a) $U(t;\ \tau)U(\tau;\ s) = U(t;\ s), \quad t \geq \tau \geq s, \quad U(t;\ t) = I;$

b) *for all* $\sigma \geq 0$ *the estimate*

$$\|U(t;\ \tau)y\|_{\mathcal{H}_\sigma} \leq \|y\|_{\mathcal{H}_\sigma} \exp\left(\frac{1}{2} b_0 (t-\tau)\right) \tag{2.22}$$

is valid;

c) *there exists a number* N_0 *depending on* γ, b_0, *and* b_1 *such that the equation*

$$\left\|(I - P_N)U(t;\ \tau)y\right\|_{\mathcal{H}_\sigma} \leq \sqrt{3}\left\|(I-P_N)y\right\|_{\mathcal{H}_\sigma} e^{-\frac{\gamma}{4}(t-\tau)}, \qquad t > \tau, \tag{2.23}$$

holds for all $N \geq N_0$, *where* P_N *is the orthoprojector onto the subspace*

$$L_N = \mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k):\ k = 1,\ 2,\ \ldots,\ N\}$$

in the space $\mathcal{H}_0$.

Proof.

Semigroup property a) follows from the uniqueness of a weak solution. The boundedness property of the operator $U(t;\ \tau)$ follows from (2.22). Let us prove relations (2.22) and (2.23). It is sufficient to consider the case $\tau = 0$. According to the definition of the evolutionary operator we have that

$$U(t;\ 0)y = (u(t);\ \dot{u}(t)), \qquad y = (u_0;\ u_1),$$

where $u(t)$ is the weak solution to problem (2.1) and (2.2) for $h(t) \equiv 0$. Due to (2.17) it can be represented as a convergent series of the form

$$u(t) = \sum_{k=1}^{\infty} g_k(t) e_k .$$

Moreover, Lemma 2.1 implies that

$$\dot{g}_k(t)^2 + \lambda_k^2 g_k(t)^2 \leq (\dot{g}_k(0)^2 + \lambda_k^2 g_k(0)^2)\, e^{b_0 t} . \tag{2.24}$$

Since

$$\|U(t;\ 0)y\|^2_{\mathcal{H}_\sigma} = \sum_{k=1}^{\infty} \left(\dot{g}_k(t)^2 + \lambda_k^2 g_k(t)^2\right) \lambda_k^{2\sigma},$$

equation (2.24) implies (2.22).

Further we use equation (2.13) to obtain that

$$\dot{g}_k(t)^2 + \lambda_k^2 g_k(t)^2 \le 3\left(g_k(0)^2 + \lambda_k^2 g_k(0)^2\right) e^{-\frac{\gamma}{2}t}, \qquad (2.25)$$

provided the conditions (cf. (2.12))

$$-\frac{1}{2}\lambda_k + \frac{\gamma^2}{\lambda_k} \le b(t) \le \frac{1}{2}\lambda_k, \qquad \dot{b}(t) + \gamma\left(\frac{\lambda_k}{4} - b(t)\right) \ge 0,$$

are fulfilled. Evidently, these conditions hold if

$$\lambda_k \ge \frac{4b_1}{\gamma} + 4b_0 + \frac{\gamma^2}{b_0},$$

where $b_0 = \max_t |b(t)|$ and $b_1 = \max_t |\dot{b}(t)|$. Since

$$\|(I - P_N)U(t;\, 0)y\|^2_{\mathcal{H}_\sigma} = \sum_{k=N+1}^{\infty} \left(\dot{g}_k(t)^2 + k^4 g_k(t)^2\right) k^{2\sigma},$$

equation (2.25) gives us (2.23) for all $N \ge N_0 - 1$, where N_0 is the smallest natural number such that

$$\lambda_{N_0} \ge \frac{4b_1}{\gamma} + 4b_0 + \frac{\gamma^2}{b_0}. \qquad (2.26)$$

Thus, **Theorem 2.2 is proved**.

— Exercise *2.8* Show that a weak solution $u(t)$ to problem (2.1) and (2.2) can be represented in the form

$$(u(t);\, \dot{u}(t)) = U(t;\, 0)y + \int_0^t U(t;\, \tau)(0;\, h(\tau))\, d\tau, \qquad (2.27)$$

where $y = (u_0;\, u_1)$ and $U(t;\, \tau)$ is defined by (2.21).

— Exercise *2.9* Use the result of Exercise 2.2 to show that Theorem 2.1 and Theorem 2.2 (a, b) with another constant in (2.22) also remain true for $\gamma < 0$. Use this fact to prove that if the hypotheses of Theorems 2.1 and 2.2 hold on the whole time axis, then problem (2.1) and (2.2) is solvable in the class of functions

$$\mathcal{W} = C(\mathbb{R};\, \mathcal{F}_1) \cap C^1(\mathbb{R};\, \mathcal{F}_0)$$

with $\gamma \ge 0$.

— Exercise *2.10* Show that the evolutionary operator $U(t,\, \tau)$ has a bounded inverse operator in every space $\mathcal{H}_\sigma$ for $\sigma \ge 0$. How is the operator $[U(t,\, \tau)]^{-1}$ for $t > \tau$ related to the solution to equation (2.1) for $h(t) \equiv 0$? Define the operator $U(t,\, \tau)$ using the formula $U(t,\, \tau) = [U(\tau,\, t)]^{-1}$ for $t < \tau$ and prove assertion (a) of Theorem 2.2 for all $t, \tau \in \mathbb{R}$.

§ 3 Theorem on Existence and Uniqueness of Solutions

In this section we use the compactness method (see, e.g., [8]) to prove the theorem on the existence and uniqueness of weak solutions to problem (0.1) and (0.2) under the assumption that

$$u_0 \in \mathcal{F}_1, \qquad u_1 \in \mathcal{F}_0, \qquad p(t) \in L^\infty(0, T; \mathcal{F}_0); \tag{3.1}$$

$$M(z) \in C^1(\mathbb{R}_+), \qquad \mathcal{M}(z) \equiv \int_0^z M(\xi)\,d\xi \ \geq -az - b, \tag{3.2}$$

where $0 \leq a < \lambda_1$, $b \in \mathbb{R}$, λ_1 is the first eigenvalue of the operator A, and the operator L is defined on $D(A)$ and satisfies the estimate

$$\|Lu\| \leq C\|Au\|, \qquad u \in D(A). \tag{3.3}$$

Similarly to the linear problem (see Section 2), the function $u(t) \in W_T$ is said to be a ***weak solution*** to problem (0.1) and (0.2) on the segment $[0, T]$ if $u(0) = u_0$ and the equation

$$-\int_0^T (\dot{u}(t) + \gamma u(t),\ \dot{v}(t))\,dt + \int_0^T \Big(Au(t) + M\Big(\|A^{1/2}u(t)\|^2\Big)u(t),\ Av(t)\Big)\,dt +$$
$$+\int_0^T (Lu(t),\ v(t))\,dt \ = \ (u_1 + \gamma u_0,\ v(0)) + \int_0^T (p(t),\ v(t))\,dt \tag{3.4}$$

holds for any function $v(t) \in W_T$ such that $v(T) = 0$. Here the space W_T is defined by equation (1.2).

Exercise *3.1* Prove the analogue of formula (2.4) for weak solutions to problem (0.1) and (0.2).

The following assertion holds.

Theorem 3.1

Assume that conditions (3.1)–(3.3) ***hold. Then on every segment*** $[0, T]$ ***problem*** (0.1) ***and*** (0.2) ***has a weak solution*** $u(t)$. ***This solution is unique. It possesses the properties***

$$u(t) \in C(0, T; \mathcal{F}_1), \qquad \dot{u}(t) \in C(0, T; \mathcal{F}_0) \tag{3.5}$$

and satisfies the energy equality

$$E(u(t),\ \dot{u}(t)) = E(u_0,\ u_1) + \int_0^t \Big(-\gamma\|\dot{u}(\tau)\|^2 + (-Lu(t) + p(t),\ \dot{u}(\tau))\Big)\,\mathrm{d}\tau, \quad (3.6)$$

where

$$E(u,\ v) = \frac{1}{2}\Big(\|v\|^2 + \|Au\|^2 + \mathcal{M}\Big(\|A^{1/2}u\|^2\Big)\Big). \quad (3.7)$$

We use the scheme from Section 2 to prove the theorem.

The Galerkin approximate solution of the order m to problem (0.1) and (0.2) with respect to the basis e_k is defined as a function of the form

$$u_m(t) = \sum_{k=1}^{m} g_k(t)e_k$$

which satisfies the equations

$$(\ddot{u}_m(t) + \gamma\dot{u}_m(t),\ e_j) + $$
$$+ \Big(Au_m(t) + M\Big(\|A^{1/2}u\|^2\Big)u_m(t),\ Ae_j\Big) + (Lu_m(t) - p(t),\ e_j) = 0 \quad (3.8)$$

for $j = 1,\ 2,\ \ldots,\ m$ with $t \in (0,\ T]$ and the initial conditions

$$(u_m(0),\ e_j) = (u_0,\ e_j), \quad (\dot{u}_m(0),\ e_j) = (u_1,\ e_j), \quad j = 1,\ 2,\ \ldots,\ m. \quad (3.9)$$

Simple calculations show that the problem of determining of approximate solutions can be reduced to solving the following system of ordinary differential equations:

$$\ddot{g}_k + \gamma\dot{g}_k + \lambda_k^2 g_k + \lambda_k M\Big(\sum_{j=1}^{m}\lambda_j g_j(t)^2\Big)g_k + \sum_{j=1}^{m}(Le_j,\ e_k)g_j = p_k(t), \quad (3.10)$$

$$g_k(0) = g_{0k} = (u_0,\ e_k), \quad \dot{g}_k(0) = g_{1k} = (u_1,\ e_k), \quad k = 1,\ 2,\ \ldots,\ m. \quad (3.11)$$

The nonlinear terms of this system are continuously differentiable with respect to g_j. Therefore, it is solvable at least locally. The global solvability follows from the a priori estimate of a solution as in the linear problem. Let us prove this estimate.

We consider an approximate solution $u_m(t)$ to problem (0.1) and (0.2) on the solvability interval $(0,\ \bar{t})$. It satisfies equations (3.8) and (3.9) on the interval $(0,\ \bar{t})$. We multiply equation (3.8) by $\dot{g}_j(t)$ and summarize these equations with respect to j from 1 to m. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}\Big(\|A^{1/2}u\|^2\Big) = 2M\Big(\|A^{1/2}u\|^2\Big)(Au(t),\ \dot{u}(t)),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u_m(t),\ \dot{u}_m(t)) = -\gamma\|\dot{u}_m(t)\|^2 - (Lu_m(t) - p(t),\ \dot{u}_m(t)) \quad (3.12)$$

as a result, where $E(u, \dot{u})$ is defined by (3.7). Equation (3.3) implies that

$$|(Lu_m, \dot{u}_m)| \le C\|Au_m\|\|\dot{u}_m\| \le C\Big(\|Au_m\|^2 + \|\dot{u}_m\|^2\Big).$$

Condition (3.2) gives us the estimate

$$\frac{1}{2}\Big(\|Au_m\|^2 + \|\dot{u}_m\|^2\Big) \le C_1 + C_2 E(u_m, \dot{u}_m) \tag{3.13}$$

with the constants independent of m. Therefore, due to Gronwall's lemma equation (3.12) implies that

$$\|\dot{u}_m(t)\|^2 + \|Au_m(t)\|^2 \le \Big(C_0 + C_1 E(u_m(0), \dot{u}_m(0))\Big)\, e^{C_2 t}, \tag{3.14}$$

with the constants C_0, C_1, and C_2 depending on the problem parameters only.

— Exercise *3.2* Use equation (3.14) to prove the global solvability of Cauchy problem (3.10) and (3.11).

It is evident that

$$\|\dot{u}_m(0)\| \le \|u_1\| \quad \text{and} \quad \|A^{1/2}u_m(0)\| \le \frac{1}{\sqrt{\lambda_1}}\|Au_m(0)\| \le \frac{1}{\sqrt{\lambda_1}}\|u_0\|_1 .$$

Therefore,

$$E(u_m(0), \dot{u}_m(0)) \le \frac{1}{2}\Big(\|u_1\|^2 + \|u_0\|_1^2 + C_M(\|u_0\|_1)\Big),$$

where $C_M(\rho) = \max\{\mathcal{M}(z):\ 0 \le z \le \rho^2/\lambda_1\}$. Consequently, equation (3.14) gives us that

$$\sup_{t \in [0, T]}\Big(\|\dot{u}_m(t)\|^2 + \|Au_m(t)\|^2\Big) \le C_T \tag{3.15}$$

for any $T > 0$, where C_T does not depend on m. Thus, the set of approximate solutions $\{u_m(t)\}$ is bounded in W_T for any $T > 0$. Hence, there exist an element $u(t) \in W_T$ and a sequence $\{m_k\}$ such that $u_{m_k}(t) \to u(t)$ weakly in W_T. Let us show that the weak limit point $u(t)$ possesses the property

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le C_T \tag{3.16}$$

for almost all $t \in [0, T]$. Indeed, the weak convergence of the sequence $\{u_{m_k}\}$ to the function u in W_T means that $\dot{u}_{m_k}$ and Au_{m_k} weakly (in $L^2(0, T; H)$) converge to $\dot{u}$ and Au respectively. Consequently, this convergence will also take place in $L^2(a, b; H)$ for any a and b from the segment $[0, T]$. Therefore, by virtue of the known property of the weak convergence we get

$$\int_a^b \Big(\|\dot{u}(t)\|^2 + \|Au(t)\|^2\Big)\,dt \le \varliminf_{k\to\infty} \int_a^b \Big(\|\dot{u}_{m_k}(t)\|^2 + \|Au_{m_k}(t)\|^2\Big)\,dt .$$

With the help of (3.15) we find that

$$\int_a^b \left(\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \right) dt \;\le\; C_T(b-a)\,.$$

Therefore, due to the arbitrariness of a and b we obtain estimate (3.16).

Lemma 3.1

For any function $v(t) \in L^2(0,\,T;\;H)$

$$\lim_{k\to\infty} J_T(u_{m_k},\; v) = J_T(u,\; v)\;,$$

where

$$J_T(u,\; v) = \int_0^T M\left(\|A^{1/2}u(t)\|^2\right)(u(t),\; v(t))\, dt\;.$$

Proof.

Since

$$\left| M\left(\left\|A^{1/2}u_{m_k}\right\|^2\right) - M\left(\|A^{1/2}u\|^2\right)\right| \;\le$$

$$\le \int_0^1 \left|\tilde{M}\left(\xi\left\|A^{1/2}u_{m_k}\right\| + (1-\xi)\|A^{1/2}u\|\right)\right| d\xi \cdot \left\|A^{1/2}(u_{m_k} - u)\right\|\;,$$

where $\tilde{M}(z) = 2zM'(z^2) \in C(\mathbb{R}_+)$, due to (3.15) and (3.16) we have

$$\left| M\left(\left\|A^{1/2}u_{m_k}\right\|^2\right) - M\left(\|A^{1/2}u\|^2\right)\right| \;\le\; C_T^{(1)}\left\|A^{1/2}(u_{m_k}(t) - u(t))\right\|\,,$$

where the constant $C_T^{(1)}$ is the maximum of the function $\tilde{M}(z)$ on the sufficiently large segment $[0,\; a_T]$, determined by the constant C_T from inequalities (3.15) and (3.16). Hence,

$$\delta_k \equiv \int_0^T \left| M\left(\|A^{1/2}u_m(t)\|^2\right) - M\left(\|A^{1/2}u(t)\|^2\right)\right| \cdot \left|(u_{m_k}(t),\; v(t))\right| dt \;\le$$

$$\le\; C_T\|v\|_{L^2(0,\,T;\;H)}\left(\int_0^T \left\|A^{1/2}(u_{m_k}(t) - u(t))\right\|^2 dt\right)^{1/2}.$$

The compactness of the embedding of W_T into $L^2(0,\,T;\;\mathscr{F}_{1/2})$ (see Exercise 1.13) implies that $\delta_k \to 0$ as $k \to \infty$. It is evident that

$$\left|J_T(u_{m_k},\, v) - J_T(u,\, v)\right| \;\le\; \delta_k + \left|\int_0^T (u_{m_k}(t) - u(t),\; v(t))\cdot M\Big(\|A^{1/2}u(t)\|^2\Big)\,\mathrm{d}t\right| .$$

Because of the weak convergence of u_{m_k} to u this gives us the assertion of the lemma.

— Exercise *3.3* Prove that the functional

$$\mathscr{L}[u] = \int_0^T (Lu(t),\, v(t))\,\mathrm{d}t$$

is continuous on W_T for any $v \in L^2(0,\, T;\, H)$.

Let us prove that the limit function $u(t)$ is a weak solution to problem (0.1) and (0.2).

Let p_l be the orthoprojector onto the span of elements e_k, $k = 1,\, 2,\, \ldots,\, l$ in the space H. We also assume that

$$\widetilde{W}_T = \{v \in W_T\colon\; v(T) = 0\}$$

and

$$\widetilde{W}_T^l \equiv p_l \widetilde{W}_T = \{p_l v\colon\; v \in \widetilde{W}_T\}.$$

It is clear that an arbitrary element of the space $\widetilde{W}_T^l$ has the form

$$v_l(x,\, t) = \sum_{k=1}^{l} \eta_k(t) e_k ,$$

where $\eta_k(t)$ is an absolutely continuous real function on $[0,\, T]$ such that

$$\eta_k(T) = 0 , \qquad \dot{\eta}_k(t) \in L^2(0,\, T).$$

If we multiply equation (3.8) by $\eta_j(t)$, summarize the result with respect to j from 1 to l, and integrate it with respect to t from 0 to T, then it is easy to find that

$$-\int_0^T (\dot{u}_m + \gamma u_m,\; \dot{v}_l)\,\mathrm{d}t + \int_0^T \Big(Au_m + M\Big(\|A^{1/2}u_m\|^2\Big)u_m,\; Av_l\Big)\,\mathrm{d}t + \int_0^T (Lu_m,\; v_l)\,\mathrm{d}t =$$

$$= (u_1 + \gamma u_0,\; v_l(0)) + \int_0^T (p,\; v_l)\,\mathrm{d}t$$

for $m \ge l$. The weak convergence of the sequence u_{m_k} to u in W_T as well as Lemma 3.1 and Exercise 3.3 enables us to pass to the limit in this equality and to show that the function $u(t)$ satisfies equation (3.4) for any function $v \in \widetilde{W}_T^l$, where $l = 1, 2, \ldots$. Further we use (cf. Exercise 2.1.11) the formula

$$\lim_{l \to \infty} \int_0^T \left(\|p_l \dot{v}(t) - \dot{v}(t)\|^2 + \|p_l A v(t) - A v(t)\|^2 \right) \mathrm{d}t = 0$$

for any function $v(t) \in \widetilde{W}_T$ in order to turn from the elements v of $\widetilde{W}_T^l$ to the functions v from the space $\widetilde{W}_T$.

Exercise 3.4 Prove that $u(t)\big|_{t=0} = u_0$.

Thus, every weak limit point $u(t)$ of the sequence of Galerkin approximations $\{u_m\}$ in the space W_T is a weak solution to problem (0.1) and (0.2).

If we compare equations (3.4) and (2.3), then we find that every weak solution $u(t)$ is simultaneously a weak solution to problem (2.1) and (2.2) with $b(t) = 0$ and

$$h(t) = -M\left(\|A^{1/2} u(t)\|^2\right) A u(t) - L u(t) + p(t). \tag{3.17}$$

It is evident that $h(t) \in L^\infty(0, T; \mathcal{F}_0)$. Therefore, due to Theorem 2.1 equations (3.5) are valid for the function $u(t)$.

To prove energy equality (3.6) it is sufficient (due to (2.6)) to verify that for $h(t)$ of form (3.17) the equality

$$\int_0^t (h(\tau), \dot{u}(\tau))\, \mathrm{d}\tau =$$

$$= \int_0^t (-L u(\tau) + p(\tau), \dot{u}(\tau))\, \mathrm{d}\tau - \frac{1}{2} \mathcal{M}\left(\|A^{1/2} u(\tau)\|^2\right)\Big|_{\tau=0}^{\tau=t} \tag{3.18}$$

holds. Here $u(t)$ is a vector-function possessing property (3.5). We can do that by first proving (3.18) for the function of the form $p_l u$ and then passing to the limit.

Exercise 3.5 Let $u(t)$ be a weak solution to problem (0.1) and (0.2). Use equation (3.6) to prove that

$$\|\dot{u}(t)\|^2 + \|A u(t)\|^2 \le (C_0 + C_1 E(u_0, u_1))\, e^{C_2 t}, \quad t > 0, \tag{3.19}$$

where $C_0, C_1, C_2 > 0$ are constants depending on the parameters of problem (0.1) and (0.2).

Let us prove the uniqueness of a weak solution to problem (0.1) and (0.2). We assume that $u_1(t)$ and $u_2(t)$ are weak solutions to problem (0.1) and (0.2) with the initial conditions $\{u_{01}, u_{11}\}$ and $\{u_{02}, u_{12}\}$, respectively. Then the function

$$u(t) = u_1(t) - u_2(t)$$

is a weak solution to problem (2.1) and (2.2) with the initial conditions $u_0 = u_{01} - u_{02}$, $u_1 = u_{11} - u_{12}$, the function $b(t) = M(\|A^{1/2} u_1(t)\|^2)$, and the right-hand side

$$h(t) = \left[M\left(\|A^{1/2} u_2(t)\|^2\right) - M\left(\|A^{1/2} u_1(t)\|^2\right)\right] A u_2(t) + L(u_2(t) - u_1(t)).$$

We use equation (3.19) to verify that

$$\|h(t)\| \le G_T(E(u_{01}, u_{11}) + E(u_{02}, u_{12})) \|A(u_1(t) - u_2(t))\|, \qquad t \in [0, T],$$

where $G_T(\xi)$ is a positive monotonely increasing function of the parameter ξ. Therefore, equation (2.20) implies that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|^2 + \|u_1(t) - u_2(t)\|_1^2 \le$$

$$\le C_T \left(\|u_{11} - u_{12}\|^2 + \|u_{01} - u_{02}\|_1^2 + \int_0^t \|u_1(\tau) - u_2(\tau)\|_1^2 \, d\tau \right),$$

where $C_T > 0$ depends on T and the problem parameters and is a function of the variables $E(u_{0j}, u_{1j})$, $j = 1, 2$. We can assume that C_T is the same for all initial data such that $E(u_{0j}, u_{1j}) \le R$, $j = 1, 2$. Using Gronwall's lemma we obtain that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|^2 + \|u_1(t) - u_2(t)\|_1^2 \le C_1\left(\|u_{11} - u_{12}\|^2 + \|u_{01} - u_{02}\|_1^2\right) e^{a C_2}, \quad (3.20)$$

where $t \in [0, T]$ and $C > 0$ is a constant depending only on T, the problem parameters and the value $R > 0$ such that $\|u_{0j}\|_1^2 + \|u_{1j}\|^2 \le R$. In particular, this estimate implies the uniqueness of weak solutions to problem (0.1) and (0.2). The proof of Theorem 3.1 is complete.

Exercise *3.6* Show that a weak solution $u(t)$ satisfies equation (0.1) if we consider this equation as an equality of elements in $\mathcal{F}_{-1}$ for almost all t. Moreover, $\ddot{u}(t) \in C(0, T; \mathcal{F}_{-1})$ (*Hint:* see Exercise 2.5).

Exercise *3.7* Assume that the hypotheses of Theorem 3.1 hold. Let $u(t)$ be a weak solution to problem (0.1) and (0.2) on the segment $[0, T]$ and let $u_m(t)$ be the corresponding Galerkin approximation of the order m. Show that

$$u_m(t) \to u(t) \quad \text{weakly in} \quad L^2(0, T; \mathcal{F}_1),$$

$$\dot{u}_m(t) \to \dot{u}(t) \quad \text{weakly in} \quad L^2(0, T; \mathcal{F}_0),$$

$$u_m(t) \to u(t) \quad \text{strongly in} \quad L^2(0, T; \mathcal{F}_s), \qquad s < 1,$$

as $m \to \infty$.

In conclusion of the section we note that in case of stationary load $p(t) \equiv p \in H$ we can construct an evolutionary operator S_t of problem (0.1) and (0.2) in the space $\mathcal{H} \equiv \mathcal{H}_0 = \mathcal{F}_1 \times \mathcal{F}_0$ supposing that

$$S_t y = (u(t);\ \dot{u}(t))$$

for $y = (u_0,\ u_1)$, where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0;\ u_1)$. Due to the uniqueness of weak solutions we have

$$S_t \circ S_\tau = S_{t+\tau}, \qquad S_0 = I, \qquad t,\ \tau \ge 0 .$$

By virtue of (3.20) the nonlinear mapping S_t is a continuous mapping of $\mathcal{H}$. Equation (3.5) implies that the vector-function $S_t y$ is strongly continuous with respect to t for any $y \in \mathcal{H}$. Moreover, for any $R > 0$ and $T > 0$ there exists a constant $C(R,\ T) > 0$ such that

$$\|S_t y_1 - S_t y_2\|_{\mathcal{H}} \le C(R,\ T) \cdot \|y_1 - y_2\|_{\mathcal{H}} \tag{3.21}$$

for all $t \in [0,\ T]$ and for all $y_j \in \{y \in \mathcal{H}:\ \|y\|_{\mathcal{H}} \le R\}$.

— Exercise *3.8* Use equation (3.21) to show that $(t,\ y) \to S_t y$ is a continuous mapping from $\mathbb{R}_+ \times \mathcal{H}$ into $\mathcal{H}$.

— Exercise *3.9* Prove the theorem on the existence and uniqueness of solutions to problem (0.1) and (0.2) for $\gamma \le 0$. Use this fact to show that the collection of operators $\{S_t\}$ is defined for negative t and forms a group (*Hint:* cf. Exercises 2.9 and 2.10).

— Exercise *3.10* Prove that the mapping S_t is a homeomorphism in $\mathcal{H}$ for every $t > 0$.

— Exercise *3.11* Let $p(t) \in L^\infty(\mathbb{R}_+,\ H)$ be a periodic function: $p(t) = p(t + t_0)$, $t_0 > 0$. Define the family of operators S_m by the formula

$$S_m y = (u(m t_0);\ \dot{u}(m t_0)), \qquad m = 0,\ 1,\ 2,\ \dots ,$$

in the space $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$. Here $u(t)$ is a solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0,\ u_1)$. Prove that the pair $(\mathcal{H},\ S_m)$ is a discrete dynamical system. Moreover, $S_m = S_1^m$ and S_1 is a homeomorphism in $\mathcal{H}$.

§ 4 Smoothness of Solutions

In the study of smoothness properties of solutions constructed in Section 3 we use some ideas presented in paper [9]. The main result of this section is the following assertion.

Theorem 4.1

Let the hypotheses of Theorem 3.1 ***hold. We assume that*** $M(z) \in C^{l+1}(\mathbb{R}_+)$ ***and the load*** $p(t)$ ***lies in*** $C^l(0, T; \mathcal{F}_0)$ ***for some*** $l \geq 1$. ***Then for a weak solution*** $u(t)$ ***to problem*** (0.1) ***and*** (0.2) ***to possess the properties***

$$u^{(k)}(t) \in C(0, T; \mathcal{F}_2), \quad k = 0, 1, 2, \dots, l-1, \qquad u^{(l)}(t) \in C(0, T; \mathcal{F}_1), \quad u^{(l+1)}(t) \in C(0, T; \mathcal{F}_0), \tag{4.1}$$

it is necessary and sufficient that the following compatibility conditions are fulfilled:

$$u^{(k)}(0) \in \mathcal{F}_2, \quad k = 0, 1, 2, \dots, l-1; \qquad u^{(l)}(0) \in \mathcal{F}_1. \tag{4.2}$$

Here $u^{(j)}(t)$ ***is a strong derivative of the function*** $u(t)$ ***with respect to*** t ***of the order*** j ***and the values*** $u^{(k)}(0)$ ***are recurrently defined by the initial conditions*** u_0 ***and*** u_1 ***with the help of equation*** (0.1):

$$u^{(0)}(0) = u_0, \qquad u^{(1)}(0) = u_1,$$

$$u^{(k)}(0) = -\Big\{\gamma u^{(k-1)}(0) + A^2 u^{(k-2)}(0) + L u^{(k-2)}(0) + \frac{d^{k-2}}{dt^{k-2}}\Big(M\big(\|A^{1/2}u(t)\|^2\big)Au(t) - p(t)\Big)\Big|_{t=0}\Big\}, \tag{4.3}$$

where $k = 2, 3, \dots$.

Proof.

It is evident that if a solution $u(t)$ possesses properties (4.1) then compatibility conditions (4.2) are fulfilled. Let us prove that conditions (4.2) are sufficient for equations (4.1) to be satisfied. We start with the case $l = 1$. The compatibility conditions have the form: $u_0 \in \mathcal{F}_2$, $u_1 \in \mathcal{F}_1$. As in the proof of Theorem 3.1 we consider the Galerkin approximation

$$u_m(t) = \sum_{k=1}^{m} g_k(t) e_k$$

of the order m for a solution to problem (0.1) and (0.2). It satisfies the equations

$$\ddot{u}_m(t) + \gamma\dot{u}_m(t) + A^2 u_m(t) +$$

$$+ M\left(\|A^{1/2}u_m(t)\|^2\right)A u_m(t) + p_m L u_m(t) \;=\; p_m p(t) \;, \tag{4.4}$$

$$u_m(0) \;=\; p_m u_0 \,, \qquad \dot{u}_m(0) \;=\; p_m u_1 \,,$$

where p_m is the orthoprojector onto the span of elements $e_1, \ldots e_m$. The structure of equation (3.10) implies that $u_m(t) \in C^2(0, T; \mathscr{F}_2)$. We differentiate equation (4.4) with respect to t to obtain that $v_m(t) = \dot{u}_m(t)$ satisfies the equation

$$\ddot{v}_m + \gamma\dot{v}_m + A^2 v_m + M\left(\|A^{1/2}u_m\|^2\right)Av_m + p_m L v_m =$$

$$= -2M'\left(\|A^{1/2}u_m\|^2\right)(Au_m,\, \dot{u}_m)Au_m + p_m\dot{p}(t) \tag{4.5}$$

and the initial conditions

$$v_m(0) \;=\; p_m u_1 \,,$$

$$\dot{v}_m(0) \;=\; -p_m\left\{\gamma u_1 + A^2 u_0 + M\left(\|A^{1/2}p_m u_0\|^2\right)Au_0 + Lp_m u_0 - p(0)\right\}. \tag{4.6}$$

It is clear that

$$\dot{v}_m(0) = p_m u^{(2)}(0) + \left[M\left(\|A^{1/2}u_0\|^2\right) - M\left(\|A^{1/2}p_m u_0\|^2\right)\right]Ap_m u_0 + L(u_0 - p_m u_0)\,,$$

where $u^{(2)}(0)$ is defined by (4.3). Therefore,

$$\|v_m(0) - p_m u^{(2)}(0)\| \;\le\; C(\|u_0\|_1)\|u_0 - p_m u_0\|_1 \,.$$

The compatibility conditions give us that $u_0 \in \mathscr{F}_2$ and hence $u^{(2)}(0) \in \mathscr{F}_0$. Thus, the initial condition $\dot{v}_m(0)$ possesses the property

$$\|\dot{v}_m(0) - u^{(2)}(0)\| \to 0\,, \qquad m \to \infty\,. \tag{4.7}$$

We multiply (4.5) by $\dot{v}_m(t)$ scalarwise in H to find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\dot{v}_m(t)\|^2 + \|Av_m(t)\|^2\right) + \gamma\|\dot{v}_m(t)\|^2 \;=\; (F_m(t),\; \dot{v}_m(t))\,, \tag{4.8}$$

where

$$F_m(t) = -M\left(\|A^{1/2}u_m\|^2\right)Av_m - p_m L v_m -$$

$$-2M'\left(\|A^{1/2}u_m\|^2\right)(Au_m,\, \dot{u}_m)Au_m + p_m\dot{p}(t)\;. \tag{4.9}$$

Using a priori estimates (3.14) for $u_m(t)$ we obtain

$$\|F_m(t)\| \;\le\; C_T\left(1 + \|Av_m(t)\|\right), \qquad t \in [0,\, T]\,.$$

Thus, equation (4.8) implies that

$$\frac{d}{dt}\left(\|\dot{v}_m(t)\|^2 + \|Av_m(t)\|^2\right) \le C_T\left(1 + \|\dot{v}_m(t)\|^2 + \|Av_m(t)\|^2\right), \qquad t \in [0,\ T].$$

Equation (4.7) and the property $u_1 \in \mathscr{F}_1$ give us that the estimate

$$\|\dot{v}_m(0)\|^2 + \|Av_m(0)\|^2 < C$$

holds uniformly with respect to m. Therefore, we reason as in Section 3 and use Gronwall's lemma to find that

$$\|Av_m(t)\|^2 + \|\dot{v}_m(t)\|^2 \le C_T, \qquad t \in [0,\ T]. \tag{4.10}$$

Consequently,

$$\|A\dot{u}_m(t)\|^2 + \|\ddot{u}_m(t)\|^2 \le C_T, \qquad t \in [0,\ T]. \tag{4.11}$$

Equation (4.4) gives us that

$$\|A^2u_m(t)\| \le \|\ddot{u}_m(t)\| + \gamma\|\dot{u}_m(t)\| + \left|M(\|A^{1/2}u_m(t)\|^2)\right|\|Au_m(t)\| + \|Lu_m(t)\| + \|p(t)\|.$$

Therefore, (3.14) and (4.11) imply that

$$\|A^2u_m(t)\| \le C_T, \qquad t \in [0,\ T]. \tag{4.12}$$

Thus, the sequence $\{u_m(t)\}$ of approximate solutions to problem (0.1) and (0.2) possesses the properties (cf. Exercise 3.7):

$$\left.\begin{aligned} u_m(t) \to u(t) \quad &\text{weakly in} \quad L^2(0, T;\ D(A^2)); \\ \dot{u}_m(t) \to \dot{u}(t) \quad &\text{weakly in} \quad L^2(0, T;\ D(A)); \\ \ddot{u}_m(t) \to \ddot{u}(t) \quad &\text{weakly in} \quad L^2(0, T;\ H); \end{aligned}\right\} \tag{4.13}$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2). Moreover (see Exercise 1.13),

$$\lim_{m\to\infty}\int_0^T \|\dot{u}_m(t) - \dot{u}(t)\|_s^2 + \|u_m(t) - u(t)\|_{1+s}^2\, dt = 0 \tag{4.14}$$

for every $s < 1$. If we use these equations and arguments similar to the ones given in Section 3, then it is easy to pass to the limit and to prove that the function $w(t) = \dot{u}(t)$ is a weak solution to the problem

$$\left\{\begin{aligned} &\ddot{w} + \gamma\dot{w} + A^2w + M\left(\|A^{1/2}u(t)\|^2\right)Aw + Lw = \\ &\quad = -2M'\left(\|A^{1/2}u(t)\|^2\right)(Au,\ \dot{u})Au + \dot{p}(t)\ , \\ &w(0) = u_1, \quad \dot{w}(0) = u^{(2)}(0)\ , \end{aligned}\right. \tag{4.15}$$

where $u^{(2)}(0)$ is defined by (4.3). Therefore, Theorem 2.1 gives us that

$$\dot{u}(t) = w(t) \in C(0, T;\ D(A)) \cap C^1(0, T;\ H).$$

This implies equation (4.1) for $l = 1$.

Further arguments are based on the following assertion.

Lemma 4.1

Let $u(t)$ be a weak solution to the linear problem

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) + A^2 u(t) + b(t)Au = h(t)\,, \\ u(0) = u_0\,, \end{cases} \tag{4.16}$$

where $b(t)$ is a scalar continuously differentiable function, $h(t) \in C^1(0, T;\ \mathscr{F}_0)$ and $u_0 \in \mathscr{F}_2$, $u_1 \in \mathscr{F}_1$. Then

$$u(t) \in C(0, T;\ \mathscr{F}_2) \cap C^1(0, T;\ \mathscr{F}_1) \cap C^2(0, T;\ \mathscr{F}_0) \tag{4.17}$$

and the function $v(t) = \dot{u}(t)$ is a weak solution to the problem obtained by the formal differentiation of (4.16) *with respect to t and equiped with the initial conditions $v(0) = u_1$ and $\dot{v}(0) = \ddot{u}(0) \equiv -(\gamma u_1 + A^2 u_0 + b(0)Au_0 - h(0))$.*

Proof.

Let $u_m(t)$ be the Galerkin approximation of the order m of a solution to problem (4.16) (see (2.7)). It is clear that $u_m(t)$ is thrice differentiable with respect to t and $v_m(t) = \dot{u}_m(t)$ satisfies the equation

$$\ddot{v}_m + \gamma\dot{v}_m + A^2 v_m + b(t)Av_m = -\dot{b}(t)Au_m + p_m h(t)\,, \qquad t > 0\,,$$

and the initial conditions

$$v_m(0) = p_m u_1\,, \qquad \dot{v}_m(0) = -p_m(\gamma u_1 + A^2 u_0 + b(0)Au_0 - h(0))\,.$$

Therefore, as above, it is easy to prove the validity of equations (4.10)–(4.14) for the case under consideration and complete the proof of Lemma 4.1.

Exercise *4.1* Assume that the hypotheses of Lemma 4.1 hold with $b(t) \in C^l(0, T)$ and $h(t) \in C^l(0, T;\ \mathscr{F}_0)$ for some $l \ge 1$. Let the compatibility conditions (4.2) be fulfilled with $u^{(0)}(0) = u_0$, $u^{(1)}(0) = u_1$, and

$$u^{(k)}(0) = -\Big\{\gamma u^{(k-1)}(0) + A^2 u^{(k-2)}(0) + \sum_{j=0}^{k-2} C^j_{k-2}\, b^{(j)}(0) A u^{(k-2-j)}(0) + L u^{(k-2)}(0) - h^{(k-2)}(0)\Big\}$$

for $k = 2, 3, \ldots, l$. Show that the weak solution $u(t)$ to problem (4.16) possesses properties (4.1) and the function $v_k(t) = u^{(k)}(t)$ is a weak solution to the equation obtained by the formal differentiation of (4.16) k times with respect to t. Here $k = 0, 1, \ldots, l$.

In order to complete the proof of Theorem 4.1 we use induction with respect to l. Assume that the hypotheses of the theorem as well as equations (4.2) for $l = n+1$ hold. Assume that the assertion of the theorem is valid for $l = n \ge 1$. Since equations (4.1) hold for the solution $u(t)$ with $l = n$, we have

$$\frac{d^k}{dt^k}\left\{M\left(\|A^{1/2}u(t)\|^2\right)Au(t)\right\} = M\left(\|A^{1/2}u(t)\|^2\right)Au^{(k)}(t) + G_k(t),$$

where $G_k(t) \in C^1(0, T; \mathcal{F}_0)$, $k = 1, \ldots, n$. Therefore, we differentiate equation (0.1) $n-1$ times with respect to t to obtain that $v(t) = u^{(n-1)}(t)$ is a weak solution to problem (4.16) with

$$b(t) = M\left(\|A^{1/2}u(t)\|^2\right) \quad \text{and} \quad h(t) = -G_{n-1}(t) + p^{(n-1)}(t).$$

Consequently, Lemma 4.1 gives us that $w(t) = \dot{v}(t)$ is a weak solution to the problem which is obtained by the formal differentiation of equation (0.1) n times with respect to t:

$$\begin{cases} \ddot{w} + \gamma\dot{w} + A^2 w + M\left(\|A^{1/2}u(t)\|^2\right)Aw = p^{(n)}(t) - G_n(t), \\ w(0) = u^{(n)}(0), \quad \dot{w}(0) = u^{(n+1)}(0). \end{cases}$$

However, the hypotheses of Lemma 4.1 hold for this problem. Therefore (see (4.17)),

$$u^{(n)}(t) = w(t) \in C(0, T; \mathcal{F}_2) \cap C^1(0, T; \mathcal{F}_1) \cap C^2(0, T; \mathcal{F}_0),$$

i.e. equations (4.1) hold for $l = n+1$. **Theorem 4.1 is proved**.

— Exercise *4.2* Show that if the hypotheses of Theorem 4.1 hold, then the function $v(t) = u^{(k)}(t)$ is a weak solution to the problem which is obtained by the formal differentiation of equation (0.1) k times with respect to t, $k = 1, 2, \ldots, l$.

— Exercise *4.3* Assume that the hypotheses of Theorem 4.1 hold and $L \equiv 0$ in equation (0.1). Show that if the conditions

$$p(t) \in C^k([0, T]; \mathcal{F}_{l-k}), \quad k = 0, 1, \ldots, l, \tag{4.18}$$

are fulfilled, then a solution $u(t)$ to problem (0.1) and (0.2) possesses the properties

$$u(t) \in C^k([0, T]; \mathcal{F}_{l+1-k}), \quad k = 0, 1, \ldots, l+1.$$

Exercise 4.4 Assume that the hypotheses of Theorem 4.1 hold. We define the sets

$$\mathscr{V}_k = \left\{ (u_0,\ u_1) \in \mathscr{H}:\ \text{ equation (4.2) holds with } l = k \right\} \qquad (4.19)$$

in the space $\mathscr{H} = \mathscr{F}_1 \times \mathscr{F}_0$. Prove that

$$\mathscr{V}_1 = \mathscr{F}_2 \times \mathscr{F}_1 \quad \text{and} \quad \mathscr{V}_1 \supset \mathscr{V}_2 \supset \ldots \supset \mathscr{V}_l\ .$$

Exercise 4.5 Show that every set $\mathscr{V}_k$ given by equality (4.19) is invariant:

$$(u_0;\ u_1) \in \mathscr{V}_k \Rightarrow (u(t);\ \dot{u}(t)) \in \mathscr{V}_k\,, \qquad k = 1,\ \ldots,\ l\,.$$

Here $u(t)$ is a weak solution to problem (0.1) and (0.2).

Exercise 4.6 Assume that $L \equiv 0$ in equation (0.1) and the load $p(t)$ possesses property (4.18). Show that for $k = 1,\ 2,\ \ldots,\ l$ the set $\mathscr{V}_k$ of form (4.19) contains the subspace $\mathscr{F}_{k+1} \times \mathscr{F}_k$.

Exercise 4.7 Assume that the hypotheses of Theorem 3.1 hold and the operator L (in equation (0.1)) possesses the property

$$\|Lu\|_s \le C\|u\|_{1+s} \quad \text{for some} \quad 0 < s < 1\,. \qquad (4.20)$$

Let $u_0 \in \mathscr{F}_{1+s}$ and let $u_1 \in \mathscr{F}_s$. Show that the estimate

$$\|\dot{u}_m(t)\|_s^2 + \|u_m(t)\|_{1+s}^2 \le C_T\,, \qquad t \in [0,\ T]\,, \qquad (4.21)$$

is valid for the approximate Galerkin solution $u_m(t)$ to problem (0.1) and (0.2). Here the constant C_T does not depend on m (*Hint:* multiply equation (3.8) by $\lambda_j^{2s}\,\dot{g}_j(t)$ and summarize the result with respect to j; then use relation (3.14) to estimate the nonlinear term).

Exercise 4.8 Show that if the hypotheses of Exercise 4.7 hold, then problem (0.1) and (0.2) possesses a weak solution $u(t)$ such that

$$u(t) \in C(0,\ T;\ \mathscr{F}_{1+s}) \cap C^1(0,\ T;\ \mathscr{F}_s) \cap C^2(0,\ T;\ \mathscr{F}_{-1+s})\,,$$

where $s \in (0,\ 1)$ is the number from Exercise 4.7.

§ 5 Dissipativity and Asymptotic Compactness

In this section we prove the dissipativity and asymptotic compactness of the dynamical system $(\mathcal{H}, S_t)$ generated by weak solutions to problem (0.1) and (0.2) for $\gamma > 0$ in the case of a stationary load $p(t) \equiv p \in H = \mathcal{F}_0$. The phase space is $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$. The evolutionary operator is defined by the formula

$$S_t y = (u(t),\ \dot{u}(t)), \tag{5.1}$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial condition $y = (u_0;\ u_1)$.

Theorem 5.1

Assume that in addition to (3.2) *the following conditions are fulfilled:*

a) *there exist numbers* $a_j > 0$ *such that*

$$zM(z) - a_1 \int_0^z M(\xi)\,\mathrm{d}\xi \ \ge\ a_2 z^{1+\alpha} - a_3, \qquad z \ge 0 \tag{5.2}$$

with a constant $\alpha > 0$*;*

b) *there exist* $0 \le \theta < 1$ *and* $C > 0$ *such that*

$$\|Lu\| \le C\|A^\theta u\|, \qquad u \in D(A^\theta). \tag{5.3}$$

Then the dynamical system $(\mathcal{H}, S_t)$ *generated by problem* (0.1) *and* (0.2) *for* $\gamma > 0$ *and for* $p(t) \equiv p \in H$ *is dissipative.*

To prove the theorem it is sufficient to verify (see Theorem 1.4.1 and Exercise 1.4.1) that there exists a functional $V(y)$ on $\mathcal{H}$ which is bounded on the bounded sets of the space $\mathcal{H}$, differentiable along the trajectories of system (0.1) and (0.2), and such that

$$V(y) \ \ge\ \alpha\|y\|^2_{\mathcal{H}} - \mathcal{D}_1, \tag{5.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(V(S_t y)) + \beta V(S_t y) \ \le\ \mathcal{D}_2, \tag{5.5}$$

where $\alpha, \beta > 0$ and $\mathcal{D}_1, \mathcal{D}_2 \ge 0$ are constants. To construct the functional $V(y)$ we use the method which is widely-applied for finite-dimensional systems (we used it in the proof of estimate (2.13)).

Let

$$V(y) = E(y) + \nu\Phi(y),$$

where $y = (u_0;\ u_1) \in \mathcal{H}$. Here $E(y) = E(u_0;\ u_1)$ is the energy of system (0.1) and (0.2) defined by the formula (3.7),

$$\Phi(y) = (u_0;\ u_1) + \frac{\gamma}{2}\|u_0\|^2 ,$$

and the parameter $\nu > 0$ will be chosen below. It is evident that

$$-\frac{1}{2\gamma}\|u_1\|^2 \le \Phi(y) \le \frac{1}{2\gamma}\|u_1\|^2 + \gamma\|u_0\|^2 .$$

For $0 < \nu < \gamma$ this implies estimate (5.4) and the inequality

$$V(y) \le b_1 E(y) + b_2 \qquad (5.6)$$

with the constants $b_1,\ b_2 > 0$ independent of ν. This inequality guarantees the boundedness of $V(y)$ on the bounded sets of the space $\mathcal{H}$.

Energy equality (3.6) implies that the function $E(y(t))$, where $y(t) = S_t y$, is continuously differentiable and

$$\frac{d}{dt}E(y(t)) = -\gamma\|\dot{u}(t)\|^2 + (-Lu(t) + p,\ \dot{u}(t)) .$$

Therefore, due to (5.3) we have that

$$\frac{d}{dt}E(y(t)) \le -\frac{\gamma}{2}\|\dot{u}\|^2 + C_1\|A^\theta u\|^2 + C_2 .$$

We use interpolation inequalities (see Exercises 2.1.12 and 2.1.13) to obtain that

$$\|A^\theta u\|^2 \le \delta\|Au\|^2 + C_\delta\|A^{1/2}u\|^2 , \qquad \delta > 0 .$$

Thus, the estimate

$$\frac{d}{dt}E(y(t)) \le -\frac{\gamma}{2}\|\dot{u}\|^2 + \frac{\varepsilon}{2}\|Au\|^2 + C_\varepsilon\|A^{1/2}u\|^2 + C_2 \qquad (5.7)$$

holds for any $\varepsilon > 0$.

Lemma 5.1

Let $u(t)$ be a weak solution to problem (0.1) *and* (0.2) *and let $y(t) = (u(t),\ \dot{u}(t))$. Then the function $\Phi(y(t))$ is continuously differentiable and*

$$\frac{d}{dt}\Phi(y(t)) = \|\dot{u}(t)\|^2 + (\ddot{u}(t) + \gamma\dot{u}(t),\ u(t)) . \qquad (5.8)$$

We note that since $\ddot{u}(t) \in C(\mathbb{R}_+,\ \mathcal{F}_{-1})$ (see Exercise 3.6), equation (5.8) is correctly defined.

Proof.

It is sufficient to verify that

$$(u(t),\ \dot{u}(t)) = (u_0,\ u_1) + \int_0^t \left\{(\ddot{u}(\tau),\ u(\tau)) + \|\dot{u}(\tau)\|^2\right\} d\tau . \qquad (5.9)$$

Let p_l be the orthoprojector onto the span of elements $\{e_k,\ k = 1,\ 2,\ \dots,\ l\}$ in $\mathcal{F}_0$. Then it is evident that the vector-function $u_l(t) = p_l u(t)$ is twice continuously differentiable with respect to t. Therefore,

$$(u_l(t),\ \dot{u}_l(t)) = (u_l(0),\ \dot{u}_l(0)) + \int_0^t \left\{(\ddot{u}_l(\tau),\ u_l(\tau)) + \|\dot{u}_l(\tau)\|^2\right\} d\tau .$$

The properties of the projector p_l (see Exercise 2.1.11) enable us to pass to the limit $l \to \infty$ and to obtain (5.9). Lemma 5.1 is proved.

Since $u(t)$ is a solution to equation (0.1), relation (5.8) implies that

$$\frac{d}{dt}\Phi(y(t)) = \|\dot{u}\|^2 - \left\{\|Au\|^2 + M\left(\|A^{1/2}u\|^2\right)\cdot\|A^{1/2}u\|^2 + (Lu - p,\ u)\right\}.$$

Therefore, equation (5.2) and the evident inequality

$$|(Lu - p,\ u)| \le \frac{1}{2}\|Au\|^2 + C_1\|u\|^2 + \|p\|^2$$

give us that

$$\frac{d}{dt}\Phi(y(t)) \le \|\dot{u}\|^2 - \frac{1}{2}\|Au\|^2 - a_1\mathcal{M}\left(\|A^{1/2}u\|^2\right) - a_2\|A^{1/2}u\|^{2+2\alpha} + C_1\|u\|^2 + C_2 .$$

Hence, (5.6) and (5.7) enable us to obtain the estimate

$$\frac{d}{dt}V(y(t)) + \delta V(y(t)) \le -\frac{1}{2}(\gamma - \delta b_1 - 2\nu)\|\dot{u}\|^2 -$$
$$-\frac{1}{2}(\nu - \delta b_1 - \varepsilon)\|Au\|^2 - \left(\nu a_1 - \frac{\delta b_1}{2}\right)\mathcal{M}\left(\|A^{1/2}u\|^2\right) + R(u;\ \nu,\ \varepsilon,\ \delta) ,$$

where $\delta > 0$ and

$$R(u;\ \nu,\ \varepsilon,\ \delta) = -\nu a_2\|A^{1/2}u\|^{2+2\alpha} + C_\varepsilon\|A^{1/2}u\|^2 + \nu C_1\|u\|^2 + C_\delta .$$

Therefore, for any $0 < \nu < \gamma/2$ estimate (5.5) holds, provided δ and ε are chosen appropriately. Thus, **Theorem 5.1 is proved**.

Exercise *5.1* Prove that if the hypotheses of Theorem 5.1 hold, then the assertion on the dissipativity of solutions to problem (0.1) and (0.2) remains true in the case of a nonstationary load $p(t) \in L^\infty(\mathbb{R}_+,\ H)$.

Theorem 5.2

***Let the hypotheses of Theorem* 5.1 *hold and assume that for some* $\sigma > 0$**

$$p \in \mathcal{F}_\sigma, \qquad LD(A) \subset D(A^\sigma), \qquad \|A^\sigma L u\| \le C\|Au\| . \tag{5.10}$$

Then there exists a positively invariant bounded set K_σ ***in the space*** $\mathcal{H}_\sigma = \mathcal{F}_{1+\sigma} \times \mathcal{F}_\sigma$ ***which is closed in*** $\mathcal{H}$ ***and such that***

$$\sup\left\{\operatorname{dist}_{\mathcal{H}}(S_t y,\ K_\sigma):\ y \in B\right\} \le C e^{-\frac{\gamma}{4}(t-t_B)} \tag{5.11}$$

for any bounded set B ***in the space*** $\mathcal{H}$, $t > t_B$.

Due to the compactness of the embedding of $\mathcal{H}_\sigma$ in $\mathcal{H}$ for $\sigma > 0$, this theorem and Lemma 1.4.1 imply that the dynamical system $(\mathcal{H},\ S_t)$ is asymptotically compact.

Proof.

Since the system $(\mathcal{H},\ S_t)$ is dissipative, there exists $R > 0$ such that for all $y \in B$ and $t \ge t_0 = t_0(B)$

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le R^2 , \tag{5.12}$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0;\ u_1) \in B$. We consider $u(t)$ as a solution to linear problem (2.1) and (2.2) with $b(t) = M(\|A^{1/2}u(t)\|^2)$ and $h(t) = -Lu(t) + p$. It is easy to verify that $b(t)$ is a continuously differentiable function and

$$|b(t)| + |\dot{b}(t)| \le C_R , \qquad t \ge t_0 .$$

Moreover, equation (2.27) implies that

$$S_t y = U(t,\ t_0)\, y(t_0) + G(t,\ t_0;\ y) , \tag{5.13}$$

where

$$G(t,\ t_0;\ y) = -\int_{t_0}^{t} U(t,\ \tau)(0;\ Lu(\tau) - p)\, d\tau .$$

Here $U(t,\ \tau)$ is the evolutionary operator of the homogeneous problem (2.1) and (2.2) with $h(t) = 0$ and $b(t) = M(\|A^{1/2}u(t)\|^2)$. By virtue of Theorem 2.2 there exists $N_0 \ge 0$ such that

$$\|(1 - P_N)\, U(t,\ \tau) h\|_{\mathcal{H}_\sigma} \le \sqrt{3}\, \|h\|_{\mathcal{H}_\sigma} \exp\left\{-\frac{\gamma}{4}(t - \tau)\right\} , \tag{5.14}$$

where $N \ge N_0$, $t \ge \tau \ge t_0$, and P_N is the orthoprojector $\mathcal{H}$ onto

$$\mathcal{L}_N = \operatorname{Lin}\{(e_k;\ 0),\ (0;\ e_k),\quad k = 1,\ 2,\ \dots,\ N\} .$$

This implies that

$$\|(1 - P_{N_0})\, G(t,\ t_0;\ y)\|_{\mathcal{H}_\sigma} \le \sqrt{3} \int_{t_0}^{t} \exp\left\{-\frac{\gamma}{4}(t - \tau)\right\} \|Lu(\tau) - p\|_\sigma\, d\tau .$$

Therefore, we use (5.10) to obtain that

$$\|(1 - P_{N_0})\, G(t,\ t_0;\ y)\|_{\mathcal{H}_\sigma} \le C(R) . \tag{5.15}$$

It is also easy to find that

$$\left\|P_{N_0} S_t y\right\|_{\mathscr{H}_\sigma} \le N_0^\sigma \left\|S_t y\right\|_{\mathscr{H}} \le R N_0^\sigma, \qquad t \ge t_0 .$$

Consequently, there exists a number R_σ depending on the radius of dissipativity R and the parameters of the problem such that the value

$$S_t y - (1 - P_{N_0})\, U(t,\ t_0) S_{t_0} y_0 = P_{N_0} S_t y + (1 - P_{N_0}) G(t,\ t_0;\ y) \tag{5.16}$$

lies in the ball

$$B_\sigma = \{y:\ \|y\|_{\mathscr{H}_\sigma} \le R_\sigma\}$$

for $t \ge t_0$. Therefore, with the help of (5.12) and (2.23) we have that

$$\operatorname{dist}_{\mathscr{H}}(S_t y,\ B_\sigma) \le \left\|(1 - P_{N_0})\, U(t,\ t_0) S_{t_0} y\right\| \le R\sqrt{3}\, e^{-\frac{\gamma}{4}(t - t_0)} . \tag{5.17}$$

Let $K_\sigma = \gamma^+(B_\sigma) \equiv \bigcup_{t \ge 0} S_t(B_\sigma)$. Evidently equation (5.11) is valid. Moreover, K_σ is positively invariant. The continuity of $S_t y$ with recpect to the both variables $(t;\ y)$ in the space $\mathscr{H}$ (see Exercise 3.8) and attraction property (5.17) imply that K_σ is a closed set in $\mathscr{H}$. Let us prove that K_σ is bounded in $\mathscr{H}_\sigma$. First we note that K_σ is bounded in $\mathscr{H}$. Indeed, by virtue of the dissipativity we have that $\left\|S_t y\right\| \le R$ for all $y \in B_\sigma$ and $t \ge t_\sigma \equiv t(B_\sigma)$. Since $S_t y$ is continuous with respect to the variables $(t;\ y)$, its maximum is attained on the compact $[0,\ t_\sigma] \times B_\sigma$. Thus, there exists $\bar{R} > 0$ such that $\|y\| \le \bar{R}$ for all $y \in K_\sigma$. Let us return to equality (5.16) for $t_0 = 0$ and $y_0 \in B_\sigma$. It is evident that the norm of the right-hand side in the space $\mathscr{H}_\sigma$ is bounded by the constant $C = C(\sigma,\ \bar{R})$. However, equation (5.14) implies that

$$\left\|(1 - P_{N_0})\, U(t,\ 0) y_0\right\|_{\mathscr{H}_\sigma} \le \sqrt{3} R_\sigma , \qquad t \ge 0 , \qquad y_0 \in B_\sigma .$$

Therefore, equation (5.16) leads to the uniform estimate

$$\left\|S_t y\right\|_{\mathscr{H}_\sigma} \le \bar{R}_\sigma , \qquad t \ge 0 , \qquad y \in B_\sigma .$$

Thus, the set K_σ is bounded in $\mathscr{H}_\sigma$. **Theorem 5.2 is proved**.

— Exercise *5.2* Show that for any $0 \le s \le \sigma$ a bounded set of $\mathscr{H}_s$ is attracted by K_σ at an exponential rate with respect to the metric of the space $\mathscr{H}_s$. Thus, we can replace $\operatorname{dist}_{\mathscr{H}}$ by $\operatorname{dist}_{\mathscr{H}_s}$ in (5.11).

— Exercise *5.3* Prove that if the hypotheses of Theorem 5.2 hold, then the assertion on the asymptotic compactness of solutions to problem (0.1) and (0.2) remains true in the case of nonstationary load $p(t) \in$ $\in L^\infty(\mathbb{R}_+,\ \mathscr{F}_\sigma)$ (see also Exercise 5.1).

— Exercise *5.4* Prove that the hypotheses of Theorem 5.1 and 5.2 hold for problem (0.3) and (0.4) for any $0 < \sigma < 1/4$, provided that $\gamma > 0$ and $p(x,\ t) \equiv p(x)$ lies in the Sobolev space $H_0^1(\Omega)$.

Let us consider the dissipativity properties of smooth solutions (see Section 4) to problem (0.1) and (0.2).

Theorem 5.3

Let the hypotheses of Theorem** 5.1 **hold. Assume that** $M(z) \in C^{l+1}(\mathbb{R}_+)$ **and the initial conditions** $y_0 = (u_0;\ u_1)$ **are such that equations** (4.1) **(and hence** (4.2)**) are valid for the solution** $(u(t);\ \dot{u}(t)) = S_t y_0$. **Then there exists** $R_l > 0$ **such that for any initial data** $y_0 = (u_0;\ u_1)$ **possessing the property

$$\|u^{(k+1)}(0)\|^2 + \|Au^{(k)}(0)\|^2 + \|A^2 u^{(k-1)}(0)\|^2 < \rho^2, \quad k = 1,\ 2,\ \dots,\ l, \tag{5.18}$$

the solution** $(u(t);\ \dot{u}(t)) = S_t y_0$ **admits the estimate

$$\|u^{(k+1)}(t)\|^2 + \|Au^{(k)}(t)\|^2 + \|A^2 u^{(k-1)}(t)\|^2 < R_l^2 \tag{5.19}$$

***for all** $k = 1,\ 2,\ \dots,\ l$ **as soon as** $t \ge t_0(\rho)$.*

We use induction to prove the theorem. The proof is based on the following assertion.

Lemma 5.2

Assume that the hypotheses of Theorem 5.3 *hold for* $l = 1$. *Then the dynamical system* $(\mathcal{H}_1,\ S_t)$ *generated by problem* (0.1) *and* (0.2) *in the space* $\mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$ *is dissipative.*

Proof.

Let $(u(t);\ \dot{u}(t)) = S_t y_0$ be a semitrajectory of the dynamical system $(\mathcal{H}_1,\ S_t)$ and let $y_0 = (u_0;\ u_1) \in \mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$. If the hypotheses of the lemma hold, then the function $w(t) = \dot{u}(t)$ is a weak solution to problem (4.15) obtained by formal differentiation of (0.1) with respect to t (as we have shown in Section 4). By virtue of Theorem 2.1 the energy equality of the form (2.6) holds for the function $w(t)$. We rewrite it in the differential form:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t;\ w(t),\ \dot{w}(t)) + \gamma \|\dot{w}(t)\|^2 = \Psi(u(t),\ w(t)), \tag{5.20}$$

where

$$F(t;\ w,\ \dot{w}) = \frac{1}{2}\Big(\|\dot{w}\|^2 + \|Aw\|^2 + M\Big(\|A^{1/2}u(t)\|^2\Big) \cdot \|A^{1/2}w\|^2\Big) \tag{5.21}$$

and

$$\Psi(u(t),\ w(t)) = -(Lw(t),\ \dot{w}(t)) +$$

$$+\ M'\Big(\|A^{1/2}u(t)\|^2\Big)(Au(t),\ \dot{u}(t))\Big[\|A^{1/2}w(t)\|^2 - 2(Au(t),\ \dot{w}(t))\Big].$$

The dissipativity property of $(\mathcal{H},\ S_t)$ given by Theorem 5.1 leads to the estimates

$$\left| M\left(\|A^{1/2}u(t)\|^2\right)\right| \|A^{1/2}w\|^2 \;\le\; C_R\|Aw(t)\|\,,$$

$$|\Psi(u(t),\; w(t))| \;\le\; \|Lw\|\,\|\dot{w}\| + C_R(\|Aw(t)\| + \|\dot{w}(t)\|)$$

for all $(u_0;\; u_1)$ with the property

$$\|Au_0\|^2 + \|u_1\|^2 \;\le\; \rho^2$$

and for all $t \ge t_0(\rho)$ large enough. Hereinafter R is the radius of dissipativity of the system $(\mathscr{H},\; S_t)$. These estimates imply that for $t \ge t_0(\rho)$ we have

$$\frac{1}{4}\left(\|\dot{w}\|^2 + \|Aw\|^2\right) - \alpha_1 \;\le\; F(t;\; w,\; \dot{w}) \;\le\; \alpha_2\left(\|w(t)\|^2 + \|\dot{w}(t)\|^2\right) + \alpha_3 \quad (5.22)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t;\; w,\; \dot{w}) + \frac{\gamma}{2}\|\dot{w}\|^2 \;\le\; \beta\|A^{\theta}w\|^2 + \alpha_4\|Aw\| + \alpha_5\,, \qquad (5.23)$$

where the constants $\alpha_j > 0$ depend on R. Here $\theta < 1$. Similarly, we use Lemma 5.1 to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(w(t),\; \dot{w}(t)) + \frac{\gamma}{2}\|w(t)\|^2\right\} \;\le\; \|\dot{w}(t)\|^2 - \frac{1}{2}\|Aw(t)\|^2 + C_R$$

for $t \ge t_0(\rho)$. Consequently, the function

$$V(t) = F(t;\; w,\, \dot{w}) + \nu\left\{(w,\; \dot{w}) + \frac{\gamma}{2}\|w\|^2\right\}$$

possesses the properties

$$\frac{\mathrm{d}V}{\mathrm{d}t} + \omega V \;\le\; C_R\,, \qquad \omega > 0\,,$$

and

$$\frac{1}{4}\left(\|\dot{w}\|^2 + \|Aw\|^2\right) - a_1 \;\le\; V \;\le\; a_2\left(\|w\|^2 + \|\dot{w}\|^2\right) + a_3$$

for $t \ge t_0(\rho)$ and for $\nu > 0$ small enough. This implies that

$$\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \;\le\; C_1\left(\|\dot{w}(t_0)\|^2 + \|Aw(t_0)\|^2\right)e^{-\omega t} + C_2 \qquad (5.24)$$

for $t \ge t_0 = t_0(\rho)$, provided that

$$\|Au_0\|^2 + \|u_1\|^2 \;\le\; \rho^2\,. \qquad (5.25)$$

If we use (5.4)–(5.6), then it is easy to find that

$$\|Au(t)\|^2 + \|\dot{u}(t)\|^2 \;\le\; C_{\rho,\,R}\,, \qquad t > 0\,,$$

under condition (5.25). Using the energy equality for the weak solutions to problem (4.15) we conclude that

$$\frac{d}{dt}\left(\|Aw(t)\|^2 + \|\dot{w}(t)\|^2\right) \le C_1(\rho, R)\cdot\|Aw\|\cdot\|\dot{w}\| + C_2(\rho, R).$$

Therefore, standard reasoning in which Gronwall's lemma is used leads to

$$\|Aw(t)\|^2 + \|\dot{w}(t)\|^2 \le \left(\|Aw(0)\|^2 + \|\dot{w}(0)\|^2 + a\right)e^{bt},$$

provided that equation (5.25) is valid. Here a and b are some positive constants depending on ρ and R. This and equation (5.24) imply that

$$\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \le C_1(\rho, R)\left((1 + \|\dot{w}(0)\|^2) + \|Aw(0)\|^2\right)e^{-\omega t} + C_2, \quad (5.26)$$

where C_2 depends on R only. Since

$$\|\dot{w}(0)\|^2 + \|Aw(0)\|^2 \le C_\rho, \qquad \|u_1\|^2 + \|Au_0\|^2 \le \left(\frac{\rho}{\lambda_1}\right)^2$$

provided that

$$\|Au_1\|^2 + \|A^2u_0\|^2 \le \rho^2,$$

equation (5.26) gives us the estimate

$$\|\ddot{u}(t)\|^2 + \|A\dot{u}(t)\|^2 \le b_R^2, \qquad t \ge \bar{t}_0(\rho).$$

This easily implies the dissipativity of the dynamical system $(\mathcal{H}_1, S_t)$. Thus, Lemma 5.2 is proved.

Exercise 5.5 Prove that the dynamical system $(\mathcal{H}_1, S_t)$ generated by problem (0.1) and (0.2) with the initial data $y_0 = (u_0; u_1) \in \mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$ is asymptotically compact provided that equations (5.10) hold.

In order to complete the proof of Theorem 5.3, we should note first that Lemma 5.2 coincides with the assertion of the theorem for $l = 1$ and second we should use the fact that the derivatives $u^{(k)}(t)$ are weak solutions to the problem obtained by differentiation of the original equation. The main steps of the reasoning are given in the following exercises.

Exercise 5.6 Assume that the hypotheses of Theorem 5.3 hold for $l = n+1$ and its assertion is valid for $l = n$. Show that $w(t) = u^{(n+1)}(t)$ is a weak solution to the problem of the form

$$\begin{cases} \ddot{w}(t) + \gamma\dot{w}(t) + A^2w(t) + M\|A^{1/2}u\|^2 Aw + Lw = G_{n+1}(t), \\ w(0) = u^{(n+1)}(0), \quad \dot{w}(0) = u^{(n+2)}(0), \end{cases}$$

where

$$\|G_{n+1}(t)\| \le C(R_n) \quad \text{for all} \quad t > t_0(\rho).$$

— Exercise *5.7* Use the result of Exercise 5.6 and the method given in the proof of Lemma 5.2 to prove that $w(t) = u^{(n+1)}(t)$ can be estimated as follows:

$$\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \le$$

$$\le C_1\Big(\|\dot{w}(0)\|^2 + \|Aw(0)\|^2 + C_2\Big)e^{-\omega t} + C_3 , \qquad (5.27)$$

where $\omega > 0$, the numbers C_j depend on ρ and R_n, $j = 1, 2$, and the constant C_3 depends on R_n only.

— Exercise *5.8* Use the induction assumption and equation (5.27) to prove the assertion of Theorem 5.3 for $l = n+1$.

§ 6 *Global Attractor and Inertial Sets*

The above given properties of the evolutionary operator S_t generated by problem (0.1) and (0.2) in the case of stationary load $p(t) \equiv p$ enable us to apply the general assertions proved in Chapter 1 (see also [10]).

Theorem 6.1

Assume that conditions (3.2), (5.2), (5.3), ***and*** (5.10) ***are fulfilled. Then the dynamical system*** $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***possesses a global attractor*** $\mathcal{A}$ ***of a finite fractal dimension. This attractor is a connected compact set in*** $\mathcal{H}$ ***and is bounded in the space*** $\mathcal{H}_\sigma = \mathcal{F}_{1+\sigma} \times \mathcal{F}_\sigma$, ***where*** $\sigma > 0$ ***is defined by condition*** (5.10).

Proof.

By virtue of Theorems 5.1, 5.2, and 1.5.1 we should prove only the finite dimensionality of the attractor. The corresponding reasoning is based on Theorem 1.8.1 and the following assertions.

Lemma 6.1.

Assume that conditions (3.2), (5.2), *and* (5.3) *are fulfilled. Let* $p \in H$. *Then for any pair of semitrajectories* $\{S_t y_j\colon t \ge 0\}$, $j = 1, 2$, *possessing the property* $\|S_t y_j\| \le R$ *for all* $t \ge 0$ *the estimate*

$$\|S_t y_1 - S_t y_2\|_{\mathcal{H}} \le \exp(a_0 t)\|y_1 - y_2\|_{\mathcal{H}} , \qquad t \ge 0 , \qquad (6.1)$$

holds with the constant a_0 *depending on* R.

Proof.

If $u_1(t)$ and $u_2(t)$ are solutions to problem (0.1) and (0.2) with the initial conditions $y_1 = (u_{01};\ u_{11})$ and $y_2 = (u_{02};\ u_{12})$, then the function $v(t) = u_1(t) - u_2(t)$ satisfies the equation

$$\ddot{v} + \gamma \dot{v} + A^2 v = \mathcal{F}(u_1,\ u_2,\ t)\,, \tag{6.2}$$

where

$$\mathcal{F}(u_1,\ u_2,\ t) = M\Big(\big\|A^{1/2}u_2(t)\big\|^2\Big)Au_2(t) - M\Big(\big\|A^{1/2}u_1(t)\big\|^2\Big)Au_1(t) - Lv(t)\,.$$

It is evident that the estimate

$$\|\mathcal{F}(u_1,\ u_2,\ t)\| \ \le\ C_R \cdot \|A(u_1(t) - u_2(t))\|$$

holds, provided that $\|y_i(t)\|^2_{\mathcal{H}} \ =\ \|\dot{u}_i(t)\|^2 + \|u_i(t)\|^2_1 \ \le\ R^2$. Therefore, (2.20) implies that

$$\|S_t y_1 - S_t y_2\|^2_{\mathcal{H}} \ \le\ \|y_1 - y_2\|^2_{\mathcal{H}} + a_1 \int_0^t \|S_t y_1 - S_t y_2\|^2_{\mathcal{H}}\, d\tau\ .$$

Gronwall's lemma gives us equation (6.1).

Lemma 6.2

Assume that the hypotheses of Theorem 6.1 *hold. Let* K_σ *be the compact positively invariant set constructed in Theorem* 5.2. *Then for any* $y_1,\ y_2 \in K_\sigma$ *the inequality*

$$\|Q_N(S_t y_1 - S_t y_2)\|_{\mathcal{H}} \ \le\ a_2\, e^{-\frac{\gamma}{4}t}\left(1 + \frac{L_\sigma}{\lambda^\sigma_{N+1}}\, e^{a_3 t}\right)\|y_1 - y_2\|_{\mathcal{H}} \tag{6.3}$$

is valid, where $Q_N = 1 - P_N$, $N \ge N_0$, *the orthoprojector* P_N *and the number* N_0 *are defined as in* (5.14), L_σ *and* a_i *are positive constants which depend on the parameters of the problem.*

Proof.

It is evident that

$$Q_N S_t y_i \ = (q_N u_i(t);\ q_N \dot{u}_i(t))\,,$$

where q_N is the orthoprojector onto the closure of the span of elements $\{e_k:\ k = N+1,\ N+2,\ \ldots\}$ in $\mathcal{F}_0 = H$. Moreover, the function $w_N(t) = q_N(u_1(t) - u_2(t))$ is a solution to the equation

$$\ddot{w}_N + \gamma \dot{w}_N + A^2 w_N + M\Big(\big\|A^{1/2}u_1\big\|^2\Big)Aw_N = \Phi_N(u_1,\ u_2,\ t)\,,$$

where

$$\Phi_N(u_1,\ u_2,\ t) = -\Big[M\Big(\big\|A^{1/2}u_1\big\|^2\Big) - M\Big(\big\|A^{1/2}u_2\big\|^2\Big)\Big]q_N A u_2 - q_N L(u_1 - u_2)\,.$$

Let us estimate the value Φ_N. Since $\|y\|_{\mathcal{H}_\sigma} \le \overline{R}_\sigma$ for $y \in K_\sigma$ (see the proof of Theorem 5.2), we have

$$\|q_N A u_2\| = \|q_N u_2\|_1 \le \lambda_{N+1}^{-\sigma} \|q_N u_2\|_{1+\sigma} \le \lambda_{N+1}^{-\sigma} \overline{R}_\sigma .$$

Using equation (5.10) we similarly obtain that

$$\|q_N L v\| \le \lambda_{N+1}^{-\sigma} \|A^\sigma L v\| \le C \lambda_{N+1}^{-\sigma} \|A v\| .$$

Therefore,

$$\|q_N L(u_1(t) - u_2(t))\| \le C \lambda_{N+1}^{-\sigma} \|S_t y_1 - S_t y_2\|_{\mathcal{H}} .$$

Consequently,

$$\|\Phi_N(u_1, u_2, t)\| \le \frac{C(\overline{R}_\sigma)}{\lambda_{N+1}^{\sigma}} \|S_t y_1 - S_t y_2\|_{\mathcal{H}} . \tag{6.4}$$

Using equation (2.27) we obtain that

$$Q_N(S_t y_1 - S_t y_2) = U(t, 0)\, Q_N(y_1 - y_2) + \int_0^t U(t, \tau)(0;\ \Phi_N(\tau))\, d\tau ,$$

where $U(t, \tau)$ is the evolutionary operator of homogeneous problem (2.1) with $b(t) = M(\|A^{1/2} u_1(t)\|^2)$ and $h(t) = 0$. Therefore, (2.23) and (6.4) imply that

$$\|Q_N(S_t y_1 - S_t y_2)\|_{\mathcal{H}} \le$$

$$\le \sqrt{3}\, e^{-\frac{\gamma}{4} t} \left\{ \|y_1 - y_2\|_{\mathcal{H}} + \frac{C(\overline{R}_\sigma)}{\lambda_{N+1}^{\sigma}} \int_0^t e^{\frac{\gamma}{4}\tau} \|S_\tau y_1 - S_\tau y_2\|\, d\tau \right\} . \tag{6.5}$$

We substitute (6.1) in this equation to obtain (6.3). Lemma 6.2 is proved.

Let us choose t_0 and $N \ge N_0$ such that

$$a_2 \exp\left\{-\frac{\gamma}{4} t_0\right\} = \frac{\delta}{2}, \qquad L_\sigma \lambda_{N+1}^{-\sigma} \exp\{a_3 t_0\} \le 1 , \qquad \delta < 1 .$$

Then Lemmata 6.1 and 6.2 enable us to state that

$$\|S_{t_0} y_1 - S_{t_0} y_2\|_{\mathcal{H}} \le l \|y_1 - y_2\|_{\mathcal{H}}$$

and

$$\|Q_N(S_{t_0} y_1 - S_{t_0} y_2)\|_{\mathcal{H}} \le \delta \|y_1 - y_2\|_{\mathcal{H}} ,$$

where $l = e^{a_0 t_0}$ and the elements y_1 and y_2 lie in the global attractor $\mathcal{A}$. Hence, we can use Theorem 1.8.1 with $M = \mathcal{A}$, $V = S_{t_0}$, and $P = P_N$. Therefore, the fractal dimension of the attractor $\mathcal{A}$ is finite. Thus, **Theorem 6.1 is proved**.

Theorem 5.2 and Lemmata 6.1 and 6.2 enable us to use Theorem 1.9.2 to obtain an assertion on the existence of the inertial set (fractal exponential attractor) for the dynamical system $(\mathcal{H}, S_t)$ generated by problem (0.1) and (0.2).

Theorem 6.2

Assume that the hypotheses of Theorem* 6.1 *hold. Then there exists a compact positively invariant set $\mathcal{A}_{\exp} \subset \mathcal{H}$ ***of the finite fractal dimension such that***

$$\sup\left\{\operatorname{dist}_{\mathcal{H}}(S_t y,\ \mathcal{A}_{\exp}):\ y \in B\right\} \le C e^{-\nu(t-t_B)}$$

for any bounded set B ***in*** $\mathcal{H}$ ***and*** $t \ge t_B$. ***Here*** C ***and*** ν ***are positive numbers. The inertial set*** $\mathcal{A}_{\exp}$ ***is bounded in the space*** $\mathcal{H}_\sigma$.

To prove the theorem we should only note that relations (5.11), (6.1), and (6.3) coincide with conditions (9.12)–(9.14) of Theorem 1.9.2.

Using (1.8.3) and (1.9.18) we can obtain estimates (involving the parameters of the problem) for the dimensions of the attractor and the inertial set by an accurate observing of the constants in the proof of Theorems 5.1 and 5.2 and Lemmata 6.1 and 6.2. However, as far as problem (0.3) and (0.4) is concerned, it is rather difficult to evaluate these estimates for the values of parameters that are very interesting from the point of view of applications. Moreover, these estimates appear to be quite overstated. Therefore, the assertions on the finite dimensionality of an attractor and inertial set should be considered as qualitative results in this case. In particular, this assertions mean that the nonlinear flutter of a plate is an essentially finite-dimensional phenomenon. The study of oscillations caused by the flutter can be reduced to the study of the structure of the global attractor of the system and the properties of inertial sets.

— Exercise *6.1* Prove that the global attractor of the dynamical system generated by problem (0.1) and (0.2) is a uniformly asymptotically stable set (*Hint:* see Theorem 1.7.1).

We note that theorems analogous to Theorems 6.1 and 6.2 also hold for a class of retarded perturbations of problem (0.1) and (0.2). For example, instead of (0.1) and (0.2) we can consider (cf. [11–13]) the following problem

$$\ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu + q(u_t) = p\ ,$$

$$u\big|_{t=0} = u_0,\quad \dot{u}\big|_{t=0} = u_1,\quad u\big|_{t\in(-r,\,0)} = \varphi(t)\,.$$

Here A, M, and L are the same as in Theorems 6.1 and 6.2, the symbol u_t denotes the function on $[-r, 0]$ which is given by the equality $u_t(\sigma) = u(t+\sigma)$ for $\sigma \in [-r, 0]$, the parameter r is a delay value, and $q(\cdot)$ is a linear mapping from $L^2(-r, 0; \mathcal{F}_{1+\alpha})$ into H possessing the property

$$\|A^\alpha q(v)\|^2 \le c_0 \int_{-r}^{0} \|A^{1+\alpha} v(\sigma)\|^2 \, d\sigma$$

for c_0 small enough and for all $\alpha \in [0, \alpha_0]$, where α_0 is a positive number. Such a formulation of the problem corresponds to the case when we use the model of the linearized potential gas flow (see [11–14]) to take into account the aerodynamic pressure in problem (0.3) and (0.4).

The following assertion gives the time smoothness of trajectories lying in the attractor of problem (0.1) and (0.2).

Theorem 6.3

Assume that conditions (3.2) ***and*** (5.2) ***are fulfilled and the linear operator*** L ***possesses the property***

$$\|A^\sigma L u\| \le C\|A^{\sigma+1/2} u\|, \qquad u \in D(A) \tag{6.6}$$

for all $\sigma \in [-1/2, 1/2)$. ***Let*** $p \in \mathcal{F}_{1/2}$. ***Then the assertions of Theorem*** 6.1 ***are valid for any*** $\sigma \in (0, 1/2)$. ***Moreover, if*** $M(z) \in C^{l+1}(\mathbb{R}_+)$ ***for some*** $l \ge 1$, ***then the trajectories*** $y = (u(t), \dot{u}(t))$ ***lying in the global attractor*** $\mathcal{A}$ ***of the system*** $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***for*** $\gamma > 0$ ***possess the property***

$$\|u^{(k+1)}(t)\|^2 + \|Au^{(k)}(t)\|^2 + \|A^2 u^{(k-1)}(t)\|^2 \le R_l^2 \tag{6.7}$$

for all $-\infty < t < \infty$, $k = 1, 2, \ldots, l$, ***where*** R_l ***is a constant depending on the problem parameters only.***

Proof.

It is evident that conditions (5.3) and (5.10) follow from (6.6). Therefore, we can apply Theorem 6.1 which guarantees the existence of a global attractor $\mathcal{A}$. Let us assume that $M(z) \in C^{l+1}(\mathbb{R}_+)$, $l \ge 1$. Let $y(t) = (u(t), \dot{u}(t))$ be a trajectory in $\mathcal{A}$, $-\infty < t < \infty$. We consider a function $u_m(t) = p_m u(t)$, where p_m is the orthoprojector onto the span of the basis vectors $\{e_1, \ldots, e_m\}$ in $\mathcal{F}_0$ for m large enough. It is clear that $u_m(t) \in C^2(\mathbb{R}; \mathcal{F}_2)$ and satisfies the equation

$$\ddot{u}_m + \gamma \dot{u}_m + A^2 u_m + M\left(\|A^{1/2} u(t)\|^2\right) A u_m + p_m L u = p_m p. \tag{6.8}$$

Equation (6.6) for $\sigma = -1/2$ implies that $p_m L u$ is a continuously differentiable function. It is also evident that $M(\|A^{1/2} u(t)\|^2) \in C^1(\mathbb{R})$. Therefore, we differentiate equation (6.8) with respect to t to obtain the equation

$$\ddot{w}_m + \gamma \dot{w}_m + A^2 w_m + M\Big(\|A^{1/2} u(t)\|^2\Big) A w_m = -p_m L \dot{u}(t) + F_m(t)$$

for the function $w_m(t) = \dot{u}_m(t) = p_m \dot{u}(t)$. Here

$$F_m(t) = -2M'\Big(\|A^{1/2} u(t)\|^2\Big)(A u(t),\ \dot{u}(t)) A u_m(t)\,.$$

Since any trajectory $y = (u(t),\ \dot{u}(t))$ lying in the attractor possesses the property

$$\|\dot{u}(t)\|^2 + \|A u(t)\|^2 \ \le\ R_0^2\,, \qquad -\infty < t < \infty\,, \tag{6.9}$$

it is clear that

$$\|F_m(t)\| \ \le\ C_{R_0}\,, \qquad -\infty < t < \infty\,. \tag{6.10}$$

Relation (6.9) also implies that the function

$$b(t) = M\Big(\|A^{1/2} u(t)\|^2\Big) \tag{6.11}$$

possesses the property

$$|b(t)| + |\dot{b}(t)| \ \le\ C_{R_0}\,, \qquad -\infty < t < \infty\,.$$

Therefore, as in the proof of Theorem 2.2, we find that there exists N_0 such that

$$\|(1-P_N)\, U(t,\ \tau) y\|_{\mathcal{H}_S} \ \le\ C\|(1-P_N) y\|_{\mathcal{H}_S} e^{-\frac{\gamma}{4}(t-\tau)} \tag{6.12}$$

for all real s, where $\mathcal{H}_s = \mathcal{F}_{1+s} \times \mathcal{F}_s$, P_N is the orthoprojector onto

$$L_N = \mathrm{Lin}\{(e_k,\ 0);\ (0,\ e_k):\ k = 1,\ 2,\ \ldots,\ N\},$$

$N \ge N_0$, and $U(t,\ \tau)$ is the evolutionary operator of the problem

$$\begin{cases} \ddot{u} + \gamma \dot{u} + A^2 u + b(t) A u = 0\ , \\ u\big|_{t=0} = u_0\,, \quad \dot{u}\big|_{t=0} = u_1\ , \end{cases}$$

with $b(t)$ of the form (6.10). Moreover, $z_m(t) = (w_m(t),\ \dot{w}_m(t))$ can be presented in the form

$$z_m(t) \ =\ U(t,\ t_0)\, z_m(t_0) + \int_{t_0}^{t} U(t,\ \tau)(0;\ \ -p_m L \dot{u}(\tau) + F_m(\tau))\, d\tau\,. \tag{6.13}$$

Then for $m > N > N_0$ we have

$$\|(1-P_N)\, z_m(t)\|_{\mathcal{H}_{-1/2}} \ \le\ C e^{-\frac{\gamma}{4}(t-t_0)} \|z_m(t_0)\|_{\mathcal{H}_{-1/2}} +$$

$$+\ C \int_{t_0}^{t} e^{-\frac{\gamma}{4}(t-\tau)} \|p_m L \dot{u}(\tau) + F_m(\tau)\|_{-1/2}\, d\tau\ .$$

Therefore, we use (6.9), (6.10), and (6.6) for $\sigma = -1/2$ to obtain that

$$\|(1-P_N)z_m(t)\|_{\mathcal{H}_{-1/2}} \le C_{R_{01}m}\, e^{-\frac{\gamma}{4}(t-t_0)} + C_{R_0}\int_{t_0}^{t} e^{-\frac{\gamma}{4}(t-\tau)}\,d\tau \,.$$

We tend $t_0 \to -\infty$ in this inequality to find that

$$\sup_{t\,\in\,R}\|(1-P_N)z_m(t)\|_{\mathcal{H}_{-1/2}} \le C\,,$$

where $C > 0$ does not depend on m. It further follows from (6.9) and equation (0.1) that

$$\sup_{t\,\in\,R}\|P_N\, z_m(t)\|_{\mathcal{H}_{-1/2}} \le C\,,$$

where the constant C can depend on N. Hence,

$$\sup_{t\,\in\,R}\left(\|p_m A^{1/2}\dot{u}(t)\|^2 + \|A^{-1/2}p_m\ddot{u}(t)\|^2\right) \le C\,.$$

We tend $m \to \infty$ to find that any trajectory $y(t) = (u(t),\ \dot{u}(t))$ lying in the attractor possesses the property

$$\|\dot{u}(t)\|_{1/2} = \|A^{1/2}\dot{u}(t)\| \le C_{R_0}\,, \qquad -\infty < t < \infty\,.$$

By virtue of (6.6) we have

$$\|L\dot{u}(t)\| \le C_{R_0}\,, \qquad -\infty < t < \infty\,.$$

Therefore, we reason as above to find that equation (6.13) implies

$$\|(1-P_N)z_m(t)\|_{\mathcal{H}} \le C_{R_0 m}\, e^{-\frac{\gamma}{4}(t-t_0)} + C_{R_0}\,.$$

Similarly we get

$$\|\ddot{u}(t)\|^2 + \|A\dot{u}(t)\|^2 \le R_1^2$$

for all $t \in (-\infty,\infty)$. Consequently, using equation (0.1) we obtain estimate (6.7) for $k = 1$. In order to prove (6.7) for the other values of k we should use induction with respect to k and similar arguments. We offer the reader to make an independent detailed study as an exercise.

Exercise *6.2* In addition to the hypotheses of Theorem 6.3 we assume that $L \equiv 0$ and $\rho \in \mathcal{F}_l \equiv D(A^l)$. Prove that the global attractor $\mathcal{A}$ of the system $(\mathcal{H},\ S_t)$ lies in $\mathcal{F}_{l+1} \times \mathcal{F}_l$.

§ 7 Conditions of Regularity of Attractor

Unfortunately, the structure of the global attractor of problem (0.1) and (0.2) can be described only under additional conditions that guarantee the existence of the Lyapunov function (see Section 1.6). These conditions require that $L \equiv 0$ and assume the stationarity of the transverse load $p(t)$. For the Berger system (0.3) and (0.4) these hypotheses correspond to $\rho = 0$ and $p(x, t) \equiv p(x)$, i.e. to the case of plate oscillations in a motionless stationary medium.

Thus, let us assume that the operator L is identically equal to zero and $p(t) \equiv p$ in (0.1). Assume that the hypotheses of Theorem 3.1 hold. Then energy equality (3.6) implies that

$$E(y(t_2)) - E(y(t_1)) = -\gamma \int_{t_1}^{t_2} \|\dot{u}(\tau)\|^2 \, d\tau + \int_{t_1}^{t_2} (p, \dot{u}(\tau))^2 \, d\tau , \tag{7.1}$$

where $y(t) = S_t y = (u(t), \dot{u}(t))$, the function $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y = (u_0; u_1)$, and $E(y)$ is the energy of the system defined by formula (3.7).

Let us prove that the functional $\Psi(y) = E(y) - (p, u_0)$ with $y = (u_0; u_1)$ is a Lyapunov function (for definition see Section 1.6) of the dynamical system $(\mathcal{H}, S_t)$. Indeed, it is evident that the functional $\Psi(y)$ is continuous on $\mathcal{H}$. By virtue of (7.1) it is monotonely increasing. If $E(y(t_0)) = E(y)$ for some $t_0 > 0$, then

$$\int_0^{t_0} \|\dot{u}(\tau)\|^2 \, d\tau = 0 .$$

Therefore, $\dot{u}(\tau) = 0$ for $\tau \in [0, t_0]$, i.e. $u(t) = \overline{u}$ is a stationary solution to problem (0.1) and (0.2). Hence, $y = (\overline{u}; 0)$ is a fixed point of the semigroup S_t.

Therefore, Theorems 1.6.1 and 6.1 give us the following assertion.

Theorem 7.1

Assume that $\gamma > 0$, $L \equiv 0$, ***and*** $p \in \mathcal{F}_\sigma$ ***for some*** $\sigma > 0$. ***We also assume that the function*** $M(z)$ ***satisfies conditions*** (3.2) ***and*** (5.2). ***Then the global attractor*** $\mathcal{A}$ ***of the dynamical system*** $(\mathcal{H}, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2) ***has the form***

$$\mathcal{A} = \boldsymbol{M}_+(\mathcal{N}) , \tag{7.2}$$

where $\mathcal{N}$ ***is the set of fixed points of the semigroup*** S_t, ***i.e.***

$$\mathcal{N} = \left\{ (u; 0) : \ u \in \mathcal{F}_1, \ \ A^2 u + M\left(\|A^{1/2} u\|^2\right) A u = p \right\} , \tag{7.3}$$

and $\boldsymbol{M}_+(\mathcal{N})$ ***is the unstable set emanating from*** $\mathcal{N}$ *(for definition see Section 1.6).*

— **Exercise** *7.1* Let $p \equiv 0$. Prove that if the hypotheses of Theorem 7.1 hold, then any fixed point z of problem (0.1) and (0.2) either equals to zero, $z = (0;\ 0)$, or has the form $z = (c \cdot e_k;\ 0)$, where the constant c is the solution to the equation $M(c^2\lambda_k) + \lambda_k = 0$.

— **Exercise** *7.2* Assume that $p \equiv 0$ and $M(z) = -\Gamma + z$. Then problem (0.1) and (0.2) has a unique fixed point $z_0 = (0;\ 0)$ for $\Gamma \le \lambda_1$. If $\lambda_n < \Gamma \le \lambda_{n+1}$, then the number of fixed points is equal to $2n+1$ and all of them have the form $z_k = (w_k;\ 0)$, $k = 0, \pm 1, \ldots, \pm n$, where

$$w_0 = 0, \qquad w_{\pm k} = \pm\sqrt{\frac{\Gamma - \lambda_k}{\lambda_k}} \cdot e_k, \qquad k = 1, 2, \ldots, n.$$

— **Exercise** *7.3* Show that if the hypotheses of Exercise 7.2 hold, then the energy $E(z_k)$ of each fixed point z_k has the form

$$E(z_0) = 0, \qquad E(z_{\pm k}) = -\tfrac{1}{4}(\Gamma - \lambda_k)^2, \qquad k = 1, 2, \ldots, n,$$

for $\lambda_n < \Gamma \le \lambda_{n+1}$.

— **Exercise** *7.4* Assume that the hypotheses of Theorem 7.1 hold. Show that if the set

$$\mathcal{D}_c = \left\{ y = (u_0;\ u_1) \in \mathcal{H}\colon\ \Psi(y) \equiv E(y) - (p, u_0) \le c \right\} \tag{7.4}$$

is not empty, then it is a closed positively invariant set of the dynamical system $(\mathcal{H}, S_t)$ generated by weak solutions to problem (0.1) and (0.2).

— **Exercise** *7.5* Assume that the hypotheses of Theorem 7.1 hold and the set $\mathcal{D}_c$ defined by equality (7.4) is not empty. Show that the dynamical system $(\mathcal{D}_c, S_t)$ possesses a compact global attractor $\mathcal{A}_c = \boldsymbol{M}_+(\mathcal{N}_c)$, where $\mathcal{N}_c$ is the set of fixed points of S_t satisfying the condition $\Psi(z) \le c$.

— **Exercise** *7.6* Show that if the hypotheses of Theorem 7.1 hold, then the global minimal attractor $\mathcal{A}_{\min}$ (for definition see Section 1.3) of problem (0.1) and (0.2) coincides with the set $\mathcal{N}$ of the fixed points (see (7.3)).

Further we prove that if the hypotheses of Theorem 7.1 hold, then the attractor $\mathcal{A}$ of problem (0.1) and (0.2) is regular in generic case. As in Section 2.5, the corresponding arguments are based on the results obtained by A. V. Babin and M. I. Vishik (see also Section 1.6). These results prove that in generic case the number of fixed points is finite and all of them are hyperbolic.

Lemma 7.1.

Assume that conditions (3.2) *and* (5.2) *are fulfilled. Then the problem*

$$\mathcal{L}[u] \equiv A^2 u + M\left(\|A^{1/2}u\|^2\right)Au = p, \qquad u \in D(A^{2+\sigma}), \tag{7.5}$$

possesses a solution for any $p \in \mathcal{F}_\sigma$, *where* $\sigma \geq 0$. *If* $\mathcal{B}$ *is a bounded set in* $\mathcal{F}_\sigma$, *then its preimage* $\mathcal{L}^{-1}(\mathcal{B})$ *is bounded in* $\mathcal{F}_{2+\sigma} = D(A^{2+\sigma})$. *If* $\mathcal{B}$ *is a compact in* $\mathcal{F}_\sigma$, *then* $\mathcal{L}^{-1}(\mathcal{B})$ *is a compact in* $D(A^{2+\sigma})$, *i.e. the mapping* $\mathcal{L}$ *is proper.*

Proof.

We follow the line of arguments given in the proof of Lemma 2.5.3. Let us consider the continuous functional

$$W(u) = \frac{1}{2}\left\{(Au,\ Au) + \mathcal{M}\left(\|A^{1/2}u\|^2\right)\right\} - (p,\ u) \tag{7.6}$$

on $\mathcal{F}_1 = D(A)$, where $\mathcal{M}(z) = \int_0^z M(\xi)\,d\xi$ is a primitive of the function $M(z)$. Equation (3.2) implies that

$$W(u) \;\geq\; \frac{1}{2}\left(\|Au\|^2 - a\,\|A^{1/2}u\|^2 - b\right) - \|A^{-1}p\| \cdot \|Au\| \;\geq$$

$$\geq\; \frac{1}{4}\left(1 - \frac{a}{\lambda_1}\right)\|Au\|^2 - \frac{b}{2} - \left(1 - \frac{a}{\lambda_1}\right)^{-1}\|A^{-1}p\|^2. \tag{7.7}$$

Thus, the functional $W(u)$ is bounded below. Let us consider it on the subspace $p_m\mathcal{F}_1$, where p_m is the orthoprojector onto $\mathrm{Lin}\{e_1,\ \ldots,\ e_m\}$ as before. Since $W(u) \to +\infty$ as $\|Au\| \to \infty$, there exists a minimum point u_m on the subspace $p_m\mathcal{F}_1$. This minimum point evidently satisfies the equation

$$A^2 u_m + M\left(\|A^{1/2}u_m\|^2\right)Au_m \;=\; p_m p\,. \tag{7.8}$$

Equation (7.7) gives us that

$$\|Au_m\|^2 \;\leq\; c_1 + c_2 \inf\left\{W(u):\ \ u \in p_m\mathcal{F}_1\right\} + c_3\|A^{-1}p\|^2$$

with the constants being independent of m. Therefore, it follows from (7.8) that $\|A^2 u_m\| \leq C_R$, provided $\|p\| \leq R$. This estimate enables us to pass to the limit in (7.8) and to prove that if $\sigma = 0$, then equation (7.5) is solvable for any $p \in \mathcal{F}_\sigma$. Equation (7.5) implies that

$$\|A^{2+\sigma}u_m\| \leq C_R \quad \text{for} \quad \|p\|_\sigma \leq R\,,$$

i.e. $\mathcal{L}^{-1}(\mathcal{B})$ is bounded in $D(A^{2+\sigma})$ if $\mathcal{B}$ is bounded. In order to prove that the mapping $\mathcal{L}$ is proper we should reason as in the proof of Lemma 2.5.3. We give the reader an opportunity to follow these reasonings individually, as an exercise. Lemma 7.1 is proved.

Lemma 7.2

Let $u \in \mathscr{F}_1$. *Then the operator* $\mathscr{L}'[u]$ *defined by the formula*

$$\mathscr{L}'[u]w = A^2 w + 2M'\left(\|A^{1/2}u\|^2\right)(Au,\ w)Au + M\left(\|A^{1/2}u\|^2\right)Aw \qquad (7.9)$$

with the domain $D(\mathscr{L}'[u]) = D(A^2)$ *is selfadjoint and* $\dim \mathscr{K}er\, \mathscr{L}'[u] < < \infty$.

Proof.

It is clear that $\mathscr{L}'[u]$ is a symmetric operator on $D(A^2)$. Moreover, it is easy to verify that

$$\|\mathscr{L}'[u]w - A^2 w\| \le C(u)\cdot\|Aw\|, \qquad w \in D(A^2), \qquad (7.10)$$

i.e. $\mathscr{L}'[W]$ is a relatively compact perturbation of the operator A^2. Therefore, $\mathscr{L}'[u]$ is selfadjoint. It is further evident that

$$\mathscr{K}er\, \mathscr{L}'[u] = \mathscr{K}er\left[I + A^{-2}\left(\mathscr{L}'[u] - A^2\right)\right].$$

However, due to (7.10) the operator $A^{-2}(\mathscr{L}'[u] - A^2)$ is compact. Therefore, $\dim \mathscr{K}er\, \mathscr{L}'[u] < \infty$. Lemma 7.2 is proved.

Exercise *7.7* Prove that for any $u \in \mathscr{F}_1$ the operator $\mathscr{L}'[u]$ is bounded below and has a discrete spectrum, i.e. there exists an orthonormal basis $\{f_k\}$ in $\mathscr{H}$ such that

$$\mathscr{L}'[u]f_k = \mu_k f_k, \quad k = 1,\ 2,\ \dots, \quad \mu_1 \le \mu_2 \le \dots \quad \lim_{n\to\infty} \mu_n = \infty.$$

Exercise *7.8* Assume that $u = c_0 e_{k_0}$, where c_0 is a constant and e_{k_0} is an element of the basis $\{e_k\}$ of eigenfunctions of the operator A. Show that $\mathscr{L}'[u]e_k = \bar{\mu}_k e_k$ for all $k = 1,\ 2,\ \dots$, where

$$\bar{\mu}_k = \lambda_k^2\left[1 + 2\delta_{k k_0} c_0^2 M'(c_0^2 \lambda_{k_0})\right] + M(c_0^2\lambda_{k_0})\lambda_k.$$

Here $\delta_{k k_0} = 1$ for $k = k_0$ and $\delta_{k k_0} = 0$ for $k \ne k_0$.

As in Section 2.5, Lemmata 7.1 and 7.2 enable us to use the Sard-Smale theorem (see, e.g., the book by A. V. Babin and M. I. Vishik [10]) and to state that the set

$$\mathscr{R}_\sigma = \left\{h \in \mathscr{F}_\sigma:\ \exists[\mathscr{L}'[u]]^{-1} \quad \text{for all} \quad u \in \mathscr{L}^{-1}[h]\right\}$$

of regular values of the operator $\mathscr{L}$ is an open everywhere dense set in $\mathscr{F}_\sigma$ for $\sigma \ge 0$.

Exercise *7.9* Show that the set of solutions to equation (7.5) is finite for $p \in \mathscr{R}_\sigma$ (*Hint:* see the proof of Lemma 2.5.5).

Let us consider the linearization of problem (0.1) and (0.2) on a solution $u \in D(A^2)$ to problem (7.5):

$$\begin{aligned} &\ddot{w} + \gamma \dot{w} + \mathcal{L}'[u]w = 0 \ , \\ &w\big|_{t=0} = w_0, \quad \dot{w}\big|_{t=0} = w_1 \ . \end{aligned} \tag{7.11}$$

Here $\mathcal{L}'[u]$ is given by formula (7.9).

— Exercise *7.10* Prove that problem (7.11) has a unique weak solution on any segment $[0, T]$ if $w_0 \in \mathcal{F}_1$, $w_1 \in \mathcal{F}_0$, and the function $M(z) \in \in C^2(\mathbb{R}_+)$ possesses property (3.2).

Thus, problem (7.11) defines a strongly continuous linear evolutionary semigroup $T_t[u]$ in the space $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$ by the formula

$$T_t[u](w_0;\ w_1) = (w(t);\ \dot{w}(t)), \tag{7.12}$$

where $w(t)$ is a weak solution to problem (7.11).

— Exercise *7.11* Let $\{f_k\}$ be the orthonormal basis of eigenelements of the operator $\mathcal{L}'[u]$ and let μ_k be the corresponding eigenvalues. Then each subspace

$$\mathcal{H}_k = \mathrm{Lin}\{(f_k;\ 0),\ (0;\ f_k)\} \subset \mathcal{H}$$

is invariant with respect to $T_t[u]$. The eigenvalues of the restriction of the operator $T_t[u]$ onto the subspace $\mathcal{H}_k$ have the form

$$\exp\left\{-\left(\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \mu_k}\right) t\right\}.$$

Lemma 7.3

Let $L \equiv 0$. Assume that $M(z) \in C^2(\mathbb{R}_+)$ possesses property (3.2). Then the evolutionary operator S_t of problem (0.1) and (0.2) is Frechét differentiable at each fixed point $\overline{y} = (\overline{u};\ 0)$. Moreover, $S_t'[u] = T_t[\overline{u}]$, where $T_t[u]$ is defined by equality (7.12).

Proof.

Let

$$z(t) = S_t[\overline{y} + h] - \overline{y} - T_t[\overline{u}]h \ ,$$

where $h = (h_0;\ h_1) \in \mathcal{H}$, $\overline{y} = (\overline{u};\ 0)$, and $\overline{u}$ is a solution to equation (7.5). It is clear that $z(t) = (v(t);\ \dot{v}(t))$, where $v(t) \equiv u(t) - \overline{u} - w(t)$ is a weak solution to problem

$$\begin{cases} \ddot{v} + \gamma v + A^2 v = F(u(t),\ \overline{u},\ w(t)) \ , \\ v\big|_{t=0} = 0 \ , \quad \dot{v}\big|_{t=0} = 0 \ . \end{cases} \tag{7.13}$$

Here

$$F(u(t),\ \overline{u},\ w(t)) = M\left(\|A^{1/2}\overline{u}\|^2\right)A\overline{u} - M\left(\|A^{1/2}u(t)\|^2\right)Au(t) +$$

$$+M\left(\|A^{1/2}\overline{u}\|^2\right)Aw(t) + 2M'\left(\|A^{1/2}\overline{u}\|\right)(A\overline{u},\ w(t))A\overline{u}\ ,$$

where $u(t)$ is a weak solution to problem (0.1) and (0.2) with the initial conditions $y_0 = \overline{y} + h \equiv (\overline{u} + h_0;\ h_1)$ and $w(t)$ is a solution to problem (7.11) with $w_0 = h_0$ and $w_1 = h_1$. It is evident that

$$F(u(t),\ \overline{u},\ w(t)) \ = \ -M\left(\|A^{1/2}\overline{u}\|^2\right)Av + F_1(t) - M'\left(\|A^{1/2}\overline{u}\|^2\right)F_2(t)\ , \quad (7.14)$$

where

$$F_1(t) \ = \ -\Big\{M\left(\|A^{1/2}u(t)\|^2\right) - M\left(\|A^{1/2}\overline{u}\|^2\right) -$$

$$- M'\left(\|A^{1/2}\overline{u}\|^2\right)\left(\|A^{1/2}u(t)\|^2 - \|A^{1/2}\overline{u}\|^2\right)\Big\}Au(t)\ ,$$

$$F_2(t) = \left(\|A^{1/2}u(t)\|^2 - \|A^{1/2}\overline{u}\|^2\right)Au(t) - 2(A\overline{u},\ w(t))A\overline{u}\,.$$

It is also evident that the value $F_1(t)$ can be estimated in the following way

$$\|F_1(t)\| \ \le \ c_1 \cdot \max\Big\{|M''(z)| :\ z \in [0,\ c_2]\Big\}\ \Big|\|A^{1/2}u(t)\|^2 - \|A^{1/2}\overline{u}\|^2\Big|^2$$

for $t \in [0,\ T]$ and for $\|h\|_{\mathcal{H}} \le R$, where the constants c_1 and c_2 depend on T, R, and $\overline{u}$. This implies that

$$\|F_1(t)\| \ \le \ C(T,\ R,\ \overline{u})\|A(u(t) - \overline{u})\|^2\ . \quad (7.15)$$

Let us rewrite the value $F_2(t)$ in the form

$$F_2(t) = (u(t) + \overline{u},\ A(u(t) - \overline{u}))\cdot A(u(t) - \overline{u}) +$$

$$+ \|A^{1/2}(u(t) - \overline{u})\|^2 A\overline{u} + 2(\overline{u},\ Av(t))A\overline{u}\ .$$

Consequently, the estimate

$$\|F_2(t)\| \ \le \ C_1(T,\ R,\ \overline{u})\|A(u(t) - \overline{u})\|^2 + C_2(\overline{u})\|Av(t)\| \quad (7.16)$$

holds for $t \in [0,\ T]$ and for $\|h\|_{\mathcal{H}} \le R$. Therefore, equations (7.14)–(7.16) give us that

$$\|F(u(t),\ \overline{u},\ w(t))\| \ \le \ C_1\|Av(t)\| + C_2\|A(u(t) - \overline{u})\|^2$$

on any segment $[0,\ T]$. Here $C_1 = C_1(\overline{u})$ and $C_2 = C_2(T,\ R,\ \overline{u})$. We use continuity property (3.20) of a solution to problem (0.1) and (0.2) with respect to the initial conditions to obtain that

$$\|A(u(t)-\overline{u})\| \le C_{T,R}\|h\|_{\mathscr{H}}, \quad t \in [0, T], \quad \|h\|_{\mathscr{H}} \le R.$$

Therefore,

$$\|F(u(t), \overline{u}, w(t))\| \le C_1\|Av(t)\| + C_2\|h\|^2_{\mathscr{H}}$$

for $t \in [0, T]$ and for $\|h\|_{\mathscr{H}} \le R$. Hence, the energy equality for the solutions to problem (7.13) gives us that

$$\frac{d}{dt}\left(\|Av(t)\|^2 + \|\dot{v}(t)\|^2\right) \le a\left(\|Av(t)\|^2 + \|\dot{v}(t)\|^2\right) + C\|h\|^4.$$

Therefore, Gronwall's lemma implies that

$$\|Av(t)\|^2 + \|\dot{v}(t)\|^2 \le C\|h\|^4, \quad t \in [0, T].$$

This equation can be rewritten in the form

$$\left\|S_t[\overline{y}+h] - \overline{y} - T_t[\overline{u}]h\right\|_{\mathscr{H}} \le C\|h\|^2.$$

Thus, Lemma 7.3 is proved.

— Exercise *7.12* Use the arguments given in the proof of Lemma 7.3 to verify that under condition (3.2) for $M(z) \in C^2(\mathbb{R}_+)$ the evolutionary operator S_t of problem (0.1) and (0.2) in $\mathscr{H}$ belongs to the class C^1 and

$$\left\|S'_t[y_1] - S'_t[y_2]\right\|_{\mathscr{L}(\mathscr{H},\, \mathscr{H})} \le C\left\|y_1 - y_2\right\|_{\mathscr{H}}$$

for any $t > 0$ and $y_j \in \mathscr{H}$.

— Exercise *7.13* Use the results of Exercises 7.7 and 7.11 to prove that for a regular value p of the mapping $\mathscr{L}[u]$ the spectrum of the operator $T_t[u]$ does not intersect the unit circumference while the eigensubspace E_+ which corresponds to the spectrum outside the unit disk does not depend on t and is finite-dimensional.

The results presented above enable us to prove the following assertion (see Chapter V of the book by A. V. Babin and M. I. Vishik [10]).

Theorem 7.2

Assume that the hypotheses of Theorem 7.1 ***hold. Then there exists an open dense set*** $\mathscr{R}_\sigma$ ***in*** $\mathscr{F}_\sigma$ ***such that the dynamical system*** $(\mathscr{H}, S_t)$ ***possesses a regular global attractor*** $\mathscr{A}$ ***for every*** $p \in \mathscr{R}_\sigma$, ***i. e.***

$$\mathscr{A} = \bigcup_{j=1}^{N} \boldsymbol{M}_+(z_j),$$

where $\boldsymbol{M}_+(z_j)$ ***is the unstable manifold of the evolutionary operator*** S_t ***emanating from the fixed point*** z_j. ***Moreover, each set*** $\boldsymbol{M}_+(z_j)$ ***is a finite-dimensional surface of the class*** C^1.

In the case of a zero transverse load ($p \equiv 0$) Theorem 7.2 is not applicable in general. However, this case can be studied by using the structure of the problem. For example, we can guarantee finiteness of the set of fixed points if we assume (see Exercise 7.1) that the equation $M(c^2\lambda_k) + \lambda_k = 0$, first, is solvable with respect to c only for a finite number of the eigenvalues λ_k and, second, possesses not more than a finite number of solutions for every k. The solutions to equation (7.5) are either $\bar{u} \equiv 0$, or $\bar{u} = c_0 e_{k_0}$, where c_0 and k_0 satisfy $M(c_0^2\lambda_{k_0}) + \lambda_{k_0} = 0$. The eigenvalues of the operator $\mathcal{L}'[u]$ have the form

$$\bar{\mu}_k = \lambda_k^2 + M(0)\lambda_k \quad \text{if} \quad \bar{u} \equiv 0$$

and

$$\bar{\mu}_k = \lambda_k\left[\lambda_k - \lambda_{k_0} + 2\delta_{k k_0}\lambda_{k_0} c_0^2 M'(c_0^2\lambda_{k_0})\right] \quad \text{if} \quad \bar{u} = c_0 e_{k_0}.$$

Therefore, the result of Exercise 7.11 implies that the fixed points are hyperbolic if all the numbers $\bar{\mu}_k$ are nonzero, i.e. if

$$M(0) \neq -\lambda_k, \quad k = 1, 2, \ldots; \qquad \lambda_k \neq \lambda_{k_0}, \quad k \neq k_0; \qquad M(c_0^2\lambda_{k_0}) \neq 0$$

for all c_0 and k_0 such that $M(c_0^2\lambda_{k_0}) + \lambda_{k_0} = 0$. In particular, if $M(z) = -\Gamma + z$, then for any real Γ there exists a finite number of fixed points (see Exercise 7.2) and all of them are hyperbolic, provided that $\Gamma \neq \lambda_k$ for all k and the eigenvalues λ_j satisfying the condition $\lambda_j < \Gamma$ are simple. Moreover, we can prove that for $\lambda_n < \Gamma < \lambda_{n+1}$ the unstable manifold $\boldsymbol{M}_+(z_k)$, $k = 0, \pm 1, \ldots, \pm n$, emanating from the fixed point z_k (see Exercise 7.2) possesses the property

$$\dim \boldsymbol{M}_+(z_0) = n, \qquad \dim \boldsymbol{M}_+(z_k) = |k| - 1.$$

§ 8 *On Singular Limit in the Problem of Oscillations of a Plate*

In this section we consider problem (0.1) and (0.2) in the following form:

$$\mu\ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu = p, \quad t > 0, \tag{8.1}$$

$$u\big|_{t=0} = u_0, \quad \dot{u}\big|_{t=0} = u_1. \tag{8.2}$$

Equation (8.1) differs from equation (0.1) in that the parameter $\mu > 0$ is introduced. It stands for the mass density of the plate material. The introduction of a new time $t' = t/\sqrt{\mu}$ transforms equation (8.1) into (0.1) with the medium resistance parameter $\gamma' = \gamma/\sqrt{\mu}$ instead of γ. Therefore, all the above results mentioned above remain true for problem (8.1) and (8.2) as well.

The main question discussed in this section is the asymptotic behaviour of the solution to problem (8.1) and (8.2) for the case when the inertial forces are small with respect to the medium resistance forces $\mu \ll \gamma$. Formally, this assumption leads to a quasistatic statement of problem (8.1) and (8.2):

$$\gamma \dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right) Au + Lu = p, \quad t > 0, \tag{8.3}$$

$$u\big|_{t=0} = u_0 . \tag{8.4}$$

Here we prove that the global attractor of problem (8.1) and (8.2) is close to the global attractor of the dynamical system generated by equations (8.3) and (8.4) in some sense.

Without loss of generality we further assume that $\gamma = 1$. We also note that problem (8.3) and (8.4) belongs to the class of equations considered in Chapter 2.

— Exercise *8.1* Assume that conditions (3.2) and (3.3) are fulfilled and $p \in \mathcal{F}_0 = H$. Show that problem (8.3) and (8.4) has a unique mild (in $\mathcal{F}_1 = D(A)$) solution on any segment $[0, T]$, i.e. there exists a unique function $u(t) \in C(0, T; \mathcal{F}_1)$ such that

$$u(t) = e^{-A^2 t} u_0 -$$

$$- \int_0^t e^{-A^2(t-\tau)} \left\{ M\left(\|A^{1/2}u(\tau)\|^2\right) Au(\tau) + Lu(\tau) - p \right\} d\tau \ .$$

(*Hint:* see Theorem 2.2.4 and Exercise 2.2.10).

Let us consider the Galerkin approximations of problem (8.3) and (8.4):

$$\dot{u}_m(t) + A^2 u_m(t) + M\left(\|A^{1/2}u_m(t)\|^2\right) Au_m + p_m L u_m(t) = p_m p, \tag{8.5}$$

$$u_m(0) = p_m u_0, \tag{8.6}$$

where p_m is the orthoprojector onto the first m eigenvectors of the operator A and $u_m(t) \in \mathrm{Lin}\{e_1, \dots, e_m\}$.

— Exercise *8.2* Assume that conditions (3.2) and (3.3) are fulfilled and $p \in \mathcal{F}_0 = H$. Then problem (8.5) and (8.6) is solvable on any segment $[0, T]$ and

$$\max_{[0, T]} \|A(u(t) - u_m(t))\| \to 0, \qquad m \to \infty . \tag{8.7}$$

Theorem 8.1

Let $p \in H$ ***and assume that conditions*** (3.2), (5.2), ***and*** (5.3) ***are fulfilled. Then the dynamical system*** $(\mathcal{F}_1, S_t)$ ***generated by weak solutions to problem*** (8.3) ***and*** (8.4) ***possesses a compact connected global attractor*** $\mathcal{A}$. ***This attractor is a bounded set in*** $\mathcal{F}_{1+\beta}$ ***for*** $0 \le \beta < 1$ ***and has a finite fractal dimension.***

Proof.

First we prove that the system $(\mathcal{F}_1, S_t)$ is dissipative. To do that we consider the Galerkin approximations (8.5) and (8.6). We multiply (8.5) by $u_m(t)$ scalarwise and find that

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \|Au_m(t)\|^2 + M\left(\|A^{1/2}u_m(t)\|^2\right)\|A^{1/2}u_m(t)\|^2 +$$
$$+ (Lu_m(t), u_m(t)) = (p, u_m(t)) .$$

Using equation (5.2) we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \|Au_m(t)\|^2 + a_1\mathcal{M}\left(\|A^{1/2}u_m(t)\|^2\right) \le$$
$$\le a_3 - a_2\|A^{1/2}u_m(t)\|^{2+2\alpha} - (Lu_m(t), u_m(t)) + (p, u_m(t)) .$$

We use equation (5.3) and reason in the same way as in the proof of Theorem 5.1 to find that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + b_0\left(\|Au_m\|^2 + \mathcal{M}\left(\|A^{1/2}u_m\|^2\right)\right) + b_1\|A^{1/2}u_m\|^{2+2\alpha} \le b_2 \qquad (8.8)$$

with some positive constants b_j, $j = 0, 1, 2$. Multiplying equation (8.5) by $\dot{u}_m(t)$ we obtain that

$$\frac{1}{2}\frac{d}{dt}\Pi(u_m(t)) + \|\dot{u}_m(t)\|^2 + (Lu_m(t), \dot{u}_m(t)) = (p, \dot{u}_m(t)),$$

where

$$\Pi(u) = \|Au\|^2 + \mathcal{M}\left(\|A^{1/2}u\|^2\right).$$

It follows that

$$\frac{d}{dt}\Pi(u_m(t)) + \|\dot{u}_m(t)\|^2 \le 2\|p\|^2 + C\|A^{\theta}u_m(t)\|^2 . \qquad (8.9)$$

If we summarize (8.8) and (8.9), then it is easy to find that

$$\frac{d}{dt}\left\{\|u_m\|^2 + \Pi(u_m)\right\} + b_0\left\{\|u_m\|^2 + \Pi(u_m)\right\} \le C .$$

This implies that

$$\|u_m(t)\|^2 + \Pi(u_m(t)) \le \left\{\|p_m u_0\|^2 + \Pi(p_m u_0)\right\} e^{-b_0 t} + C .$$

We use (8.7) to pass to the limit as $m \to \infty$ and to obtain that

$$\|u(t)\|^2 + \Pi(u(t)) \le \left\{\|u_0\|^2 + \Pi(u_0)\right\} e^{-b_0 t} + C .$$

This implies the dissipativity of the dynamical system $(\mathcal{F}_1, S_t)$ generated by problem (8.3) and (8.4). In order to complete the proof of the theorem we use Theorem 2.4.1.

We note that the dissipativity also implies that the dynamical system $(\mathcal{F}_1,\ S_t)$ possesses a fractal exponential attractor (see Theorem 2.4.2).

Exercise *8.3* Assume that the hypotheses of Theorem 8.1 hold and $L \equiv 0$. Show that for generic $p \in H$ the attractor of the dynamical system $(\mathcal{F}_1,\ S_t)$ generated by equations (8.3) and (8.4) is regular (see the definition in the statement of Theorem 7.2). *Hint:* see Section 2.5.

We assume that $M(z) \in C^2(\mathbb{R}_+)$ and conditions (3.2), (5.2), (5.3), and (5.10) are fulfilled. Let us consider the dynamical system $(\mathcal{H}_1,\ S_t^\mu)$ generated by problem (8.1) and (8.2) in the space $\mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1 \equiv D(A^2) \times D(A)$. Lemma 5.2 and Exercise 5.5 imply that $(\mathcal{H}_1,\ S_t^\mu)$ possesses a compact global attractor $\mathcal{A}_\mu$ for any $\mu > 0$.

The main result of this section is the following assertion on the closeness of attractors of problem (8.1) and (8.2) and problem (8.3) and (8.4) for small $\mu > 0$.

Theorem 8.2

Assume that $M(z) \in C^2(\mathbb{R}_+)$ ***and conditions*** (3.2), (5.2), (5.3), ***and*** (5.10) ***concerning*** $M(z)$, L, ***and*** p ***are fulfilled. Then the equation***

$$\lim_{\mu \to 0} \sup\{\operatorname{dist}_{\mathcal{H}}(y,\ \mathcal{A}^*):\ y \in \mathcal{A}_\mu\} = 0 \tag{8.10}$$

is valid, where $\mathcal{A}_\mu$ ***is a global attractor of the dynamical system*** $(\mathcal{H}_1,\ S_t^\mu)$ ***generated by problem*** (8.1) ***and*** (8.2),

$$\mathcal{A}^* = \left\{(z_0;\ z_1):\ z_0 \in \mathcal{A},\ \ z_1 = -A^2 z_0 - M\left(\left\|A^{1/2} z_0\right\|^2\right) A z_0 - L z_0 + p\right\}.$$

Here $\mathcal{A}$ ***is a global attractor of problem*** (8.3) ***and*** (8.4) ***in*** $\mathcal{F}_1$ ***and*** $\operatorname{dist}_{\mathcal{H}}(y, A)$ ***is the distance between the element*** y ***and the set*** A ***in the space*** $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$. ***We remind that*** $\gamma = 1$ ***in equations*** (8.1) ***and*** (8.3).

The proof of the theorem is based on the following lemmata.

Lemma 8.1

The dynamical system $(\mathcal{H}_1,\ S_t^\mu)$ *is uniformly dissipative in* $\mathcal{H}$ *with respect to* $\mu \in (0,\ \mu_0]$ *for some* $\mu_0 > 0$, *i.e. there exists* $\mu_0 > 0$ *and* $R > 0$ *such that for any set* $B \subset \mathcal{H}_1$ *which is bounded in* $\mathcal{H}$ *we have*

$$S_t^\mu B \ \subset \ \left\{y = (u_0;\ u_1):\ \ \mu\left\|u_1\right\|^2 + \left\|A u_0\right\|^2 \le R^2\right\} \tag{8.11}$$

for all $t \ge t(B,\ \mu)$, $\mu \in (0,\ \mu_0]$.

Proof.

We use the arguments from the proof of Theorem 5.1 slightly modifying them. Let

$$V(y) = E(y) + \nu \Phi(y), \qquad y = (u_0;\ u_1),$$

where

$$E(y) = \frac{1}{2}\Big(\mu \|u_1\|^2 + \|Au_0\|^2 + \mathcal{M}\Big(\|A^{1/2}u_0\|^2\Big)\Big)$$

and

$$\Phi(y) = \mu(u_0,\ u_1) + \frac{1}{2}\|u_0\|^2 .$$

As in the proof of Theorem 5.1 it is easy to find that the inequalities

$$\frac{d}{dt}E(y(t)) \le -\frac{1}{2}\|u\|^2 + \frac{\varepsilon}{2}\|Au\|^2 + C_\varepsilon \|A^{1/2}u\|^2 + C_1 \tag{8.12}$$

and

$$\frac{d}{dt}\Phi(y(t)) \le \mu\|\dot{u}\|^2 - \frac{1}{2}\|Au\|^2 - a_1 \mathcal{M}\Big(\|A^{1/2}u\|^2\Big) -$$

$$- a_2 \|A^{1/2}u\|^{2+2\alpha} + C_2\|u\|^2 + C_3 \tag{8.13}$$

are valid for $y(t) = (u(t);\ \dot{u}(t)) = S_t^{\mu} y_0$. Here $\varepsilon > 0$ is an arbitrary number, the constants a_j and c_j do not depend on μ. Moreover, it is also evident that

$$V(y) \le \frac{1}{2}\beta_0\Big(\mu\|u_1\|^2 + \|Au_0\|^2 + \mathcal{M}\Big(\|A^{1/2}u_0\|^2\Big)\Big) + \beta_1 \tag{8.14}$$

for $\mu \in (0,\ \mu_0]$ and for any μ_0. Here β_0 and β_1 do not depend on $\mu \in$ $\in (0,\ \mu_0]$ and $\nu \le 1$. Equations (8.12)–(8.14) lead us to the inequality

$$\frac{d}{dt}V(y(t)) + \delta V(y(t)) \le -\frac{1}{2}(1 - (\delta\beta_0 + \nu)\mu)\|\dot{u}\|^2 -$$

$$-\frac{1}{2}(\nu - \delta\beta_0 - \varepsilon)\|Au\|^2 - \Big(\nu a_1 - \frac{\delta\beta_0}{2}\Big)\mathcal{M}\Big(\|A^{1/2}u\|^2\Big) + \mathcal{D},$$

where the constant $\mathcal{D}$ does not depend on $\mu \in (0,\ \mu_0]$. If we choose μ_0 small enough, then we can take $\delta > 0$ and $\nu > 0$ independent of $\mu \in (0,\ \mu_0]$ and such that

$$\frac{d}{dt}V(y(t)) + \delta V(y(t)) \le \mathcal{D}_1, \tag{8.15}$$

where $\mathcal{D}_1 > 0$ does not depend on $\mu \in (0,\ \mu_0]$. Moreover, we can assume (due to the choice of μ_0) that

$$V(y(t)) \ge \beta_3\Big(\mu\|\dot{u}(t)\|^2 + \|Au(t)\|^2\Big) - C, \tag{8.16}$$

where β_3 and C do not depend on $\mu \in (0,\ \mu_0]$. Using equations (8.14)–(8.16) we obtain the assertion of Lemma 8.1.

Lemma 8.2

Let $u(t)$ be a solution to problem (8.1) *and* (8.2) *such that* $\|Au(t)\| \le R$ *for all* $t \ge 0$. *Then the estimate*

$$\frac{1}{2}\int_0^t \|\dot{u}(\tau)\|^2 e^{-\beta(t-\tau)}\,d\tau \;\le\; \left(\mu\|\dot{u}(0)\|^2 + \|Au(0)\|^2\right)e^{-\beta t} + C(R,\,\beta)$$

is valid for $t \ge 0$, *where* β *is a positive constant such that* $\beta\mu \le 1/2$.

Proof.

It is evident that the estimate

$$\left\|M\left(\|A^{1/2}u(t)\|^2\right)Au(t) + Lu(t) - p\right\| \;\le\; C_R$$

holds, provided $\|Au(t)\| \le R$. Therefore, equaiton (8.1) easily implies the estimate

$$\frac{1}{2}\frac{d}{dt}\left(\mu\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right) + \frac{1}{2}\|\dot{u}(t)\|^2 \;\le\; C_R$$

for the solution $u(t)$. We multiply this inequality by $2\exp(\beta t)$. Then by virtue of the fact that $\|Au(t)\| \le R$ we have

$$\frac{d}{dt}\left[e^{\beta t}\left(\mu\|\dot{u}(t)\|^2 + \|Au(t)\|^2\right)\right] + \frac{1}{2}\|\dot{u}(t)\|^2 e^{\beta t} \;\le\; C_{R,\,\beta}\, e^{\beta t}$$

for $\mu\beta \le 1/2$. We integrate this equation from 0 to t to obtain the assertion of the lemma.

Lemma 8.3

Let $u(t)$ be a solution to problem (8.1) *and* (8.2) *with the initial conditions* $(u_0;\,u_1) \in \mathcal{H}_1 = \mathcal{F}_2 \times \mathcal{F}_1$ *and such that* $\|Au(t)\| \le R$ *for* $t \ge 0$. *Then the estimate*

$$\mu\|\dot{w}(t)\|^2 + \|Aw(t)\|^2 \;\le\; C_1\left(1 + \|\dot{w}(0)\|^2 + \|Aw(0)\|^2\right)e^{-\beta_0 t} + C_2 \qquad (8.17)$$

is valid for the function $w(t) = \dot{u}(t)$. *Here* $\mu \in (0,\,\mu_0]$, μ_0 *is small enough,* $\beta_0 > 0$, *and the numbers* C_1 *and* C_2 *do not depend on* $\mu \in (0,\,\mu_0]$.

Proof.

Let us consider the function

$$W(t) = \frac{1}{2}\left(\mu\|\dot{w}(t)\|^2 + \|Aw(t)\|^2\right) + \nu\left(\mu(\dot{w},\,w) + \frac{1}{2}\|w\|^2\right)$$

for $\nu > 0$. It is clear that

$$\frac{1}{2}\mu(1-\nu\mu)\|\dot{w}\|^2+\frac{1}{2}\|Aw\|^2 \le W(t) \le$$
$$\le \frac{1}{2}\mu(1+\nu\mu)\|\dot{w}\|^2+\left(\frac{1}{2}+\nu\lambda_1^{-2}\right)\|Aw\|^2 . \qquad (8.18)$$

Since the function $w(t)$ is a weak solution to the equation obtained by the differentiation of (8.1) with respect to t (cf. (4.15)):

$$\mu\ddot{w}(t)+\dot{w}(t)+A^2w(t)+M\left(\|A^{1/2}u(t)\|^2\right)Aw(t)+Lw(t) = F(t),$$

where

$$F(t) = -2M'\left(\|A^{1/2}u(t)\|^2\right)(Au(t),\ w(t))\,Au(t),$$

then we have that

$$\frac{d}{dt}W(t) = -(1-\nu\mu)\|\dot{w}\|^2 - M(\|A^{1/2}u(t)\|^2)(Aw,\ \dot{w})-(Lw,\ \dot{w})+(F,\ \dot{w})-$$
$$-\ \nu\left\{\|A^2w\|^2+M(\|A^{1/2}u(t)\|^2)\|A^{1/2}w\|^2+(Lw,\ w)-(F,\ w)\right\}.$$

It follows that

$$\frac{d}{dt}W(t) \le -\left(\frac{1}{2}-\nu\mu\right)\|\dot{w}\|^2-\frac{1}{2}(\nu-C_R^{(1)})\|Aw\|^2+\nu C_R^{(2)}\|w\|^2 .$$

We take $\nu = C_R^{(1)}+2$ and choose μ_0 small enough to obtain with the help of (8.18) that

$$\frac{d}{dt}W(t)+\beta_0 W(t) \le C\|w(t)\|^2 , \qquad t\ge 0, \qquad \mu\in(0,\ \mu_0],$$

where the constants $\beta_0>0$ and C do not depend on μ. Consequently,

$$W(t) \le W(0)\,e^{-\beta_0 t}+C\int_0^t \|\dot{u}(\tau)\|^2 e^{-\beta_0(t-\tau)}\,d\tau .$$

Therefore, estimate (8.17) follows from equation (8.18) and Lemma 8.2. Thus, Lemma 8.3 is proved.

Lemma 8.3 and equations (8.11) imply the existence of a constant R_1 such that for any bounded set B in $\mathcal{H}_1$ there exists $t_0 = t_0(B,\ \mu)$ such that

$$\mu\|\ddot{u}(t)\|^2+\|A\dot{u}(t)\|^2 \le R_1^2 , \qquad \mu\in(0,\ \mu_0), \qquad (8.19)$$

where $u(t)$ is a solution to problem (8.1) and (8.2) with the initial conditions from B. However, due to (8.1) equations (8.11) and (8.19) imply that $\|A^2u(t)\| \le C$ for $t\ge t_0(B,\ \mu)$. Thus, there exists $R_2>0$ such that

$$\mu\|\ddot{u}(t)\|^2+\|A\dot{u}(t)\|^2+\|A^2u(t)\|^2 \le R_2^2\,, \quad t \ge t_0(B,\ \mu)\,, \tag{8.20}$$

where $u(t)$ is a solution to system (8.1) and (8.2) with the initial conditions from the bounded set B in $\mathcal{H}_1$, R_2 does not depend on $\mu \in (0,\ \mu_0)$, and μ_0 is small enough. Equation (8.20) and the invariance property of the attractor $\mathcal{A}_\mu$ imply the estimate

$$\mu\left\|\ddot{u}_\mu(t)\right\|^2+\left\|A\dot{u}_\mu(t)\right\|^2+\left\|A^2u_\mu(t)\right\|^2 \le R_2^2 \tag{8.21}$$

for any trajectory $S_t^\mu y_0 = (u_\mu(t);\ \dot{u}_\mu(t))$ lying in $\mathcal{A}_\mu$ for all $t \in (-\infty, \infty)$.

Let us prove (8.10). It is evident that there exists an element $y_\mu = (u_{0\,\mu};\ u_{1\,\mu})$ from $\mathcal{A}_\mu$ such that

$$d(y_\mu) \equiv \operatorname{dist}_{\mathcal{H}}(y_\mu,\ \mathcal{A}^*) = \sup\{\operatorname{dist}_{\mathcal{H}}(y,\ \mathcal{A}^*):\ y \in \mathcal{A}_\mu\}\,.$$

Let $y_\mu(t) = (u_\mu(t);\ \dot{u}_\mu(t))$ be a trajectory of system (8.1) and (8.2) lying in the attractor $\mathcal{A}_\mu$ and such that $y_\mu(0) = y_\mu$. Equation (8.21) implies that there exist a subsequence $\{y_{\mu_n}(t)\}$ and an element $y(t)=(u(t);\ \dot{u}(t)) \in L^\infty(-\infty, \infty;\ \mathcal{H}_1)$ such that for any segment $[a,\ b]$ the sequence $y_{\mu_n}(t)$ converges to $y(t)$ in the $*$-weak topology of the space $L^\infty(a,\ b;\ \mathcal{H}_1)$ as $\mu_n \to 0$. Equation (8.21) gives us that the subsequence $\{Au_{\mu_n}(t)\}$ is uniformly continuous and uniformly bounded in H. Therefore (cf. Exercise 1.14),

$$\lim_{\mu_n \to 0}\ \max_{t\, \in [a,\, b]}\left\|A(u_{\mu_n}(t) - u(t))\right\| = 0 \tag{8.22}$$

for any $a < b$. However, it follows from (8.21) that $\mu\left\|\ddot{u}_\mu(t)\right\| \to 0$ as $\mu \to 0$. Therefore, we pass to the limit $\mu \to 0$ in equation (8.1) and obtain that the function $u(t)$ is a bounded (on the whole axis) solution to problem (8.3) and (8.4). Hence, it lies in the attractor $\mathcal{A}$ of the system $(\mathcal{F}_1,\ S_t)$. With the help of (8.21) and (8.22) it is easy to find that

$$d(y_{\mu_n}) \le \left\|y_{\mu_n} - y_0\right\|_{\mathcal{H}} \to 0\,, \qquad \mu_n \to 0\,,$$

where

$$y_0 = \Big(u(0);\ -Au(0) - M\Big(\|A^{1/2}u(0)\|^2\Big)Au_0 - Lu_0 + p\Big) \in \mathcal{A}^*\,.$$

Thus, **Theorem 8.2 is proved**.

§ 9 On Inertial and Approximate Inertial Manifolds

The considerations of this section are based on the results presented in Sections 3.7, 3.8, and 3.9. For the sake of simplicity we further assume that $p(t) \equiv g \in H$.

Theorem 9.1

Assume that conditions (3.2), (5.2), *and* (5.3) *are fulfilled. We also assume that eigenvalues of the operator* A *possess the properties*

$$\inf_N \frac{\lambda_N}{\lambda_{N+1}} > 0 \quad \textit{and} \quad \lambda_{N(k)+1} = c_0 k^\rho (1 + o(1)), \qquad \rho > 0, \quad k \to \infty, \tag{9.1}$$

for some sequence $\{N(k)\} \to \infty$. *Then there exist numbers* $\gamma_0 > 0$ *and* $k_0 > 0$ *such that the conditions*

$$\gamma > \gamma_0 \quad \textit{and} \quad \lambda^2_{N(k)+1} - \lambda^2_{N(k)} \ge k_0 \lambda_{N(k)+1} \tag{9.2}$$

imply that the dynamical system $(\mathcal{H}, S_t)$ *generated by problem* (0.1) *and* (0.2) *possesses a local inertial manifold, i.e. there exists a finite-dimensional manifold* $\mathcal{M}$ *in* $\mathcal{H} = \mathcal{F}_1 \times \mathcal{F}_0$ *of the form*

$$\mathcal{M} = \{y = w + \Phi(w): \quad w \in P\mathcal{H}, \ \Phi(w) \in (1-P)\mathcal{H}\}, \tag{9.3}$$

where $\Phi(\,\cdot\,)$ *is a Lipschitzian mapping from* $P\mathcal{H}$ *into* $(1-P)\mathcal{H}$ *and* P *is a finite-dimensional projector in* $\mathcal{H}$. *This manifold possesses the properties:*

1) *for any bounded set* B *in* $\mathcal{H}$ *and for* $t \ge t_0(B)$
$$\sup\{\operatorname{dist}(S_t y, \mathcal{M}): y \in \mathcal{B}\} \le C \exp\{-\beta(t - t_0(\boldsymbol{B}))\}\ ; \tag{9.4}$$
2) *there exists* $R > 0$ *such that the conditions* $y \in \mathcal{M}$ *and* $\|S_t y\|_{\mathcal{H}} \le R$ *for* $t \in [0, t_0]$ *imply that* $S_t y \in \mathcal{M}$ *for* $t \in [0, t_0]$;
3) *if the global attractor of the system* $(\mathcal{H}, S_t)$ *exists, then the set* $\mathcal{M}$ *contains it (see* Theorem 6.1*).*

Proof.

Conditions (3.2), (5.2), and (5.3) imply (see Theorem 5.1) that the dynamical system $(\mathcal{H}, S_t)$ is dissipative, i.e. there exists $R > 0$ such that

$$\|S_t y\|_{\mathcal{H}} \le R\,, \qquad y \in B\,, \qquad t \ge t_0(B) \tag{9.5}$$

for any bounded set $B \in \mathcal{H}$. This enables us to use the dynamical system $(\mathcal{H}, \tilde{S}_t)$ generated by an equation of the type

$$\begin{cases} \ddot{u} + \gamma \dot{u} + A^2 u = B_R(u)\,, \\ u\big|_{t=0} = u_0, \quad \dot{u}\big|_{t=0} = u_1\,, \end{cases} \tag{9.6}$$

to describe the asymptotic behaviour of solutions to problem (0.1) and (0.2). Here

$$B_R(u) = \chi\Big((2R)^{-1}\|Au\|\Big) \cdot \Big\{g - M\Big(\|A^{1/2}u\|^2\Big)Au - Lu\Big\}$$

and $\chi(s)$ is an infinitely differentiable function on $\mathbb{R}_+$ possessing the properties

$$0 \le \chi(s) \le 1\,; \qquad |\chi'(s)| \le 2\,;$$
$$\chi(s) = 1\,, \quad 0 \le s \le 1\,; \qquad \chi(s) = 0\,, \quad s \ge 2\,.$$

It is easy to find that there exists a constant C_R such that

$$\|B_R(u)\| \le C_R$$

and

$$\|B_R(u_1) - B_R(u_2)\| \ \le\ C_R \|A(u_1 - u_2)\|\,.$$

Therefore, we can apply Theorem 3.7.2 to the dynamical system $(\mathcal{H}, \tilde{S}_t)$ generated by equation (9.6). This theorem guarantees the existence of an inertial manifold of the system $(\mathcal{H}, \tilde{S}_t)$ if the hypotheses of Theorem 9.1 hold. However, inside the dissipativity ball $\{y\colon \|y\|_{\mathcal{H}} \le R\}$ problem (9.6) coincides with problem (0.1) and (0.2). This easily implies the assertion of Theorem 9.1.

— Exercise *9.1* Show that the hypotheses of Theorem 9.1 hold for the problem on oscillations of an infinite panel in a supersonic flow of gas:

$$\begin{cases} \partial_t^2 u + \gamma \partial_t u + \partial_x^4 u + \left(\Gamma - \displaystyle\int_0^\pi |\partial_x u(x,\, t)|^2 \, dx \right) \partial_x^2 u + \rho \partial_x u = g(x)\,, \quad x \in (0,\, \pi),\ t > 0\,, \\ u\big|_{x=0,\ x=\pi} = \partial_x^2 u\big|_{x=0,\ x=\pi} = 0\,, \quad u\big|_{t=0} = u_0(x)\,, \quad \partial_t u\big|_{t=0} = u_1(x)\,. \end{cases}$$

Here Γ and ρ are real parameters and $g(x) \in L^2(0,\, \pi)$.

It is evident that the most essential assumption of Theorem 9.1 that restricts its application is condition (9.2). In this connection the following assertion concerning the case when problem (0.1) and (0.2) possesses a regular attractor is of some interest.

Theorem 9.2

Assume that in equation (0.1) ***we have*** $L \equiv 0$ ***and*** $p(t) \equiv g \in \mathrm{Lin}\{e_1,\, \ldots,\, e_{N_0}\}$ ***for some*** N_0. ***We also assume that conditions*** (3.2) ***and*** (5.2) ***are fulfilled. Then there exists*** $N_1 \ge N_0$ ***such that for all*** $N \ge N_1$ ***the subspace***

$$\mathcal{H}_N = \mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k)\colon\ k = 1,\ 2,\ \ldots,\ N\} \tag{9.7}$$

is an invariant and exponentially attracting set of the dynamical system $(\mathcal{H},\, S_t)$ ***generated by problem*** (0.1) ***and*** (0.2):

$$\mathrm{dist}(S_t y,\ \mathcal{H}_N) \ \le\ C_B \|(1 - P_N) y\|_{\mathcal{H}}\, e^{-\frac{\gamma}{4}(t - t_0(B))}, \qquad y \in B \tag{9.8}$$

for $t \ge t_0(B)$ ***and for any bounded set*** B ***in*** $\mathcal{H}$. ***Here*** P_N ***is the orthoprojector onto*** $\mathcal{H}_N$.

Proof.

Since $p_N g = g$ for $N \geq N_0$, where p_N is the orthoprojector onto the span of $\{e_1, \ldots, e_{N_0}\}$, the uniqueness theorem implies the invariance of $\mathcal{H}_N$. Let us prove attraction property (9.8). It is sufficient to consider a trajectory $(u(t), \dot{u}(t))$ lying in the ball of dissipativity $\{y : \|y\|_{\mathcal{H}} \leq R\}$. Evidently the function $v(t) = (1 - p_N) u(t)$ satisfies the equation

$$\begin{cases} \ddot{v} + \gamma \dot{v} + A^2 v + M\left(\|A^{1/2} u(t)\|^2\right) A v = 0 \, , \\ v\big|_{t=0} = (1 - p_N) u_0 , \quad \dot{v}\big|_{t=0} = (1 - p_N) u_1 \, . \end{cases} \tag{9.9}$$

It is also clear that the conditions

$$\left| M\left(\|A^{1/2} u(t)\|^2\right) \right| \leq b_0(R) \quad \text{and} \quad \left| \frac{\mathrm{d}}{\mathrm{d}t} M\left(\|A^{1/2} u(t)\|^2\right) \right| \leq b_1(R)$$

hold in the ball of dissipativity. This fact enables us to use Theorem 2.2 with $b(t) = M(\|A^{1/2} u(t)\|^2)$. In particular, equation (2.23) guarantees the existence of a number $N_1 \geq N_0$ which depends on γ, $b_0(R)$, and $b_1(R)$ and such that

$$\|y(t)\|_{\mathcal{H}} = \|(1 - P_N) y(t)\|_{\mathcal{H}} \leq \sqrt{3} \, \|(1 - P_N) y(0)\| \, e^{-\frac{\gamma}{4} t} \, , \qquad t > 0 \, ,$$

for all $N \geq N_1$, where P_N is the orthoprojector onto $\mathcal{H}_N$ and $y(t) = (v(t), \dot{v}(t))$. This implies estimate (9.8). **Theorem 9.2 is proved**.

— Exercise *9.2* Assume that the hypotheses of Theorem 9.2 hold. Show that for any semitrajectory $S_t y$ there exists an induced trajectory in $\mathcal{H}_N$, i.e. there exists $\bar{y} \in \mathcal{H}_N$ such that

$$\|S_t y - S_t \bar{y}\|_{\mathcal{H}} \leq C_R \, e^{-\frac{\gamma}{4}(t - t_0)}$$

for $t \geq t_0$ and for some $t_0 = t_0(\|y\|_{\mathcal{H}})$.

— Exercise *9.3* Write down an inertial form of problem (0.1) and (0.2) in the subspace $\mathcal{H}_N$, provided the hypotheses of Theorem 9.2 hold. Prove that the inertial form coincides with the Galerkin approximation of the order N of problem (0.1) and (0.2).

— Exercise *9.4* Show that if the hypotheses of Theorem 9.2 hold, then the global attractor of problem (0.1) and (0.2) coincides with the global attractor of its Galerkin approximation of a sufficiently large order.

Let us now turn to the question on the construction of approximate inertial manifolds for problem (0.1) and (0.2). In this case we can use the results of Section 3.8 and the theorems on the regularity proved in Sections 4 and 5.

— Exercise *9.5* Assume that $M(z) \in C^{m+1}(\mathbb{R}_+)$, $m \ge 1$ and

$$B(u) = p - M\Big(\|A^{1/2}u\|^2\Big)Au - Lu\,,$$

where $p \in H$ and $u \in \mathcal{D}(A) = \mathcal{F}_1$. Show that the mapping $B(\cdot)$ has the Frechét derivatives $B^{(k)}$ up to the order l inclusive. Moreover, the estimates

$$\Big\|\langle B^{(k)}[u];\ w_1,\ \ldots,\ w_k\rangle\Big\| \ \le\ C_R \prod_{j=1}^{k} \|Aw_j\| \tag{9.10}$$

and

$$\Big\|\langle B^{(k)}[u] - B^{(k)}[u^*];\ w_1,\ \ldots,\ w_k\rangle\Big\| \ \le$$

$$\le\ C_R\|A(u-u^*)\| \prod_{j=1}^{k} \|Aw_j\| \tag{9.11}$$

are valid, where $k = 0,\ 1,\ \ldots,\ m$, $\|Au\| \le R$, $\|Au^*\| \le R$, and $w_j \in \mathcal{D}(A)$. Here $\langle B^{(k)}[u];\ w_1,\ \ldots,\ w_k\rangle$ is the value of the Frechét derivative on the elements $w_1,\ \ldots,\ w_k$.

We consider equations (9.10) and (9.11) as well as Theorem 5.3 which guarantees nonemptiness of the classes $L_{m,\,R}$ corresponding to the problem considered when $R > 0$ is large enough. They enable us to apply the results of Section 3.8.

Let P be the orthoprojector onto the span of elements $\{e_1,\ \ldots,\ e_N\}$ in H and let $Q = 1 - P$. We define the sequences $\{h_n(p,\ \dot p)\}_{n=0}^{\infty}$ and $\{l_n(p,\ \dot p)\}_{n=0}^{\infty}$ of mappings from $PH \times PH$ into QH by the formulae

$$h_0(p,\ \dot p)\ =\ l_0(p,\ \dot p)\ =0\,, \tag{9.12}$$

$$A^2 h_k(p,\ \dot p)\ =\ a_0 - M_{k-1}(p,\ \dot p)Ah_{k-1} - QL(p+h_{k-1}) - \gamma\, l_{\nu(k)} -$$

$$-\langle \delta_p l_{k-1};\ \dot p\rangle + \langle \delta_{\dot p} l_{k-1};\ \gamma\dot p + A^2 p - b_0 + M_{k-1}(p,\ \dot p)Ap + PL(p+h_{k-1})\rangle\,, \tag{9.13}$$

$$l_k(p,\ \dot p)\ =\ \langle \delta_p h_{k-1};\ \dot p\rangle\ -$$

$$-\ \langle \delta_{\dot p} h_{k-1};\ \gamma\dot p + A^2 p - b_0 + M_{k-1}(p,\ \dot p)Ap + PL(p+h_{k-1})\rangle\,. \tag{9.14}$$

Here $M_k(p,\ \dot p) = M\Big(\|A^{1/2}p\|^2 + \big\|A^{1/2}h_k(p,\ \dot p)\big\|^2\Big)$, δ_p and $\delta_{\dot p}$ are the Frechét derivatives with respect to the corresponding variables, $a_0 = Qg$, $b_0 = Pg$, where $g \equiv p(t)$ is a stationary transverse load in (0.1), $k = 1,\ 2,\ \ldots,\ m$, the numbers $\nu(k)$ are chosen to fulfil the inequality $k-1\ \le\ \nu(k) \le k$.

— Exercise *9.6* Evaluate the functions $h_1(p,\ \dot p)$ and $l_1(p,\ \dot p)$.

Theorem 3.8.2 implies the following assertion.

Theorem 9.3

Assume that $p(t) \equiv g \in H$, $M(z) \in C^{m+1}(\mathbb{R}_+)$, $m \geq 2$, *and conditions* (3.2), (5.2), *and* (5.3) *are fulfilled. Then for all* $k = 1, \ldots, m$ *the collection of mappings* (h_n, l_n) *given by equalities* (9.12)–(9.14) *possesses the properties*

1) *there exist constants* $M_j = M_j(n, \rho)$ *and* $L_j(n, \rho)$, $j = 1, 2$ *such that*

$$\|A^2 h_n(p_0, \dot{p}_0)\| \leq M_1, \qquad \|A l_n(p_0, \dot{p}_0)\| \leq M_2,$$

$$\|A^2(h_n(p_1, \dot{p}_1) - h_n(p_2, \dot{p}_2))\| \leq L_1(\|A^2(p_1 - p_2)\| + \|A(\dot{p}_1 - \dot{p}_2)\|),$$

$$\|A(l_n(p_1, \dot{p}_1) - l_n(p_2, \dot{p}_2))\| \leq L_2(\|A^2(p_1 - p_2)\| + \|A(\dot{p}_1 - \dot{p}_2)\|),$$

for all p_j *and* $\dot{p}_j$ *from* PH *and such that*

$$\|A^2 p_j\|^2 + \|A\dot{p}_j\|^2 \leq \rho, \quad j = 0, 1, 2, \quad \rho > 0;$$

2) *for any solution* $u(t)$ *to problem* (0.1) *and* (0.2) *which satisfies compatibility conditions* (4.3) *with* $l = m$ *the estimate*

$$\left\{\|A^2(u(t) - u_n(t))\|^2 + \|A(u(t) - \bar{u}_n(t))\|^2\right\}^{1/2} \leq C_n \lambda_{N+1}^{-n}$$

is valid for $n \leq m - 1$ *and for* t *large enough. Here*

$$u_n(t) = p(t) + h_n(p(t), \dot{p}(t)),$$

$$\bar{u}_n(t) = \dot{p}(t) + l_n(p(t), \dot{p}(t)),$$

λ_N *is the* N*-th eigenvalue of the operator* A *and the constant* C_n *depends on the radius of dissipativity.*

In particular, Theorem 9.3 means that the manifold

$$\mathcal{M}_n = \{(p + h_n(p, \dot{p});\ \dot{p} + l_n(p, \dot{p})) : \quad p, \dot{p} \in PH\}$$

attracts sufficiently smooth trajectories of the dynamical system $(\mathcal{H}, S_t)$ generated by problem (0.1) and (0.2) into a small vicinity (of the order $C_n \lambda_{N+1}^{-n}$) of $\mathcal{M}_n$.

Exercise *9.7* Assume that the hypotheses of Theorem 6.3 hold (this theorem guarantees the existence of the global attractor $\mathcal{A}$ consisting of smooth trajectories of problem (0.1) and (0.2)). Prove that

$$\sup\{\operatorname{dist}(y, \mathcal{M}_n) : y \in \mathcal{A}\} \leq C_n \lambda_{N+1}^{-n}$$

for all $n \leq m - 1$ (the number m is defined by the condition $M(z) \in C^{m+1}(\mathbb{R}_+)$).

Exercise *9.8* Prove the analogue of Theorem 3.9.1 on properties of the nonlinear Galerkin method for problem (0.1) and (0.2).

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Chapter 5

Theory of Functionals that Uniquely Determine Long-Time Dynamics

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The results presented in previous chapters show that in many cases the asymptotic behaviour of infinite-dimensional dissipative systems can be described by a finite-dimensional global attractor. However, a detailed study of the structure of attractor has been carried out only for a very limited number of problems. In this regard it is of importance to search for minimal (or close to minimal) sets of natural parameters of the problem that uniquely determine the long-time behaviour of a system. This problem was first discussed by Foias and Prodi [1] and by Ladyzhenskaya [2] for the 2D Navier-Stokes equations. They have proved that the long-time behaviour of solutions is completely determined by the dynamics of the first N Fourier modes if N is sufficiently large. Later on, similar results have been obtained for other parameters and equations. The concepts of determining nodes and determining local volume averages have been introduced. A general approach to the problem on the existence of a finite number of determining parameters has been discussed (see survey [3]).

In this chapter we develop a general theory of determining functionals. This theory enables us, first, to cover all the results mentioned above from a unified point of view and, second, to suggest rather simple conditions under which a set of functionals on the phase space uniquely determines the asymptotic behaviour of the system by its values on the trajectories. The approach presented here relies on the concept of completeness defect of a set of functionals and involves some ideas and results from the approximation theory of infinite-dimensional spaces.

§ 1 Concept of a Set of Determining Functionals

Let us consider a nonautonomous differential equation in a real reflexive Banach space H of the type

$$\frac{du}{dt} = F(u, t), \qquad t > 0, \qquad u\big|_{t=0} = u_0 \,. \tag{1.1}$$

Let $\mathscr{W}$ be a class of solutions to (1.1) defined on the semiaxis $\mathbb{R}_+ \equiv \{t : t \ge 0\}$ such that for any $u(t) \in \mathscr{W}$ there exists a point of time $t_0 > 0$ such that

$$u(t) \in C(t_0, +\infty; H) \cap L^2_{loc}(t_0, +\infty, V), \tag{1.2}$$

where V is a reflexive Banach space which is continuously embedded into H. Hereinafter $C(a, b; X)$ is the space of strongly continuous functions on $[a, b]$ with the values in X and $L^2_{loc}(a, b; X)$ has a similar meaning. The symbols $\|\cdot\|_H$ and $\|\cdot\|_V$ stand for the norms in the spaces H and V, $\|\cdot\|_H \le \|\cdot\|_V$.

The following definition is based on the property established in [1] for the Fourier modes of solutions to the 2D Navier-Stokes system with periodic boundary conditions.

Let $\mathcal{L} = \{l_j :\ j = 1, \ldots, N\}$ be a set of continuous linear functionals on V. Then $\mathcal{L}$ is said to be a ***set of asymptotically*** $(V, H, \mathcal{W})$***-determining functionals*** (or elements) for problem (1.1) if for any two solutions $u, v \in \mathcal{W}$ the condition

$$\lim_{t \to \infty} \int_t^{t+1} |l_j(u(\tau)) - l_j(v(\tau))|^2 \, d\tau = 0 \quad \text{for} \quad j = 1, \ldots, N \tag{1.3}$$

implies that

$$\lim_{t \to \infty} \|u(t) - v(t)\|_H = 0 . \tag{1.4}$$

Thus, if $\mathcal{L}$ is a set of asymptotically determining functionals for problem (1.1), the asymptotic behaviour of a solution $u(t)$ is completely determined by the behaviour of a finite number of scalar values $\{l_j(u(t)) :\ j = 1, 2, \ldots, N\}$. Further, if no ambiguity results, we will sometimes omit the spaces V, H, and $\mathcal{W}$ in the description of determining functionals.

Exercise *1.1* Show that condition (1.3) is equivalent to

$$\lim_{t \to \infty} \int_t^{t+1} [\mathcal{N}_{\mathcal{L}}(u(\tau) - v(\tau))]^2 \, d\tau = 0 ,$$

where $\mathcal{N}_{\mathcal{L}}(u)$ is a seminorm in H defined by the equation

$$\mathcal{N}_{\mathcal{L}}(u) = \max_{l \in \mathcal{L}} |l(u)| .$$

Exercise *1.2* Let u_1 and u_2 be stationary (time-independent) solutions to problem (1.1) lying in the class $\mathcal{W}$. Let $\mathcal{L} = \{l_j :\ j = 1, \ldots, N\}$ be a set of asymptotically determining functionals. Show that condition $l_j(u_1) = l_j(u_2)$ for all $j = 1, 2, \ldots, N$ implies that $u_1 = u_2$.

The following theorem forms the basis for all assertions known to date on the existence of finite sets of asymptotically determining functionals.

Theorem 1.1.

Let $\mathcal{L} = \{l_j :\ j = 1, \ldots, N\}$ ***be a family of continuous linear functionals on*** V***. Suppose that there exists a continuous function*** $\mathcal{V}(u, t)$ ***on*** $H \times \mathbb{R}_+$ ***with the values in*** $\mathbb{R}_+$ ***which possesses the following properties:***

a) ***there exist positive numbers*** α ***and*** σ ***such that***

$$\mathcal{V}(u, t) \geq \alpha \cdot \|u\|^\sigma \quad \textit{for all} \quad u \in H, \quad t \in \mathbb{R}_+ ; \tag{1.5}$$

b) ***for any two solutions*** $u(t)$, $v(t) \in \mathscr{W}$ ***to problem*** (1.1) ***there exist*** (i) ***a point of time*** $t_0 > 0$, (ii) ***a function*** $\psi(t)$ ***that is locally integrable over the half-interval*** $[t_0, \infty)$ ***and such that***

$$\gamma_\psi^+ = \underline{\lim}_{t \to \infty} \int_t^{t+a} \psi(\tau)\, d\tau \ > \ 0 \tag{1.6}$$

and

$$\Gamma_\psi^+ = \overline{\lim}_{t \to \infty} \int_t^{t+a} \max\{0, \ -\psi(\tau)\}\, d\tau \ < \ \infty \tag{1.7}$$

for some $a > 0$, ***and*** (iii) ***a positive constant*** C ***such that for all*** $t \geq s \geq t_0$ ***we have***

$$\mathscr{V}(u(t) - v(t), \ t) + \int_s^t \psi(\tau) \cdot \mathscr{V}(u(\tau) - v((\tau), \ \tau))\, d\tau \ \leq$$

$$\leq \ \mathscr{V}(u(s) - v(s), \ s) + C \cdot \int_s^t \max_{j = 1, \ \ldots, \ N} \left| l_j(u(\tau)) - l_j(v(\tau)) \right|^2 d\tau \ . \tag{1.8}$$

Then $\mathscr{L}$ ***is a set of asymptotically*** $(V, \ H, \ \mathscr{W})$***-determining functionals for problem*** (1.1).

It is evident that the proof of this theorem follows from a version of Gronwall's lemma stated below.

Lemma 1.1.

Let $\psi(t)$ *and* $g(t)$ *be two functions that are locally integrable over some half-interval* $[t_0, \ \infty)$. *Assume that* (1.6) *and* (1.7) *hold and* $g(t)$ *is nonnegative and possesses the property*

$$\lim_{t \to \infty} \int_t^{t+a} g(\tau)\, d\tau \ = \ 0, \qquad a > 0. \tag{1.9}$$

Suppose that $w(t)$ *is a nonnegative continuous function satisfying the inequality*

$$w(t) + \int_s^t \psi(\tau) \cdot w(\tau)\, d\tau \ \leq \ w(s) + \int_s^t g(\tau)\, d\tau \tag{1.10}$$

for all $t \geq s \geq t_0$. *Then* $w(t) \to 0$ *as* $t \to \infty$.

It should be noted that this version of Gronwall's lemma has been used by many authors (see the references in the survey [3]).

Proof.

Let us first show that equation (1.10) implies the inequality

$$w(t) \;\le\; w(s)\exp\left\{-\int_s^t \psi(\sigma)\,d\sigma\right\} + \int_s^t g(\tau)\exp\left\{-\int_\tau^t \psi(\sigma)\,d\sigma\right\}d\tau \quad (1.11)$$

for all $t \ge s \ge t_0$. It follows from (1.10) that the function $w(t)$ is absolutely continuous on any finite interval and therefore possesses a derivative $\dot{w}(t)$ almost everywhere. Therewith, equation (1.10) gives us

$$\dot{w}(t) + \psi(t)w(t) \;\le\; g(t) \quad (1.12)$$

for almost all t. Multiplying this inequality by

$$e(t) = \exp\left\{\int_s^t \psi(\sigma)\,d\sigma\right\},$$

we find that

$$\frac{d}{dt}(w(t)\,e(t)) \;\le\; g(t)\,e(t)$$

almost everywhere. Integration gives us equation (1.11).

Let us choose the value s such that

$$\int_\tau^{\tau+a} \max\{-\psi(\sigma),\ 0\}\,d\sigma \;\le\; \Gamma + 1\ , \qquad \Gamma \equiv \Gamma_\psi^+ \quad (1.13)$$

and

$$\int_\tau^{\tau+a} \psi(\sigma)\,d\sigma \;\ge\; \frac{\gamma}{2}\,, \qquad \gamma \equiv \gamma_\psi^+ \quad (1.14)$$

for all $\tau \ge s$. It is evident that if $t \ge \tau \ge s$ and $k = \left[\frac{t-\tau}{a}\right]$, where $[\,\cdot\,]$ is the integer part of a number, then

$$\int_\tau^t \psi(\sigma)\,d\sigma = \int_\tau^{ka+\tau} \psi(\sigma)\,d\sigma + \int_{ka+\tau}^t \psi(\sigma)\,d\sigma \;\ge$$

$$\ge\; \frac{\gamma}{2}k - \int_{ka+\tau}^{(k+1)a+\tau} \max\{-\psi(\sigma),\ 0\}\,d\sigma \;\ge\; \frac{\gamma}{2}k - (\Gamma+1)\ .$$

Thus, for all $t \ge \tau \ge s$

$$\int_{\tau}^{t} \psi(\sigma)\, d\sigma \;\geq\; \frac{\gamma}{2a}(t-\tau) - \left(1 + \Gamma + \frac{\gamma}{2}\right).$$

Consequently, equation (1.11) gives us that

$$w(t) \;\leq\; C(\Gamma,\, \gamma)\left[w(s)\exp\left\{-\frac{\gamma}{2a}(t-s)\right\} + \int_{s}^{t} g(r)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\} d\tau \right],$$

where $C(\Gamma,\, \gamma) = \exp\left\{1 + \Gamma + \frac{\gamma}{2}\right\}$. Therefore,

$$\overline{\lim_{t\to\infty}}\, w(t) \;\leq\; C(\Gamma,\, \gamma)\cdot \overline{\lim_{t\to\infty}} \int_{s}^{t} g(\tau)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\} d\tau\,. \qquad (1.15)$$

It is evident that

$$G(t,\, s) \;\equiv\; \int_{s}^{t} g(\tau)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\} d\tau \;\leq$$

$$\leq\; \sum_{k=0}^{N} \int_{s+ka}^{s+(k+1)a} g(\tau)\exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\} d\tau\,,$$

where $N = \left[\frac{t-s}{a}\right]$ is the integer part of the number $\frac{t-s}{a}$. Therefore,

$$G(t,\, s) \;\leq\; \sup_{\sigma \geq s} \int_{\sigma}^{\sigma+a} g(\tau)\, d\tau \cdot \sum_{k=0}^{N} \int_{s+ka}^{s+(k+1)a} \exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\} d\tau \;=$$

$$=\; \sup_{\sigma \geq s} \int_{\sigma}^{\sigma+a} g(\tau)\, d\tau \int_{s}^{s+(N+1)a} \exp\left\{-\frac{\gamma}{2a}(t-\tau)\right\} d\tau \;=$$

$$=\; \frac{2a}{\gamma}\cdot \sup_{\sigma \geq s} \int_{\sigma}^{\sigma+a} g(\tau)\, d\tau \,\cdot\, \exp\left\{-\frac{\gamma}{2a}(t-s)\right\}\left[e^{\frac{\gamma}{2}(N+1)} - 1\right].$$

Since $N = \left[\frac{t-s}{a}\right]$, this implies that

$$\lim_{t\to\infty} G(t,\, s) \;\leq\; C(a,\, \gamma)\cdot \sup_{\sigma \geq s} \int_{\sigma}^{\sigma+a} g(\tau)\, d\tau \qquad (1.16)$$

for any s such that equations (1.13) and (1.14) hold. Hence, equations (1.15) and (1.16) give us that

$$\overline{\lim_{t \to \infty}}\, w(t) \le C(\Gamma, \gamma, a) \cdot \sup_{\sigma \ge s} \int_{\sigma}^{\sigma + a} g(\tau)\, d\tau .$$

If we tend $s \to \infty$, then with the help of (1.9) we obtain

$$\overline{\lim_{t \to \infty}}\, w(t) = 0 .$$

This implies the assertion of Lemma 1.1.

In cases when problem (1.1) is the Cauchy problem for a quasilinear partial differential equation, we usually take some norm of the phase space as the function $\mathcal{V}(u, t)$ when we try to prove the existence of a finite set of asymptotically determining functionals. For example, the next assertion which follows from Theorem 1.1 is often used for parabolic problems.

Corollary 1.1.

Let V and H be reflexive Banach spaces such that V is continuously and densely embedded into H. Assume that for any two solutions $u_1(t), u_2(t) \in \mathcal{W}$ to problem (1.1) *we have*

$$\|u_1(t) - u_2(t)\|_V^2 + \int_s^t \psi(\tau) \|u_1(\tau) - u_2(\tau)\|_V^2\, d\tau \le$$

$$\le \|u_1(s) - u_2(s)\|_V^2 + K \int_s^t \|u_1 - u_2\|_H^2\, d\tau \tag{1.17}$$

for $t \ge s \ge t_0$, where K is a constant and the function $\psi(t)$ depends on $u_1(t)$ and $u_2(t)$ in general and possesses properties (1.6) *and* (1.7). *Assume that the family $\mathcal{L} = \{l_j :\ j = 1, \ldots, N\}$ on V possesses the property*

$$\|v\|_H \le C \cdot \max_{j = 1 \ldots N} |l_j(v)| + \varepsilon_{\mathcal{L}} \|v\|_V \tag{1.18}$$

for any $v \in V$, where C and $\varepsilon_{\mathcal{L}}$ are positive constants depending on $\mathcal{L}$. Then $\mathcal{L}$ is a set of asymptotically determining functionals for problem (1.1), *provided*

$$\varepsilon_{\mathcal{L}}^2 < \frac{1}{K} \cdot \underline{\lim_{t \to \infty}}\, \frac{1}{a} \cdot \int_t^{t+a} \psi(\tau)\, d\tau \equiv \gamma_\psi^+ a^{-1} K^{-1} . \tag{1.19}$$

Proof.

Using the obvious inequality

$$(a+b)^2 \le (1+\delta)a^2 + \left(1+\frac{1}{\delta}\right)b^2, \qquad \delta > 0,$$

we find from equation (1.18) that

$$\|v\|_H^2 \le (1+\delta)\varepsilon_{\mathcal{L}}^2 \|v\|_V^2 + C_\delta \cdot \max_{j=1\dots N} |l_j(v)|^2 \tag{1.20}$$

for any $\delta > 0$. Therefore, equation (1.17) implies that

$$\|u_1(t) - u_2(t)\|_V^2 + \int_s^t \left(\psi(\tau) - (1+\delta)K\varepsilon_{\mathcal{L}}^2\right)\|u_1(\tau) - u_2(\tau)\|_V^2 \, d\tau \le$$

$$\le \|u_1(s) - u_2(s)\|_V^2 + C\int_s^t [\mathcal{N}_{\mathcal{L}}(u_1(\tau) - u_2(\tau))]^2 \, d\tau ,$$

where $\mathcal{N}_{\mathcal{L}}(v) = \max_{j=1\dots N} |l_j(v)|$. Consequently, if for some $\delta > 0$ the function

$$\tilde{\psi}(t) = \psi(t) - (1+\delta)K\varepsilon_{\mathcal{L}}^2$$

possesses properties (1.6) and (1.7) with some constants γ and $\Gamma > 0$, then Theorem 1.1 is applicable. A simple verification shows that it is sufficient to require that equation (1.19) be fulfilled. Thus, Corollary 1.1 is proved.

Another variant of Corollary 1.1 useful for applications can be formulated as follows.

Corollary 1.2.

Let V and H be reflexive Banach spaces such that V is continuously embedded into H. Assume that for any two solutions $u(t), v(t) \in \mathcal{W}$ to problem (1.1) there exists a moment $t_0 > 0$ such that for all $t \ge s \ge t_0$ the equation

$$\|u(t) - v(t)\|_H^2 + \nu\int_s^t \|u(\tau) - v(\tau)\|_V^2 \, d\tau \le$$

$$\le \|u(s) - v(s)\|_H^2 + \int_s^t \phi(\tau)\cdot\|u(\tau) - v(\tau)\|_H^2 \, d\tau \tag{1.21}$$

holds. Here $\nu > 0$ and the positive function $\phi(t)$ is locally integrable over the half-interval $[t_0, \infty)$ and satisfies the relation

$$\overline{\lim_{t\to\infty}} \frac{1}{a}\int_t^{t+a} \phi(\tau)\, d\tau \le R \tag{1.22}$$

for some $a > 0$, *where the constant* $R > 0$ *is independent of* $u(t)$ *and* $v(t)$. *Let* $\mathscr{L} = \{l_j\colon\ j = 1, \ldots, N\}$ *be a family of continuous linear functionals on* V *possessing the property*

$$\|w\|_H \ \le\ C_{\mathscr{L}} \cdot \max_{j=1,\ \ldots,\ N} |l_j(w)| + \varepsilon_{\mathscr{L}} \cdot \|w\|_V$$

for any $w \in V$. *Here* $C_{\mathscr{L}}$ *and* $\varepsilon_{\mathscr{L}}$ *are positive constants. Then* $\mathscr{L}$ *is a set of asymptotically* $(V, H, \mathscr{W})$*-determining functionals for problem* (1.1), *provided that* $\varepsilon_{\mathscr{L}} < \sqrt{\nu / R}$.

Proof.

Equation (1.20) implies that

$$\|w\|_V^2 \ \ge\ (1+\delta)^{-1} \cdot \varepsilon_{\mathscr{L}}^{-2} \cdot \|w\|_H^2 \ -\ C_{\mathscr{L},\,\delta} \cdot \max_{j=1,\ \ldots,\ N} |l_j(w)|^2$$

for any $\delta > 0$. Therefore, (1.21) implies that

$$\|u(t) - v(t)\|_H^2 + \int_s^t \psi(\tau) \cdot \|u(\tau) - v(\tau)\|_H^2 \, \mathrm{d}\tau \ \le$$

$$\le\ \|u(s) - v(s)\|_H^2 \ +\ \nu C_{\mathscr{L},\,\delta} \cdot \int_s^t \max_{j=1,\ \ldots,\ N} |l_j(u(\tau) - v(\tau))|^2 \, \mathrm{d}\tau\ ,$$

where $\psi(t) = \nu(1+\delta)^{-1}\varepsilon_{\mathscr{L}}^{-2} - \phi(t)$. Using (1.22) and applying Theorem 1.1 with $\mathscr{V}(u, t) = \|u\|^2$, we complete the proof of Corollary 1.2.

Other approaches of introduction of the concept of determining functionals are also possible. The definition below is an extension to a more general situation of the property proved by O.A. Ladyzhenskaya [2] for trajectories lying in the global attractor of the 2D Navier-Stokes equations.

Let $\overline{\mathscr{W}}$ be a class of solutions to problem (1.1) on the real axis $\mathbb{R}$ such that $\overline{\mathscr{W}} \subset L^2_{\mathrm{loc}}(-\infty, +\infty;\ V)$. A family $\mathscr{L} = \{l_j\colon\ j = 1, \ldots, N\}$ of continuous linear functionals on V is said to be a ***set of*** $(V, \overline{\mathscr{W}})$***-determining functionals*** (or elements) for problem (1.1) if for any two solutions $u,\ v \in \overline{\mathscr{W}}$ the condition

$$l_j(u(t)) = l_j(v(t)) \ \text{ for } \ j = 1, \ldots, N \ \text{ and almost all } \ t \in R \tag{1.23}$$

implies that $u(t) \equiv v(t)$.

It is easy to establish the following analogue of Theorem 1.1.

Theorem 1.2.

Let $\mathscr{L} = \{l_j\colon\ j = 1, \ldots, N\}$ ***be a family of continuous linear functionals on*** V***. Let*** $\overline{\mathscr{W}}$ ***be a class of solutions to problem*** (1.1) ***on the real axis*** $\mathbb{R}$ ***such that***

$$\overline{\mathscr{W}} \subset C(-\infty, +\infty;\ H) \cap L^2_{\text{loc}}(-\infty, +\infty;\ V)\ . \tag{1.24}$$

Assume that there exists a continuous function $\mathscr{V}(u, t)$ ***on*** $H \times \mathbb{R}$ ***with the values in*** $\mathbb{R}$ ***which possesses the following properties:***

a) ***there exist positive numbers*** α ***and*** σ ***such that***

$$\mathscr{V}(u, t) \ge \alpha \cdot \|u\|^{\sigma} \quad \textit{for all} \quad u \in H, \quad t \in \mathbb{R}\ ; \tag{1.25}$$

b) ***for any*** $u(t), v(t) \in \overline{\mathscr{W}}$

$$\sup_{t \in \mathbb{R}} \mathscr{V}(u(t) - v(t),\ t) < \infty\ ; \tag{1.26}$$

c) ***for any two solutions*** $u(t), v(t) \in \overline{\mathscr{W}}$ ***to problem*** (1.1) ***there exist*** (i) ***a function*** $\psi(t)$ ***locally integrable over the axis*** $\mathbb{R}$ ***with the properties***

$$\gamma_{\psi}^{-} \equiv \varliminf_{t \to -\infty} \int_t^{t+a} \psi(\tau)\, d\tau > 0 \tag{1.27}$$

and

$$\Gamma_{\psi}^{-} \equiv \varlimsup_{t \to -\infty} \int_t^{t+a} \max\{0,\ -\psi(\tau)\} d\tau < \infty \tag{1.28}$$

for some $a > 0$***, and*** (ii) ***a positive constant*** C ***such that equation*** (1.8) ***holds for all*** $t \ge s$. ***Then*** $\mathscr{L}$ ***is a set of*** $(V, \overline{\mathscr{W}})$***-determining functionals for problem*** (1.1).

Proof.

It follows from (1.23), (1.8), and (1.11) that the function $w(t) = \mathscr{V}(u(t) - v(t),\ t)$ satisfies the inequality

$$w(t) \le w(s) \cdot \exp\left\{ -\int_s^t \psi(\tau)\, d\tau \right\} \tag{1.29}$$

for all $t \ge s$. Using properties (1.27) and (1.28) it is easy to find that there exist numbers s^*, $a_0 > 0$, and $b_0 > 0$ such that

$$\int_{s_1}^{s_2} \psi(\tau)\, d\tau \ \ge\ a_0 \cdot (s_2 - s_1) - b_0\,, \qquad s_1 \le s_2 \le s^*\,.$$

This equation and boundedness property (1.26) enable us to pass to the limit in (1.29) for fixed t as $s \to -\infty$ and to obtain the required assertion.

Using Theorem 1.2 with $\mathscr{V}(u,\ t) = \|u\|^2$ as above, we obtain the following assertion.

Corollary 1.3.

Let V and H be reflexive Banach spaces such that V is continuously embedded into H. Let $\overline{\mathcal{W}}$ be a class of solutions to problem (1.1) *on the real axis $\mathbb{R}$ possessing property* (1.24) *and such that*

$$\sup_{t\in R}\|u(t)\|_H < \infty \quad \text{for all} \quad u(t)\in\overline{\mathcal{W}}. \tag{1.30}$$

Assume that for any $u(t),\ v(t)\in\overline{\mathcal{W}}$ and for all real $t\ge s$ equation (1.21) *holds with $\nu>0$ and a positive function $\phi(t)$ locally integrable over the axis $\mathbb{R}$ and satisfying the condition*

$$\overline{\lim_{s\to-\infty}}\ \frac{1}{a}\int_s^{s+a}\phi(\tau)\,d\tau \ \le\ R \tag{1.31}$$

for some $a>0$. Here $R>0$ is a constant independent of $u(t)$ and $v(t)$. Let $\mathcal{L}=\{l_j\colon j=1,\,\dots,\,N\}$ be a family of continuous linear functionals on V possessing property (1.18) *with $\varepsilon_{\mathcal{L}}<\sqrt{\nu/R}$. Then $\mathcal{L}$ is a set of asymptotically $(V,\overline{\mathcal{W}})$-determining functionals for problem* (1.1).

Proof.

As in the proof of Corollary 1.2 equations (1.20) and (1.21) imply that

$$\|u(t)-v(t)\|_H+\int_s^t\psi(\tau)\|u(\tau)-v(\tau)\|_H^2\,d\tau \ \le\ \|u(s)-v(s)\|_H^2$$

for all $t\ge s$, where $\psi(\tau)=\nu(1+\delta)^{-1}\varepsilon_{\mathcal{L}}^{-2}-\phi(\tau)$ and δ is an arbitrary positive number. Hence

$$\|u(t)-v(t)\|_H^2\ \le\ \|u(s)-v(s)\|_H^2\exp\left\{-\int_s^t\psi(\tau)\,d\tau\right\} \tag{1.32}$$

for all $t\ge s$. Using (1.31) it is easy to find that for any $\eta>0$ there exists $M_\eta>0$ such that

$$\int_{s_1}^{s_2}\phi(\tau)\,d\tau\ \le\ (R+\eta)(s_2-s_1+a)$$

for all $s_1\le s_2\le -M_\eta$. This equation and boundedness property (1.30) enable us to pass to the limit as $s\to-\infty$ in (1.32), provided $\varepsilon_{\mathcal{L}}<\sqrt{\nu/R}$, and to obtain the required assertion.

We now give one more general result on the finiteness of the number of determining functionals. This result does not use Lemma 1.1 and requires only the convergence of functionals on a certain sequence of moments of time.

Theorem 1.3.

Let V *and* H *be reflexive Banach spaces such that* V *is continuously embedded into* H. *Assume that* $\mathscr{W}$ *is a class of solutions to problem* (1.1) *possessing property* (1.2). *Assume that there exist constants* $C, K \ge 0$, $\beta > \alpha > 0$, *and* $0 < q < 1$ *such that for any pair of solutions* $u_1(t)$ *and* $u_2(t)$ *from* $\mathscr{W}$ *we have*

$$\|u_1(t) - u_2(t)\|_V \le C\|u_1(s) - u_2(s)\|_V, \qquad s \le t \le s + \beta, \tag{1.33}$$

and

$$\|u_1(t) - u_2(t)\|_V \le K\|u_1(t) - u_2(t)\|_H + q\|u_1(s) - u_2(s)\|_V, \quad s+\alpha \le t \le s+\beta \tag{1.34}$$

for s *large enough. Let* $\mathscr{L}$ *be a finite set of continuous linear functionals on* V *possessing property* (1.18) *with* $\varepsilon_{\mathscr{L}} < (1-q)K^{-1}$. *Assume that* $\{t_k\}$ *is a sequence of positive numbers such that* $t_k \to +\infty$ *and* $\alpha \le t_{k+1} - t_k \le \beta$. *Assume that*

$$\lim_{k \to \infty} l(u_1(t_k) - u_2(t_k)) = 0, \qquad l \in \mathscr{L}. \tag{1.35}$$

Then

$$\|u_1(t) - u_2(t)\|_V \to 0 \quad \textit{as} \quad t \to +\infty. \tag{1.36}$$

It should be noted that relations like (1.33) and (1.34) can be obtained for a wide class of equations (see, e.g., Sections 1.9, 2.2, and 4.6).

Proof.

Let $u(t) = u_1(t) - u_2(t)$. Then equations (1.34) and (1.18) give us

$$\|u(t_k)\|_V \le q_{\mathscr{L}}\|u(t_{k-1})\|_V + C \max_j |l_j(u(t_k))|,$$

where $q_{\mathscr{L}} = q(1 - \varepsilon_{\mathscr{L}}K)^{-1} < 1$. Therefore, after iterations we obtain that

$$\|u(t_n)\|_V \le q_{\mathscr{L}}^n\|u(t_0)\|_V + C \cdot \sum_{k=1}^{n} q_{\mathscr{L}}^{n-k} \max_j |l_j(u(t_k))|.$$

Hence, equation (1.35) implies that $\|u(t_n)\|_V \to 0$ as $n \to \infty$. Therefore, (1.36) follows from equation (1.33). **Theorem 1.3 is proved**.

Application of Corollaries 1.1–1.3 and Theorem 1.3 to the proof of finiteness of a set $\mathscr{L}$ of determining elements requires that the inequalities of the type (1.17) and (1.21), or (1.30) and (1.33), or (1.33) and (1.34), as well as (1.18) with the constant $\varepsilon_{\mathscr{L}}$ small enough be fulfilled. As the analysis of particular examples shows, the fulfilment of estimates (1.17), (1.21), (1.30), (1.31), (1.33), and (1.34) is mainly connected with the dissipativity properties of the system. Methods for obtaining them are rather well-developed (see Chapters 1 and 2 and the references therein) and in many cases the corresponding constants ν, R, K, and q either are close to optimal or can be estimated explicitly in terms of the parameters of equations. Therefore, the problem of description of finite families of functionals that asymptotically determine the dynamics of the process can be reduced to the study of sets of functionals for which estimate (1.18) holds with $\varepsilon_{\mathscr{L}}$ small enough. It is convenient to base this study on the concept of completeness defect of a family of functionals with respect to a pair of spaces.

§ 2 Completeness Defect

Let V and H be reflexive Banach spaces such that V is continuously and densely embedded into H. The ***completeness defect*** of a set $\mathscr{L}$ of linear functionals on V with respect to H is defined as

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\left\{\|w\|_H : \ w \in V, \ l(w) = 0, \ l \in \mathscr{L}, \ \|w\|_V \le 1\right\}. \tag{2.1}$$

It should be noted that the finite dimensionality of Lin $\mathscr{L}$ is not assumed here.

— Exercise *2.1* Prove that the value $\varepsilon_{\mathscr{L}}(V, H)$ can also be defined by one of the following formulae:

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\left\{\|w\|_H : \ w \in V, \ l(w) = 0, \ \|w\|_V = 1\right\}, \tag{2.2}$$

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\left\{\frac{\|w\|_H}{\|w\|_V} : \ w \in V, \ w \ne 0, \ l(w) = 0\right\}, \tag{2.3}$$

$$\varepsilon_{\mathscr{L}}(V, H) = \inf\left\{C : \ \|w\|_H \le C\|w\|_V, \ w \in V, \ l(w) = 0\right\}. \tag{2.4}$$

— Exercise *2.2* Let $\mathscr{L}_1 \subset \mathscr{L}_2$ be two sets in the space V^* of linear functionals on V. Show that $\varepsilon_{\mathscr{L}_1}(V, H) \ge \varepsilon_{\mathscr{L}_2}(V, H)$.

— Exercise *2.3* Let $\mathscr{L} \subset V^*$ and let $\hat{\mathscr{L}}$ be a weakly closed span of the set $\mathscr{L}$ in the space V^*. Show that $\varepsilon_{\mathscr{L}}(V, H) = \varepsilon_{\hat{\mathscr{L}}}(V, H)$.

The following fact explains the name of the value $\varepsilon_{\mathscr{L}}(V, H)$. We remind that a set $\mathscr{L}$ of functionals on V is said to be complete if the condition $l(w) = 0$ for all $l \in \mathscr{L}$ implies that $w = 0$.

— Exercise *2.4* Show that for a set $\mathscr{L}$ of functionals on V to be complete it is necessary and sufficient that $\varepsilon_{\mathscr{L}}(V, H) = 0$.

The following assertion plays an important role in the construction of a set of determining functionals.

Theorem 2.1.

Let $\varepsilon_{\mathcal{L}} = \varepsilon_{\mathcal{L}}(V, H)$ ***be the completeness defect of a set*** $\mathcal{L}$ ***of linear functionals on*** V ***with respect to*** H. ***Then there exists a constant*** $C_{\mathcal{L}} > 0$ ***such that***

$$\|u\|_H \le \varepsilon_L \|u\|_V + C_L \cdot \sup\left\{|l(u)|: \ l \in \hat{\mathcal{L}}, \ \|l\|_* \le 1\right\} \tag{2.5}$$

for any element $u \in V$, ***where*** $\hat{\mathcal{L}}$ ***is a weakly closed span of the set*** $\mathcal{L}$ ***in*** V^*.

Proof.

Let

$$\mathcal{L}^{\perp} = \{v \in V: \ l(v) = 0, \ l \in \mathcal{L}\} \tag{2.6}$$

be the annihilator of $\mathcal{L}$. If $u \in \mathcal{L}^{\perp}$, then it is evident that $l(u) = 0$ for all $l \in \hat{\mathcal{L}}$. Therefore, equation (2.4) implies that

$$\|u\|_H \le \varepsilon_{\mathcal{L}} \|u\|_V \quad \text{for all} \quad u \in \mathcal{L}^{\perp}, \tag{2.7}$$

i.e. for $u \in \mathcal{L}^{\perp}$ equation (2.5) is valid.

Assume that $u \notin \mathcal{L}^{\perp}$. Since $\mathcal{L}^{\perp}$ is a subspace in V, it is easy to verify that there exists an element $w \in \mathcal{L}^{\perp}$ such that

$$\|u - w\|_V = \operatorname{dist}_V(u, \ \mathcal{L}^{\perp}) = \inf\left\{\|u - v\|_V: \ v \in \mathcal{L}^{\perp}\right\} . \tag{2.8}$$

Indeed, let the sequence $\{v_n\} \subset \mathcal{L}^{\perp}$ be such that

$$d \equiv \operatorname{dist}_V(u, \ \mathcal{L}^{\perp}) = \lim_{n \to \infty} \|u - v_n\|_V .$$

It is clear that $\{v_n\}$ is a bounded sequence in V. Therefore, by virtue of the reflexivity of the space V, there exist an element w from $\mathcal{L}^{\perp}$ and a subsequence $\{v_{n_k}\}$ such that v_{n_k} weakly converges to w as $k \to \infty$, i.e. for any functional $f \in V^*$ the equation

$$f(u - w) = \lim_{k \to \infty} f(u - v_{n_k})$$

holds. It follows that

$$|f(u - w)| \le \lim_{k \to \infty} \|u - v_{n_k}\|_V \cdot \|f\|_* \le d \|f\|_* .$$

Therefore, we use the reflexivity of V once again to find that

$$\|u - w\|_V = \sup\left\{|f(u - w)|: \ f \in V^*, \ \|f\|_* = 1\right\} \le d .$$

However, $\|u - w\|_V \ge d$. Hence, $\|u - w\|_V = d$. Thus, equation (2.8) holds.

Equation (2.7) and the continuity of the embedding of V into H imply that

$$\|u\|_H \le \|w\|_H + \|u-w\|_H \le \varepsilon_{\mathscr{L}}\|w\|_V + C\|u-w\|_V .$$

It is clear that

$$\|w\|_V \le \|u\|_V + \|u-w\|_V .$$

Therefore,

$$\|u\|_H \le \varepsilon_{\mathscr{L}}\|u\|_V + (\varepsilon_{\mathscr{L}} + C)\|u-w\|_V . \tag{2.9}$$

Let us now prove that there exists a continuous linear functional l_0 on the space V possessing the properties

$$l_0(u-w) = \|u-w\|_V, \qquad \|l_0\|_* = 1, \qquad l_0(v) = 0 \quad \text{for} \quad v \in \mathscr{L}^\perp . \tag{2.10}$$

To do that, we define the functional $\tilde{l}_0$ by the formula

$$\tilde{l}_0(m) = a\|u-w\|_V, \qquad m = v + a(u-w),$$

on the subspace

$$M = \left\{ m = v + a(u-w):\ v \in \mathscr{L}^\perp,\ a \in \mathbb{R} \right\}.$$

It is clear that $\tilde{l}_0$ is a linear functional on M and $\tilde{l}_0(m) = 0$ for $m \in \mathscr{L}^\perp$. Let us calculate its norm. Evidently

$$\|m\|_V \equiv \|v + a(u-w)\|_V = |a| \cdot \left\| u - w + \frac{1}{a} v \right\|_V, \qquad a \ne 0 .$$

Since $w - a^{-1}v \in \mathscr{L}^\perp$, equation (2.8) implies that

$$\|m\|_V \ge |a| \cdot \|u-w\|_V = \left|\tilde{l}_0(m)\right|, \quad m = v + a(u-w), \quad a \ne 0 .$$

Consequently, for any $m \in M$

$$\left|\tilde{l}_0(m)\right| \le \|m\|_V .$$

This implies that $\tilde{l}_0$ has a unit norm as a functional on M. By virtue of the Hahn-Banach theorem the functional $\tilde{l}_0$ can be extended on V without increase of the norm. Therefore, there exists a functional l_0 on V possessing properties (2.10). Therewith l_0 lies in a weakly closed span $\hat{\mathscr{L}}$ of the set $\mathscr{L}$. Indeed, if $l_0 \notin \hat{\mathscr{L}}$, then using the reflexivity of V and reasoning as in the construction of the functional l_0 it is easy to verify that there exists an element $x \in V$ such that $l_0(x) \ne 0$ and $l(x) = 0$ for all $l \in \mathscr{L}$. It is impossible due to (2.10) .

In order to complete the proof of Theorem 2.1 we use equations (2.9) and (2.10). As a result, we obtain that

$$\|u\|_H \le \varepsilon_{\mathscr{L}}\|u\|_V + (\varepsilon_{\mathscr{L}} + C)\, l_0(u-w) .$$

However, $l_0 \in \hat{\mathscr{L}}$, $l_0(u-w) = l_0(u)$, and $\|l_0\|_* = 1$. Therefore, equation (2.5) holds. **Theorem 2.1 is proved**.

— Exercise *2.5* Assume that $\mathscr{L} = \{l_j : j = 1, \ldots, N\}$ is a finite set in V^*. Show that there exists a constant $C_{\mathscr{L}}$ such that

$$\|u\|_H \le \varepsilon_{\mathscr{L}}(V, H)\|u\|_V + C_{\mathscr{L}} \max_{j=1,\ldots,N} |l_j(u)| \tag{2.11}$$

for all $u \in V$.

In particular, if the hypotheses of Corollaries 1.1 and 1.3 hold, then Theorem 2.1 and equation (2.11) enable us to get rid of assumption (1.18) by replacing it with the corresponding assumption on the smallness of the completeness defect $\varepsilon_{\mathscr{L}}(V, H)$.

The following assertion provides a way of calculating the completeness defect when we are dealing with Hilbert spaces.

Theorem 2.2.

Let V and H be separable Hilbert spaces such that V is compactly and densely embedded into H. Let K be a selfadjoint positive compact operator in the space V defined by the equality

$$(Ku, v)_V = (u, v)_H, \qquad u, v \in V.$$

Then the completeness defect of a set $\mathscr{L}$ of functionals on V can be evaluated by the formula

$$\varepsilon_{\mathscr{L}}(V, H) = \sqrt{\mu_{\max}(P_{\mathscr{L}} K P_{\mathscr{L}})}\,, \tag{2.12}$$

where $P_{\mathscr{L}}$ is the orthoprojector in the space V onto the annihilator

$$\mathscr{L}^{\perp} = \{v \in V : l(v) = 0,\ l \in \mathscr{L}\}$$

and $\mu_{\max}(S)$ is the maximal eigenvalue of the operator S.

Proof.

It follows from definition (2.1) that

$$\varepsilon_{\mathscr{L}}(V, H) = \sup\{\|u\|_H : u \in B_{\mathscr{L}}\}$$

where $B_{\mathscr{L}} = \mathscr{L}^{\perp} \cap \{v : \|v\|_V \le 1\}$ is the unit ball in $\mathscr{L}^{\perp}$. Due to the compactness of the embedding of V into H, the set $B_{\mathscr{L}}$ is compact in H. Therefore, there exists an element $u_0 \in B_{\mathscr{L}}$ such that

$$\varepsilon_{\mathscr{L}}(V, H)^2 = \|u_0\|_H^2 = (Ku_0, u_0)_V \equiv \mu\,.$$

Therewith u_0 is the maximum point of the function $(Ku, u)_V$ on the set $B_{\mathscr{L}}$. Hence, for any $v \in \mathscr{L}^{\perp}$ and $s \in \mathbb{R}^1$ we have

$$\frac{(K(u_0 + sv),\ u_0 + sv)_V}{\|u_0 + sv\|_V^2} \le \mu \equiv (Ku_0, u_0)_V\,.$$

It follows that

$$(K(u_0+sv),\ u_0+sv)_V - \mu\|u_0+sv\|_V^2 \ \le\ 0\ .$$

It is also clear that $\|u_0\|_V = 1$. Therefore,

$$s^2\{(Kv,\ v)_V - \mu\|v\|^2\} + 2s\{(Ku_0,\ v)_V - \mu(u_0,\ v)_V\} \ \le\ 0$$

for all $s \in \mathbb{R}$. This implies that

$$(Ku_0,\ v) - \mu(u_0,\ v)_V \ \le\ 0$$

for any $v \in \mathcal{L}^\perp$. If we take $-v$ instead of v in this equation, then we obtain the opposite inequality. Therefore,

$$(Ku_0,\ v) - \mu(u_0,\ v) \ =\ 0\ , \quad v \in \mathcal{L}^\perp\ .$$

Consequently,

$$P_{\mathcal{L}} K P_{\mathcal{L}} u_0 \ =\ \mu u_0\ ,$$

i.e. $\mu = (Ku_0,\ u_0)_V = \|u_0\|_H^2$ is an eigenvalue of the operator $P_{\mathcal{L}}KP_{\mathcal{L}}$. It is evident that this eigenvalue is maximal. Thus, **Theorem 2.2 is proved**.

Corollary 2.1.

Assume that the hypotheses of Theorem 2.2 *hold. Let* $\{e_j\}$ *be an orthonormal basis in the space* V *that consists of eigenvectors of the operator* K*:*

$$Ke_j \ =\ \mu_j e_j\ , \quad (e_i,\ e_j)_V = \delta_{ij}\ , \quad \mu_1 \ge \mu_2 \ge \dots\ , \quad \mu_N \to 0\ .$$

Then the completeness defect of the system of functionals

$$\mathcal{L} = \{l_j \in V^*:\ \ l_j(v) = (v,\ e_j)_V:\ j = 1,\ 2,\ \dots,\ N\}$$

is given by the formula $\varepsilon_{\mathcal{L}}(V, H) = \sqrt{\mu_{N+1}}$.

To prove this assertion, it is just sufficient to note that $P_{\mathcal{L}}$ is the orthoprojector onto the closure of the span of elements $\{e_j:\ j \ge N+1\}$ and that $P_{\mathcal{L}}$ commutes with K.

Exercise *2.6* Let A be a positive operator with discrete spectrum in the space H:

$$Ae_k = \lambda_k e_k\ , \quad \lambda_1 \le \lambda_2 \le \dots\ , \quad \lambda_N \to +\infty\ , \quad (e_k,\ e_j)_H = \delta_{kj}\ ,$$

and let $\mathcal{F}_s = D(A^s)$, $s \in \mathbb{R}$, be a scale of spaces generated by the operator A (see Section 2.1). Assume that

$$\mathcal{L} = \{l_j:\ \ l_j(v) = (v,\ e_j)_H:\ j = 1,\ 2,\ \dots,\ N\}\ . \tag{2.13}$$

Prove that $\varepsilon_{\mathcal{L}}(\mathcal{F}_\sigma,\ \mathcal{F}_s) = \lambda_{N+1}^{-(\sigma-s)}$ for all $\sigma > s$.

It should be noted that the functionals in Exercise 2.6 are often called ***modes***.

Let us give several more facts on general properties of the completeness defect.

Theorem 2.3.

Assume that the hypotheses of Theorem* 2.2 *hold. Assume that* $\mathcal{L}$ *is a set of linear functionals on* V *and* $\mathcal{K}_{\mathcal{L}}$ *is a family of linear bounded operators* R *that map* V *into* H *and are such that* $Ru = 0$ *for all* $u \in \mathcal{L}^{\perp}$. *Let

$$e_V^H(R) = \sup\left\{\|u - Ru\|_H : \|u\|_V \le 1\right\} \tag{2.14}$$

be the global approximation error in* H *arising from the approximation of elements* $v \in V$ *by elements* Rv. *Then

$$\varepsilon_{\mathcal{L}}(V;\, H) = \min\left\{e_V^H(R) : R \in \mathcal{K}_{\mathcal{L}}\right\}. \tag{2.15}$$

Proof.

Let $R \in \mathcal{K}_{\mathcal{L}}$. Equation (2.14) implies that

$$\|u - Ru\|_H \le e_V^H(R)\|u\|_V, \qquad u \in V.$$

Therefore, for $u \in \mathcal{L}^{\perp}$ we have $\|u\|_H \le e_V^H(R)\|u\|_V$, i.e. $\varepsilon_{\mathcal{L}}(V;\, H) \le e_V^H(R)$ for all $R \in \mathcal{K}_{\mathcal{L}}$. Let us show that there exists an operator $R_0 \in \mathcal{K}_{\mathcal{L}}$ such that $\varepsilon_{\mathcal{L}}(V;\, H) = e_V^H(R_0)$. Equation (2.12) implies that

$$\varepsilon_{\mathcal{L}}(V;\, H) = \left\|K^{1/2}P_{\mathcal{L}}\right\|_{L(V;\, V)} = \sup\left\{\left\|K^{1/2}P_{\mathcal{L}}\,u\right\|_V : \|u\|_V \le 1\right\},$$

where $P_{\mathcal{L}}$ is the orthoprojector in the space V onto $\mathcal{L}^{\perp}$ and $\left\|K^{1/2}P_{\mathcal{L}}\right\|_{L(V,\, V)}$ is the norm of the operator $K^{1/2}P_{\mathcal{L}}$ in the space $L(V,\, V)$ of bounded linear operators in V. Therefore, the definition of the operator K implies that

$$\varepsilon_{\mathcal{L}}(V;\, H) = \sup\left\{\|P_{\mathcal{L}}\,u\|_H : \|u\|_V \le 1\right\} = e_V^H(I - P_{\mathcal{L}}). \tag{2.16}$$

It is evident that the orthoprojector $Q_{\mathcal{L}} = I - P_{\mathcal{L}}$ belongs to $\mathcal{K}_{\mathcal{L}}$ (it projects onto the subspace that is orthogonal to $\mathcal{L}^{\perp}$ in V). **Theorem 2.3 is proved**.

Exercise *2.7* Assume that $\mathcal{L} = \{l_j : \; j = 1, \ldots, N\}$ is a finite set. Show that the family $\mathcal{K}_{\mathcal{L}}$ consists of finite-dimensional operators R of the form

$$Ru = \sum_{j=1}^{N} l_j(u)\,\varphi_j, \qquad u \in V,$$

where $\{\varphi_j : \; j = 1, \ldots, N\}$ is an arbitrary collection of elements of the space V (they do not need to be distinct). How should the choice of elements $\{\varphi_j\}$ be made for the operator $Q_{\mathcal{L}}$ from the proof of Theorem 2.3?

Theorem 2.3 will be used further (see Section 3) to obtain upper estimates of the completeness defect for some specific sets of functionals. The simplest situation is presented in the following example.

Example *2.1*

Let $H = L^2(0, l)$ and let $V = (H^2 \cap H_0^1)(0, l)$. As usual, here $H^s(0, l)$ is the Sobolev space of the order s and $H_0^s(0, l)$ is the closure of the set $C_0^\infty(0, l)$ in $H^s(0, l)$. We define the norms in H and V by the equalities

$$\|u\|_H^2 = \|u\|^2 \equiv \int_0^l (u(x))^2 \, dx , \qquad \|u\|_V^2 = \|u''\|^2 .$$

Let $h = l/N$, $x_j = jh$, $j = 1, \ldots, N-1$. Consider a set of functionals

$$\mathscr{L} = \{ l(u) = u(x_j) : \; j = 1, \ldots, N-1 \}$$

on V. Assume that R is a transformation that maps a function $u \in V$ into its linear interpolating spline

$$s(x) = \sum_{j=1}^{N-1} u(x_j) \, \chi\left(\frac{x}{h} - j\right).$$

Here $\chi(x) = 1 - |x|$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| > 1$. We apply Theorem 2.3 and obtain

$$\varepsilon_{\mathscr{L}}(V; H) \le \sup\left\{ \|u - s\| : \; u \in V, \; \|u''\| \le 1 \right\}.$$

We use an easy verifiable equation

$$u(x) - s(x) = -\frac{1}{h} \int_{x_j}^{x} d\tau \int_{x_j}^{x_{j+1}} d\xi \int_{\tau}^{\xi} u''(y) \, dy , \qquad x \in [x_j, x_{j+1}],$$

to obtain the estimate

$$\|u - s\| \le \frac{h^2}{\sqrt{3}} \|u''\| .$$

This implies that $\varepsilon_{\mathscr{L}}(H; V) \le h^2/\sqrt{3}$.

The assertion on the interdependence of the completeness defect $\varepsilon_{\mathscr{L}}$ and the Kolmogorov N-width made below enables us to obtain effective lower estimates for $\varepsilon_{\mathscr{L}}(V; H)$.

Let V and H be separable Hilbert spaces such that V is continuously and densely embedded into H. Then the ***Kolmogorov*** N***-width of the embedding of*** V ***into*** H is defined by the relation

$$\varkappa_N = \varkappa_N(V;\, H) = \inf\left\{e_V^H(F):\ F \in \mathscr{F}_N\right\}, \tag{2.17}$$

where $\mathscr{F}_N$ is the family of all N-dimensional subspaces F of the space V and

$$e_V^H(F) = \sup\left\{\mathrm{dist}_H(v,\, F):\ \|v\|_V \le 1\right\}$$

is the global error of approximation of elements $v \in V$ in H by elements of the subspace F. Here

$$\mathrm{dist}_H(v,\, F) = \inf\left\{\|v - f\|_H:\ f \in F\right\}.$$

In other words, the Kolmogorov N-width $\varkappa_N$ of the embedding of V into H is the minimal possible global error of approximation of elements of V in H by elements of some N-dimensional subspace.

Theorem 2.4.

Let V and H be separable Hilbert spaces such that V is continuously and densely embedded into H. Then

$$\varkappa_N(V;\, H) \;=\; \min\left\{\varepsilon_{\mathscr{L}}(V;\, H):\ \ \mathscr{L} \subset V^*;\ \ \dim \mathrm{Lin}\, \mathscr{L} = N\right\} = \sqrt{\mu_{N+1}}\ , \tag{2.18}$$

where $\{\mu_j\}$ is the nonincreasing sequence of eigenvalues of the operator K defined by the equality $(u,\, v)_H = (Ku,\, v)_V$.

The proof of the theorem is based on the lemma given below as well as on the fact that $(u,\, v)_H = (Ku,\, v)_V$, where K is a compact positive operator in the space V (see Theorem 2.2). Further the notation $\{e_j\}_{j=1}^{\infty}$ stands for the proper basis of the operator K in the space V while the notation μ_j stands for the corresponding eigenvalues:

$$Ke_j = \mu_j e_j\,, \quad \mu_1 \ge \mu_2 \ge \ldots\,, \quad \mu_n \to 0\,, \quad (e_i,\, e_j)_V = \delta_{ij}\ .$$

It is evident that $\left\{\frac{1}{\sqrt{\mu_i}}\, e_i\right\}$ is an orthonormalized basis in the space H.

Lemma 2.1.

Assume that the hypotheses of Theorem 2.4 hold. Then

$$\min\left\{\varepsilon_{\mathscr{L}}(V;\, H):\ \ \mathscr{L} \subset V^*,\ \ \dim \mathrm{Lin}\, \mathscr{L} = N\right\} = \sqrt{\mu_{N+1}}\ . \tag{2.19}$$

Proof.

By virtue of Corollary 2.1 it is sufficient to verify that

$$\varepsilon_{\mathscr{L}}(V;\, H) \geq \sqrt{\mu_{N+1}}$$

for all $\mathscr{L} = \{l_j :\ j = 1,\ \dots,\ N\}$, where l_j are linearly independent functionals on V. Definition (2.1) implies that

$$[\varepsilon_{\mathscr{L}}(V;\, H)]^2 \ \geq\ \|u\|_H^2 = \sum_{j=1}^{\infty} \mu_j (u,\, e_j)_V^2 \tag{2.20}$$

for all $u \in V$ such that $\|u\|_V = 1$ and $l_j(u) = 0$, $j = 1,\ \dots,\ N$. Let us substitute in (2.20) the vector

$$u = \sum_{j=1}^{N+1} c_j\, e_j\ ,$$

where the constants c_j are choosen such that $l_j(u) = 0$ for $j = 1,\ \dots,\ N$ and $\|u\|_V = 1$. Therewith equation (2.20) implies that

$$[\varepsilon_{\mathscr{L}}(V;\, H)]^2 \ \geq\ \sum_{j=1}^{N+1} \mu_j c_j^2 \ \geq\ \mu_{N+1} \sum_{j=1}^{N+1} c_j^2 \ = \mu_{N+1} \|u\|_V^2 \ = \mu_{N+1}\,.$$

Thus, Lemma 2.1 is proved.

We now prove that $\varkappa_N = \min\{\varepsilon_{\mathscr{L}}\}$. Let us use equation (2.16)

$$\varepsilon_{\mathscr{L}}(V;\, H) = \ \sup \left\{ \|u - Q_{\mathscr{L}}\, u\|_H :\ \|u\|_V \ \leq\ 1 \right\}. \tag{2.21}$$

Here $Q_{\mathscr{L}}$ is the orthoprojector onto the subspace $Q_{\mathscr{L}} V$ orthogonal to $\mathscr{L}^\perp$ in V. It is evident that $Q_{\mathscr{L}} V$ is isomorphic to $\operatorname{Lin}\mathscr{L}$. Therefore, $\dim Q_{\mathscr{L}} V = N$. Hence, equation (2.21) gives us that

$$\varepsilon_{\mathscr{L}}(V;\, H) \ \geq\ \sup \left\{ \operatorname{dist}_H(u,\, Q_{\mathscr{L}} V) :\ \|u\|_V \leq 1 \right\} \ \geq\ \varkappa_N \tag{2.22}$$

for all $\mathscr{L} \in V^*$ such that $\dim \operatorname{Lin} \mathscr{L} = N$. Conversely, let F be an N-dimensional subspace in V and let $\{ f_j :\ j = 1,\ \dots,\ N\}$ be a orthonormalized basis in the space H. Assume that

$$\mathscr{L}_F \ = \ \{ l_j :\ \ l_j(v) = (f_j,\, v)_H :\ j = 1,\ \dots,\ N \}\,.$$

Let $Q_{H,\,F}$ be the orthoprojector in the space H onto F. It is clear that

$$Q_{H,\,F}\, u \ = \sum_{j=1}^{N} (u,\, f_j)_H\, f_j\ .$$

Therefore, if $u \in \mathscr{L}_F^\perp$, then $Q_{H,\,F}\, u \ = \ 0$. It is clear that $Q_{H,\,F}$ is a bounded operator from V into H. Using Theorem 2.3 we find that

$$\varepsilon_{\mathcal{L}_F}(V;\,H) \;\le\; e_V^H(Q_{H,\,F}) \;=\; \sup\left\{\|u - Q_{H,\,F}\,u\|_H:\; \|u\|_V \le 1\right\}.$$

However, $\|u - Q_{H,\,F}\,u\|_H = \mathrm{dist}_H(u,\,F)$. Hence,

$$\min\{\varepsilon_{\mathcal{L}}\} \;\le\; \varepsilon_{\mathcal{L}_F}(V;\,H) \;\le\; e_V^H(F) \tag{2.23}$$

for any N-dimensional subspace F in V. Equations (2.22) and (2.23) imply that

$$\varkappa_N(V;\,H) = \min\left\{\varepsilon_{\mathcal{L}}(V;\,H):\; \mathcal{L} \subset V^*,\; \dim \mathrm{Lin}\,\mathcal{L} = N\right\}.$$

This equation together with Lemma 2.1 **completes the proof of Theorem 2.4**.

— Exercise *2.8* Let V_k and H_k be reflexive Banach spaces such that V_k is continuously and densely embedded into H_k, let $\mathcal{L}_k$ be a set of linear functionals on V_k, $k = 1,\,2$. Assume that

$$\mathcal{L} = \overline{\mathcal{L}}_1 \cup \overline{\mathcal{L}}_2 \subset (V_1 \times V_2)^*,$$

where

$$\overline{\mathcal{L}}_k = \left\{ l \in (V_1 \times V_2)^*:\;\; l(v_1,\,v_2) = l(v_k),\;\; l \in \mathcal{L}_k \right\}, \qquad k = 1,\,2.$$

Prove that

$$\varepsilon_{\mathcal{L}}(V_1 \times V_2,\,H_1 \times H_2) = \max\left\{\varepsilon_{\mathcal{L}_1}(V_1,\,H_1),\; \varepsilon_{\mathcal{L}_2}(V_2,\,H_2)\right\}.$$

— Exercise *2.9* Use Lemma 2.1 and Corollary 2.1 to calculate the Kolmogorov N-width of the embedding of the space $\mathcal{F}_s = D(A^s)$ into $\mathcal{F}_\sigma = D(A^\sigma)$ for $s > \sigma$, where A is a positive operator with discrete spectrum.

— Exercise *2.10* Show that in Example 2.1 $\varkappa_N(V,\,H) = l^2 \cdot [\pi(N+1)]^{-2}$. Prove that $\pi^{-2} h^2 \;\le\; \varepsilon_{\mathcal{L}}(V;\,H) \;\le\; h^2/\sqrt{3}$.

— Exercise *2.11* Assume that there are three reflexive Banach spaces $V \subset W \subset H$ such that all embeddings are dense and continuous. Let $\mathcal{L}$ be a set of functionals on W. Prove that $\varepsilon_{\mathcal{L}}(V,\,H) \le \varepsilon_{\mathcal{L}}(V,\,W) \cdot \varepsilon_{\mathcal{L}}(W,\,H)$ (*Hint*: see (2.3)).

— Exercise *2.12* In addition to the hypotheses of Exercise 2.11, assume that the inequality

$$\|u\|_W \;\le\; a_\theta \|u\|_H^\theta \cdot \|u\|_V^{1-\theta}, \qquad u \in V,$$

holds for some constants $a_\theta > 0$ and $\theta \in (0,\,1)$. Show that

$$\left[a_\theta^{-1}\,\varepsilon_{\mathcal{L}}(V,\,W)\right]^{\frac{1}{\theta}} \;\le\; \varepsilon_{\mathcal{L}}(V,\,H) \;\le\; \left[a_\theta\,\varepsilon_{\mathcal{L}}(W,\,H)\right]^{\frac{1}{1-\theta}}.$$

§ 3 Estimates of Completeness Defect in Sobolev Spaces

In this section we consider several families of functionals on Sobolev spaces that are important from the point of view of applications. We also give estimates of the corresponding completeness defects. The exposition is quite brief here. We recommend that the reader who does not master the theory of Sobolev spaces just get acquainted with the statements of Theorems 3.1 and 3.2 and the results of Examples 3.1 and 3.2 and Exercises 3.2–3.6.

We remind some definitions (see, e.g., the book by Lions-Magenes [4]). Let Ω be a domain in $\mathbb{R}^\nu$. The Sobolev space $H^m(\Omega)$ of the order m $(m = 0, 1, 2, \dots)$ is a set of functions

$$H^m(\Omega) = \left\{ f \in L^2(\Omega): \ \mathcal{D}^j f(x) \in L^2(\Omega), \ |j| \le m \right\},$$

where $j = (j_1, \dots, j_\nu)$, $j_k = 0, 1, 2, \dots$, $|j| = j_1 + \dots + j_\nu$ and

$$\mathcal{D}^j f(x) = \frac{\partial^{|j|} f}{\partial x_1^{j_1} \cdot \partial x_2^{j_2} \dots \partial x_\nu^{j_\nu}} . \tag{3.1}$$

The space $H^m(\Omega)$ is a separable Hilbert space with the inner product

$$(u, v)_m = \sum_{|j| \le m} \int_\Omega \mathcal{D}^j u \cdot \mathcal{D}^j v \, dx .$$

Further we also use the space $H_0^m(\Omega)$ which is the closure (in $H^m(\Omega)$) of the set $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω and the space $H^s(\mathbb{R}^\nu)$ which is defined as follows:

$$H^s(\mathbb{R}^\nu) = \left\{ u(x) \in L^2(\mathbb{R}^\nu): \int_{\mathbb{R}^\nu} (1 + |y|^2)^s \, |\hat{u}(y)|^2 \, dy \equiv \|u\|_s^2 < \infty \right\},$$

where $s \ge 0$, $\hat{u}(y)$ is the Fourier transform of the function $u(x)$,

$$\hat{u}(y) = \int_{\mathbb{R}^\nu} e^{ixy} u(x) \, dx ,$$

$|y|^2 = y_1^2 + \dots + y_2^2$, and $xy = x_1 y_1 + \dots + x_\nu y_\nu$. Evidently this definition coincides with the previous one for natural s and $\Omega = \mathbb{R}^\nu$.

Exercise *3.1* Show that the norms in the spaces $H^s(\mathbb{R}^\nu)$ possess the property

$$\|u\|_{s(\theta)} \le \|u\|_{s_1}^\theta \cdot \|u\|_{s_2}^{1-\theta}, \quad s(\theta) = \theta s_1 + (1-\theta) s_2 ,$$
$$0 \le \theta \le 1 , \quad s_1, s_2 \ge 0 .$$

We can also define the space $H^s(\Omega)$ as restriction (to Ω) of functions from $H^s(\mathbb{R}^\nu)$ with the norm

$$\|u\|_{s,\,\Omega} = \inf\left\{\|v\|_{s,\,\mathbb{R}^\nu} :\ v(x) = u(x) \text{ in } \Omega,\ v \in H^s(\mathbb{R}^\nu)\right\}$$

and the space $H_0^s(\Omega)$ as the closure of the set $C_0^\infty(\Omega)$ in $H^s(\Omega)$. The spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ are separable Hilbert spaces. More detailed information on the Sobolev spaces can be found in textbooks on the theory of such spaces (see, e.g., [4], [5]).

The following version of the Sobolev integral representation will be used further.

Lemma 3.1.

Let Ω be a domain in $\mathbb{R}^\nu$ and let $\lambda(x)$ be a function from $L^\infty(\mathbb{R}^\nu)$ such that

$$\operatorname{supp}\lambda \subset\subset \Omega, \qquad \int\limits_{\mathbb{R}^\nu} \lambda(x)\,\mathrm{d}x = 1\,. \tag{3.2}$$

Assume that Ω is a star-like domain with respect to the support supp λ *of the function λ. This means that for any point $x \in \Omega$ the cone*

$$V_x = \{z = \tau x + (1-\tau)y;\quad 0 \le \tau \le 1,\ y \in \operatorname{supp}\lambda\} \tag{3.3}$$

belongs to the domain Ω. Then for any function $u(x) \in H^m(\Omega)$ the representation

$$u(x) = P_{m-1}(x;\ u) + \sum_{|\alpha| = m} \frac{m}{\alpha!} \int\limits_{V_x} (x-y)^\alpha K(x,\ y)\,\mathcal{D}^\alpha u(y)\,\mathrm{d}y \tag{3.4}$$

is valid, where

$$P_{m-1}(x,\ u) = \sum_{|\alpha| < m} \frac{1}{\alpha!} \int\limits_{\Omega} \lambda(y)(x-y)^\alpha\, \mathcal{D}^\alpha u(y)\,\mathrm{d}y\ , \tag{3.5}$$

$$\alpha = (\alpha_1,\ \dots,\ \alpha_\nu), \qquad \alpha! = \alpha_1!\dots\alpha_\nu!\,, \qquad z^\alpha = z_1^{\alpha_1}\dots z_\nu^{\alpha_\nu},$$

$$K(x,\ y) = \int\limits_0^1 s^{-\nu-1}\,\lambda\!\left(x + \frac{y-x}{s}\right)\mathrm{d}s\,. \tag{3.6}$$

Proof.

If we multiply Taylor's formula

$$u(x) = \sum_{|\alpha| < m} \frac{(x-y)^\alpha}{\alpha!}\,\mathcal{D}^\alpha u(y)\ +$$

$$+\ m \sum_{|\alpha| = m} \frac{(x-y)^\alpha}{\alpha!} \int\limits_0^1 s^{m-1}\,\mathcal{D}^\alpha u(x + s\,(y-x))\,\mathrm{d}s$$

by $\lambda(y)$ and integrate it over y, then after introducing a new variable $z = x + s(y - x)$ we obtain the assertion of the lemma.

Integral representation (3.4) enables us to obtain the following generalization of the Poincaré inequality.

Lemma 3.2.

Let the hypotheses of Lemma 3.1 *be valid for a bounded domain* $\Omega \subset \mathbb{R}^\nu$ *and for a function* $\lambda(x)$. *Then for any function* $u(x) \in H^1(\Omega)$ *the inequality*

$$\left\| u - \langle u \rangle_\lambda \right\|_{L^2(\Omega)} \le \frac{\sigma_\nu}{\nu} d^{\nu+1} \|\lambda\|_{L^\infty(\Omega)} \cdot \|\nabla u\|_{L^2(\Omega)} \tag{3.7}$$

is valid, where $\langle u \rangle_\lambda = \int_\Omega \lambda(x) u(x)\,\mathrm{d}x$, σ_ν *is the surface measure of the unit sphere in* $\mathbb{R}^\nu$ *and* $d = \operatorname{diam}\Omega \equiv \sup\{|x-y| :\ x, y \in \Omega\}$.

Proof.

We use formula (3.4) for $m = 1$:

$$u(x) = \langle u \rangle_\lambda + \sum_{j=1}^{\nu} \int_{V_x} K(x, y)(x_j - y_j) \cdot \frac{\partial u}{\partial y_j}(y)\,\mathrm{d}y\ . \tag{3.8}$$

It is clear that $\lambda(x + (1/s)(y - x)) = 0$ when $s^{-1}|x - y| \ge d \equiv \operatorname{diam}\Omega$. Therefore,

$$K(x, y) = \int_{\mathrm{d}^{-1}|x-y|}^{1} s^{-\nu-1} \lambda\Big(x + \frac{1}{s}(y - x)\Big)\,\mathrm{d}s\,.$$

Consequently,

$$|K(x, y)| \le \frac{\delta}{|x - y|^\nu}\,, \qquad \delta \equiv \delta(\lambda, \nu) = \frac{d^\nu}{\nu} \|\lambda\|_{L^\infty(\Omega)}\,. \tag{3.9}$$

Thus, it follows from (3.8) that

$$\left| u(x) - \langle u \rangle_\lambda \right| \le \delta \int_\Omega \frac{|\nabla u(y)|}{|x - y|^{\nu-1}}\,\mathrm{d}y \le$$

$$\le \delta \left(\int_\Omega \frac{|\nabla u(y)|^2}{|x - y|^{\nu-1}}\,\mathrm{d}y \right)^{1/2} \left(\int_\Omega \frac{\mathrm{d}y}{|x - y|^{\nu-1}} \right)^{1/2}. \tag{3.10}$$

Let $B_r(x) = \{y :\ |x - y| \le r\}$. Then it is evident that

$$\int_\Omega \frac{\mathrm{d}y}{|x - y|^{\nu-1}} \le \int_{B_\mathrm{d}(x)} \frac{\mathrm{d}y}{|x - y|^{\nu-1}} \le \sigma_\nu \cdot d\,, \tag{3.11}$$

where σ_ν is the surface measure of the unit sphere in $\mathbb{R}^\nu$. Therefore, equation (3.10) implies that

$$\left|u(x) - \langle u\rangle_\lambda\right|^2 \;\le\; \delta^2 \cdot \sigma_\nu \cdot d \int\limits_\Omega \frac{|\nabla u(y)|^2}{|x-y|^{\nu-1}}\, \mathrm{d}y \;.$$

After integration with respect to x and using (3.11) we obtain (3.7). Lemma 3.2 is proved.

Lemma 3.3.

Assume that the hypotheses of Lemma 3.1 *are valid for a bounded domain* Ω *from* $\mathbb{R}^\nu$ *and for a function* $\lambda(x)$. *Let* $m_\nu = [\nu/2] + 1$, *where* $[\cdot]$ *is a sign of the integer part of a number. Then for any function* $u(x) \in H^{m_\nu}(\Omega)$ *we have the inequality*

$$\max_{x\in\Omega}\left|u(x) - P_{m_\nu - 1}(x;\, u)\right| \;\le$$

$$\le\; \frac{c_\nu}{\nu}\, d^{\,m_\nu + \frac{\nu}{2}}\, \|\lambda\|_{L^\infty(\Omega)} \sum_{|\alpha| = m_\nu} \frac{m_\nu}{\alpha!}\, \|\mathscr{D}^\alpha u\|_{L^2(\Omega)} \;, \tag{3.12}$$

where $P_{m-1}(x;\, u)$ *is defined by formula* (3.5), $c_\nu = [\sigma_\nu/(2m_\nu - \nu)]^{1/2}$, σ_ν *is the surface measure of the unit sphere in* $\mathbb{R}^\nu$, *and* $d = \operatorname{diam}\Omega$.

Proof.

Using (3.4) and (3.9) we find that

$$\left|u(x) - P_{m_\nu-1}(x;\, u)\right| \;\le\; \sum_{|\alpha| = m_\nu} \frac{m_\nu}{\alpha!} \int\limits_{V_x} |x-y|^{m_\nu} |K(x,\, y)| \cdot \left|\mathscr{D}^\alpha u(y)\right| \mathrm{d}y \;\le$$

$$\le\; \sum_{|\alpha| = m_\nu} \frac{\delta\, m_\nu}{\alpha!} \int\limits_\Omega |x-y|^{m_\nu - \nu} \cdot \left|\mathscr{D}^\alpha u(y)\right| \mathrm{d}y \;\le$$

$$\le\; \sum_{|\alpha| = m_\nu} \frac{\delta\, m_\nu}{\alpha!}\, \|\mathscr{D}^\alpha u\|_{L^2(\Omega)} \cdot \left(\int\limits_\Omega |x-y|^{2(m_\nu - \nu)}\, \mathrm{d}y\right)^{1/2} .$$

As above, we obtain that

$$\int\limits_\Omega |x-y|^{2(m_\nu-\nu)}\, \mathrm{d}y \;\le\; \sigma_\nu \int\limits_0^d r^{2m_\nu - \nu - 1}\, \mathrm{d}r \;\le\; \sigma_\nu d^{2m_\nu - \nu} \cdot (2m_\nu - \nu)^{-1} .$$

Thus, equation (3.12) is valid. Lemma 3.3 is proved.

Lemma 3.4.

Assume that the hypotheses of Lemma 3.3 hold. Then for any function $u(x) \in H^{m_\nu}(\Omega)$ *and for any* $x_* \in \Omega$ *the inequality*

$$\|u - u(x_*)\|_{L^2(\Omega)} \equiv \left(\int_\Omega |u(x) - u(x_*)|^2 \, \mathrm{d}x \right)^{1/2} \le$$

$$\le C_\nu \cdot (d^\nu \|\lambda\|_{L^\infty(\Omega)}) \cdot \sum_{j=1}^{m_\nu} d^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \|\mathscr{D}^\alpha u\|_{L^2(\Omega)} \tag{3.13}$$

is valid, where C_ν *is a constant that depends on* ν *only and* d *is the diameter of the domain* Ω.

Proof.

It is evident that

$$\|u - u(x_*)\|_{L^2(\Omega)} \le \|u - \langle u \rangle_\lambda\|_{L^2(\Omega)} + d^{\nu/2} |\langle u \rangle_\lambda - u(x_*)|, \tag{3.14}$$

where

$$\langle u \rangle_\lambda = \int_\Omega \lambda(y) \, u(y) \, \mathrm{d}y \, .$$

The structure of the polynomial $P_{m-1}(x, u)$ implies that

$$|\langle u \rangle_\lambda - u(x)| \le$$

$$\le |u(x) - P_{m_\nu - 1}(x, u)| + \sum_{1 \le |\alpha| \le m_\nu - 1} \frac{1}{\alpha!} \int_\Omega |\lambda(y)(x-y)^\alpha \, \mathscr{D}^\alpha u(y)| \, \mathrm{d}y \le$$

$$\le |u(x) - P_{m_\nu - 1}(x, u)| + \sum_{1 \le |\alpha| \le m_\nu - 1} \frac{d^{|\alpha|}}{\alpha!} \|\lambda\|_{L^2(\Omega)} \cdot \|\mathscr{D}^\alpha u\|_{L^2(\Omega)}$$

for all $x \in \Omega$. Therefore, estimate (3.13) follows from (3.14) and Lemmata 3.2 and 3.3. Lemma 3.4 is proved.

These lemmata enable us to estimate the completeness defect of two families of functionals that are important from the point of view of applications. We consider these families of functionals on the Sobolev spaces in the case when the domain is strongly Lipschitzian, i.e. the domain $\Omega \subset \mathbb{R}^\nu$ possesses the property: for every $x \in \partial\Omega$ there exists a vicinity U such that

$$U \cap \Omega = \{x = (x_1, \ldots, x_\nu) : \; x_\nu < f(x_1, \ldots, x_{\nu-1})\}$$

in some system of Cartesian coordinates, where $f(x)$ is a Lipschitzian function. For strongly Lipschitzian domains the space $H^s(\Omega)$ consists of restrictions to Ω of functions from $H^s(\mathbb{R}^\nu)$, $s > 0$ (see [5] or [6]).

Theorem 3.1.

Assume that a bounded strongly Lipschitzian domain Ω in $\mathbb{R}^\nu$ can be divided into subdomains $\{\Omega_j\colon\ j = 1,\ 2,\ \ldots,\ N\}$ such that

$$\overline{\Omega} = \bigcup_{j=1}^{N} \overline{\Omega}_j\,, \quad \Omega_i \cap \Omega_j = \varnothing \quad \textit{for} \quad i \neq j\,. \tag{3.15}$$

Here the bar stands for the closure of a set. Assume that $\lambda_j(x)$ is a function in $L^\infty(\Omega_j)$ such that

$$\operatorname{supp}\lambda_j \subset\subset \Omega_j\,, \qquad \int\limits_{\Omega_j} \lambda_j(x)\,\mathrm{d}x = 1 \tag{3.16}$$

and Ω_j is a star-like domain with respect to $\operatorname{supp}\lambda_j$. We define the set $\mathscr{L}$ of generalized local volume averages corresponding to the collection

$$\mathscr{T} = \{(\Omega_j;\ \lambda_j)\colon\ j = 1,\ 2,\ \ldots,\ N\}$$

as the family of functionals

$$\mathscr{L} = \left\{ l_j(u) = \int\limits_{\Omega_j} \lambda_j(x)\,u(x)\,\mathrm{d}x\,,\ \ j = 1,\ 2,\ \ldots,\ N \right\}. \tag{3.17}$$

Then the estimate

$$\varepsilon_{\mathscr{L}}(H^s(\Omega),\ H^\sigma(\Omega)) \ \le \begin{cases} C(\nu,\ s)(\Lambda\,d)^{s-\sigma}, & s > 1\,,\ \ 0 \le \sigma \le s\,, \\ C_\nu \Lambda^{1-\frac{\sigma}{s}} \cdot\, d^{\,s-\sigma}, & 0 \le \sigma \le s \le 1\,,\ \ s \neq 0\,, \end{cases} \tag{3.18}$$

holds, where $\Lambda = \max\left\{ d_j^\nu\, \|\lambda_j\|_{L^\infty(\Omega)}\colon\ j = 1,\ 2,\ \ldots,\ N \right\}$, $\quad d = \max\limits_j d_j$,

$$d_j \equiv \operatorname{diam}\Omega = \sup\{|x - y|\colon\ x,\, y \in \Omega_j\}\,,$$

$C(\nu,\ s)$ ***and*** C_ν ***are constants.***

Proof.

Let us define the interpolation operator $\mathscr{R}_{\mathscr{T}}$ for the collection $\mathscr{T}$ by the formula

$$(\mathscr{R}_{\mathscr{T}}u)(x) = l_j(u) = \int\limits_{\Omega_j} \lambda_j(x)\,u(x)\,\mathrm{d}x\,, \qquad x \in \Omega_j\,, \quad j = 1,\ 2,\ \ldots,\ N\,.$$

Chapter 5

It is easy to check that

$$\|u - \mathcal{R}_{\mathcal{T}} u\|_{L^2(\Omega)} \le C_\nu \Lambda \|u\|_{L^2(\Omega)}, \quad u \in L^2(\Omega).$$

It further follows from Lemma 3.2 that

$$\|u - l_j(u)\|_{L^2(\Omega_j)} \le \frac{\sigma_\nu}{\nu} d_j^{\nu+1} \|\lambda_j\|_{L^\infty(\Omega_j)} \|\nabla u\|_{L^2(\Omega_j)}.$$

This implies the estimate

$$\|u - \mathcal{R}_{\mathcal{T}} u\|_{L^2(\Omega)} \le \frac{\sigma_\nu}{\nu} d\Lambda \|u\|_{H^1(\Omega)}.$$

Using the fact (see [4, 6]) that

$$\|u\|_{H^s(\Omega)} \le C\|u\|_{L^2(\Omega)}^{1-s} \cdot \|u\|_{H^1(\Omega)}^{s}, \quad 0 \le s \le 1,$$

and the interpolation theorem for operators [4] we find that

$$\|u - R_T u\|_{L^2(\Omega)} \le (C_\nu \Lambda)^{1-s} \left(\frac{\sigma_\nu}{\nu} d\Lambda\right)^s \|u\|_{H^s(\Omega)}$$

for all $0 \le s \le 1$. Consequently, Theorem 2.3 gives the equation

$$\varepsilon_{\mathcal{L}}(H^s(\Omega);\ L^2(\Omega)) \le C_\nu \Lambda d^s, \quad 0 \le s \le 1. \tag{3.19}$$

Using the result of Exercise 2.12 and the interpolation inequalities (see, e.g., [4,6])

$$\|u\|_{H^{s(\theta)}(\Omega)} \le C\|u\|_{H^{s_1}(\Omega)}^{\theta} \cdot \|u\|_{H^{s_2}(\Omega)}^{1-\theta}, \quad s(\theta) = s_1\theta + s_2(1-\theta), \quad 0 \le \theta \le 1, \tag{3.20}$$

it is easy to obtain equation (3.18) from (3.19).

Let us illustrate this theorem by the following example.

Example *3.1*

Let $\Omega = (0,\ l)^\nu$ be a cube in $\mathbb{R}^\nu$ with the edge of the length l. We construct a collection $\mathcal{T}$ which defines local volume averages in the following way. Let $K = (0,\ 1)^\nu$ be the standard unit cube in $\mathbb{R}^\nu$ and let ω be a measurable set in K with the positive Lebesgue measure, $\operatorname{mes}\omega > 0$. We define the function $\lambda(\omega,\ x)$ on K by the formula

$$\lambda(\omega,\ x) = \begin{cases} [\operatorname{mes}\omega]^{-1}, & x \in \omega, \\ 0, & x \in K \backslash \omega. \end{cases}$$

Assume that

$$\Omega_j = h \cdot (j + K) \equiv \left\{ x = (x_1,\ \ldots,\ x_\nu):\ j_i < \frac{x_i}{h} < j_i + 1,\quad i = 1,\ 2,\ \ldots,\ \nu \right\},$$

$$\lambda_j(x) = \frac{1}{h^\nu} \lambda\left(\omega,\ \frac{x}{h} - j\right), \quad x \in \Omega_j,$$

for any multi-index $j = (j_1, \ldots, j_\nu)$, where $j_\nu = 0, 1, \ldots, N-1$, $h = l/N$. It is clear that the hypotheses of Theorem 3.1 are valid for the collection $\mathcal{T} = \{\Omega_j, \lambda_j\}$. Moreover,

$$d_j = \operatorname{diam} \Omega_j = \sqrt{\nu} \cdot h, \qquad \Lambda = \frac{\nu^{\nu/2}}{\operatorname{mes} \omega}$$

and hence in this case we have

$$\varepsilon_{\mathcal{L}}(H^s(\Omega);\ H^\sigma(\Omega)) \ \le\ C_{\nu,\,s}\left(\frac{h}{\operatorname{mes}\omega}\right)^{s-\sigma} \tag{3.21}$$

for the set $\mathcal{L}$ of functionals of the form (3.17) when $s \ge 1$ and $0 \le \sigma \le s$. It should be noted that in this case the number of functionals in the set $\mathcal{L}$ is equal to $\mathcal{N}_{\mathcal{L}} = N^\nu$. Thus, estimate (3.21) can be rewritten as

$$\varepsilon_{\mathcal{L}}(H^s(\Omega),\ H^\sigma(\Omega)) \ \le\ C_{\nu,\,s}\left(\frac{l}{\operatorname{mes}\omega}\right)^{s-\sigma} \cdot \left(\frac{1}{\mathcal{N}_{\mathcal{L}}}\right)^{\frac{s-\sigma}{\nu}}.$$

However, one can show (see, e.g., [6]) that the Kolmogorov $N_{\mathcal{L}}$-width of the embedding of $H^s(\Omega)$ into $H^\sigma(\Omega)$ has the same order in $\mathcal{N}_{\mathcal{L}}$, i.e.

$$\varkappa_{\mathcal{N}_{\mathcal{L}}}(H^s(\Omega),\ H^\sigma(\Omega)) \ =\ c_0\left(\frac{1}{N_{\mathcal{L}}}\right)^{\frac{s-\sigma}{\nu}}.$$

Thus, it follows from Theorem 2.4 that local volume averages have a completeness defect that is close (when the number of functionals is fixed) to the minimal. In the example under consideration this fact yields a double inequality

$$c_1 h^{s-\sigma} \ \le\ \varepsilon_L(H^s(\Omega),\ H^\sigma(\Omega)) \ \le\ c_2 h^{s-\sigma}, \qquad \sigma < s, \tag{3.22}$$

where c_1 and c_2 are positive constants that may depend on s, ν, ω, and Ω. Similar relations are valid for domains of a more general type.

Another important example of functionals is given in the following assertion.

Theorem 3.2.

Assume the hypotheses of Theorem 3.1. ***Let us choose a point*** x_j ***(called a node) in every set*** Ω_j ***and define a set of functionals on*** $H^m(\Omega)$, $m = [\nu/2] + 1$, ***by***

$$\mathcal{L} = \left\{ l_j(u) = u(x_j):\ x_j \in \Omega_j,\ j = 1, \ldots, N \right\}. \tag{3.23}$$

Then for all $s \ge m$ ***and*** $0 \le \sigma \le s$ ***the estimate***

$$\varepsilon_{\mathcal{L}}(H^s(\Omega);\ H^\sigma(\Omega)) \ \le\ C(\nu, s)(d\Lambda)^{s-\sigma} \tag{3.24}$$

is valid for the completeness defect of this set of functionals, where

$$d = \max\left\{d_j:\ j = 1,\ 2,\ \dots,\ N\right\}, \qquad d_j = \operatorname{diam}\Omega\,,$$

$$\Lambda = \max\left\{d_j^{\nu}\,\|\lambda_j\|_{L^\infty(\Omega)}:\ j = 1,\ 2,\ \dots,\ N\right\}.$$

Proof.

Let $u \in H^m(\Omega)$ and let $l_j(u) = u(x_j) = 0$, $j = 1,\ 2,\ \dots,\ N$. Then using (3.13) for $\Omega = \Omega_j$ and $x_* = x_j$ we obtain that

$$\|u\|_{L^2(\Omega_j)} \le C_\nu\Big(d_j^\nu\,\|\lambda_j\|_{L^\infty(\Omega_j)}\Big)\sum_{l=1}^{m} d_j^l\,\|u\|_{l,\,\Omega_j}\,,$$

where

$$\|u\|_{l,\,\Omega_j} = \sum_{|\alpha| = l}\frac{1}{\alpha!}\,\|\mathscr{D}^\alpha u\|_{L^2(\Omega_j)}\,.$$

It follows that

$$\|u\|_{L^2(\Omega)} \le C(\nu)\cdot\Lambda\cdot\sum_{l=1}^{m} d^l\,\|u\|_{H^l(\Omega)}$$

for all $u \in H^m(\Omega)$ such that $l_j(u) = u(x_j) = 0$, $j = 1,\ 2,\ \dots,\ N$. Using interpolation inequality (3.20) we find that

$$\|u\|_{L^2(\Omega)} \le C_\nu\cdot\Lambda\cdot\sum_{l=1}^{m} d^l\,\|u\|_{L^2(\Omega)}^{1-\frac{l}{m}}\cdot\|u\|_{H^m(\Omega)}^{l/m}\,. \tag{3.25}$$

By virtue of the inequality

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}\,, \qquad \frac{1}{p} + \frac{1}{q} = 1\,, \qquad x,\ y \ge 0\,,$$

we get

$$\Lambda\cdot d^l\cdot\|u\|_{L^2(\Omega)}^{1-\frac{l}{m}}\cdot\|u\|_{H^m(\Omega)}^{l/m} = \|u\|_{L^2}^{1-\frac{l}{m}}\cdot\left(d^m\Lambda^{\frac{m}{l}}\,\|u\|_{H^m}\right)^{\frac{l}{m}} \le$$

$$\le (1 - \frac{l}{m})\,\delta^{\frac{m}{m-l}}\,\|u\|_{L^2(\Omega)} + \frac{l}{m}\cdot\delta^{-\frac{m}{l}}\,d^m\Lambda^{\frac{m}{l}}\,\|u\|_{H^m(\Omega)}$$

for $l = 1,\ 2,\ \dots,\ m-1$ and for all $\delta > 0$. We substitute these inequalities in (3.25) to obtain that

$$\|u\|_{L^2(\Omega)} \le C_\nu\sum_{l=1}^{m-1}(1-\frac{l}{m})\,\delta^{\frac{m}{m-l}}\cdot\|u\|_{L^2(\Omega)} +$$

$$+\, C_\nu\left(\sum_{l=1}^{m-1}\frac{l}{m}\,\delta^{-\frac{m}{l}}\,\Lambda^{\frac{m}{l}} + \Lambda\right)d^m\|u\|_{H^m(\Omega)}\,. \tag{3.26}$$

We choose $\delta = \delta(\nu, m)$ such that

$$C_\nu \sum_{l=1}^{m-1} \left(1 - \frac{l}{m}\right) \delta^{\frac{m}{m-l}} \le \frac{1}{2} .$$

Then equation (3.26) gives us that

$$\|u\|_{L^2(\Omega)} \le C(\nu) \cdot \sum_{l=1}^{m} \frac{l}{m} \Lambda^{\frac{m}{l}} \cdot d^m \|u\|_{H^m(\Omega)}$$

for all $u \in H^m(\Omega)$ such that $u(x_j) = 0$, $j = 1, 2, \ldots, N$. Hence, the estimate

$$\varepsilon_{\mathscr{L}}(H^m(\Omega); L^2(\Omega)) \le C(\nu) d^m \cdot \sum_{l=1}^{m} \frac{l}{m} \Lambda^{\frac{m}{l}}$$

is valid. Since $\Lambda \ge 1$, this implies inequality (3.24) for $\sigma = 0$ and $s = m = [\nu/2] + 1$. As in Theorem 3.1 further arguments rely on Lemma 2.1 and interpolation inequalities (3.20). **Theorem 3.2 is proved**.

Example *3.2*

We return to the case described in Example 3.1. Let us choose nodes $x_j \in \Omega_j$ and assume that $\omega = K$. Then for a set $\mathscr{L}$ of functionals of the form (3.23) we have

$$\varepsilon_{\mathscr{L}}(H^s(\Omega); H^\sigma(\Omega)) \le C_{\nu, s} h^{s-\sigma}$$

for all $s \ge [\nu/2] + 1$, $0 \le \sigma \le s$ and for any location of the nodes x_j inside the Ω_j. In the case under consideration double estimate (3.22) is preserved.

In the exercises below several one-dimensional situations are given.

Exercise *3.2* Prove that

$$\varepsilon_{\mathscr{L}}(H^1(0, l); L^2(0, l)) = \varkappa_N(H^1(0, l); L^2(0, l)) = \left[1 + \left(\frac{\pi N}{l}\right)^2\right]^{-\frac{1}{2}}$$

for

$$\mathscr{L} = \left\{ l_j^c(u) = \int_0^l u(x) \cos\frac{j\pi}{l} x \, \mathrm{d}x, \; j = 0, 1, \ldots, N-1 \right\}.$$

Exercise *3.3* Verify that

$$\varepsilon_{\mathscr{L}}(H_0^1(0, l); L^2(0, l)) = \varkappa_{N-1}(H_0^1(0, l); L^2(0, l)) = \left[1 + \left(\frac{\pi N}{l}\right)^2\right]^{-\frac{1}{2}}$$

for

$$\mathscr{L} = \left\{ l_j^s(u) = \int_0^l u(x) \sin\frac{j\pi x}{l} \, \mathrm{d}x; \; j = 1, 2, \ldots, N-1 \right\}.$$

Exercise 3.4 Let

$$H^1_{\text{per}}(0,\, l) = \{u \in H^1(0,\, l):\ u(0) = u(l)\}.$$

Show that

$$\varepsilon_{\mathscr{L}}(H^1_{\text{per}}(0,\, l);\ L^2(0,\, l)) =$$

$$= \varkappa_{2N-1}(H^1_{\text{per}}(0,\, l);\ L^2(0,\, l)) = \left[1 + \left(\frac{2\pi N}{l}\right)^2\right]^{-\frac{1}{2}},$$

where $\mathscr{L}$ consists of functionals l^c_{2k} and l^s_{2k} for $k = 0, 1, \ldots, N-1$ (the functionals l^c_j and l^s_j are defined in Exercises 3.2 and 3.3).

Exercise 3.5 Consider the functionals

$$l_j(u) = \frac{1}{h}\int_0^h u(x_j + \tau)\, d\tau\,,$$

$$x_j = jh\,,\quad h = \frac{l}{N},\quad j = 0,\ 1,\ \ldots,\ N-1\,,$$

on the space $L^2(0,\, l)$. Assume that an interpolation operator R_h maps an element $u \in L^2(0,\, l)$ into a step-function equal to $l_j(u)$ on the segment $[x_j,\, x_{j+1}]$. Show that

$$\|u - R_h u\|_{L^2(0,\, l)} \le h\|u'\|_{L^2(0,\, l)}\,.$$

Prove the estimate

$$\frac{h}{\sqrt{\pi^2 + h^2}} \le \varepsilon_{\mathscr{L}}(H^1(0,\, l);\ L^2(0,\, l)) \le h\,.$$

Exercise 3.6 Consider a set $\mathscr{L}$ of functionals

$$l_j(u) = u(x_j)\,,\quad x_j = jh\,,\quad h = \frac{l}{N},\quad j = 0,\ 1,\ \ldots,\ N-1\,,$$

on the space $H^1(0,\, l)$. Assume that an interpolation operator R_h maps an element $u \in H^1(0,\, l)$ into a step-function equal to $l_j(u)$ on the segment $[x_j,\, x_{j+1}]$. Show that

$$\|u - R_h u\|_{L^2(0,\, l)} \le \frac{h}{\sqrt{2}}\|u'\|_{L^2(0,\, l)}\,.$$

Prove the estimate

$$\frac{h}{\sqrt{\pi^2 + h^2}} \le \varepsilon_{\mathscr{L}}(H^1(0,\, l);\ L^2(0,\, l)) \le \frac{h}{\sqrt{2}}\,.$$

§ 4 Determining Functionals for Abstract Semilinear Parabolic Equations

In this section we prove a number of assertions on the existence and properties of determining functionals for processes generated in some separable Hilbert space H by an equation of the form

$$\frac{du}{dt} + Au = B(u, t), \qquad t > 0, \quad u\big|_{t=0} = u_0. \tag{4.1}$$

Here A is a positive operator with discrete spectrum (for definition see Section 2.1) and $B(u, t)$ is a continuous mapping from $D(A^{1/2}) \times \mathbb{R}$ into H possessing the properties

$$\|B(0, t)\| \le M_0, \qquad \big\|B(u_1, t) - B(u_2, t)\big\| \le M(\rho)\big\|A^{1/2}(u_1 - u_2)\big\| \tag{4.2}$$

for all t and for all $u_j \in D(A^{1/2})$ such that $\big\|A^{1/2}u_j\big\| \le \rho$, where ρ is an arbitrary positive number, M_0 and $M(\rho)$ are positive numbers.

Assume that problem (4.1) is uniquely solvable in the class of functions

$$\mathscr{W} = C([0, +\infty); H) \cap C([0, +\infty); D(A^{1/2}))$$

and is pointwise dissipative, i.e. there exists $R > 0$ such that

$$\big\|A^{1/2}u(t)\big\| \le R \quad \text{when} \quad t \ge t_0(u) \tag{4.3}$$

for all $u(t) \in \mathscr{W}$. Examples of problems of the type (4.1) with the properties listed above can be found in Chapter 2, for example.

The results obtained in Sections 1 and 2 enable us to establish the following assertion.

Theorem 4.1.

For the set of linear functionals $\mathscr{L} = \{l_j : j = 1, 2, \ldots, N\}$ ***on the space*** $V = D(A^{1/2})$ ***with the norm*** $\|\cdot\|_V = \|A^{1/2} \cdot\|_H$ ***to be*** $(V, V; \mathscr{W})$***-asymptotically determining for problem*** (4.1) ***under conditions*** (4.2) ***and*** (4.3), ***it is sufficient that the completeness defect*** $\varepsilon_{\mathscr{L}}(V, H)$ ***satisfies the inequality***

$$\varepsilon_{\mathscr{L}} \equiv \varepsilon_{\mathscr{L}}(V, H) < M(R)^{-1}, \tag{4.4}$$

where $M(\rho)$ ***and*** R ***are the same as in*** (4.2) ***and*** (4.3).

Proof.

We consider two solutions $u_1(t)$ and $u_2(t)$ to problem (4.1) that lie in $\mathscr{W}$. By virtue of dissipativity property (4.3) we can suppose that

$$\big\|A^{1/2}u_j(t)\big\| \le R, \qquad t \ge 0, \quad j = 1, 2. \tag{4.5}$$

Let $u(t) = u_1(t) - u_2(t)$. If we consider $u(t)$ as a solution to the linear problem

$$\frac{du}{dt} + Au = f(t) \equiv B(u_1(t),\ t) - B(u_2(t),\ t),$$

then it is easy to find that

$$\frac{1}{2}\|u(t)\|^2 + \int_s^t (Au(\tau),\ u(\tau))\,d\tau \ \le\ \frac{1}{2}\|u(s)\|^2 + M(R)\int_s^t \|A^{1/2}u(\tau)\|\cdot\|u(\tau)\|\,d\tau$$

for all $t \ge s \ge 0$. We use (2.11) to obtain that

$$M(R)\cdot\|A^{1/2}u\|\cdot\|u\| \ \le\ \varepsilon_{\mathscr{L}}\cdot M(R)\cdot\|A^{1/2}u\|^2 + \delta\|A^{1/2}u\|^2 + C(R,\ \mathscr{L},\ \delta)[N(u)]^2$$

for any $\delta > 0$, where

$$N(u) = \max\{|l_j(u)|:\ j = 1,\ 2,\ \dots,\ N\}.$$

Therefore,

$$\|u(t)\|^2 + 2(1 - \delta - \varepsilon_{\mathscr{L}}M(R))\int_s^t \|A^{1/2}u(\tau)\|^2\,d\tau \ \le$$

$$\le\ \|u(s)\|^2 + C(R,\ \mathscr{L},\ \delta)\int_s^t [N(u(\tau))]^2\,d\tau\ . \tag{4.6}$$

Using (4.4) we can choose the parameter $\delta > 0$ such that $1 - \delta - \varepsilon_{\mathscr{L}}M(R) > 0$. Thus, we can apply Theorem 1.1 and find that under condition (4.4) equation

$$\lim_{t\to\infty}\int_t^{t+1}[N(u_1(\tau) - u_2(\tau))]^2\,d\tau = 0$$

implies the equality

$$\lim_{t\to\infty}\|u_1(t) - u_2(t)\| = 0\,. \tag{4.7}$$

In order to complete the proof of the theorem we should obtain

$$\lim_{t\to\infty}\|A^{1/2}(u_1(t) - u_2(t))\| = 0 \tag{4.8}$$

from (4.7). To prove (4.8) it should be first noted that

$$\lim_{t\to\infty}\|A^{\theta}(u_1(t) - u_2(t))\| = 0 \tag{4.9}$$

for any $0 \le \theta < 1/2$. Indeed, the interpolation inequality (see Exercise 2.1.12)

$$\|A^{\theta}u\| \ \le\ \|u\|^{1-2\theta}\cdot\|A^{1/2}u\|^{2\theta}, \qquad 0 \le \theta < 1/2\,,$$

and dissipativity property (4.5) enable us to obtain (4.9) from equation (4.7). Now we use the integral representation of a weak solution (see (2.2.3)) and the method applied in the proof of Lemma 2.4.1 to show (do it yourself) that

$$\|A^\beta u_j(t)\| \le C(R, M_0), \quad \frac{1}{2} < \beta < 1, \quad t \ge t_0 .$$

Therefore, using the interpolation inequality

$$\|A^{1/2}u\| \le \|A^{1/2-\delta}u\| \cdot \|A^{1/2+\delta}u\|, \qquad 0 < \delta < \frac{1}{2},$$

we obtain (4.8) from (4.9). **Theorem 4.1 is proved**.

Exercise *4.1* Show that if the hypotheses of Theorem 4.1 hold, then equation (4.9) is valid for all $0 \le \theta < 1$.

The reasonings in the proof of Theorem 4.1 lead us to the following assertion.

Corollary 4.1.

Assume that the hypotheses of Theorem 4.1 *hold. Then for any two weak (in $D(A^{1/2})$) solutions $u_1(t)$ and $u_2(t)$ to problem* (4.1) *that are bounded on the whole axis,*

$$\sup\left\{\|A^{1/2}u_i(t)\| : -\infty < t < \infty\right\} \le R, \qquad i = 1, 2, \tag{4.10}$$

the condition $l_j(u_1(t)) = l_j(u_2(t))$ for $l_j \in \mathcal{L}$ and $t \in \mathbb{R}$ implies that $u_1(t) \equiv u_2(t)$.

Proof.

In the situation considered equation (4.6) implies that

$$\|u(t)\|^2 + \beta_{\mathcal{L}} \int_s^t \|A^{1/2}u(\tau)\|^2 d\tau \le \|u(s)\|^2$$

for all $t > s$ and some $\beta_{\mathcal{L}} > 0$. It follows that

$$\|u(t)\| \le e^{-\beta_{\mathcal{L}}(t-s)}\|u(s)\|, \qquad t \ge s .$$

Therefore, if we tend $s \to -\infty$, then using (4.10) we obtain that $\|u(t)\| = 0$ for all $t \in \mathbb{R}$, i.e. $u_1(t) \equiv u_2(t)$.

It should be noted that Corollary 4.1 means that solutions to problem (4.1) that are bounded on the whole axis are uniquely determined by their values on the functionals l_j. It was this property of the functionals $\{l_j\}$ which was used by Ladyzhenskaya [2] to define the notion of determining modes for the two-dimensional Navier-Stokes system. We also note that a more general variant of Theorem 4.1 can be found in [3].

Exercise *4.2* Assume that problem (4.1) is autonomous, i.e. $B(u, t) \equiv B(u)$. Let $\mathcal{A}$ be a global attractor of the dynamical system (V, S_t) generated by weak (in $V = D(A^{1/2})$) solutions to problem (4.1) and assume that a set of functionals $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ possesses property (4.4). Then for any pair of trajectories $u_1(t)$ and $u_2(t)$ lying in the attractor $\mathscr{A}$ the condition $l_j(u_1(t)) = l_j(u_2(t))$ implies that $u_1(t) \equiv u_2(t)$ for all $t \in \mathbb{R}$ and $j = 1, 2, \ldots, N$.

Theorems 4.1 and 2.4 enable us to obtain conditions on the existence of N determining functionals.

Corollary 4.2.

Assume that the Kolmogorov N-width of the embedding of the space $V = D(A^{1/2})$ into H possesses the property $\varkappa_N(V; H) < M(R)^{-1}$. Then there exists a set of asymptotically $(V, V; \mathscr{W})$-determining functionals for problem (4.1) *consisting of N elements.*

Theorem 2.4, Corollary 2.1, and Exercise 2.6 imply that if the hypotheses of Theorem 4.1 hold, then the family of functionals $\mathscr{L}$ given by equation (2.13) is a $(V, V; \mathscr{W})$-determining set for problem (4.1), provided $\lambda_{N+1} > M(R)^2$. Here $M(R)$ and R are the constants from (4.2) and (4.3). It should be noted that the set $\mathscr{L}$ of the form (2.13) for problem (4.1) is often called a set of ***determining modes.*** Thus, Theorem 4.1 and Exercise 2.6 imply that semilinear parabolic equation (4.1) possesses a finite number of determining modes.

When condition (4.2) holds uniformly with respect to ρ, we can omit the requirement of dissipativity (4.3) in Theorem 4.1. Then the following assertion is valid.

Theorem 4.2.

Assume that a continuous mapping $B(u, t)$ ***from*** $D(A^{1/2}) \times \mathbb{R}$ ***into*** H ***possesses the properties***

$$\|B(0, t)\| \le M_0, \qquad \|B(u_1, t) - B(u_2, t)\| \le M\|A^{1/2}(u_1 - u_2)\| \tag{4.11}$$

for all $u_j \in D(A^{1/2})$. ***Then a set of linear functionals*** $\mathscr{L} = \{l_j : j = 1, 2, \ldots, N\}$ ***on*** $V = D(A^{1/2})$ ***is asymptotically*** $(V, V; \mathscr{W})$***-determining for problem*** (4.1)***, provided*** $\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}}(V, H) < M^{-1}$.

Proof.

If we reason as in the proof of Theorem 4.1, we obtain that if $\varepsilon_{\mathscr{L}} < M^{-1}$, then for an arbitrary pair of solutions $u_1(t)$ and $u_2(t)$ emanating from the points u_1 and u_2 at a moment s the equation (see (4.6))

$$\|u(t)\|^2 + \beta \int_s^t \|A^{1/2} u(\tau)\|^2 \, d\tau \le \|u(s)\|^2 + C \int_s^t [N(u(\tau))]^2 \, d\tau \tag{4.12}$$

is valid. Here $u(t) = u_1(t) - u_2(t)$, $\beta = \beta(\varepsilon_{\mathscr{L}}, M)$ and $C(\mathscr{L}, M)$ are positive constants, and

$$N(u) = \max\{|l_j(u)| : j = 1, 2, \ldots, N\}.$$

Therefore, using Theorem 1.1 we conclude that the condition

$$\lim_{t \to \infty} \int_t^{t+1} [N(u(\tau))]^2 \, d\tau = 0 \tag{4.13}$$

implies that

$$\lim_{t \to \infty} \left\{ \|u(t)\|^2 + \int_t^{t+1} \|A^{1/2} u(\tau)\|^2 \, d\tau \right\} = 0 . \tag{4.14}$$

Since $u(t) = u_1(t) - u_2(t)$ is a solution to the linear equation

$$\frac{du}{dt} + Au = f(t) \equiv B(u_1(t), t) - B(u_2(t), t), \tag{4.15}$$

it is easy to verify that

$$\|A^{1/2} u(t)\|^2 \le \|A^{1/2} u(s)\|^2 + \frac{1}{2} \int_s^t \|f(\tau)\|^2 \, d\tau \tag{4.16}$$

for $t \ge s$. It should be noted that equation (4.16) can be obtained with the help of formal multiplication of (4.15) by $\dot{u}(t)$ with subsequent integration. This conversion can be grounded using the Galerkin approximations. If we integrate equation (4.16) with respect to s from $t-1$ to t, then it is easy to see that

$$\|A^{1/2} u(t)\| \le \int_{t-1}^t \|A^{1/2} u(s)\|^2 \, ds + \frac{1}{2} \int_{t-1}^t \|f(\tau)\|^2 \, d\tau .$$

Using the structure of the function $f(t)$ and inequality (4.11), we obtain that

$$\|A^{1/2} u(t)\| \le \left(1 + \frac{M^2}{2}\right) \int_{t-1}^t \|A^{1/2} u(\tau)\|^2 \, d\tau .$$

Consequently, (4.14) gives us that

$$\lim_{t \to \infty} \left\|A^{1/2}(u_1(t) - u_2(t))\right\| = 0 .$$

Therefore, **Theorem 4.2 is proved**.

Further considerations in this section are related to the problems possessing inertial manifolds (see Chapter 3). In order to cover a wider class of problems, it is convenient to introduce the notion of a process.

Let H be a real reflexive Banach space. A two-parameter family $\{S(t, \tau);\ t \ge \tau;\ \ \tau, t \in \mathbb{R}\}$ of continuous mappings acting in H is said to be ***evolutionary***, if the following conditions hold:

(a) $S(t, s) \cdot S(s, \tau) = S(t, \tau), \quad t \ge s \ge \tau, \quad S(t, t) = I$.

(b) $S(t, s) u_0$ is a strongly continuous function of the variable t.

A pair $(H, S(t, \tau))$ with $S(t, \tau)$ being an evolutionary family in H is often called a ***process***. Therewith the space H is said to be a phase space and the family of mappings $S(t, \tau)$ is called an evolutionary operator. A curve

$$\gamma_s[u_0] = \{u(t) \equiv S(t+s, s)u_0 : \ t \geq 0\}$$

is said to be a trajectory of the process emanating from the point u_0 at the moment s.

It is evident that every dynamical system (H, S_t) is a process. However, main examples of processes are given by evolutionary equations of the form (1.1). Therewith the evolutionary operator is defined by the obvious formula

$$S(t, s)u_0 = u(t, s; \ u_0),$$

where $u(t) = u(t, s; \ u_0)$ is the solution to problem (1.1) with the initial condition u_0 at the moment s.

Exercise *4.3* Assume that the conditions of Section 2.2 and the hypotheses of Theorem 2.2.3 hold for problem (4.1). Show that weak solutions to problem (4.1) generate a process in H.

Similar to the definitions of Chapter 3 we will say that a process $(V; \ S(t, \tau))$ acting in a separable Hilbert space V possesses an asymptotically complete finite-dimensional inertial manifold $\{M_t\}$ if there exist a finite-dimensional orthoprojector P in the space V and a continuous function $\Phi(p, t)\colon \ PV \times \mathbb{R} \to (1-P)V$ such that

(a)
$$\|\Phi(p_1, t) - \Phi(p_2, t)\|_V \ \leq \ L\|p_1 - p_2\|_V \tag{4.17}$$

for all $p_j \in PV$, $t \in \mathbb{R}$, where L is a positive constant;

(b) the surface

$$M_t = \{p + \Phi(p, t)\colon \ p \in PV\} \subset V \tag{4.18}$$

is invariant: $S(t, \tau)M_\tau \subset M_t$;

(c) the condition of asymptotical completeness holds: for any $s \in \mathbb{R}$ and $u_0 \in V$ there exists $u_0^* \in M_s$ such that

$$\|S(t, s)u_0 - S(t, s)u_0^*\|_V \ \leq \ Ce^{-\gamma(t-s)}, \qquad t > s, \tag{4.19}$$

where C and γ are positive constants which may depend on u_0 and $s \in \mathbb{R}$.

Exercise *4.4* Show that for any two elements $u_1, \ u_2 \in M_s$, $s \in \mathbb{R}$ the following inequality holds

$$(1+L^2)^{-1/2}\|u_1 - u_2\|_V \ \leq \ \|P(u_1 - u_2)\|_V \ \leq \ \|u_1 - u_2\|_V.$$

Exercise *4.5* Using equation (4.19) prove that

$$\|(1-P)S(t, s)u_0 - \Phi(PS(t, s)u_0, \ t)\|_V \ \leq \ (1+L)\cdot Ce^{-\gamma(t-s)}$$

for $t \geq s$.

Exercise *4.6* Show that for any two trajectories $u_j(t) = S(t, s)u_j$, $j = 1, 2$, of the process $(V;\ S(t, \tau))$ the condition

$$\lim_{t \to \infty} \|P(u_1(t) - u_2(t))\|_V = 0$$

implies that

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_V = 0 .$$

In particular, the results of these exercises mean that M_s is homeomorphic to a subset in $\mathbb{R}^N$, $N = \dim P$, for every $s \in \mathbb{R}$. The corresponding homeomorphism $r\colon\ V \to \mathbb{R}^N$ can be defined by the equality $ru = \{(u, \phi_j)_V\}_{j=1}^N$, where $\{\phi_j\}$ is a basis in PV. Therewith the set of functionals $\{l_j(u) = (u, \varphi_j)_V\colon\ j = 1, \ldots, N\}$ appears to be asymptotically determining for the process. The following theorem contains a sufficient condition of the fact that a set of functionals $\{l_j\}$ possesses the properties mentioned above.

Theorem 4.3.

Assume that V ***and*** H ***are separable Hilbert spaces such that*** V ***is continuously and densely embedded into*** H. ***Let a process*** $(V;\ S(t, \tau))$ ***possess an asymptotically complete finite-dimensional inertial manifold*** $\{M_t\}$. ***Assume that the orthoprojector*** P ***from the definition of*** $\{M_t\}$ ***can be continuously extended to the mapping from*** H ***into*** V, ***i.e. there exists a constant*** $\Lambda = \Lambda(P) > 0$ ***such that***

$$\|Pv\|_V \le \Lambda \cdot \|v\|_H , \qquad v \in V. \tag{4.20}$$

If $\mathcal{L} = \{l_j\colon\ j = 1, \ldots, N\}$ ***is a set of linear functionals on*** V ***such that***

$$\varepsilon_{\mathcal{L}}(V;\ H) < (1 + L^2)^{-1/2}\Lambda^{-1} , \tag{4.21}$$

then the following conditions are valid:

1) ***there exist positive constants*** c_1 ***and*** c_2 ***depending on*** $\mathcal{L}$ ***such that***

$$c_1\|u_1 - u_2\|_H \le \max\{|l_j(u_1 - u_2)|\colon\ j = 1, \ldots, N\} \le c_2\|u_1 - u_2\|_H \tag{4.22}$$

for all $u_1, u_2 \in M_t$, $t \in \mathbb{R}$; ***i.e. the mapping*** $r_{\mathcal{L}}$ ***acting from*** V ***into*** $\mathbb{R}^N$ ***according to the formula*** $r_{\mathcal{L}}u = \{l_j(u)\}_{j=1}^N$ ***is a Lipschitzian homeomorphism from*** M_t ***into*** $\mathbb{R}^N$ ***for every*** $t \in \mathbb{R}$;

2) ***the set of functionals*** $\mathcal{L}$ ***is determining for the process*** $(V;\ S(t, \tau))$ ***in the sense that for any two trajectories*** $u_j(t) = S(t, s)u_j$ ***the condition***

$$\lim_{t \to \infty} (l_j(u_1(t)) - l_j(u_2(t))) = 0$$

implies that

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_V = 0 .$$

Proof.

Let $u_1,\ u_2 \in M_t$. Then

$$u_j = Pu_j + \Phi(Pu_j,\ t)\,, \qquad j = 1,\ 2\,.$$

Therewith equation (4.17) gives us that

$$\|u_1 - u_2\|_V \ \le\ (1+L^2)^{1/2}\,\|Pu_1 - Pu_2\|_V\,, \qquad u_j \in M_t\,. \tag{4.23}$$

Consequently, using Theorem 2.1 and inequality (4.20) we obtain that

$$\|u_1 - u_2\|_H \ \le\ C_{\mathscr{L}}\,N_{\mathscr{L}}(u_1 - u_2) + \varepsilon_{\mathscr{L}}(1+L^2)^{1/2}\cdot\Lambda\|u_1 - u_2\|_H\,,$$

where $N_{\mathscr{L}}(u) = \max\{|l_j(u)|:\ j = 1,\ \ldots,\ N\}$ and $\varepsilon_{\mathscr{L}} = \varepsilon_{\mathscr{L}}(V;\ H)$. Therefore, equation (4.21) implies that

$$\|u_1 - u_2\|_H \ \le\ C_1(\mathscr{L})\cdot N_{\mathscr{L}}(u_1 - u_2)\,, \qquad \text{where}\quad C_1(\mathscr{L}) = C_{\mathscr{L}}\cdot\left(1 - \varepsilon_{\mathscr{L}}(1+L^2)^{1/2}\Lambda\right)^{-1}. \tag{4.24}$$

On the other hand, (4.20) and (4.23) give us that

$$|l_j(u_1 - u_2)| \ \le\ C_{\mathscr{L}}\|u_1 - u_2\|_V \ \le\ C_{\mathscr{L}}(1+L^2)^{1/2}\,\Lambda\|u_1 - u_2\|_H\,. \tag{4.25}$$

Equations (4.24) and (4.25) imply estimate (4.22). Hence, assertion 1 of the theorem is proved.

Let us prove the second assertion of the theorem. Let $u_j(t) = S(t,\ s)u_j$, $t \ge s$, be trajectories of the process. Since

$$u_j(t) = (Pu_j(t) + \Phi(Pu_j(t),\ t)) + ((1-P)u_j(t) - \Phi(Pu_j(t),\ t))\,,$$

using (4.17) it is easy to find that

$$\|u_1(t) - u_2(t)\|_V \ \le\ (1+L^2)^{1/2}\,\|P(u_1(t) - u_2(t))\|_V + \sum_{j=1,\ 2}\|(1-P)u_j(t) - \Phi(Pu_j(t),\ t)\|_V\,.$$

The property of asymptotical completeness (4.19) implies (see Exercise 4.5) that

$$\|(1-P)u_j(t) - \Phi(Pu_j(t),\ t)\| \ \le\ Ce^{-\gamma(t-s)}\,, \qquad t \ge s\,.$$

Therefore, equation (4.20) gives us the estimate

$$\|u_1(t) - u_2(t)\|_V \ \le\ (1+L^2)^{1/2}\,\Lambda\|u_1(t) - u_2(t)\|_H + Ce^{-\gamma(t-s)}\,. \tag{4.26}$$

It follows from Theorem 2.1 that

$$\|u_1(t) - u_2(t)\|_H \ \le\ C_{\mathscr{L}}N_{\mathscr{L}}(u_1(t) - u_2(t)) + \varepsilon_{\mathscr{L}}\|u_1(t) - u_2(t)\|_V\,.$$

Therefore, provided (4.21) holds, equation (4.26) implies that

$$\|u_1(t) - u_2(t)\|_V \ \le\ A_{\mathscr{L}}\cdot N_{\mathscr{L}}(u_1(t) - u_2(t)) + B_{\mathscr{L}}\,e^{-\gamma(t-s)}\,, \qquad t \ge s\,,$$

where $A_{\mathscr{L}}$ and $B_{\mathscr{L}}$ are positive numbers. Hence, the condition

$$\lim_{t \to \infty} N_{\mathscr{L}}(u_1(t) - u_2(t)) = 0$$

implies that

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_V = 0\,.$$

Thus, **Theorem 4.3 is proved**.

Exercise *4.7* Assume that the hypotheses of Theorem 4.3 hold. Let $u_1,\ u_2 \in M_s$ be such that $l_j(u_1) = l_j(u_2)$ for all $l_j \in \mathscr{L}$. Show that

$$S(t,\ s)u_1 \equiv S(t,\ s)u_2 \quad \text{for } t \geq s\,.$$

Exercise *4.8* Prove that if the hypotheses of Theorem 4.3 hold, then inequality (4.22) as well as the equation

$$c_1\|u_1 - u_2\|_V \ \leq\ \max\left\{\left|l_j(u_1 - u_2)\right|:\ j = 1,\ 2,\ \ldots,\ N\right\} \ \leq\ c_2\|u_1 - u_2\|_V$$

is valid for any $u_1,\ u_2 \in M_t$ and $t \in \mathbb{R}$, where $c_1,\ c_2 > 0$ are constants depending on $\mathscr{L}$.

Let us return to problem (4.1). Assume that $B(u,\ t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H, $0 \leq \theta < 1$, possessing the properties

$$\|B(u_0,\ t)\| \ \leq\ M\left(1 + \left\|A^\theta u_0\right\|\right), \qquad \|B(u_1,\ t) - B(u_2,\ t)\| \ \leq\ M\left\|A^\theta(u_1 - u_2)\right\|$$

for all $u_j \in D(A^\theta)$, $0 \leq \theta < 1$. Assume that the spectral gap condition

$$\lambda_{n+1} - \lambda_n \ \geq\ \frac{2M}{q}\left((1+k)\,\lambda_{n+1}^\theta + \lambda_n^\theta\right)$$

holds for some n and $0 < q < 2 - \sqrt{2}$. Here $\{\lambda_n\}$ are the eigenvalues of the operator A indexed in the increasing order and k is a constant defined by (3.1.7). Under these conditions there exists (see Chapter 2) a process $(D(A^\theta);\ S(t,\ s))$ generated by problem (4.1). By virtue of Theorems 3.2.1 and 3.3.1 this process possesses an asymptotically complete finite-dimensional inertial manifold $\{M_t\}$ and the corresponding orthoprojector P is a projector onto the span of the first n eigenvectors of the operator A. Therefore,

$$\|A^\theta P u\| \leq \lambda_n^{\theta - s}\|A^s u\|\,, \qquad -\infty < s < \theta\,.$$

Therewith the Lipschitz constant L for $\Phi(p,\ t)$ can be estimated by the value $q/(1-q)$. Thus, if $\mathscr{L}$ is a set of functionals on $V = D(A^\theta)$, then in order to apply Theorem 4.3 with $H = D(A^s)$, $-\infty < s < \theta$, it is sufficient to require that

$$\varepsilon_{\mathscr{L}}(D(A^\theta);\ D(A^s)) \ \leq\ \frac{1-q}{\sqrt{2 - 2q + 2q^2}} \cdot \frac{1}{\lambda_n^{\theta - s}}\ .$$

Due to Theorem 2.4 this estimate can be rewritten as follows:

$$\varepsilon_{\mathcal{L}}(D(A^\theta);\ D(A^s)) \le \frac{1-q}{\sqrt{2-2q+2q^2}} \cdot \left(\frac{\lambda_{n+1}}{\lambda_n}\right)^{\theta-s} \cdot \varkappa_n(D(A^\theta);\ D(A^s)),$$

where $\varkappa_n(D(A^\theta);\ D(A^s))$ is the Kolmogorov n-width of the embedding of $D(A^\theta)$ into $D(A^s)$, $-\infty < s < \theta$, $0 \le \theta < 1$.

It should be noted that the assertion similar to Theorem 4.3 was first established for the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + u_x u = 0, \qquad x \in (0, L), \qquad t > 0,$$

with the periodic boundary conditions on $[0, L]$ in the case when $\{l_j\}$ is a set of uniformly distributed nodes on $[0, L]$, i.e.

$$l_j(u) = u(jh), \quad \text{where} \quad h = \frac{L}{N}, \quad j = 0, 1, \ldots, N-1.$$

For the references and discussion of general case see survey [3].

In conclusion of this section we give one more theorem on the existence of determining functionals for problem (4.1). The theorem shows that in some cases we can require that the values of functionals on the difference of two solutions tend to zero only on a sequence of moments of time (cf. Theorem 1.3).

Theorem 4.4.

As before, assume that A is a positive operator with discrete spectrum:

$$Ae_k = \lambda_k e_k, \quad 0 < \lambda_1 \le \lambda_2 \le \ldots, \quad \lim_{k\to\infty} \lambda_k \to \infty.$$

Assume that $B(u, t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H for some $0 < \theta < 1$ and the estimate

$$\|B(u_1, t) - B(u_2, t)\| \le M\|A^\theta(u_1 - u_2)\|, \qquad u_1, u_2 \in D(A^\theta),$$

holds. Let $\mathcal{L} = \{l_j\}$ be a finite set of linear functionals on $D(A^\theta)$. Then for any $0 < \alpha < \beta$ there exists $n = n(\alpha, \beta, \theta, \lambda_1, M)$ such that the condition

$$\varepsilon_{\mathcal{L}} \equiv \varepsilon_{\mathcal{L}}(D(A^\theta), H) < \frac{1}{2}\lambda_n^{-\theta}$$

implies that the set of functionals $\mathcal{L}$ is determining for problem (4.1) in the sense that if for some pair of solutions $\{u_1(t);\ u_2(t)\}$ and for some sequence $\{t_k\}$ such that

$$\lim_{k\to\infty} t_k = \infty, \qquad \alpha \le t_{k+1} - t_k \le \beta, \qquad k \ge 1,$$

the condition

$$\lim_{k\to\infty} l_j(u_1(t_k) - u_2(t_k)) = 0, \qquad l_j \in \mathcal{L},$$

holds, then

$$\lim_{t\to\infty} \|A^\theta(u_1(t) - u_2(t))\| = 0. \tag{4.27}$$

Proof.

Let $u(t) = u_1(t) - u_2(t)$. Then the results of Chapter 2 (see Theorem 2.2.3 and Exercise 2.2.7) imply that

$$\|A^\theta u(t)\| \le a_1 e^{a_2(t-s)} \|A^\theta u(s)\|, \tag{4.28}$$

$$\|(1-P_n)A^\theta u(t)\| \le \left\{ e^{-\lambda_{n+1}(t-s)} + \frac{a_3}{\lambda_{n+1}^{1-\theta}} e^{a_2(t-s)} \right\} \|A^\theta u(s)\| \tag{4.29}$$

for $t \ge s \ge 0$, where a_1, a_2, and a_3 are positive numbers depending on θ, λ_1, and M and P_n is the orthoprojector onto $\mathrm{Lin}\{e_1, \dots, e_n\}$. It follows form (4.29) that

$$\|(1-P_n)A^\theta u(t)\| \le q_{\alpha, \beta} \|A^\theta u(s)\|, \quad s+\alpha \le t \le s+\beta, \tag{4.30}$$

where

$$q_{\alpha, \beta} = e^{-\lambda_{n+1}\alpha} + \frac{a_3}{\lambda_{n+1}^{1-\theta}} e^{a_2\beta}.$$

Let us choose $n = n(\alpha, \beta, \theta, \lambda_1, M)$ such that $q_{\alpha, \beta} \le 1/2$. Then equation (4.30) gives us that

$$\|A^\theta u(t)\| \le \lambda_n^\theta \|u(t)\| + \frac{1}{2}\|A^\theta u(s)\|, \quad s+\alpha \le t \le s+\beta.$$

This inequality as well as estimate (4.28) enables us to use Theorem 1.3 with $V = D(A^\theta)$ and **to complete the proof of Theorem 4.4**.

Exercise *4.9* Assume that n is chosen such that $q_{\alpha, \beta} < 1$ in the proof of Theorem 4.4. Show that the condition $\|P_n(u_1(t_k) - u_2(t_k))\| \to 0$ as $k \to \infty$ implies (4.27).

The results presented in this section can also be proved for semilinear retarded equations. For example, we can consider a retarded perturbation of problem (4.1) of the following form

$$\begin{cases} \dfrac{du}{dt} + Au = B(u, t) + Q(u_t, t), \\ u\big|_{t \in [-r, 0]} = \varphi(t) \in C_r \equiv C([-r, 0], D(A^{1/2})), \end{cases}$$

where, as usual (see Section 2.8), u_t is an element of C_r defined with the help of $u(t)$ by the equality $u_t(\theta) = u(t+\theta)$, $\theta \in [-r, 0]$, and Q is a continuous mapping from $C_r \times \mathbb{R}$ into H possessing the property

$$\|Q(v_1, t) - Q(v_2, t)\|^2 \le M_1 \cdot \int_{-r}^{0} \|A^{1/2}(v_1(\theta) - v_2(\theta))\|^2 \, d\theta$$

for any $v_1, v_2 \in C_r$. The corresponding scheme of reasoning is similar to the method used in [3], where the second order in time retarded equations are considered.

§ 5 Determining Functionals for Reaction-Diffusion Systems

In this section we consider systems of parabolic equations of the reaction-diffusion type and find conditions under which a finite set of linear functionals given on the phase space uniquely determines the asymptotic behaviour of solutions. In particular, the results obtained enable us to prove the existence of finite collections of determining modes, nodes, and local volume averages for the class of systems under consideration. It also appears that in some cases determining functionals can be given only on a part of components of the state vector. As an example, we consider a system of equations which describes the Belousov-Zhabotinsky reaction and the Navier-Stokes equations.

Assume that Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \ge 1$, $H^s(\Omega)$ is the Sobolev space of the order s on Ω, and $H^s_0(\Omega)$ is the closure (in $H^s(\Omega)$) of the set of infinitely differentiable functions with compact support in Ω. Let $\|\cdot\|_s$ be a norm in $H^s(\Omega)$ and let $\|\cdot\|$ and $(\cdot, \cdot)$ be a norm and an inner product in $L^2(\Omega)$, respectively. Further we also use the spaces

$$\boldsymbol{H}^s = (H^s(\Omega))^m \equiv H^s(\Omega) \times \ldots \times H^s(\Omega), \qquad m \ge 1 .$$

Notations $\boldsymbol{L}^2$ and $\boldsymbol{H}^s_0$ have a similar meaning. We denote the norms and the inner products in $\boldsymbol{L}^2$ and $\boldsymbol{H}^s$ as in $L^2(\Omega)$ and $H^s(\Omega)$.

We consider the following system of equations

$$\partial_t u = a(x, t)\Delta u - f(x, u, \nabla u;\ t), \qquad x \in \Omega, \qquad t > 0, \tag{5.1}$$

$$u\big|_{\partial\Omega} = 0, \qquad u(x,\ 0) = u_0(x),$$

as the main model. Here $u(x,\ t) = (u_1(x,\ t);\ \ldots;\ u_m(x,\ t))$, Δ is the Laplace operator, $\nabla u_k = (\partial_{x_1} u_k, \ldots, \partial_{x_n} u_k)$, and $a(x,\ t)$ is an m-by-m matrix with the elements from $L^\infty(\overline{\Omega} \times \mathbb{R}_+)$ such that for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}_+$

$$a_+(x,\ t) \equiv \frac{1}{2}\cdot(a + a^*) \ge \mu_0 \cdot I, \qquad \mu_0 > 0 . \tag{5.2}$$

We also assume that the continuous function

$$f = (f_1;\ \ldots;\ f_m):\quad \overline{\Omega} \times \mathbb{R}^{(n+1)m} \times \mathbb{R}_+ \to \mathbb{R}^m$$

is such that problem (5.1) has solutions which belong to a class $\mathscr{W}$ of functions on $\mathbb{R}_+$ with the following properties:

a) for any $u \in \mathscr{W}$ there exists $t_0 > 0$ such that

$$u(t) \in C(t_0,\ +\infty;\ \boldsymbol{H}^2 \cap \boldsymbol{H}^1_0), \qquad \partial_t u(t) \in C(t_0,\ +\infty;\ \boldsymbol{L}^2), \tag{5.3}$$

where $C(a, b;\ X)$ is the space of strongly continuous functions on $[a,\ b]$ with the values in X;

b) there exists a constant $k > 0$ such that for any $u, v \in \mathscr{W}$ there exists $t_1 > 0$ such that for $t > t_1$ we have

$$\|f(u, \nabla u;\ t) - f(v, \nabla v;\ t)\| \ \le\ k \cdot \Big(\|u - v\| + \|\nabla u - \nabla v\|\Big). \tag{5.4}$$

It should be noted that if $a(x, t)$ is a diagonal matrix with the elements from $C^2(\overline{\Omega} \times \mathbb{R}_+)$ and f is a continuously differentiable mapping such that

$$|f(x, u, p;\ t) - f(x, v, q;\ t)| \ \le\ k \cdot (|u - v| + |p - q|) \tag{5.5}$$

for all $u, v \in \mathbb{R}^m$, $p, q \in \mathbb{R}^{nm}$, $x \in \overline{\Omega}$, and $t \in \mathbb{R}_+$, then under natural compatibility conditions problem (5.1) has a unique classical solution [7] which evidently possesses properties (5.3) and (5.4). In cases when the dynamical system generated by equations (5.1) is dissipative, the global Lipschitz condition (5.5) can be weakened. For example (see [8]), if a is a constant matrix and

$$f(x, u, \nabla u;\ t) = \bar{f}(x, u) + \sum_{j=1}^{n} b_j(x)\partial_{x_j} u + g(x)\,,$$

where $b_j = (b_j^1, \ldots, b_j^m) \in \boldsymbol{L}^\infty$, $g = (g_1, \ldots, g_m) \in \boldsymbol{L}^2$, and $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_m)$ is continuously differentiable and satisfies the conditions

$$\bar{f}(x, u)u \ \ge\ \mu_1 |u|^{p_0}, \qquad |\bar{f}(x, u)| \ \le\ \mu_2 |u|^{p_0 - 1} + C\,, \qquad p_0 > 2\,,$$

$$\left|\frac{\partial \bar{f}_k}{\partial u_j}\right| \ \le\ C \cdot (1 + |u|^{p_1})\,, \qquad 1 \le k \le m\,, \qquad 1 \le j \le n\,,$$

where $\mu_1, \mu_2 > 0$ and $p_1 < \min(4/n,\ 2/(n-2))$ for $n > 2$, then any solution to problem (5.1) with the initial condition from $\boldsymbol{L}^2$ is unique and possesses properties (5.3) and (5.4).

Let us formulate our main assertion.

Theorem 5.1.

A set $\mathscr{L} = \{l_j\colon\ j = 1, \ldots, N\}$ ***of linearly independent continuous linear functionals on*** $\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ ***is an asymptotically determining set with respect to the space*** $\boldsymbol{H}_0^1$ ***for problem*** (5.1) ***in the class*** $\mathscr{W}$ ***if***

$$\varepsilon_{\mathscr{L}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1,\ L^2) \ <\ \frac{c_0\,\mu_0}{k\sqrt{2}} \cdot \left(1 + \frac{27}{4} \cdot \left(\frac{k}{\mu_0}\right)^2\right)^{-1/2} \equiv p(k,\ \mu_0)\,, \tag{5.6}$$

where $c_0^{-1} = \sup\{\|w\|_2\colon\ w \in (\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1)(\Omega),\ \|\Delta w\| \le 1\}$***, and*** μ_0 ***and*** k ***are constants from*** (5.2) ***and*** (5.4)***. This means that if inequality*** (5.6) ***holds, then for some pair of solutions*** $u, v \in \mathscr{W}$ ***the equation***

$$\lim_{t \to \infty} \int_t^{t+1} |l_j(u(\tau)) - l_j(v(\tau))|^2\, d\tau = 0, \qquad j = 1, \ldots, N, \tag{5.7}$$

implies their asymptotic closeness in the space H^1:

$$\lim_{t \to \infty} \|u(t) - v(t)\|_1 = 0. \tag{5.8}$$

Proof.

Let $u, v \in \mathscr{W}$. Then equation (5.1) for $w = u - v$ gives us that

$$\partial_t w = a(x, t)\Delta w - (f(x, u, \nabla u;\ t) - f(x, v, \nabla v;\ t)).$$

If we multiply this by Δw in L^2 scalarwise and use equation (5.4), then we find that

$$\frac{1}{2}\cdot\frac{d}{dt}\|\Delta w(t)\|^2 + \mu_0\|\Delta w(t)\|^2 \le k\Big(\|w(t)\| + \|(\nabla w)(t)\|\Big)\cdot\|\Delta w(t)\|$$

for $t > 0$ large enough. Therefore, the inequality $\|\nabla w\|^2 = |(w,\ \Delta w)| \le \|w\|\cdot\|\Delta w\|$ enables us to obtain the estimate

$$\frac{d}{dt}\|\nabla w\|^2 + \mu_0\|\Delta w\|^2 \le \frac{2k^2}{\mu_0}\left(1 + \frac{27}{4}\left(\frac{k}{\mu_0}\right)^2\right)\|w\|^2. \tag{5.9}$$

Theorem 2.1 implies that

$$\|w\|^2 \le C(N,\ \delta)\max_j|l_j(w)|^2 + (1+\delta)\cdot\varepsilon_{\mathscr{L}}^2\cdot\|w\|_2^2 \tag{5.10}$$

for all $w \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1$ and for any $\delta > 0$, where $C(N,\ \delta) > 0$ is a constant and $\varepsilon_{\mathscr{L}} \equiv \equiv \varepsilon_{\mathscr{L}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1,\ L^2)$. Consequently, estimate (5.9) gives us that

$$\frac{d}{dt}\|\nabla w(t)\|^2 + \mu_0\cdot\left(1 - \frac{(1+\delta)\varepsilon_{\mathscr{L}}^2}{p(k,\ \mu_0)^2}\right)\cdot\|\Delta w(t)\|^2 \le C\cdot\max_j|l_j(w)|^2,$$

where $p(k,\ \mu_0)$ is defined by equation (5.6). It follows that if estimate (5.6) is valid, then there exists $\beta > 0$ such that

$$\|\nabla w(t)\|^2 \le e^{-\beta(t-t_0)}\|\nabla w(t_0)\|^2 + C\cdot\int_{t_0}^{t} e^{-\beta(t-\tau)}\max_j|l_j(w(\tau))|^2\,d\tau$$

for all $t \ge t_0$, where t_0 is large enough. Therefore, equation (5.7) implies (5.8). Thus, **Theorem 5.1 is proved**.

Exercise *5.1* Assume that $u(t)$ and $v(t)$ are two solutions to equation (5.1) defined for all $t \in \mathbb{R}$. Let (5.3) and (5.4) hold for every $t_0 \in \mathbb{R}$ and let

$$\sup_{t<0}\Big(\|\nabla u(t)\| + \|\nabla v(t)\|\Big) < \infty.$$

Prove that if the hypotheses of Theorem 5.1 hold, then equalities $l_j(u(t)) = l_j(v(t))$ for $j = 1, \ldots, N$ and $t < s$ for some $s \le \infty$ imply that $u(t) \equiv v(t)$ for all $t < s$.

Let us give several examples of sets of determining functionals for problem (5.1).

Example *5.1* (determining modes, $m \ge 1$)

Let $\{e_k\}$ be eigenelements of the operator $-\Delta$ in $\boldsymbol{L}^2$ with the Dirichlet boundary conditions on $\partial\Omega$ and let $0 < \lambda_1 \le \lambda_2 \le \dots$ be the corresponding eigenvalues. Then the completeness defect of the set

$$\mathscr{L} = \left\{ l_j : \quad l_j(w) = \int_\Omega w(x) \cdot e_j(x)\, \mathrm{d}x\,, \; j = 1, \dots, N \right\}$$

can be easily estimated as follows: $\varepsilon_{\mathscr{L}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1, \boldsymbol{L}^2) \le \sqrt{n} \cdot \lambda_{N+1}^{-1}$ (see Exercise 2.6). Thus, if N possesses the property $\lambda_{N+1} > \sqrt{n} \cdot p(k, \mu_0)^{-1}$, then $\mathscr{L}$ is a set of asymptotically $(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1, \boldsymbol{H}_0^1, \mathscr{W})$-determining functionals for problem (5.1).

Considerations of Section 5.3 also enable us to give the following examples.

Example *5.2* (determining generalized local volume averages)

Assume that the domain Ω is divided into local Lipschitzian subdomains $\{\Omega_j : j = 1, \dots, N\}$, with diameters not exceeding some given number $h > 0$. Assume that on every domain Ω_j a function $\lambda_j(x) \in L^\infty(\Omega_j)$ is given such that the domain Ω_j is star-like with respect to $\operatorname{supp} \lambda_j$ and the conditions

$$\int_{\Omega_j} \lambda_j(x)\, \mathrm{d}x = 1\,, \qquad \operatorname*{ess\,sup}_{x \in \Omega_j} |\lambda_j(x)| \le \frac{\Lambda}{h^n}\,,$$

hold, where the constant $\Lambda > 0$ does not depend on h and j. Theorem 3.1 implies that

$$\varepsilon_{\mathscr{L}_h}(H^2(\Omega) \cap H_0^1(\Omega);\; L^2(\Omega)) \le c_n h^2 \cdot \Lambda^2$$

for the set of functionals

$$\mathscr{L}_h = \left\{ l_j(u) = \int_{\Omega_j} \lambda_j(x)\, u(x)\, \mathrm{d}x\,, \; j = 1, 2, \dots, N \right\}.$$

Therewith the reasonings in the proof of Theorem 3.1 imply that $c_n = \sigma_n^2\, n^{-2}$, where σ_n is the area of the unit sphere in $\mathbb{R}^n$. For every $l_j \in \mathscr{L}_h$ we define the functionals $l_j^{(k)}$ on $\boldsymbol{H}^2$ by the formula

$$l_j^{(k)}(w) = l_j(w_k)\,, \qquad w = (w_1, \dots, w_m) \in \boldsymbol{H}^2\,, \qquad k = 1, 2, \dots, m\,. \tag{5.11}$$

Let

$$\mathscr{L}_h^{(m)} = \{ l_j^{(k)}(u) :\; k = 1, 2, \dots, m\,,\;\; l_j \in \mathscr{L}_h \}\,. \tag{5.12}$$

One can check (see Exercise 2.8) that

$$\varepsilon_{\mathscr{L}_h^{(m)}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1;\ \boldsymbol{L}^2) \le c_m \cdot h^2 \cdot \Lambda^2 .$$

Therefore, if h is small enough, then $\mathscr{L}_h^{(m)}$ is a set of asymptotically determining functionals for problem (5.1).

Example 5.3 (determining nodes, $n \le 3$)

Let Ω be a convex smooth domain in $\mathbb{R}^n$, $n \le 3$. Let $h > 0$ and let

$$\Omega_j = \Omega \cap \{x = (x_1, \dots, x_n):\ j_k h < x_k < (j_k + 1)h,\ k = 1, 2, 3\},$$

where $j = (j_1, \dots, j_n) \in \mathbb{Z}^n \cap \Omega$. Let us choose a point x_j in every subdomain Ω_j and define the set of nodes

$$\mathscr{L}_h = \{l_j(u) = u(x_j):\ j \in \mathbb{Z}^n \cap \Omega\}, \qquad n \le 3. \tag{5.13}$$

Theorem 3.2 enables us to state that

$$\varepsilon_{\mathscr{L}_h}(H^2(\Omega) \cap H_0^1(\Omega);\ L^2(\Omega)) \le c \cdot h^2 ,$$

where c is an absolute constant. Therefore, the set of functionals $\mathscr{L}_h^{(m)}$ defined by formulae (5.11) and (5.12) with $\mathscr{L}_h$ given by equality (5.13) possesses the property

$$\varepsilon_{\mathscr{L}_h^{(m)}}(\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1;\ \boldsymbol{L}^2) \le c \cdot h^2 .$$

Consequently, $\mathscr{L}_h^{(m)}$ is a set of asymptotically determining functionals for problem (5.1) in the class $\mathscr{W}$, provided that h is small enough.

It is also clear that the result of Exercise 2.8 enables us to construct mixed determining functionals: they are determining nodes or local volume averages depending on the components of a state vector. Other variants are also possible.

However, it is possible that not all the components of the solution vector $u(x, t)$ appear to be essential for the asymptotic behaviour to be uniquely determined. A theorem below shows when this situation can occur.

Let I be a subset of $\{1, \dots, m\}$. Let us introduce the spaces

$$\boldsymbol{H}_I^s = \{w = (w_1, \dots, w_m) \in \boldsymbol{H}^s:\ w_k \equiv 0, k \notin I\}, \qquad s \in \mathbb{R}.$$

We identify these spaces with $(H^s(\Omega))^{|I|}$, where $|I|$ is the number of elements of the set I. Notations $\boldsymbol{L}_I^2$ and $\boldsymbol{H}_{0,I}^s$ have the similar meaning. The set $\mathscr{L}$ of linear functionals on $\boldsymbol{H}_I^2$ is said to be determining if $p_I^* \mathscr{L}$ is determining, where p_I is the natural projection of $\boldsymbol{H}^s$ onto $\boldsymbol{H}_I^s$.

Theorem 5.2.

Let $a = \mathrm{diag}(d_1, \ldots, d_m)$ *be a diagonal matrix with constant elements and let* $\{I, I'\}$ *be a partition of the set* $\{1, \ldots, m\}$ *into two disjoint subsets. Assume that there exist positive constants* ω, k^*, θ_i, *where* $i = 1, \ldots, m$, *such that for any pair of solutions* $u, v \in \mathcal{W}$ *the following inequality holds (hereinafter* $w = u - v$*):*

$$\sum_{i \in I} \theta_i \left\{ -\frac{d_i}{2} \|\Delta w_i\|^2 + (f_i(u, \nabla u;\ t) - f_i(v, \nabla v;\ t),\ \Delta w_i) \right\} +$$

$$+ \sum_{i \in I'} \theta_i \left\{ -d_i \|\nabla w_i\|^2 - (f_i(u, \nabla u;\ t) - f_i(v, \nabla v;\ t),\ w_i) \right\} \le$$

$$\le -\omega \sum_{i \in I'} \|w_i\|^2 + k^* \sum_{i \in I} \|w_i\|^2 . \qquad (5.14)$$

Then a set $= \{l_j:\ j = 340(1), \ldots,$ *of linearly independent continuous linear functionals on* $H_I^2 \cap H_{0,I}^1$ *is an asymptotically determining set with respect to the space* H_0^1 *for problem* (5.1) *in the class* $\mathcal{W}$ *if*

$$\varepsilon_{\mathcal{L}} \equiv \varepsilon_{\mathcal{L}}(H_I^2 \cap H_{0,I}^1,\ L_I^2) < c_0 \cdot \min_{i \in I} \sqrt{\frac{d_i \theta_i}{2k^*}}\ , \qquad (5.15)$$

where $c_0 > 0$ *is defined as in* (5.6). *This means that if two solutions* $u, v \in \mathcal{W}$ *possess the property*

$$\lim_{t \to \infty} \int_t^{t+1} |l_j(p_I u(\tau)) - l_j(p_I v(\tau))|^2 \, d\tau = 0 \quad \textit{for} \quad j = 1, \ldots, N, \qquad (5.16)$$

where p_I *is the natural projection of* H^s *onto* H_I^s, *then equation* (5.8) *holds.*

Proof.

Let $u, v \in \mathcal{W}$ and $w = u - v$. Then

$$\partial_t w_i = d_i \Delta w_i - (f_i(x, u, \nabla u;\ t) - f_i(x, v, \nabla v;\ t)) \qquad (5.17)$$

for $i = 1, \ldots, m$. In $L^2(\Omega)$ we scalarwise multiply equations (5.17) by $-\theta_i \Delta w_i$ for $i \in I$ and by $\theta_i w_i$ for $i \in I'$ and summarize the results. Using inequality (5.14) we find that

$$\frac{1}{2} \cdot \frac{d}{dt} \Phi(w(t)) + \frac{1}{2} \sum_{i \in I} d_i \theta_i \|\Delta w_i\|^2 + \omega \sum_{i \in I'} \|w_i\|^2 \le k^* \sum_{i \in I} \|w_i\|^2 ,$$

where

$$\Phi(w) = \sum_{i \in I} \theta_i \|\nabla w_i\|^2 + \sum_{i \in I'} \theta_i \|w_i\|^2 .$$

As in the proof of Theorem 5.1 (see (5.10)) we have

$$\sum_{i \in I} \|w_i\|^2 \le C(N, \delta) \max_j |l_j(p_I w)|^2 + \frac{1+\delta}{c_0^2} \varepsilon_{\mathscr{L}}^2 \cdot \sum_{i \in I} \|\Delta w_i\|^2$$

for every $\delta > 0$. Therefore, provided (5.15) holds, we obtain that

$$\frac{d}{dt}\Phi(w(t)) + \beta\,\Phi(w(t)) \le C \cdot \max_j |l_j(p_I w(t))|^2$$

with some constant $\beta > 0$. Equation (5.16) implies that $\Phi(w(t)) \to 0$ as $t \to 0$. Hence,

$$\lim_{t \to \infty} \|w(t)\| = 0 . \tag{5.18}$$

However, equation (5.9) and the inequality

$$\|\nabla w\| \le C \cdot \|\Delta w\| , \qquad w \in \boldsymbol{H}^2 \cap \boldsymbol{H}_0^1 ,$$

imply that

$$\|\nabla w(t)\|^2 \le e^{-\beta(t-t_0)} \|\nabla w(t_0)\|^2 + C \int_{t_0}^{t} e^{-\beta(t-t_0)} \|w(\tau)\|^2 \, d\tau$$

for all $t \ge t_0$ with t_0 large enough and for $\beta > 0$. Therefore, equation (5.8) follows from (5.18). **Theorem 5.2 is proved**.

The abstract form of Theorem 5.2 can be found in [3].

As an application of Theorem 5.2 we consider a system of equations which describe the Belousov-Zhabotinsky reaction. This system (see [9], [10], and the references therein) can be obtained from (5.1) if we take $n \le 3$, $m = 3$, $a(x, t) = \mathrm{diag}(d_1, d_2, d_3)$ and

$$f(x, u, \nabla u;\ t) \equiv f(u) = (f_1(u);\ f_2(u);\ f_3(u)) ,$$

where

$$f_1(u) = -\alpha(u_2 - u_1 u_2 + u_1 - \beta u_1^2) ,$$

$$f_2(u) = -\frac{1}{\alpha}(\gamma u_3 - u_2 - u_1 u_2) , \qquad f_3(u) = -\delta(u_1 - u_3) .$$

Here α, β, γ, and δ are positive numbers. The theorem on the existence of classical solutions can be proved without any difficulty (see, e.g., [7]). It is well-known [10] that if $a_3 > a_1 > \max(1, \beta^{-1})$ and $a_2 > \gamma a_3$, then the domain

$$D = \{u \equiv (u_1, u_2, u_3):\ 0 \le u_j \le a_j,\ j = 1, 2, 3\} \subset \mathbb{R}^3$$

is invariant (if the initial condition vector lies in D for all $x \in \Omega$, then $u(x, t) \in D$ for $x \in \Omega$ and $t > 0$). Let $\mathscr{W} \equiv \mathscr{W}_D$ be a set of classical solutions the initial conditions of which have the values in D. It is clear that assumptions (5.3) and (5.4) are valid for $\mathscr{W}$. Simple calculations show that the numbers ω, k^*, θ_2, and θ_3 can be chosen such that equation (5.14) holds for $I = \{1\}$, $I' = \{2, 3\}$, and $\theta_1 = 1$. Indeed, let smooth functions $u(x)$ and $v(x)$ be such that $u(x), v(x) \in D$ for all $x \in \Omega$ and let $w(x) = u(x) - v(x)$. Then it is evident that there exist constants $C_j > 0$ such that

$$\Phi_1 \equiv (f_1(u) - f_1(v),\ \Delta w_1) \le \frac{d_1}{2}\|\Delta w_1\|^2 + C_1 \cdot \left(\|w_1\|^2 + \|w_2\|^2\right),$$

$$\Phi_2 \equiv -(f_2(u) - f_2(v),\ w_2) \le -\frac{1}{2\alpha}\|w_2\|^2 + C_2 \cdot \left(\|w_1\|^2 + \|w_3\|^2\right),$$

$$\Phi_3 \equiv -(f_3(u) - f_3(v),\ w_3) \le -\frac{\delta}{2}\|w_3\|^2 + C_3 \cdot \|w_1\|^2 .$$

Consequently, for any $\theta_2,\ \theta_3 > 0$ we have

$$\Phi_1 + \theta_2\Phi_2 + \theta_3\Phi_3 \le \frac{d_1}{2}\|\Delta w_1\|^2 + (C_1 + \theta_2 C_2 + \theta_3 C_3)\|w_1\|^2 +$$
$$+ \left(C_1 - \frac{\theta_2}{2\alpha}\right)\|w_2\|^2 + \left(\theta_2 C_2 - \frac{\theta_3\delta}{2}\right)\|w_3\|^2 .$$

It follows that there is a possibility to choose the parameters θ_2 and θ_3 such that

$$\Phi_1 + \theta_2\Phi_2 + \theta_3\Phi_3 \le \frac{d_1}{2}\|\Delta w_1\|^2 + k^*\|w_1\|^2 - \omega\left(\|w_2\|^2 + \|w_3\|^2\right)$$

with positive constants k^* and ω. This enables us to prove (5.14) and, hence, the validity of the assertions of Theorem 5.2 for the system of Belousov-Zhabotinsky equations. Therefore, if $\mathscr{L} = \{l_j:\ j = 1, \ldots, N\}$ is a set of linear functionals on $H^2(\Omega) \cap H_0^1(\Omega)$ such that $\varepsilon_{\mathscr{L}}((H^2 \cap H_0^1)(\Omega),\ L^2(\Omega))$ is small enough, then the condition

$$\lim_{t \to \infty} \int_t^{t+1} \left|l_j(u_1(\tau)) - l_j(v_1(\tau))\right|^2 d\tau = 0 , \qquad j = 1, \ldots, N ,$$

for some pair of solutions $u(t) = (u_1(t),\ u_2(t),\ u_3(t))$ and $v(t) = (v_1(t),\ v_2(t),\ v_3(t))$ which lie in $\mathscr{W}$ implies that

$$\|u(t) - v(t)\|_1 \to 0 \quad \text{as} \quad t \to \infty .$$

In particular, this means that the asymptotic behaviour of solutions to the Belousov-Zhabotinsky system is uniquely determined by the behaviour of one of the components of the state vector. A similar effect for the other equations is discussed in the sections to follow.

The approach presented above can also be used in the study of the Navier-Stokes system. As an example, let us consider equations that describe the dynamics of a viscous incompressible fluid in the domain $\Omega \equiv T^2 = (0, L) \times (0, L)$ with periodic boundary conditions:

$$\partial_t u - \nu\Delta u + (u, \nabla)u + \nabla p = F(x, t), \quad x \in T^2, \quad t > 0,$$
$$\nabla u = 0, \quad x \in T^2, \quad u(x, 0) = u_0(x), \tag{5.19}$$

where the unknown velocity vector $u(x, t) = (u_1(x, t);\ u_2(x, t))$ and pressure $p(x, t)$ are L-periodic functions with respect to spatial variables, $\nu > 0$, and $F(x, t)$ is the external force.

Let us introduce some definitions. Let $\mathcal{V}$ be a space of trigonometric polynomials $v(x)$ of the period L with the values in $\mathbb{R}^2$ such that $\operatorname{div} v = 0$ and $\int_{T^2} v(x)\,dx = 0$. Let H be the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$, let Π be the orthoprojector onto H in $\boldsymbol{L}^2$, let $A = -\Pi\Delta u = -\Delta u$ and $B(u, v) = \Pi(u, \nabla)v$ for all u and v from $D(A) = H \cap \boldsymbol{H}^2$. We remind (see, e.g., [11]) that A is a positive operator with discrete spectrum and the bilinear operator $B(u, v)$ is a continuous mapping from $D(A) \times D(A)$ into H. In this case problem (5.19) can be rewritten in the form

$$\partial_t u + \nu A u + B(u, u) = \Pi F(t), \quad u\big|_{t=0} = u_0 \in H. \tag{5.20}$$

It is well-known (see, e.g., [11]) that if $u_0 \in H$ and $\Pi F(t) \in L^\infty(\mathbb{R}_+;\ H)$, then problem (5.20) has a unique solution $u(t)$ such that

$$u(t) \in C(\mathbb{R}_+;\ H) \cap C(t_0, +\infty;\ D(A)), \quad t_0 > 0. \tag{5.21}$$

One can prove (see [9] and [12]) that it possesses the property

$$\overline{\lim_{t\to\infty}}\ \frac{1}{a}\int_t^{t+a} \|Au(\tau)\|^2\,d\tau \ \le\ \frac{F^2}{\nu^2}\left(1 + \frac{1}{a\nu\lambda_1}\right) \tag{5.22}$$

for any $a > 0$. Here $\lambda_1 = (2\pi/L)^2$ is the first eigenvalue of the operator A in H and

$$F = \overline{\lim_{t\to\infty}}\ \|\Pi F(t)\|.$$

Lemma 5.1.

Let $u, v \in D(A)$ and let $w = u - v$. Then

$$|(B(u, u) - B(v, v), Aw)| \ \le\ \sqrt{2}\,\|w\|_\infty \|\nabla w\|\,\|Au\|, \tag{5.23}$$

$$|(B(u, u) - B(v, v), Aw)| \ \le\ c\sqrt{L}\,\|\nabla w_k\| \cdot \|\nabla w\|^{1/2} \cdot \|\Delta w\|^{1/2} \cdot \|Au\|, \tag{5.24}$$

where $\|\cdot\|_\infty$ is the L^∞-norm, w_k is the k-th component of the vector w, $k = 1, 2$, and c is an absolute constant.

Proof.

Using the identity (see [12])

$$(B(u, w), Aw) + (B(w, v), Aw) + (B(w, w), Au) = 0$$

for $u, v \in D(A)$ and $w = u - v$, it is easy to find that

$$(B(u, u) - B(v, v), Aw) = -(B(w, w), Au).$$

Therefore, it is sufficient to estimate the norm $\|B(w, w)\|$. The incompressibility condition $\nabla w = 0$ gives us that

$$(w, \nabla)w = (-w_1 \partial_2 w_2 + w_2 \partial_2 w_1, \; w_1 \partial_1 w_2 - w_2 \partial_1 w_1),$$

where ∂_i is the derivative with respect to the variable x_i. Consequently,

$$\|(w, \nabla)w\|^2 \le 2\Big(\|(w_1, \nabla)w_2\|^2 + \|(w_2, \nabla)w_1\|^2\Big). \qquad (5.25)$$

This implies (5.23). Let us prove (5.24). For the sake of definiteness we let $k = 1$. We can also assume that $w \in \mathscr{V}$. Then (5.25) gives us that

$$\|(w, \nabla)w\| \le \sqrt{2}\Big(\|w_1\|_{L^4}\|\nabla w_2\|_{L^4} + \|w_2\|_\infty \|\nabla w_1\|\Big).$$

We use the inequalities (see, e.g., [11], [12])

$$\|v\|_{L^\infty} \le a_0 \|v\|^{1/2} \cdot \|\Delta v\|^{1/2}$$

and

$$\|v\|_{L^4} \le a_1 \|v\|^{1/2} \cdot \|\nabla v\|^{1/2},$$

where a_0 and a_1 are absolute constants (their explicit equations can be found in [12]). These inequalities as well as a simply verifiable estimate $\|v\| \le$ $\le (L/2\pi) \cdot \|\nabla v\|$ imply that

$$\|(w, \nabla)w\| \le \sqrt{\frac{L}{\pi}}(a_1^2 + a_0)\|\nabla w_1\| \cdot \|\nabla w\|^{1/2} \cdot \|\Delta w\|^{1/2}.$$

This proves (5.24) for $k = 1$.

Theorem 5.3.

1. ***A set*** $\mathscr{L} = \{l_j\colon j = 1, 2, \dots, N\}$ ***of linearly independent continuous linear functionals on*** $D(A) = \boldsymbol{H}^2 \cap \boldsymbol{H}$ ***is an asymptotically determining set with respect to*** $\boldsymbol{H}^1$ ***for problem*** (5.20) ***in the class of solutions with property*** (5.21) ***if***

$$\varepsilon_{\mathscr{L}} \equiv \varepsilon_{\mathscr{L}}(D(A), H) < c_1 G \equiv c_1 \nu^2 \Big(\overline{\lim_{t\to\infty}} \|\Pi F(t)\|\Big)^{-1}, \qquad (5.26)$$

where c_1 ***is an absolute constant.***

2. ***Let*** $\mathscr{L} = \{l_j\colon j = 1, \dots, N\}$ ***be a set of linear functionals on*** $H^2(\Omega)$ ***and let***

$$\varepsilon'_{\mathscr{L}} \equiv \varepsilon_{\mathscr{L}}(H^2(\Omega), L^2(\Omega)) < c_2 \nu^4 L^{-3} \Big(\overline{\lim_{t\to\infty}} \|\Pi F(t)\|\Big)^{-2},$$

where c_2 ***is an absolute constant. Then every set*** $p_k^* \mathscr{L}$ ***is an asymptotically determining set with respect to*** $\boldsymbol{H}^1$ ***for problem*** (5.20) ***in the class of solutions with property*** (5.21)***. Here*** p_k ***is the natural projection onto the*** k***-th component of the velocity vector,*** $p_k(u_1; u_2) = u_k$, $k = 1, 2$.

Proof.

Let $u(t)$ and $v(t)$ be solutions to problem (5.20) possessing property (5.21). Then equations (5.20) and (5.23) imply that

$$\frac{1}{2}\cdot\frac{d}{dt}\|\nabla w\|^2+\nu\|\Delta w\|^2 \le \sqrt{2}\|w\|_\infty\cdot\|\nabla w\|\cdot\|Au\|$$

for $w=u-v$. As above, Theorem 2.1 gives us the estimate

$$\|w\| \le C\eta(w)+\varepsilon_{\mathscr{L}}\|\Delta w\|, \qquad \eta(w)=\max_j|l_j(w)|.$$

Therefore,

$$\|w\|_\infty \le a_0\cdot\|w\|^{1/2}\cdot\|\Delta w\|^{1/2} \le c_1\sqrt{\eta(w)}\cdot\|\Delta w\|^{1/2}+a_0\sqrt{\varepsilon_{\mathscr{L}}}\cdot\|\Delta w\|.$$

The inequality

$$\|\nabla w\|^2 \le \|w\|\cdot\|\Delta w\|$$

enables us to obtain the estimate (see Exercise 2.12) $\varepsilon_{\mathscr{L}}(D(A),\ D(A^{1/2}))\le\varepsilon_{\mathscr{L}}^{1/2}$ and hence

$$\|\Delta w\|^2 \ge -C_\delta\,\eta(w)^2+(1-\delta)\varepsilon_{\mathscr{L}}^{-1}\cdot\|\nabla w\|^2$$

for every $\delta>0$. Therefore, we use dissipativity properties of solutions (see [8] and [9] as well as Chapter 2) to obtain that

$$\frac{d}{dt}\|\nabla w(t)\|^2+\alpha(t)\cdot\|\nabla w\|^2 \le C\cdot\left[\eta(w)^2+\sqrt{\eta(w)}\right],$$

where

$$\alpha(t)=\nu(1-\delta)\varepsilon_{\mathscr{L}}^{-1}-2a_0^2\varepsilon_{\mathscr{L}}\nu^{-1}\|Au(t)\|^2.$$

Equation (5.22) for $a=(\nu\lambda_1)^{-1}$ implies that the function $\alpha(t)$ possesses properties (1.6) and (1.7), provided (5.26) holds. Therefore, we apply Lemma 1.1 to obtain that $\|\nabla w(t)\|\to 0$ as $t\to\infty$, provided

$$\lim_{t\to\infty}\int_t^{t+1}[\eta(w(\tau))]^2\,d\tau=0.$$

In order to prove the second part of the theorem, we use similar arguments. For the sake of definiteness let us consider the case $k=1$ only. It follows from (5.20) and (5.24) that

$$\frac{1}{2}\cdot\frac{d}{dt}\|\nabla w\|^2+\nu\|\Delta w\|^2 \le c\sqrt{L}\|\nabla w_1\|^{1/2}\cdot\|\nabla w\|\cdot\|\Delta w\|^{1/2}\cdot\|Au\|. \qquad (5.27)$$

The definition of completeness defect implies that

$$\|\nabla w_1\|^{1/2} \le C\sqrt{\eta(w_1)}+\sqrt{\varepsilon'_{\mathscr{L}}}\|\Delta w_1\|^{1/2} \le C\sqrt{\eta(w_1)}+\sqrt{\varepsilon'_{\mathscr{L}}}\|\Delta w\|^{1/2},$$

where $\eta(w_1)=\max_j|l_j(w_1)|$. Therefore,

$$c\sqrt{L}\|\nabla w_1\|^{1/2}\|\nabla w\|\cdot\|\Delta w\|^{1/2}\|Au\| \le$$

$$\le c\sqrt{L}\cdot\sqrt{\varepsilon'_{\mathscr{L}}}\cdot\|\nabla w\|\cdot\|\Delta w\|\cdot\|Au\| + C\sqrt{\eta(w_1)}\|\nabla w\|\cdot\|\Delta w\|^{1/2}\cdot\|Au\| \le$$

$$\le C[\eta(w_1)]^2 + \left(\varkappa + \frac{Lc^2}{\nu}\varepsilon'_{\mathscr{L}}\right)\cdot\|\nabla w\|^2\cdot\|Au\|^2 + \frac{\nu}{2}\|\Delta w\|^2$$

for any $\varkappa > 0$, where the constant C depends on $\varkappa$, ν, $\mathscr{L}$, and L. Consequently, from (5.27) we obtain that

$$\frac{d}{dt}\|\nabla w\|^2 + \nu\|\Delta w\|^2 \le C[\eta(w_1)]^2 + 2\left(\varkappa + \frac{Lc^2}{\nu}\varepsilon'_{\mathscr{L}}\right)\cdot\|\nabla w\|^2\|Au\|^2 .$$

Therefore, we can choose $\varkappa = \delta\cdot L\cdot c^2\cdot\nu^{-1}\cdot\varepsilon'_{\mathscr{L}}$ and find that

$$\frac{d}{dt}\|\nabla w\|^2 + \left[\nu\left(\frac{2\pi}{L}\right)^2 - 2(1+\delta)\frac{Lc^2}{\nu}\varepsilon'_{\mathscr{L}}\cdot\|Au\|^2\right]\|\nabla w\|^2 \le C[\eta(w_1)]^2 ,$$

where $\delta > 0$ is an arbitrary number. Further arguments repeat those in the proof of the first assertion. **Theorem 5.3 is proved**.

It should be noted that assertion 1 of Theorem 5.3 and the results of Section 3 enable us to obtain estimates for the number of determining nodes and local volume averages that are close to optimal (see the references in the survey [3]). At the same time, although assertion 2 uses only one component of the velocity vector, in general it makes it necessary to consider a much greater number of determining functionals in comparison with assertion 1. It should also be noted that assertion 2 remains true if instead of w_k we consider the projections of the velocity vector onto an arbitrary a priori chosen direction [3]. Furthermore, analogues of Theorems 1.3 and 4.4 can be proved for the Navier-Stokes system (5.19) (the corresponding variants of estimates (4.28) and (4.29) can be derived from the arguments in [2], [8], and [9]).

§ 6 *Determining Functionals in the Problem of Nerve Impulse Transmission*

We consider the following system of partial differential equations suggested by Hodgkin and Huxley for the description of the mechanism of nerve impulse transmission:

$$\partial_t u - d_0\partial_x^2 u + g(u, v_1, v_2, v_3) = 0 , \quad x \in (0, L) , \quad t > 0 , \tag{6.1}$$

$$\partial_t v_j - d_j\partial_x^2 v_j + k_j(u)\cdot(v_j - h_j(u)) = 0 , \quad x \in (0, L) , \quad t > 0 , \quad j = 1, 2, 3 . \tag{6.2}$$

Here $d_0 > 0$, $d_j \geq 0$, and

$$g(u, v_1, v_2, v_3) = -\gamma_1 v_1^3 v_2 (\delta_1 - u) - \gamma_2 v_3^4 (\delta_2 - u) - \gamma_3 (\delta_3 - u) \,, \qquad (6.3)$$

where $\gamma_j > 0$ and $\delta_1 > \delta_3 > 0 > \delta_2$. We also assume that $k_j(u)$ and $h_j(u)$ are the given continuously differentiable functions such that $k_j(u) > 0$ and $0 < h_j(u) < 1$, $j = 1, 2, 3$. In this model u describes the electric potential in the nerve and v_j is the density of a chemical matter and can vary between 0 and 1. Problem (6.1) and (6.2) has been studied by many authors (see, e.g., [9], [10], [13] and the references therein) for different boundary conditions. The results of numerical simulation given in [13] show that the asymptotic behaviour of solutions to this problem can be quite complicated. In this chapter we focus on the existence and the structure of determining functionals for problem (6.1) and (6.2). In particular, we prove that the asymptotic behaviour of densities v_j is uniquely determined by sets of functionals defined on the electric potential u only. Thus, the component u of the state vector (u, v_1, v_2, v_3) is leading in some sense.

We equip equations (6.1) and (6.2) with the initial data

$$u\big|_{t=0} = u_0(x) \,, \qquad v_j\big|_{t=0} = v_{j0}(x) \,, \qquad j = 1, 2, 3 \,, \qquad (6.4)$$

and with one of the following boundary conditions:

$$u\big|_{x=0} = u\big|_{x=L} = d_j v_j\big|_{x=0} = d_j v_j\big|_{x=L} = 0 \,, \qquad t > 0 \,, \qquad (6.5a)$$

$$\partial_x u\big|_{x=0} = \partial_x u\big|_{x=L} = d_j \cdot \partial_x v_j\big|_{x=0} = d_j \cdot \partial_x v_j\big|_{x=L} = 0 \,, \qquad t > 0 \,, \qquad (6.5b)$$

$$u(x+L, t) - u(x, t) = d_j \cdot (v_j(x+L, t) - v_j(x, t)) = 0 \,, \qquad x \in \mathbb{R}^1 \,, \quad t > 0 \,, \qquad (6.5c)$$

where $j = 1, 2, 3$. Thus, we have no boundary conditions for the function $v_j(x, t)$ when the corresponding diffusion coefficient d_j is equal to zero for some $j = 1, 2, 3$.

Let us now describe some properties of solutions to problem (6.1)–(6.5). First of all it should be noted (see, e.g., [10]) that the parallelepiped

$$D = \left\{ U \equiv (u, v_1, v_2, v_3) \colon \ \delta_2 \leq u \leq \delta_1 \,, \ 0 \leq v_j \leq 1 \,, \ j = 1, 2, 3 \right\} \subset \mathbb{R}^4$$

is a positively invariant set for problem (6.1)–(6.5). This means that if the initial data $U_0(x) = (u_0(x), v_{10}(x), v_{20}(x), v_{30}(x))$ belongs to D for almost all $x \in [0, L]$, then

$$U(t) \equiv (u(x, t), v_1(x, t), v_2(x, t), v_3(x, t)) \in D$$

for $x \in [0, L]$ and for all $t > 0$ for which the solution to problem (6.1)–(6.5) exists.

Let $\mathbb{H}_0 \equiv [H]^4 \equiv [L^2(0, L)]^4$ be the space consisting of vector-functions $U(x) \equiv (u, v_1, v_2, v_3)$, where $u \in L^2(0, L)$, $v_j \in L^2(0, L)$, $j = 1, 2, 3$. We equip it with the standard norm. Let

$$\mathbb{H}_0(D) = \left\{ U(x) \in \mathbb{H}_0 : \ U(x) \in D \ \text{ for almost all } \ x \in (0, L) \right\} .$$

Depending upon the boundary conditions (6.5 a, b, or c) we use the following notations $\mathbb{H}_1 = [V_1]^4$ and $\mathbb{H}_2 = [V_2]^4$, where

$$V_1 = H_0^1(0, L), \quad \text{or} \quad H^1(0, L), \quad \text{or} \quad H^1_{\text{per}}(0, L) \tag{6.6}$$

and

$$V_2 = H^2(0, L) \cap H_0^1(0, L), \quad \text{or}$$

$$\left\{ u(x) \in H^2(0, L) : \partial_x u \big|_{x=0, \ x=L} = 0 \right\}, \quad \text{or} \quad H^2_{\text{per}}(0, L), \tag{6.7}$$

respectively. Hereinafter $H^s(0, L)$ is the Sobolev space of the order s on $(0, L)$, H_0^1 and H^1_{per} are subspaces in $H^1(0, L)$ corresponding to the boundary conditions (6.5a) and (6.5c). The norm in $H^s(0, L)$ is defined by the equality

$$\|u\|_s^2 = \left\|\partial_x^s u\right\|^2 + \|u\|^2 = \int_0^L \left(\left|\partial_x^s u(x)\right|^2 + |u(x)|^2 \right) \mathrm{d}x , \qquad s = 1, 2, \ldots$$

We use notations $\| \cdot \|$ and $(\cdot , \cdot)$ for the norm and the inner product in $H \equiv$ $\equiv L^2(0, L)$. Further we assume that $C(0, T; X)$ is the space of strongly continuous functions on $[0, T]$ with the values in X. The notation $L^p(0, T; X)$ has a similar meaning.

Let $d_j > 0$ for all $j = 1, 2, 3$. Then for every vector $U_0 \in \mathbb{H}_0(D)$ problem (6.1)–(6.5) has a unique solution $U(t) \in \mathbb{H}_0(D)$ defined for all t (see, e.g., [9], [10]). This solution lies in

$$C(0, T; \mathbb{H}_0(D)) \cap L^2(0, T; \mathbb{H}_1)$$

for any segment $[0, T]$ and if $U_0 \in \mathbb{H}_0(D) \cap \mathbb{H}_1$, then

$$U(t) \in C(0, T; \mathbb{H}_0(D) \cap \mathbb{H}_1) \cap L^2(0, T; \mathbb{H}_2). \tag{6.8}$$

Therefore, we can define the evolutionary semigroup S_t in the space $\mathbb{H}_0(D) \cap \mathbb{H}_1$ by the formula

$$S_t U_0 = U(t) \equiv (u(x, t), v_1(x, t), v_2(x, t), v_3(x, t)),$$

where $U(t)$ is a solution to problem (6.1)–(6.5) with the initial conditions

$$U_0 \equiv (u_0(x), v_{10}(x), v_{20}(x), v_{30}(x)).$$

The dynamical system $(\mathbb{H}_0(D) \cap \mathbb{H}_1; S_t)$ has been studied by many authors. In particular, it has been proved that it possesses a finite-dimensional global attractor [9].

If $d_1 = d_2 = d_3 = 0$, then the corresponding evolutionary semigroup can be defined in the space $\mathbb{V}_1 = (V_1 \times [H^1(0, L)]^3) \cap \mathbb{H}_0(D)$. In this case for any segment $[0, T]$ we have

$$S_t U_0 = U(t) \in C(0, T;\ \mathbb{V}_1) \cap L^2(0, T;\ V_2 \times (H^1(0, L))^3), \tag{6.9}$$

if $U_0 \in \mathbb{V}_1$. This assertion can be easily obtained by using the general methods of Chapter 2.

The following assertion is the main result of this section.

Theorem 6.1.

Let $\mathcal{L} = \{l_j:\ j = 1, \ldots, N\}$ ***be a finite set of continuous linear functionals on the space*** V_s, $s = 1, 2$ ***(see*** (6.6) ***and*** (6.7)***). Assume that***

$$\varepsilon_{\mathcal{L}}^{(1)} \equiv \varepsilon_{\mathcal{L}}(V_1, H) \le \sqrt{\frac{d_0}{d_0 + K_1}} \tag{6.10}$$

or

$$\varepsilon_{\mathcal{L}}^{(2)} \equiv \varepsilon_{\mathcal{L}}(V_2, H) \le \frac{d_0}{\sqrt{d_0^2 + K_2^2}}, \tag{6.11}$$

where

$$K_1 = (\delta_1 - \delta_2) \cdot \sum_{j=1}^{3} \frac{\beta_j (A_j + B_j)}{k_j^*} \tag{6.12}$$

and

$$K_2 = 2 \cdot \left((\gamma_1 + \gamma_2)^2 + 5(\delta_1 - \delta_2)^2 \cdot \sum_{j=1}^{3} \left(\frac{\beta_j (A_j + B_j)}{2 k_j^*} \right)^2 \right)^{1/2} \tag{6.13}$$

with $\beta_1 = 3\gamma_1$, $\beta_2 = \gamma_1$, $\beta_3 = 4\gamma_2$, ***and***

$$A_j = \max\left\{ |\partial_u k_j(u)|:\ \delta_2 \le u \le \delta_1 \right\}, \qquad B_j = \max\left\{ |\partial_u (k_j h_j)(u)|:\ \delta_2 \le u \le \delta_1 \right\},$$

$$k_j^* = \min\left\{ k_j(u):\ \delta_2 \le u \le \delta_1 \right\}.$$

Then $\mathcal{L}$ ***is an asymptotically determining set with respect to the space*** $\mathbb{H}_0$ ***for problem*** (6.1)–(6.5) ***in the sense that for any two solutions***

$$U(t) = (u(x, t),\ v_1(x, t),\ v_2(x, t),\ v_3(x, t))$$

and

$$U^*(t) = (u^*(x, t),\ v_1^*(x, t),\ v_2^*(x, t),\ v_3^*(x, t))$$

satisfying either (6.8) ***with*** $d_j > 0$***, or (6.9) with*** $d_j = 0$, $j = 1, 2, 3$***, the condition***

$$\lim_{t \to \infty} \int_t^{t+1} \left| l_j(u(\tau)) - l_j(u^*(\tau)) \right|^2 d\tau = 0 \quad \textit{for} \quad j = 1, \ldots, N \tag{6.14}$$

implies that

$$\lim_{t \to \infty} \left\{ \|u(t) - u^*(t)\|^2 + \sum_{j=1}^{3} \left\| v_j(t) - v_j^*(t) \right\|^2 \right\} = 0 \ . \tag{6.15}$$

Proof.

Assume that (6.10) is valid. Let

$$U(t) = (u(x, t),\ v_1(x, t),\ v_2(x, t),\ v_3(x, t))$$

and

$$U^*(t) = (u^*(x, t),\ v_1^*(x, t),\ v_2^*(x, t),\ v_3^*(x, t))$$

be solutions satisfying either (6.8) with $d_j > 0$, or (6.9) with $d_j = 0$, $j = 1, 2, 3$. It is clear that

$$G(U, U^*) \equiv g(u, v_1, v_2, v_3) - g(u^*, v_1^*, v_2^*, v_3^*) =$$
$$= (\gamma_1 v_1^3 v_2 + \gamma_2 v_3^4 + \gamma_3)(u - u^*) + h(U, U^*) \ ,$$

where

$$h(U, U^*) = -\gamma_1 (v_1^3 v_2 - v_1^{*3} v_2^*)(\delta_1 - u^*) - \gamma_2 (v_3^4 - v_3^{*4})(\delta_2 - u^*) .$$

Since $U, U^* \in D$, it is evident that

$$\left| h(U, U^*) \right| \le \sum_{j=1}^{3} a_j \left| v_j - v_j^* \right| \ ,$$

where $a_j = (\delta_1 - \delta_2) \cdot \beta_j$. Let $w = u - u^*$ and $\psi_j = v_j - v_j^*$, $j = 1, 2, 3$. It follows from (6.1) that

$$\partial_t w - d_0 \partial_x^2 w + G(U, U^*) = 0 , \qquad x \in (0, L) , \qquad t > 0 . \tag{6.16}$$

If we multiply (6.16) by w in $L^2(0, L)$, then it is easy to find that

$$\frac{1}{2} \cdot \frac{d}{dt} \|w\|^2 + d_0 \|\partial_x w\|^2 + \gamma_3 \|w\|^2 \le \sum_{j=1}^{3} a_j \|\psi_j\| \cdot \|w\| \ . \tag{6.17}$$

Equation (6.2) also implies that

$$\frac{1}{2} \cdot \frac{d}{dt} \|\psi_j\|^2 + d_j \|\partial_x \psi_j\|^2 + k_j^* \|\psi_j\|^2 \le (A_j + B_j) \|\psi_j\| \cdot \|w\| . \tag{6.18}$$

Thus, for any $0 < \varepsilon < 1$ and $\theta_j > 0$, $j = 1, 2, 3$, we obtain that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right)+d_0\|\partial_x w\|^2+\gamma_3\|w\|^2+\varepsilon\sum_{j=1}^{3}k_j^*\theta_j\|\psi_j\|^2 \le$$

$$\le -\sum_{j=1}^{3}(1-\varepsilon)k_j^*\theta_j\|\psi_j\|^2+\sum_{j=1}^{3}[a_j+\theta_j(A_j+B_j)]\|\psi_j\|\cdot\|w\| \le$$

$$\le \sum_{j=1}^{3}\frac{[a_j+\theta_j(A_j+B_j)]^2}{4(1-\varepsilon)\theta_j k_j^*}\cdot\|w\|^2 . \tag{6.19}$$

Theorem 2.1 gives us that

$$\|w\|^2 \le C_{\mathscr{L},\,\eta}\cdot\max_j|l_j(w)|^2+(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2\cdot\left(\|\partial_x w\|^2+\|w\|^2\right)$$

for any $\eta > 0$. Therefore,

$$\|\partial_x w\|^2 \ge \left(\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2}-1\right)\cdot\|w\|^2 - C_{\mathscr{L},\,\eta}\cdot\max_j|l_j(w)|^2 . \tag{6.20}$$

Consequently, it follows from (6.19) that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right)+d_0\cdot\omega(\mathscr{L},\,\varepsilon,\,\theta,\,\eta)\|w\|^2+$$

$$+\varepsilon\sum_{j=1}^{3}k_j^*\theta_j\|\psi_j\|^2 \le C_{\mathscr{L},\,\eta}\cdot\max_j|l_j(w)|^2 \tag{6.21}$$

for any $0<\varepsilon<1$, $\theta_j>0$, and $\eta>0$, where

$$\omega(\mathscr{L},\,\varepsilon,\,\theta,\,\eta) = \frac{\gamma_3}{d_0}+\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2}-1-\sum_{j=1}^{3}\frac{[a_j+\theta_j(A_j+B_j)]^2}{4(1-\varepsilon)\theta_j k_j^* d_0} .$$

We choose $\theta_j = a_j\cdot(A_j+B_j)^{-1}$ and obtain that

$$\omega(\mathscr{L},\,\varepsilon,\,\theta,\,\eta) = \frac{\gamma_3}{d_0}+\frac{1}{(1+\eta)[\varepsilon_{\mathscr{L}}^{(1)}]^2}-1-\frac{K_1}{(1-\varepsilon)d_0} .$$

It is easy to see that if (6.10) is valid, then there exist $0<\varepsilon<1$ and $\eta>0$ such that $\omega(\mathscr{L},\,\varepsilon,\,\theta,\,\eta)>0$. Therefore, equation (6.21) gives us that

$$\|w(t)\|^2+\sum_{j=1}^{3}\theta_j\|\psi_j(t)\|^2 \le$$

$$\le \left(\|w_0\|^2+\sum_{j=1}^{3}\theta_j\|\psi_{j0}\|^2\right)\cdot e^{-\omega^* t}+C_{\mathscr{L},\,\eta}\cdot\int_0^t e^{-\omega^*(t-\tau)}\max_j|l_j(w(\tau))|^2\,\mathrm{d}\tau ,$$

where $\omega^*>0$. Thus, if (6.10) is valid, then equation (6.14) implies (6.15).

Let us now assume that (6.11) is valid. Since

$$-(G(U, U^*), \partial_x^2 w) \leq -\gamma_3 \|\partial_x w\|^2 + \Big((\gamma_1+\gamma_2)\cdot\|w\| + \|h(U, U^*)\|\Big)\cdot\|\partial_x^2 w\|,$$

then it follows from (6.16) that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x w\|^2 + \frac{d_0}{2}\|\partial_x^2 w\|^2 + \gamma_3\|\partial_x w\|^2 \leq$$

$$\leq \frac{2}{d_0}\cdot(\gamma_1+\gamma_2)^2\cdot\|w\|^2 + 2\cdot\sum_{j=1}^{3}\frac{a_j^2}{d_0}\cdot\|\psi_j\|^2 . \tag{6.22}$$

Therefore, we can use equation (6.18) and obtain (cf. (6.19)) that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x w\|^2 + \sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right) + \frac{d_0}{2}\|\partial_x^2 w\|^2 + \gamma_3\|\partial_x w\|^2 + \varepsilon\sum_{j+1}^{3}k_j^*\theta_j\|\psi_j\|^2 \leq$$

$$\leq \left(\frac{2}{d_0}\cdot(\gamma_1+\gamma_2)^2 + \frac{9}{4}\cdot\sum_{j=1}^{3}\frac{a_j^2(A_j+B_j)^2}{(1-\varepsilon)^2[k_j^*]^2 d_0}\right)\cdot\|w\|^2$$

for any $0 < \varepsilon < 1$ and $\theta_j = 3a_j^2\cdot[(1-\varepsilon)k_j^* d_0]^{-1}$. As above, we find that

$$\|\partial_x^2 w\|^2 \geq \left(\frac{1}{(1+\eta)[\varepsilon_{\mathcal{L}}^{(2)}]^2} - 1\right)\cdot\|w\|^2 - C_{\mathcal{L},\eta}\cdot\max_j|l_j(w)|^2$$

for any $\eta > 0$. Therefore,

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\partial_x w\|^2 + \sum_{j=1}^{3}\theta_j\|\psi_j\|^2\right) + \gamma_3\|\partial_x w\|^2 + \varepsilon\sum_{j+1}^{3}k_j^*\theta_j\|\psi_j\|^2 \leq C_{\mathcal{L},\eta}\cdot\max_j|l_j(w)|^2,$$

provided that

$$\frac{1}{(1+\eta)[\varepsilon_{\mathcal{L}}^{(2)}]^2} - 1 - \frac{4}{d_0^2}\cdot(\gamma_1+\gamma_2)^2 - \sum_{j=1}^{3}\frac{9a_j^2(A_j+B_j)^2}{2(1-\varepsilon)^2[k_j^*]^2 d_0^2} \geq 0 .$$

As in the first part of the proof, we can now conclude that if (6.11) is valid, then (6.14) implies the equations

$$\lim_{t\to\infty}\|\partial_x w(t)\| = 0 \quad \text{and} \quad \lim_{t\to\infty}\|\psi_j(t)\| = 0, \quad j = 1, 2, 3. \tag{6.23}$$

It follows from (6.17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \gamma_3\|w\|^2 \leq \frac{3}{\gamma_3}\sum_{j=1}^{3}a_j^2\|\psi_j(t)\|^2 .$$

Therefore, as above, we obtain equation (6.15) for the case (6.11). **Theorem 6.1 is proved**.

As in the previous sections, modes, nodes and generalized local volume averages can be choosen as determining functionals in Theorem 6.1.

— Exercise *6.1* Let $\{e_k\}$ be a basis in $L^2(0, L)$ which consists of eigenvectors of the operator ∂_x^2 with one of the boundary conditions (6.5). Show that the set

$$\mathscr{L} = \left\{ l_j(u) = \int_0^L e_j(x) u(x)\, \mathrm{d}x\,, \quad j = 1, \ldots, N \right\}$$

is determining (in the sense of (6.14) and (6.15)) for problem (6.1)–(6.5) for N large enough.

— Exercise *6.2* Show that in the case of the Neumann boundary conditions (6.5b) it is sufficient to choose the number N in Exercise 6.1 such that

$$N > \frac{L}{\pi} \sqrt{\frac{K_j}{d_0}}\,, \qquad j = 1 \text{ or } 2\,. \tag{6.24}$$

Obtain a similar estimate for the other boundary conditions (*Hint*: see Exercises 3.2–3.4).

— Exercise *6.3* Let

$$\mathscr{L} = \left\{ l_j : \quad l_j(u) = u(x_j)\,, \quad x_j = jh\,, \quad h = \frac{L}{N}, \quad j = 1, \ldots, N \right\}. \tag{6.25}$$

Show that for every $w \in V_1$ the estimate

$$\|\partial_x w\|^2 \ \ge\ \frac{2}{(1+\eta)h^2}\|w\|^2 - C_{N,\,\eta} \max|l_j(w)|^2 \tag{6.26}$$

holds for any $\eta > 0$ (*Hint*: see Exercise 3.6).

— Exercise *6.4* Use estimate (6.26) instead of (6.20) in the proof of Theorem 6.1 to show that the set of functionals (6.25) is determining for problem (6.1)–(6.5), provided that $N > L \cdot \sqrt{(2K_1)/d_0}$.

— Exercise *6.5* Obtain the assertions similar to those given in Exercises 6.3 and 6.4 for the following set of functionals

$$\mathscr{L} = \left\{ l_j(w) = \frac{1}{h} \int_0^h \lambda\left(\frac{\tau}{h}\right) u(x_j + \tau)\, \mathrm{d}\tau\,, \right.$$

$$\left. x_j = jh\,, \quad h = \frac{L}{N}, \quad j = 0, \ldots, N-1 \right\},$$

where the function $\lambda(x) \in L^\infty(\mathbb{R}^1)$ possesses the properties

$$\int_{-\infty}^{\infty} \lambda(x)\,\mathrm{d}x = 1\,, \qquad \operatorname{supp} \lambda(x) \subset [0,\ 1]\,.$$

It should be noted that in their work Hodgkin and Huxley used the following expressions (see [13]) for $k_j(u)$ and $h_j(u)$:

$$k_j(u) = \alpha_j(u) + \beta_j(u)\,, \qquad h_j(u) = \frac{\alpha_j(u)}{\alpha_j(u) + \beta_j(u)}\,,$$

where
$$\alpha_1(u) = e(-0.1\,u + 2.5)\,, \qquad \beta_1(u) = 4\exp(-u/18)\,;$$

$$\alpha_2(u) = \frac{0.07}{e(-0.05\,u)}\,, \qquad \beta_2(u) = \frac{1}{1 + \exp(-0.1\,u + 3)}\,;$$

$$\alpha_3(u) = 0.1\,e(-0.1\,u + 1)\,, \qquad \beta_3(u) = 0.125\exp(-u/80)\,.$$

Here $e(z) = z/(e^z - 1)$. They also supposed that $\delta_1 = 115$, $\delta_2 = -12$, $\gamma_1 = 120$, and $\gamma_2 = 36$. As calculations show, in this case $K_1 \le 5.2 \cdot 10^4$ and $K_2 \le 7.4 \cdot 10^4$. Therefore (see Exercise 6.4), the nodes $\{x_j = jh,\ \ h = l/N,\ \ j = 0,\ 1,\ 2,\ \dots,\ N\}$ are determining for problem (6.1)–(6.5) when $N \ge 2.3 \cdot 10^2 \cdot L/\sqrt{d_0}$. Of course, similar estimates are valid for modes and generalized volume averages.

Thus, for the asymptotic dynamics of the system to be determined by a small number of functionals, we should require the smallness of the parameter $L/\sqrt{d_0}$. However, using the results available (see [14]) on the analyticity of solutions to problem (6.1)–(6.5) one can show (see Theorem 6.2 below) that the values of all components of the state vector $U = (u,\ v_1,\ v_2,\ v_3)$ in two sufficiently close nodes uniquely determine the asymptotic dynamics of the system considered not depending on the value of the parameter $L/\sqrt{d_0}$. Therewith some regularity conditions for the coefficients of equations (6.1) and (6.2) are necessary.

Let us consider the periodic initial-boundary value problem (6.1)–(6.5c). Assume that $d_j > 0$ for all j and the functions $k_j(u)$ and $h_j(u)$ are polynomials such that $k_j(u) > 0$ and $0 \le h_j(u) \le 1$ for $u \in [\delta_2,\ \delta_1]$. In this case (see [14]) every solution

$$U(t) = (u(x,\ t),\ v_1(x,\ t),\ v_2(x,\ t),\ v_3(x,\ t))$$

possesses the following ***Gevrey regularity*** property: there exists $t_* > 0$ such that

$$\sum_{l=-\infty}^{\infty} \left(|F_l(u(t))|^2 + \sum_{j=1}^{3} |F_l(v_j(t))|^2 \right) \cdot e^{\tau|l|} \ \le\ C \tag{6.27}$$

for some $\tau > 0$ and for all $t \ge t_*$. Here $F_l(w)$ are the Fourier coefficients of the function $w(x)$:

$$F_l(w) = \frac{1}{L} \cdot \int_0^L w(x) \cdot \exp\left\{i\,\frac{2\pi l}{L}\,x\right\} \mathrm{d}x\,, \qquad l = 0,\ \pm 1,\ \pm 2,\ \ldots$$

In particular, property (6.27) implies that every solution to problem (6.1)–(6.5c) becomes a real analytic function for all t large enough. This property enables us to prove the following assertion.

Theorem 6.2.

Let $d_j > 0$ ***for all*** j ***and let*** $k_j(u)$ ***and*** $h_j(u)$ ***be the polynomials possessing the properties***

$$k_j(u) > 0, \quad 0 \le h_j(u) \le 1 \quad \textit{for} \quad u \in [\delta_2,\ \delta_1].$$

Let x_1 ***and*** x_2 ***be two nodes such that*** $0 \le x_1 < x_2 \le L$ ***and*** $x_2 - x_1 < \sqrt{2d_0/K_1}$, ***where*** K_1 ***is defined by formula*** (6.12). ***Then for every two solutions***

$$U(t) = (u(x,t),\ v_1(x,t),\ v_2(x,t),\ v_3(x,t))$$

and

$$U^*(t) = (u^*(x,t),\ v_1^*(x,t),\ v_2^*(x,t),\ v_3^*(x,t))$$

to problem (6.1)–(6.5 c) ***the condition***

$$\lim_{t\to\infty}\ \max_{l=1,\,2} \left\{ |u(x_l,t) - u^*(x_l,t)| + \sum_{j=1}^{3} |v_j(x_l,t) - v_j^*(x_l,t)| \right\} = 0$$

implies their asymptotic closeness in the space $\mathbb{H}_0$:

$$\lim_{t\to\infty} \left\{ \|u(t) - u^*(t)\|^2 + \sum_{j=1}^{3} \|v_j(t) - v_j^*(t)\|^2 \right\} = 0\,. \tag{6.28}$$

Proof.

Let $w = u - u^*$ and let $\psi_j = v_j - v_j^*$, $j = 1,\ 2,\ 3$. We introduce the notations:

$$\Delta = \{x:\ x_1 \le x \le x_2\}, \qquad |\Delta| = x_2 - x_1\,, \quad \text{and} \quad \|w\|_\Delta^2 = \int_{x_1}^{x_2} |w(x)|^2\,\mathrm{d}x\,.$$

Let

$$m(t,\ \Delta) = \max_{l=1,\,2} \left\{ |w(x_l,t)| + \sum_{j=1}^{3} |\psi_j(x_l,t)| \right\}.$$

As in the proof of Theorem 6.1, it is easy to find that

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_\Delta^2 + d_0\|\partial_x w\|_\Delta^2 + \gamma_3\|w\|_\Delta^2 \ \le$$

$$\leq \sum_{l=1,\,2} \left| w(x_l,\,t) \cdot \partial_x w(x_l,\,t) \right| + \sum_{j=1}^{3} a_j \|\psi_j\|_\Delta \cdot \|w\|_\Delta$$

and

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|\psi_j\|_\Delta^2 + k_j^* \|\psi_j\|_\Delta^2 \leq \sum_{l=1,\,2} \left| \psi_j(x_l,\,t) \cdot \partial_x \psi_j(x_l,\,t) \right| + (A_j + B_j) \|\psi_j\|_\Delta \cdot \|w\|_\Delta .$$

It follows from (6.27) that

$$\sup_{t>0} \max_{x \in [0,\,L]} \left\{ \left| \partial_x w(x,\,t) \right| + \sum_{j=1}^{3} \left| \partial_x \psi_j(x,\,t) \right| \right\} < \infty .$$

Therefore, for any $0 < \varepsilon < 1$ and $\theta_j = a_j \cdot (A_j + B_j)^{-1}$, $j = 1,\,2,\,3$, we have

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\|w(t)\|_\Delta^2 + \sum_{j=1}^{3} \theta_j \|\psi_j\|_\Delta^2 \right) + d_0 \|\partial_x w\|_\Delta^2 + \gamma_3 \|w\|_\Delta^2 + \varepsilon \sum_{j+1}^{3} k_j^* \theta_j \|\psi_j\|_\Delta^2 \leq$$

$$\leq K_1 \cdot (1-\varepsilon)^{-1} \cdot \|w\|_\Delta^2 + m(t,\,\Delta) ,$$

where K_1 is defined by formula (6.12). Simple calculations give us that

$$\|w\|_\Delta^2 \equiv \int_{x_1}^{x_2} |w(x)|^2 \mathrm{d}x \leq (1+\eta) \frac{|\Delta|^2}{2} \cdot \int_{x_1}^{x_2} \left| \partial_x w(x) \right|^2 \mathrm{d}x + C_\eta \cdot |\Delta| \cdot \left| w(x_1) \right|^2$$

for any $\eta > 0$. Consequently, if $|\Delta|^2 < 2\,d_0/K_1$, then there exist $0 < \varepsilon < 1$ and $\eta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|w\|_\Delta^2 + \sum_{j=1}^{3} \theta_j \|\psi_j\|_\Delta^2 \right) + \omega \cdot \left(\|w\|_\Delta^2 + \sum_{j=1}^{3} \theta_j \|\psi_j\|_\Delta^2 \right) \leq C \cdot m(t,\,\Delta) ,$$

where ω is a positive constant. As in the proof of Theorem 6.1, it follows that condition $m(t,\,\Delta) \to 0$ as $t \to \infty$ implies that

$$\lim_{t \to \infty} \|U(t) - U^*(t)\|_\Delta \equiv \lim_{t \to \infty} \left(\|w\|_\Delta^2 + \sum_{j=1}^{3} \theta_j \|\psi_j\|_\Delta^2 \right) = 0 . \tag{6.29}$$

Let us now prove (6.28). We do it by reductio ad absurdum. Assume that there exists a sequence $t_n \to +\infty$ such that

$$\overline{\lim_{n \to \infty}} \left\| U(t_n) - U^*(t_n) \right\| > 0 . \tag{6.30}$$

Let $\{V_n\}$ and $\{V_n^*\}$ be sequences lying in the attractor $\mathcal{A}$ of the dynamical system generated by equations (6.1)–(6.5c) and such that

$$\left\| U(t_n) - V_n \right\| \to 0 , \qquad \left\| U^*(t_n) - V_n^* \right\| \to 0 . \tag{6.31}$$

Using the compactness of the attractor we obtain that there exist a sequence $\{n_k\}$ and elements $V,\ V^* \in \mathcal{A}$ such that $V_{n_k} \to V$ and $V^*_{n_k} \to V^*$. Since

$$\|V_n - V^*_n\|_\Delta \le \|U(t_n) - U^*(t_n)\|_\Delta + \|U(t_n) - V_n\| + \|U^*(t_n) - V^*_n\|,$$

it follows from (6.29) and (6.31) that

$$\|V - V^*\|_\Delta = \lim_{k\to\infty} \|V_{n_k} - V^*_{n_k}\|_\Delta = 0\,.$$

Therefore, $V(x) = V^*(x)$ for $x \in \Delta$. However, the Gevrey regularity property implies that elements of the attractor are real analytic functions. The theorem on the uniqueness of such functions gives us that $V(x) \equiv V^*(x)$ for $x \in [0, L]$. Hence, $\|V_{n_k} - V^*_{n_k}\| \to 0$ as $k \to \infty$. Therefore, equation (6.31) implies that

$$\lim_{k\to\infty} \|U(t_{n_k}) - U^*(t_{n_k})\| = 0\,.$$

This contradicts assumption (6.30). **Theorem 6.2 is proved**.

It should be noted that the connection between the Gevrey regularity and the existence of two determining nodes was established in the paper [15] for the first time. The results similar to Theorem 6.2 can also be obtained for other equations (see the references in [3]). However, the requirements of the spatial unidimensionality of the problem and the Gevrey regularity of its solutions are crucial.

§ 7 Determining Functionals for Second Order in Time Equations

In a separable Hilbert space H we consider the problem

$$\ddot{u} + \gamma\dot{u} + Au = B(u, t), \qquad u\big|_{t=s} = u_0, \qquad \dot{u}\big|_{t=s} = u_1, \tag{7.1}$$

where the dot over u stands for the derivative with respect to t, A is a positive operator with discrete spectrum, $\gamma > 0$ is a constant, and $B(u, t)$ is a continuous mapping from $D(A^\theta) \times \mathbb{R}$ into H with the property

$$\|B(u_1, t) - B(u_2, t)\| \le M(\rho)\cdot\|A^\theta(u_1 - u_2)\| \tag{7.2}$$

for some $0 \le \theta < 1/2$ and for all $u_j \in D(A^{1/2})$ such that $\|A^{1/2}u_j\| \le \rho$. Assume that for any $s \in \mathbb{R}$, $u_0 \in D(A^{1/2})$, and $u_1 \in H$ problem (7.1) is uniquely solvable in the class of functions

$$C([s, +\infty); D(A^{1/2})) \cap C^1([s, +\infty), H) \tag{7.3}$$

and defines a process $(\mathcal{H};\ S(t, \tau))$ in the space $\mathcal{H} = D(A^{1/2}) \times H$ with the evolutionary operator given by the formula

$$S(t, s)(u_0;\ u_1) = (u(t);\ \dot{u}(t)), \tag{7.4}$$

where $u(t)$ is a solution to problem (7.1) in the class (7.3). Assume that the process $(\mathcal{H};\ S(t,\ \tau))$ is pointwise dissipative, i.e. there exists $R > 0$ such that

$$\|S(t,\ s)y_0\|_{\mathcal{H}} \le R\ , \qquad t \ge s + t_0(y_0) \tag{7.5}$$

for all initial data $y_0 = (u_0;\ u_1) \in \mathcal{H}$. The nonlinear wave equation (see the book by A. V. Babin and M. I. Vishik [8])

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \gamma \dfrac{\partial u}{\partial t} - \Delta u = f(u), \quad x \in \Omega,\ t > 0\ , \\ u\big|_{\partial\Omega} = 0,\ u\big|_{t=0} = u_0(x),\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = u_1(x)\ , \end{cases}$$

is an example of problem (7.1) which possesses all the properties listed above. Here Ω is a bounded domain in $\mathbb{R}^d$ and the function $f(u) \in C^1(\mathbb{R})$ possesses the properties:

$$-(\lambda_1 - \varepsilon)u^2 - C_1 \le \int_0^u f(v)\,\mathrm{d}v \le C_2 u f(u) + C_3 + \tfrac{1}{2}(\lambda_1 - \varepsilon)u^2\ ,$$

$$|f'(u)| \le C_4(1 + |u|^\beta),$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary conditions on $\partial\Omega$, $C_j > 0$ and $\varepsilon > 0$ are constants, $\beta \le 2(d-2)^{-1}$ for $d \ge 3$ and β is arbitrary for $d = 2$.

Theorem 7.1.

Let $\mathcal{L} = \{l_j\colon\ j = 1,\ \dots,\ N\}$ ***be a set of continuous linear functionals on*** $D(A^{1/2})$. ***Assume that***

$$\varepsilon \equiv \varepsilon_{\mathcal{L}}(D(A^{1/2});\ H) < \left[\frac{\gamma}{M(R)(4 + 4\lambda_1^{-1}\gamma^2)^{3/2}}\right]^{\frac{1}{1-2\theta}}, \tag{7.6}$$

where R ***is the radius of dissipativity (see*** (7.5)***),*** $M(\rho)$ ***and*** $\theta \in [0,\ 1/2)$ ***are the constants from*** (7.2)***, and*** λ_1 ***is the first eigenvalue of the operator*** A. ***Then*** $\mathcal{L}$ ***is an asymptotically determining set for problem*** (7.1) ***in the sense that for a pair of solutions*** $u_1(t)$ ***and*** $u_2(t)$ ***from the class*** (7.3) ***the condition***

$$\lim_{t\to\infty} \int_t^{t+1} |l_j(u_1(\tau) - u_2(\tau))|\,\mathrm{d}\tau = 0 \quad \textit{for} \quad j = 1,\ \dots,\ N \tag{7.7}$$

implies that

$$\lim_{t\to\infty} \left\{\|A^{1/2}(u_1(t) - u_2(t))\|^2 + \|\dot u_1(t) - \dot u_2(t)\|^2\right\} = 0. \tag{7.8}$$

Proof.

We rewrite problem (7.1) in the form

$$\frac{dy}{dt} + \mathcal{A}y = \mathcal{B}(y,\ t), \qquad y\big|_{t=s} = y_0, \tag{7.9}$$

where

$$y = (u;\ \dot{u}), \quad \mathcal{A}y = (-\dot{u};\ \gamma\dot{u} + Au), \quad \mathcal{B}(y,\ t) = (0;\ B(u,\ t)).$$

Lemma 7.1.

There exists an exponential operator $\exp\{-t\mathcal{A}\}$ *in the space* $\mathcal{H} = D(A^{1/2}) \times H$ *and*

$$\|\exp\{-t\mathcal{A}\}y\|_{\mathcal{H}} \le 2\sqrt{1 + \frac{\gamma^2}{\lambda_1}} \cdot \exp\left\{-\frac{\lambda_1 \gamma t}{4\lambda_1 + 2\gamma^2}\right\} \|y\|_{\mathcal{H}}, \tag{7.10}$$

where λ_1 *is the first eigenvalue of the operator* A.

Proof.

Let $y_0 = (u_0;\ u_1)$. Then it is evident that $y(t) = e^{-t\mathcal{A}}y_0 = (u(t);\ \dot{u}(t))$, where $u(t)$ is a solution to the problem

$$\ddot{u} + \gamma\dot{u} + Au = 0, \quad u\big|_{t=0} = u_0, \quad \dot{u}\big|_{t=0} = u_1, \tag{7.11}$$

(see Section 3.7 for the solvability of this problem and the properties of solutions). Let us consider the functional

$$V(u) = \frac{1}{2}\left(\|u\|^2 + \|A^{1/2}u\|^2\right) + \nu\left(u,\ \dot{u} + \frac{\gamma}{2}u\right), \qquad 0 < \nu < \gamma,$$

on the space $\mathcal{H} = D(A^{1/2}) \times H$. It is clear that

$$\frac{1}{2}\left(1 - \frac{\nu}{\gamma}\right)\|\dot{u}\|^2 + \frac{1}{2}\|A^{1/2}u\|^2 \le V(u) \le$$

$$\le \left(\frac{1}{2} + \frac{\nu}{2\gamma}\right)\|\dot{u}\|^2 + \left(\frac{1}{2} + \lambda_1^{-1}\nu\gamma\right)\|A^{1/2}u\|^2 . \tag{7.12}$$

Moreover, for a solution $u(t)$ to problem (7.11) from the class (7.3) with $s = 0$ we have that

$$\frac{d}{dt}V(u(t)) = -(\gamma - \nu)\|\dot{u}\|^2 - \nu\|A^{1/2}u\|^2 . \tag{7.13}$$

Therefore, it follows from (7.12) and (7.13) that

$$\frac{d}{dt}V(u(t)) + \beta V(u(t)) \le$$

$$\le -\left(\gamma - \nu - \left(\frac{1}{2} + \frac{\nu}{2\gamma}\right)\beta\right)\|\dot{u}\|^2 - \left(\nu - \left(\frac{1}{2} + \lambda_1^{-1}\nu\gamma\right)\beta\right)\|A^{1/2}u\|^2 .$$

Hence, for $\nu = (1/2)\gamma$ and $\beta = (2 + \lambda_1^{-1}\gamma^2)^{-1} \cdot \gamma$ we obtain that

$$\frac{d}{dt}V(u(t)) + \beta V(u(t)) \leq 0 ,$$

$$\frac{1}{4}\left(\|\dot{u}\|^2 + \|A^{1/2}u\|^2\right) \leq V(u) \leq \left(\frac{3}{4} + \lambda_1^{-1}\gamma^2\right)\left(\|\dot{u}\|^2 + \|A^{1/2}u\|^2\right).$$

This implies estimate (7.10). Lemma 7.1 is proved.

It follows from (7.9) that

$$y(t) = e^{-(t-s)\mathcal{A}}y(s) + \int_s^t e^{-(t-\tau)\mathcal{A}}\,\mathcal{B}(y(\tau),\ \tau)\,d\tau\ .$$

Therefore, with the help of Lemma 7.1 for the difference of two solutions $y_j(t) = (u_j(t);\ \dot{u}_j(t))$, $j = 1,\ 2$, we obtain the estimate

$$\|y(t)\|_{\mathcal{H}} \leq De^{-\beta(t-s)}\|y(s)\|_{\mathcal{H}} + D\int_s^t e^{-\beta(t-\tau)}\left\|B(u_1(\tau)) - B(u_2(\tau))\right\|d\tau\ , \quad (7.14)$$

where $y(t) = y_1(t) - y_2(t)$ and the constants D and β have the form

$$D = 2\sqrt{1 + \lambda_1^{-1}\gamma^2}\ , \qquad \beta = \frac{\gamma\lambda_1}{4\lambda_1 + 2\gamma^2}\ .$$

By virtue of the dissipativity (7.5) we can assume that $\left\|y_j(t)\right\|_{\mathcal{H}} \leq R$ for all $t \geq s \geq s_0$. Therefore, equations (7.14) and (7.2) imply that

$$\|y(t)\|_{\mathcal{H}} \leq De^{-\beta(t-s)}\|y(s)\|_{\mathcal{H}} + DM(R)\int_s^t e^{-\beta(t-\tau)}\left\|A^{\theta}(u_1(\tau) - u_2(\tau))\right\|d\tau\ .$$

The interpolation inequality (see Exercise 2.1.12)

$$\|A^{\theta}u\| \leq \|u\|^{1-2\theta}\|A^{1/2}u\|^{2\theta}, \qquad 0 \leq \theta \leq \frac{1}{2},$$

Theorem 2.1, and the result of Exercise 2.12 give us that

$$\|A^{\theta}u\| \leq C_{\mathcal{L}}\max_j\left|l_j(u)\right| + \varepsilon_{\mathcal{L}}^{1-2\theta}\|A^{1/2}u\|,$$

where $\varepsilon_{\mathcal{L}} = \varepsilon_{\mathcal{L}}(D(A^{1/2}),\ H)$. Therefore,

$$\|y(t)\|_{\mathcal{H}} \leq De^{-\beta(t-s)}\|y(s)\|_{\mathcal{H}} + DM(R)\cdot\varepsilon_{\mathcal{L}}^{1-2\theta}\int_s^t e^{-\beta(t-\tau)}\|y(\tau)\|_{\mathcal{H}}\,d\tau +$$

$$+ C(\mathcal{L},\ D,\ R)\cdot\int_s^t e^{-\beta(t-\tau)}N_{\mathcal{L}}(u(\tau))\,d\tau\ ,$$

where $N_{\mathcal{L}}(u) = \max\{\left|l_j(u)\right| :\ j = 1,\ 2,\ \ldots,\ N\}$. If we introduce a new unknown function $\psi(t) = e^{\beta t}\|y(t)\|_{\mathcal{H}}$ in this inequality, then we obtain the equation

$$\psi(t) \le D\psi(s) + \alpha\int_s^t \psi(\tau)\,d\tau + C\int_s^t e^{\beta\tau} N_{\mathscr{L}}(u(\tau))\,d\tau\,,$$

where $\alpha = DM(R)\varepsilon_{\mathscr{L}}^{1-2\theta}$. We apply Gronwall's lemma to obtain

$$\psi(t) \le D\psi(s)e^{\alpha(t-s)} + Ce^{\alpha t}\int_s^t e^{-\alpha\tau}\left\{\int_s^\tau e^{\beta\xi} N_{\mathscr{L}}(u(\xi))\,d\xi\right\} d\tau\,.$$

After integration by parts we get

$$\|y(t)\|_{\mathscr{H}} \le D\|y(s)\|_{\mathscr{H}} \exp\{-(\beta-\alpha)(t-s)\} + C\int_s^t e^{-(\beta-\alpha)(t-\tau)} N_{\mathscr{L}}(u(\tau))\,d\tau\,.$$

If equation (7.6) holds, then $\omega = \beta - \alpha > 0$. Therewith it is evident that for $0 < a < t-s$ we have the estimate

$$\|y(t)\|_{\mathscr{H}} \le D\|y(s)\|_{\mathscr{H}} e^{-\omega(t-s)} + C\int_{t-a}^t e^{-\omega(t-\tau)} N_{\mathscr{L}}(u(\tau))\,d\tau + $$
$$+ C\int_s^{t-a} e^{-\omega(t-\tau)} N_{\mathscr{L}}(u(\tau))\,d\tau\,.$$

Therefore, using the dissipativity property we obtain that

$$\|y(t)\|_{\mathscr{H}} \le DR\cdot e^{-\omega(t-s)} + C\int_{t-a}^t N_{\mathscr{L}}(u(\tau))\,d\tau + C(R,\,\mathscr{L})e^{-\omega a}$$

for $t \ge s$ and $0 < a < t-s$. If we fix a and tend the parameter t to infinity, then with the help of (7.7) we find that

$$\overline{\lim_{t\to\infty}}\,\|y(t)\|_{\mathscr{H}} \le C(R,\,\mathscr{L})e^{-\omega a}$$

for any $a > 0$. This implies (7.8). **Theorem 7.1 is proved**.

Unfortunately, because of the fact that condition (7.2) is assumed to hold only for $0 \le \theta < 1/2$, Theorem 7.1 cannot be applied to the problem on nonlinear plate oscillations considered in Chapter 4. However, the arguments in the proof of Theorem 7.1 can be slightly modified and the theorem can still be proved for this case using the properties of solutions to linear nonautonomous problems (see Section 4.2). However, instead of a modification we suggest another approach (see also [3]) which helps us to prove the assertions on the existence of sets of determining functionals for second order in time equations. As an example, let us consider a problem of plate oscillations .

Thus, in a separable Hilbert space H we consider the equation

$$\begin{cases} \ddot{u} + \gamma\dot{u} + A^2 u + M\left(\|A^{1/2}u\|^2\right)Au + Lu = p(t)\,, & (7.15)\\ u\big|_{t=0} = u_0\,, \quad \dot{u}\big|_{t=0} = u_1\,. & (7.16) \end{cases}$$

We assume that A is an operator with discrete spectrum and the function $M(z)$ lies in $C^1(\mathbb{R}_+)$ and possesses the properties:

a) $$\mathcal{M}(z) \equiv \int_0^z M(\xi)\,d\xi \;\geq\; -az - b\,, \tag{7.17}$$

where $0 \leq a < \lambda_1$, $b \in \mathbb{R}$, and λ_1 is the first eigenvalue of the operator A;

b) there exist numbers $a_j > 0$ such that

$$zM(z) - a_1\,\mathcal{M}(z) \;\geq\; a_2\,z^{1+\alpha} - a_3\,, \qquad z \geq 0 \tag{7.18}$$

with some constant $\alpha > 0$.

We also require the existence of $0 \leq \theta < 1$ and $C > 0$ such that

$$\|Lu\| \leq C\|A^\theta u\|\,, \qquad u \in D(A^\theta)\,. \tag{7.19}$$

These assumptions enable us to state (see Sections 4.3 and 4.5) that if

$$u_0 \in D(A)\,, \qquad u_1 \in H\,, \qquad p(t) \in L^\infty(\mathbb{R}_+, H)\,, \qquad \gamma > 0\,, \tag{7.20}$$

then problem (7.15) and (7.16) is uniquely solvable in the class of functions

$$\mathcal{W} = C(\mathbb{R}_+;\; D(A)) \cap C^1(\mathbb{R}_+;\; H)\,. \tag{7.21}$$

Therewith there exists $R > 0$ such that

$$\|Au(t)\|^2 + \|\dot{u}(t)\|^2 \;\leq\; R^2\,, \qquad t \;\geq\; t_0(u_0,\, u_1) \tag{7.22}$$

for any solution $u(t) \in \mathcal{W}$ to problem (7.15) and (7.16).

Theorem 7.2.

Assume that conditions (7.17)–(7.20) ***hold. Let*** $\mathcal{L} = \{l_j\colon\; j = 1,\, 2,\, \ldots,\, N\}$ ***be a set of continuous linear functionals on*** $D(A)$. ***Then there exists*** $\varepsilon_0 > 0$ ***depending both on*** R ***and the parameter of equation*** (7.15) ***such that the condition*** $\varepsilon \equiv \varepsilon_{\mathcal{L}}(D(A);\, H) < \varepsilon_0$ ***implies that*** $\mathcal{L}$ ***is an asymptotically determining set of functionals with respect to*** $D(A) \times H$ ***for problem*** (7.15) ***and*** (7.16) ***in the class of solutions*** $\mathcal{W}$, ***i.e. for two solutions*** $u_1(t)$ ***and*** $u_2(t)$ ***from*** $\mathcal{W}$ ***the condition***

$$\lim_{t\to\infty} \int_t^{t+1} \left|l_j(u_1(\tau) - u_2(\tau))\right|^2 d\tau = 0\,, \qquad l_j \in \mathcal{L} \tag{7.23}$$

implies that

$$\lim_{t\to\infty}\left\{\|\dot u_1(t)-\dot u_2(t)\|^2+\|A(u_1(t)-u_2(t))\|^2\right\}=0. \tag{7.24}$$

Proof.

Let $u_1(t)$ and $u_2(t)$ be solutions to problem (7.15) and (7.16) lying in $\mathcal{W}$. Due to equation (7.22) we can assume that these solutions possess the property

$$\|Au_j(t)\|^2+\|\dot u_j(t)\|^2 \le R^2, \qquad t\ge 0, \qquad j=1,\ 2. \tag{7.25}$$

Let us consider the function $u(t)=u_1(t)-u_2(t)$ as a solution to equation

$$\ddot u+\gamma\dot u+A^2u+M\left(\|A^{1/2}u_1(t)\|^2\right)Au=F(u_1(t),\ u_2(t)), \tag{7.26}$$

where

$$F(u_1,\ u_2)=\left[M\left(\|A^{1/2}u_2\|^2\right)-M\left(\|A^{1/2}u_1\|^2\right)\right]Au_2-L(u_1-u_2).$$

It follows from (7.19) and (7.25) that

$$\|F(u_1(t),\ u_2(t))\| \le C_R\left(\|A^{1/2}u\|+\|A^\theta u\|\right). \tag{7.27}$$

Let us consider the functional

$$V(u,\ \dot u;\ t) = \frac{1}{2}E(u,\ \dot u;\ t)+\nu\left\{(u,\ \dot u)+\frac{\gamma}{2}\|u\|^2\right\} \tag{7.28}$$

on the space $\mathcal{H}=D(A)\times H$, where

$$E(u,\ \dot u;\ t) = \|\dot u\|^2+\|Au\|^2+M\left(\|A^{1/2}u_1(t)\|^2\right)\|A^{1/2}u\|^2+\mu\|u\|^2$$

and the positive parameters μ and ν will be chosen below. It is clear that for $(u;\ \dot u)\in\mathcal{H}$ we have

$$E(u,\ \dot u;\ t) \ge \|\dot u\|^2+\|Au\|^2+m_R\|A^{1/2}u\|^2+\mu\|u\|^2,$$

where $m_R=\min\{M(z):\ 0\le z\le \lambda_1^{-1}R^2\}$. Moreover,

$$-\frac{1}{2\gamma}\|\dot u\|^2 \le (u,\ \dot u)+\frac{\gamma}{2}\|\dot u\|^2 \le \frac{1}{2\gamma}\|\dot u\|^2+\gamma\|u\|^2.$$

Therefore, the value μ can be chosen such that

$$\alpha_1\left(\|Au\|^2+\|\dot u\|^2\right) \le V(u,\ t) \le \alpha_2\left(\|Au\|^2+\|\dot u\|^2\right) \tag{7.29}$$

for all $0<\nu<\gamma$, where α_1 and α_2 are positive numbers depending on R. Let us now estimate the value $(\mathrm{d}/\mathrm{d}t)V(u(t),\ \dot u(t);\ t)$. Due to (7.26) we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u(t),\ \dot u(t);\ t) = -\gamma\|\dot u(t)\|^2 + \mu(u(t),\ \dot u(t)) +$$

$$+\ M'\left(\|A^{1/2}u_1(t)\|^2\right)(Au_1(t),\ \dot u_1(t))\|A^{1/2}u(t)\|^2 + (F(u_1(t),\ u_2(t)),\ \dot u(t)).$$

With the help of (7.25) and (7.27) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(u,\,\dot{u};\;t)\;\le\;-\frac{\gamma}{2}\|\dot{u}(t)\|^2+C_R\left(\|A^{1/2}u\|^2+\|A^{\theta}u\|^2\right).$$

Using (7.26) and (7.27) it is also easy to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{(u,\,\dot{u})+\frac{\gamma}{2}(u,\,u)\right\}\;=\;\|\dot{u}\|^2+(u,\;\ddot{u}+\gamma\dot{u})\;\le$$

$$\le\;\|\dot{u}\|^2-\|Au\|^2+M_R\|A^{1/2}u\|^2+\;C_R\left(\|A^{1/2}u\|^2+\|A^{\theta}u\|^2\right)\|u\|\;,$$

where

$$M_R=\max\left\{|M(z)|:\;\;0\le z\le\lambda_1^{-1}R^2\right\}.$$

We choose $\nu=\gamma/4$ and use the estimate of the form

$$\|A^{\beta}u\|\;\le\;\|Au\|^{\beta}\cdot\|u\|^{1-\beta}\;\le\;\varepsilon\|Au\|+C_{\varepsilon}\|u\|\,,\qquad 0<\beta<1\,,\quad\varepsilon>0\,,$$

to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u,\,\dot{u};\;t)\;=\;\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}E(u,\,\dot{u};\;t)+\nu\left(u,\;\dot{u}+\frac{\gamma}{2}u\right)\right\}\;\le$$

$$\le\;-\frac{\gamma}{8}\left(\|Au\|^2+\|\dot{u}\|^2\right)+C_R\|u(t)\|^2\;.$$

Therefore, using the estimate

$$\|u(t)\|^2\;\le\;C_{\mathscr{L}}\max_j\left|l_j\,(u)\right|^2\;+\;2\,\varepsilon_{\mathscr{L}}(D(A),\;H)\|Au\|^2$$

and equation (7.29) we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u(t),\;\dot{u}(t);\;t)+\omega V(u(t),\;\dot{u}(t);\;t)\;\le\;C\,\max_j\left|l_j(u(t))\right|^2\;,$$

provided $\varepsilon_{\mathscr{L}}(D(A),\,H)<\varepsilon_0=\frac{\gamma}{16}\,C_R^{-1}$. Here ω is a positive constant. As above, this easily implies (7.24), provided (7.23) holds. **Theorem 7.2 is proved**.

— Exercise *7.1* Show that the method used in the proof of Theorem 7.2 also enables us to obtain the assertion of Theorem 7.1 for problem (7.1).

— Exercise *7.2* Using the results of Section 4.2 related to the linear variant of equation (7.15), prove that the method of the proof of Theorem 7.1 can also be applied in the proof of Theorem 7.2.

Thus, the methods presented in the proofs of Theorems 7.1 and 7.2 are close to each other. The same methods with slight modifications can also be used in the study of problems like (7.1) with additional retarded terms (see [3]).

Exercise *7.3* Using the estimates for the difference of two solutions to equation (7.15) proved in Lemmata 4.6.1 and 4.6.2, find an analogue of Theorems 1.3 and 4.4 for the problem (7.15) and (7.16).

§ 8 On Boundary Determining Functionals

The fact (see Sections 5–7 as well as paper [3]) that in some cases determining functionals can be defined on some auxiliary space admits in our opinion an interesting generalization which leads to the concept of boundary determining functionals. We now clarify this by giving the following simple example.

In a smooth bounded domain $\Omega \subset \mathbb{R}^d$ we consider a parabolic equation with the nonlinear boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \Delta u - f(u), \quad x \in \Omega,\ t > 0\ , \\[2ex] \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} + h(u)\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0(x)\ . \end{cases} \tag{8.1}$$

Assume that ν is a positive parameter, $f(z)$ and $h(z)$ are continuously differentiable functions on $\mathbb{R}^1$ such that

$$f'(z) \ge -\alpha\,, \qquad |h'(z)| \le \beta\,, \tag{8.2}$$

where $\alpha \ge 0$ and $\beta > 0$ are constants. Let

$$\mathcal{W} = C^{2,1}(\Omega \times \mathbb{R}_+) \cap C^{1,0}(\overline{\Omega \times \mathbb{R}_+})\,. \tag{8.3}$$

Here $C^{2,1}(\Omega \times \mathbb{R}_+)$ is a set of functions $u(x, t)$ on $\Omega \times \mathbb{R}_+$ that are twice continuously differentiable with respect to x and continuously differentiable with respect to t. The notation $C^{1,0}(\overline{\Omega \times \mathbb{R}_+})$ has a similar meaning, the bar denotes the closure of a set.

Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions to problem (8.1) lying in the class $\mathcal{W}$ (we do not discuss the existence of such solutions here and refer the reader to the book [7]). We consider the difference $u(t) = u_1(t) - u_2(t)$. Then (8.1) evidently implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)} + (f(u_1(t)) - f(u_2(t)),\ u(t))_{L^2(\Omega)} =$$

$$= -\nu \int\limits_{\partial\Omega} u(t)(h(u_1(t)) - h(u_2(t)))\,\mathrm{d}\sigma\ .$$

Using (8.2) we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} + \nu\|\nabla u(t)\|^2_{L^2(\Omega)} - \alpha\|u(t)\|^2_{L^2(\Omega)} \le \nu\beta\cdot\|u(t)\|^2_{L^2(\partial\Omega)} . \quad (8.4)$$

One can show that there exist constants c_1 and c_2 depending on the domain Ω only and such that

$$\|u\|^2_{L^2(\Omega)} \le c_1\Big(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big), \quad (8.5)$$

$$\|u\|^2_{H^{1/2}(\partial\Omega)} \le c_2\Big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big). \quad (8.6)$$

Here $H^s(\partial\Omega)$ is the Sobolev space of the order s on the boundary of the domain Ω. Equations (8.4)–(8.6) enable us to obtain the following assertion.

Theorem 8.1.

Let $\mathscr{L} = \{l_j \colon j = 1, \dots, N\}$ ***be a set of continuous linear functionals on the space*** $H^{1/2}(\partial\Omega)$. ***Assume that*** $\alpha c_1 < \nu$ ***and***

$$\varepsilon_{\mathscr{L}} \equiv \varepsilon_{\mathscr{L}}(H^{1/2}(\partial\Omega),\ L^2(\partial\Omega)) < \left[\frac{\nu - \alpha c_1}{\nu(1+c_1)\,c_2(1+\beta)}\right]^{1/2} \equiv \varepsilon_0 , \quad (8.7)$$

where the constants ν, α, β, c_1, ***and*** c_2 ***are defined in equations*** (8.1), (8.2), (8.5), ***and*** (8.6). ***Then*** $\mathscr{L}$ ***is an asymptotically determining set with respect to*** $L^2(\Omega)$ ***for problem*** (8.1) ***in the class of classical solutions*** $\mathscr{W}$.

Proof.

Let $u(t) = u_1(t) - u_2(t)$, where $u_j(t) \in \mathscr{W}$ are solutions to problem (8.1). Theorem 2.1 implies that

$$\|u\|^2_{L^2(\partial\Omega)} \le C_{\mathscr{L},\,\delta} \max_j |l_j(u)|^2 + (1+\delta)\varepsilon^2_{\mathscr{L}}\|u\|^2_{H^{1/2}(\Omega)} \quad (8.8)$$

for any $\delta > 0$. Equations (8.5) and (8.6) imply that

$$\|u\|^2_{L^2(\partial\Omega)} \le C_{\mathscr{L},\,\delta} \max_j |l_j(u)|^2 + (1+c_1)c_2(1+\delta)\,\varepsilon^2_{\mathscr{L}}\Big(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big).$$

Therefore, equation (8.4) gives us that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2(\Omega)} + \nu\Big[\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big] - \alpha\|u(t)\|^2_{L^2(\Omega)} \le$$

$$\le \nu(1+\beta)\|u(t)\|_{L^2(\partial\Omega)} \le \nu(1+c_1)\,c_2(1+\delta)(1+\beta)\,\varepsilon^2_{\mathscr{L}}\Big(\|u\|^2_{L^2(\partial\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\Big) +$$

$$+ C \max_j |l_j(u)|^2 .$$

Using estimate (8.5) once again we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{\nu}{c_1}\left\{1 - (1+c_1)c_2(1+\delta)(1+\beta)\varepsilon^2_{\mathscr{L}} - \alpha\frac{c_1}{\nu}\right\}\|u\|^2 \le C \max_j |l_j(u)|^2, \quad (8.9)$$

provided that

$$1-(1+c_1)c_2(1+\delta)(1+\beta)\varepsilon_{\mathscr{L}}^2 - \alpha\frac{c_1}{\nu} > 0\,. \tag{8.10}$$

It is evident that (8.10) with some $\delta > 0$ follows from (8.7). Therefore, inequality (8.9) enables us to **complete the proof** of the theorem.

Thus, the analogue of Theorem 3.1 for smooth surfaces enables us to state that problem (6.1) has finite determining sets of boundary local surface averages.

An assertion similar to Theorem 8.1 can also be obtained (see [3]) for a nonlinear wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = \Delta u - f(u)\,, \qquad x \in \Omega \subset \mathbb{R}^d\,, \qquad t > 0\,,$$

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma} = -\alpha\left.\frac{\partial u}{\partial t}\right|_{\Gamma} - \varphi\left(u|_{\Gamma}\right), \quad u|_{\partial\Omega\setminus\Gamma} = 0\,, \quad u|_{t=0} = u_0(x)\,, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x)\,.$$

Here Γ is a smooth open subset on the boundary of Ω, $f(u)$ and $\varphi(u)$ are bounded continuously differentiable functions, and α and γ are positive parameters.

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Chapter 6

Homoclinic Chaos in Infinite-Dimensional Systems

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In this chapter we consider some questions on the asymptotic behaviour of a discrete dynamical system. We remind (see Chapter 1) that a discrete dynamical system is defined as a pair (X, S) consisting of a metric space X and a continuous mapping of X into itself. Most assertions on the existence and properties of attractors given in Chapter 1 remain true for these systems. It should be noted that the following examples of discrete dynamical systems are the most interesting from the point of view of applications: a) systems generated by monodromy operators (period mappings) of evolutionary equations, with coefficients being periodic in time; b) systems generated by difference schemes of the type $\tau^{-1}(u_{n+1} - u_n) = F(u_n)$, $n = 0, 1, 2, \ldots$ in a Banach space X (see Examples 1.5 and 1.6 of Chapter 1).

The main goal of this chapter is to give a strict mathematical description of one of the mechanisms of a complicated (irregular, chaotic) behaviour of trajectories. We deal with the phenomenon of the so-called homoclinic chaos. This phenomenon is well-known and is described by the famous Smale theorem (see, e.g., [1–3]) for finite-dimensional systems. This theorem is of general nature and can be proved for infinite-dimensional systems. Its proof given in Section 5 is based on an infinite-dimensional variant of Anosov's lemma on ε-trajectories (see Section 4). The considerations of this Chapter are based on the paper [4] devoted to the finite-dimensional case as well as on the results concerning exponential dichotomies of infinite-dimensional systems given in Chapter 7 of the book [5]. We follow the arguments given in [6] while proving Anosov's lemma.

§ 1 *Bernoulli Shift as a Model of Chaos*

Mathematical simulation of complicated dynamical processes which take place in real systems requires that the notion of a state of chaos be formalized. One of the possible approaches to the introduction of this notion relies on a selection of a class of explicitly solvable models with complicated (in some sense) behaviour of trajectories. Then we can associate every model of the class with a definite type of chaotic behaviour and use these models as standard ones comparing their dynamical structure with a qualitative behaviour of the dynamical system considered. A discrete dynamical system known as *the Bernoulli shift* is one of these explicitly solvable models.

Let $m \geq 2$ and let

$$\Sigma_m = \left\{ x = (\ldots, x_{-1}, x_0, x_1, \ldots):\ \ x_j \in \{1, 2, \ldots, m\},\ j \in \mathbb{Z} \right\},$$

i.e. Σ_m is a set of two-sided infinite sequences the elements of which are the integers $1, 2, \ldots, m$. Let us equip the set Σ_m with a metric

$$d(x, y) = \sum_{i=-\infty}^{\infty} 2^{-|i|} \frac{|x_i - y_i|}{1 + |x_i - y_i|} . \tag{1.1}$$

Here $x = \{x_i: \ i \in \mathbb{Z}\}$ and $y = \{y_i: \ i \in \mathbb{Z}\}$ are elements of Σ_m. Other methods of introduction of a metric in Σ_m are given in Example 1.1.7 and Exercise 1.1.5.

— Exercise *1.1* Show that the function $d(x, y)$ satisfies all the axioms of a metric.

— Exercise *1.2* Let $x = \{x_i\}$ and $y = \{y_i\}$ be elements of the set Σ_m. Assume that $x_i = y_i$ for $|i| \le N$ and for some integer N. Prove that $d(x, y) \le 2^{-N+1}$.

— Exercise *1.3* Assume that equation $d(x, y) < 2^{-N}$ holds for $x, y \in \Sigma_m$, where N is a natural number. Show that $x_i = y_i$ for all $|i| \le N-1$ (*Hint*: $d(x, y) \ge 2^{-|i|-1}$ if $x_i \ne y_i$).

— Exercise *1.4* Let $x \in \Sigma_m$ and let

$$\mathcal{U}^N(x) = \{y \in \Sigma_m: \ x_i = y_i \quad \text{for} \ |i| \le N\} . \tag{1.2}$$

Prove that for any $0 < \varepsilon < 1$ the relation

$$\mathcal{U}^{N(\varepsilon)+2}(x) \subset \{y: \ d(x, y) < \varepsilon\} \subset \mathcal{U}^{N(\varepsilon)-1}(x)$$

holds, where $N(\varepsilon)$ is an integer with the property

$$N(\varepsilon) < \frac{\ln 1/\varepsilon}{\ln 2} \le N(\varepsilon) + 1 .$$

— Exercise *1.5* Show that the space Σ_m with metric (1.1) is a compact metric space.

In the space Σ_m we define a mapping S which shifts every sequence one symbol left, i.e.

$$[Sx]_i = x_{i+1} , \qquad i \in \mathbb{Z} , \qquad x = \{x_i\} \in \Sigma_m .$$

Evidently, S is invertible and the relations

$$d(Sx, Sy) \le 2d(x, y) , \qquad d(S^{-1}x, S^{-1}y) \le 2d(x, y)$$

hold for all $x, y \in \Sigma_m$. Therefore, the mapping S is a homeomorphism.

The discrete dynamical system (Σ_m, S) is called ***the Bernoulli shift*** of the space of sequences of m symbols. Let us study the dynamical properties of the system (Σ_m, S).

— Exercise *1.6* Prove that (Σ_m, S) has m fixed points exactly. What structure do they have?

We call an arbitrary ordered collection $a = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_j \in \{1, \ldots, m\}$ a segment (of the length N). Each element $x \in \Sigma_m$ can be considered as an ordered infinite family of finite segments while the elements of the set Σ_m can be constructed from segments. In particular, using the segment $a = (\alpha_1, \ldots, \alpha_N)$ we can construct a periodic element $\bar{a} \in \Sigma_m$ by the formula

$$\bar{a}_{Nk+j} = \alpha_j, \quad j \in \{1, \ldots, N\}, \quad k \in \mathbb{Z}. \tag{1.3}$$

— Exercise *1.7* Let $a = (\alpha_1, \ldots, \alpha_N)$ be a segment of the length N and let $\bar{a} \in \Sigma_m$ be an element defined by (1.3). Prove that $\bar{a}$ is a periodic point of the period N of the dynamical system (Σ_m, S), i.e. $S^N \bar{a} = \bar{a}$.

— Exercise *1.8* Prove that for any natural N there exists a periodic point of the minimal period equal to N.

— Exercise *1.9* Prove that the set of all periodic points is dense in Σ_m, i.e. for every $x \in \Sigma_m$ and $\varepsilon > 0$ there exists a periodic point a with the property $d(x, a) < \varepsilon$ (*Hint*: use the result of Exercise 1.4).

— Exercise *1.10* Prove that the set of nonperiodic points is not countable.

— Exercise *1.11* Let $a = (\ldots, \alpha, \alpha, \alpha, \ldots)$ and $b = (\ldots, \beta, \beta, \beta, \ldots)$ be fixed points of the system (Σ_m, S). Let $C = \{c_i\}$ be an element of Σ_m such that $c_i = \alpha$ for $i \le -M_1$ and $c_i = \beta$ for $i \ge M_2$, where M_1 and M_2 are natural numbers. Prove that

$$\lim_{n \to -\infty} S^n c = a, \qquad \lim_{n \to \infty} S^n c = b. \tag{1.4}$$

Assume that an element $c \in \Sigma_m$ possesses property (1.4) with $c \neq a$ and $c \neq b$. If $a \neq b$, then the set

$$\gamma_{a,b} = \{S^n c: \ n \in \mathbb{Z}\}$$

is called a ***heteroclinic trajectory*** that connects the fixed points a and b. If $a = b$, then $\gamma_a = \gamma_{a,a}$ is called a ***homoclinic trajectory*** of the point a. The elements of a heteroclinic (homoclinic, respectively) trajectory are called heteroclinic (homoclinic, respectively) points.

— Exercise *1.12* Prove that for any pair of fixed points there exists an infinite number of heteroclinic trajectories connecting them whereas the corresponding set of heteroclinic points is dense in Σ_m.

— Exercise *1.13* Let

$$\gamma_1 = \{S^n a: \ n \in \mathbb{Z}\} = \{S^n a: \ n = 0, 1, \ldots, N_1 - 1\}$$

and

$$\gamma_2 = \{S^n b: \ n \in \mathbb{Z}\} = \{S^n b: \ n = 0, 1, \ldots, N_2 - 1\}$$

be cycles (periodic trajectories). Prove that there exists a heteroclinic trajectory $\gamma_{1,2} = \{S^n c:\ n \in \mathbb{Z}\}$ that connects the cycles γ_1 and γ_2, i.e. such that

$$\operatorname{dist}(S^n c,\ \gamma_1) \equiv \inf_{x\in\gamma_1} d(S^n c,\ x) \to 0\,, \qquad n \to -\infty$$

and

$$\operatorname{dist}(S^n c,\ \gamma_2) = \inf_{x\in\gamma_2} d(S^n c,\ x) \to 0\,, \qquad n \to +\infty\,.$$

For every N there exists only a finite number of segments of the length N. Therefore, the set $\mathcal{L}$ of all segments is countable, i.e. we can assume that $\mathcal{L} = \{a_k:\ k = 1, 2, \dots\}$, therewith the length of the segment a_{k+1} is not less than the length of a_k. Let us construct an element $b = \{b_i:\ i \in \mathbb{Z}\}$ from Σ_m taking $b_i = 1$ for $i \le 0$ and sequentially putting all the segments a_k to the right of the zeroth position. As a result, we obtain an element of the form

$$b = (\dots,\ 1,\ 1,\ 1,\ a_1,\ a_2,\ a_3,\ \dots)\,, \qquad a_j \in \mathcal{L}. \tag{1.5}$$

— Exercise *1.14* Prove that a positive semitrajectory $\gamma_+ = \{S^n b,\ n \ge 0\}$ with b having the form (1.5) is dense in Σ_m, i.e. for every $x \in \Sigma_m$ and $\varepsilon > 0$ there exists $n = n(x, \varepsilon)$ such that $d(x,\ S^n b) < \varepsilon$.

— Exercise *1.15* Prove that the semitrajectory γ_+ constructed in Exercise 1.14 returns to an ε-vicinity of every point $x \in \Sigma_m$ infinite number of times (*Hint*: see Exercises 1.4 and 1.9).

— Exercise *1.16* Construct a negative semitrajectory $\gamma_- = \{S^n c:\ n \le 0\}$ which is dense in Σ_m.

Thus, summing up the results of the exercises given above, we obtain the following assertion.

Theorem 1.1.

The dynamical system (Σ_m, S) of the Bernoulli shift of sequences of m symbols possesses the properties:

1) ***there exists a finite number of fixed points;***
2) ***there exist periodic orbits of any minimal period and the set of these orbits is dense in the phase space Σ_m;***
3) ***the set of nonperiodic points is uncountable;***
4) ***heteroclinic and homoclinic points are dense in the phase space;***
5) ***there exist everywhere dense trajectories.***

All these properties clearly imply the extraordinarity and complexity of the dynamics in the system (Σ_m, S). They also give a motivation for the following definitions.

Let (X, f) be a discrete dynamical system. The dynamics of the system (X, f) is called ***chaotic*** if there exists a natural number k such that the mapping f^k is topologically conjugate to the Bernoulli shift for some m, i.e. there exists a homeomorphism $h\colon X \to \Sigma_m$ such that $h(f^k(x)) = S(h(x))$ for all $x \in X$. We also say that chaotic dynamics is observed in the system (X, f) if there exist a number k and a set Y in X invariant with respect to f^k $(f^k Y \subset Y)$ such that the restriction of f^k to Y is topologically conjugate to the Bernoulli shift.

It turns out that if a dynamical system (X, f) has a fixed point and a corresponding homoclinic trajectory, then chaotic dynamics can be observed in this system under some additional conditions (this assertion is the core of the Smale theorem). Therefore, we often speak about homoclinic chaos in this situation. It should also be noted that the approach presented here is just one of the possible methods used to describe chaotic behaviour (for example, other approaches can be found in [1] as well as in book [7], the latter contains a survey of methods used to study the dynamics of complicated systems and processes).

§ 2 Exponential Dichotomy and Difference Equations

This is an auxiliary section. Nonautonomous linear difference equations of the form

$$x_{n+1} = A_n x_n + h_n, \qquad n \in \mathbb{Z}, \tag{2.1}$$

in a Banach space X are considered here. We assume that $\{A_n\}$ is a family of linear bounded operators in X, h_n is a sequence of vectors from X. Some results both on the dichotomy (splitting) of solutions to homogeneous $(h_n \equiv 0)$ equation (2.1) and on the existence and properties of bounded solutions to nonhomogeneous equation are given here. We mostly follow the arguments given in book [5] as well as in paper [4] devoted to the finite-dimensional case.

Thus, let $\{A_n\colon\ n \in \mathbb{Z}\}$ be a sequence of linear bounded operators in a Banach space X. Let us consider a homogeneous difference equation

$$x_{n+1} = A_n x_n, \qquad n \in J, \tag{2.2}$$

where J is an interval in $\mathbb{Z}$, i.e. a set of integers of the form

$$J = \{n \in \mathbb{Z}\colon\ m_1 < n < m_2\},$$

where m_1 and m_2 are given numbers, we allow the cases $m_1 = -\infty$ and $m_2 = +\infty$. Evidently, any solution $\{x_n\colon\ n \in J\}$ to difference equation (2.2) possesses the property

$$x_m = \Phi(m, n) x_n, \qquad m \ge n, \qquad m, n \in J,$$

where $\Phi(m, n) = A_{m-1} \cdot \ldots \cdot A_n$ for $m > n$ and $\Phi(m, m) = I$. The mapping $\Phi(m, n)$ is called an ***evolutionary operator*** of problem (2.2).

— Exercise *2.1* Prove that for all $m \geq n \geq k$ we have

$$\Phi(m, k) = \Phi(m, n)\Phi(n, k).$$

— Exercise *2.2* Let $\{P_n : n \in J\}$ be a family of projectors (i.e. $P_n^2 = P_n$) in X such that $P_{n+1}A_n = A_n P_n$. Show that

$$P_m \Phi(m, n) = \Phi(m, n)P_n, \qquad m \geq n, \qquad m, n \in J,$$

i.e. the evolutionary operator $\Phi(m, n)$ maps $P_n X$ into $P_m X$.

— Exercise *2.3* Prove that solutions $\{x_n\}$ to nonhomogeneous difference equation (2.1) possess the property

$$x_m = \Phi(m, n)x_n + \sum_{k=n}^{m-1} \Phi(m, k+1)h_k, \qquad m > n.$$

Let us give the following definition. A family of linear bounded operators $\{A_n\}$ is said to possess an ***exponential dichotomy*** over an interval J with constants $K > 0$ and $0 < q < 1$ if there exists a family of projectors $\{P_n : n \in J\}$ such that

a) $$P_{n+1}A_n = A_n P_n, \qquad n, n+1 \in J;$$

b) $$\|\Phi(m, n)P_n\| \leq Kq^{m-n}, \qquad m \geq n, \quad m, n \in J;$$

c) for $n \geq m$ the evolutionary operator $\Phi(n, m)$ is a one-to-one mapping of the subspace $(1-P_m)X$ onto $(1-P_n)X$ and the following estimate holds:

$$\left\|\Phi(n, m)^{-1}(1-P_n)\right\| \leq Kq^{n-m}, \quad m \leq n, \quad m, n \in J.$$

If these conditions are fulfilled, then it is also said that difference equation (2.2) admits an exponential dichotomy over J. It should be noted that the cases $J = \mathbb{Z}$ and $J = \mathbb{Z}_{\pm}$ are the most interesting for further considerations, where $\mathbb{Z}_+$ ($\mathbb{Z}_-$) is the set of all nonnegative (nonpositive) integers.

The simplest case when difference equation (2.2) admits an exponential dichotomy is described in the following example.

— Example *2.1* (autonomous case)

Assume that equation (2.2) is autonomous, i.e. $A_n \equiv A$ for all n, and the spectrum $\sigma(A)$ does not intersect the unit circumference $\{z \in \mathbb{C} : |z| = 1\}$. Linear operators possessing this property are often called hyperbolic (with respect to the fixed point $x = 0$). It is well-known (see, e.g., [8]) that in this case there exists a projector P with the properties:

a) $AP = PA$, i.e. the subspaces PX and $(1-P)X$ are invariant with respect to A;
b) the spectrum $\sigma(A|_{PX})$ of the restriction of the operator A to PX lies strictly inside of the unit disc;
c) the spectrum $\sigma(A|_{(1-P)X})$ of the restriction of A to the subspace $(1-P)X$ lies outside the unit disc.

— Exercise *2.4* Let C be a linear bounded operator in a Banach space X and let $\rho \equiv \max\{|z| : z \in \sigma(C)\}$ be its spectral radius. Show that for any $q > \rho$ there exists a constant $M_q \geq 1$ such that

$$\|C^n\| \leq M_q q^n , \qquad n = 0,\ 1,\ 2,\ \ldots$$

(*Hint*: use the formula $\rho = \lim_{n \to \infty} \|C^n\|^{1/n}$ the proof of which can be found in [9], for example).

Applying the result of Exercise 2.4 to the restriction of the operator A to PX, we obtain that there exist $K > 0$ and $0 < q < 1$ such that

$$\|A^n P\| \leq K q^n , \qquad n \geq 0 . \tag{2.3}$$

It is also evident that the restriction of the operator A to $(1-P)X$ is invertible and the spectrum of the inverse operator lies inside the unit disc. Therefore,

$$\|A^{-n}(1-P)\| \leq K q^n , \qquad n \geq 0 , \tag{2.4}$$

where the constants $K > 0$ and $0 < q < 1$ can be chosen the same as in (2.3). The evolutionary operator $\Phi(m, n)$ of the difference equation $x_{n+1} = Ax_n$ has the form $\Phi(m, n) = A^{m-n}$, $m \geq n$. Therefore, the equality $AP = PA$ and estimates (2.3) and (2.4) imply that the equation $x_{n+1} = Ax_n$ admits an exponential dichotomy over $\mathbb{Z}$, provided the spectrum of the operator A does not intersect the unit circumference.

— Exercise *2.5* Assume that for the operator A there exists a projector P such that $AP = PA$ and estimates (2.3) and (2.4) hold with $0 < q < 1$. Show that the spectrum of the operator A does not intersect the unit circumference, i.e. A is hyperbolic.

Thus, the hyperbolicity of the linear operator A is equivalent to the exponential dichotomy over $\mathbb{Z}$ of the difference equation $x_{n+1} = Ax_n$ with the projectors P_n independent of n. Therefore, the dichotomy property of difference equation (2.2) should be considered as an extension of the notion of hyperbolicity to the nonautonomous case. The meaning of this notion is explained in the following two exercises.

— Exercise *2.6* Let A be a hyperbolic operator. Show that the space X can be decomposed into a direct sum of stable X^s and unstable X^u subspaces, i.e. $X = X^s + X^u$ therewith

$$\|A^n x\| \le K q^n \|x\|, \qquad x \in X^s, \qquad n \ge 0,$$

$$\|A^n x\| \ge K^{-1} q^{-n} \|x\|, \qquad x \in X^u, \qquad n \ge 0,$$

with some constants $K > 0$ and $0 < q < 1$.

Exercise *2.7* Let $X = \mathbb{R}^2$ be a plane and let A be an operator defined by the formula

$$A(x_1, x_2) = (2x_1 + x_2;\ x_1 + x_2), \qquad x = (x_1;\ x_2) \in \mathbb{R}^2.$$

Show that the operator A is hyperbolic. Evaluate and display graphically stable X^s and unstable X^u subspaces on the plane. Display graphically the trajectory $\{A^n x:\ n \in \mathbb{Z}\}$ of some point x that lies neither in X^s, nor in X^u.

The next assertion (its proof can be found in the book [5]) plays an important role in the study of existence conditions of exponential dichotomy of a family of operators $\{A_n:\ n \in \mathbb{Z}\}$.

Theorem 2.1.

Let $\{A_n:\ n \in \mathbb{Z}\}$ be a sequence of linear bounded operators in a Banach space X. Then the following assertions are equivalent:

(i) ***the sequence $\{A_n:\ n \in \mathbb{Z}\}$ possesses an exponential dichotomy over $\mathbb{Z}$,***

(ii) ***for any bounded sequence $\{h_n:\ n \in \mathbb{Z}\}$ from X there exists a unique bounded solution $\{x_n:\ n \in \mathbb{Z}\}$ to the nonhomogeneous difference equation***

$$x_{n+1} = A_n x_n + h_n, \qquad n \in \mathbb{Z}. \tag{2.5}$$

In the case when the sequence $\{A_n\}$ possesses an exponential dichotomy, solutions to difference equation (2.5) can be constructed using ***the Green function***:

$$G(n, m) = \begin{cases} \Phi(n, m) P_m, & n \ge m, \\ -[\Phi(m, n)]^{-1}(1 - P_m), & n < m. \end{cases}$$

Exercise *2.8* Prove that $\|G(n, m)\| \le K q^{|n-m|}$.

Exercise *2.9* Prove that for any bounded sequence $\{h_n:\ n \in \mathbb{Z}\}$ from X a solution to equation (2.5) has the form

$$x_n = \sum_{m \in \mathbb{Z}} G(n, m+1) h_m, \qquad n \in \mathbb{Z}.$$

Moreover, the following estimate is valid:

$$\sup_n \|x_n\| \le K \frac{1+q}{1-q} \sup_n \|h_n\| .$$

The properties of the Green function enable us to prove the following assertion on the uniqueness of the family of projectors $\{P_n\}$.

Lemma 2.1.

Let a sequence $\{A_n\}$ possess an exponential dichotomy over $\mathbb{Z}$. Then the projectors $\{P_n : n \in \mathbb{Z}\}$ are uniquely defined.

Proof.

Assume that there exist two collections of projectors $\{P_n\}$ and $\{Q_n\}$ for which the sequence $\{A_n\}$ possesses an exponential dichotomy. Let $G_P(n, m)$ and $G_Q(n, m)$ be Green functions constructed with the help of these collections. Then Theorem 2.1 enables us to state (see Exercise 2.9) that

$$\sum_{m \in \mathbb{Z}} G_P(n, m+1) h_m = \sum_{m \in \mathbb{Z}} G_Q(n, m+1) h_m$$

for all $n \in \mathbb{Z}$ and for any bounded sequence $\{h_n\} \subset X$. Assuming that $h_m = 0$ for $m \neq k-1$ and $h_m = h$ for $m = k-1$, we find that

$$G_P(n, k)h = G_Q(n, k)h , \qquad h \in X, \quad n, k \in \mathbb{Z}, \quad n \ge k .$$

This equality with $n = k$ gives us that $P_n h = Q_n h$. Thus, the lemma is proved.

In particular, Theorem 2.1 implies that in order to prove the existence of an exponential dichotomy it is sufficient to make sure that equation (2.5) is uniquely solvable for any bounded right-hand side. It is convenient to consider this difference equation in the space $l_X^\infty \equiv l^\infty(\mathbb{Z}, X)$ of sequences $\boldsymbol{x} = \{x_n : n \in \mathbb{Z}\}$ of elements of X for which the norm

$$|\boldsymbol{x}|_{l^\infty} = |\{x_n\}|_{l^\infty} = \sup\left\{\|x_n\| : n \in \mathbb{Z}\right\} \tag{2.6}$$

is finite. Assume that the condition

$$\sup\left\{\|A_n\| : n \in \mathbb{Z}\right\} < \infty \tag{2.7}$$

is valid. Then for any $\boldsymbol{x} = \{x_n\} \in l_X^\infty$ the sequence $\{y_n = x_n - A_{n-1}x_{n-1}\}$ lies in l_X^∞. Consequently, equation

$$(L\boldsymbol{x})_n = x_n - A_{n-1}x_{n-1} , \qquad \boldsymbol{x} = \{x_n\} \in l_X^\infty \tag{2.8}$$

defines a linear bounded operator acting in the space $l_X^\infty = l^\infty(\mathbb{Z}, X)$. Therewith assertion (ii) of Theorem 2.1 is equivalent to the assertion on the invertibility of the operator L given by equation (2.8).

The assertion given below provides a sufficient condition of invertibility of the operator L. Due to Theorem 2.1 this condition guarantees the existence of an exponential dichotomy for the corresponding difference equation. This assertion will be used in Section 4 in the proof of Anosov's lemma. It is a slightly weakened variant of a lemma proved in [6].

Theorem 2.2.

Assume that a sequence of operators $\{A_n\colon n \in \mathbb{Z}\}$ ***satisfies condition*** (2.7). ***Let there exist a family of projectors*** $\{Q_n\colon n \in \mathbb{Z}\}$ ***such that***

$$\|Q_n\| \le K, \qquad \|1 - Q_n\| \le K, \tag{2.9}$$

$$\|Q_{n+1} A_n (1 - Q_n)\| \le \delta, \qquad \|(1 - Q_{n+1}) A_n Q_n\| \le \delta, \tag{2.10}$$

for all $n \in \mathbb{Z}$. ***We also assume that the operator*** $(1 - Q_{n+1}) A_n$ ***is invertible as a mapping from*** $(1 - Q_n)X$ ***into*** $(1 - Q_{n+1})X$ ***and the estimates***

$$\|A_n Q_n\| \le \lambda, \qquad \|[(1 - Q_{n+1}) A_n]^{-1} (1 - Q_{n+1})\| \le \lambda \tag{2.11}$$

are valid for every $n \in \mathbb{Z}$. ***If***

$$K\lambda \le \frac{1}{8}, \qquad \delta \le \frac{1}{8}, \tag{2.12}$$

then the operator L ***acting in*** l_X^∞ ***according to formula*** (2.8) ***is invertible and*** $\|L^{-1}\| \le 2K + 1$.

Proof.

Let us first prove the injectivity of the mapping L. Assume that there exists a nonzero element $\boldsymbol{x} = \{x_n\} \in l_X^\infty$ such that $L\boldsymbol{x} = 0$, i.e. $x_n = A_{n-1} x_{n-1}$ for all $n \in \mathbb{Z}$. Let us prove that the sequence $\{x_n\}$ possesses the property

$$\|(1 - Q_n) x_n\| \le \|Q_n x_n\| \tag{2.13}$$

for all $n \in \mathbb{Z}$. Indeed, let there exist $m \in \mathbb{Z}$ such that

$$\|(1 - Q_m) x_m\| > \|Q_m x_m\|. \tag{2.14}$$

It is evident that this equation is only possible when $\|(1 - Q_m) x_m\| > 0$. Let us consider the value

$$N_{m+1} \equiv \|(1 - Q_{m+1}) x_{m+1}\| - \|Q_{m+1} x_{m+1}\| = \\ = \|(1 - Q_{m+1}) A_m x_m\| - \|Q_{m+1} A_m x_m\|. \tag{2.15}$$

It is clear that

$$\|(1 - Q_{n+1}) A_n x_n\| \ge \|(1 - Q_{n+1}) A_n (1 - Q_n) x_n\| - \|(1 - Q_{n+1}) A_n Q_n x_n\|.$$

Since

$$[(1 - Q_{n+1}) A_n]^{-1} (1 - Q_{n+1}) A_n (1 - Q_n) = (1 - Q_n),$$

it follows from (2.11) that

$$\|(1-Q_n)x\| \le \lambda\|(1-Q_{n+1})A_n(1-Q_n)x\|$$

for every $x \in X$ and for all $n \in \mathbb{Z}$. Therefore, we use estimates (2.10) to find that

$$\|(1-Q_{n+1})A_n x_n\| \ge \lambda^{-1}\|(1-Q_n)x_n\| - \delta\|x_n\| . \tag{2.16}$$

Then it is evident that

$$\|Q_{n+1}A_n x_n\| \le \|Q_{n+1}A_n Q_n x_n\| + \|Q_{n+1} A_n(1-Q_n)x_n\| \le$$

$$\le \left(\|Q_{n+1}\|\cdot\|A_n Q_n\| + \|Q_{n+1}A_n(1-Q_n)\|\right)\|x_n\| .$$

Therefore, estimates (2.9)–(2.11) imply that

$$\|Q_{n+1}A_n x_n\| \le (K\lambda+\delta)\|x_n\|, \qquad n \in \mathbb{Z}. \tag{2.17}$$

Thus, equations (2.15)–(2.17) lead us to the estimate

$$N_{m+1} \ge \lambda^{-1}\|(1-Q_m)x_m\| - (2\delta + K\lambda)\|x_m\| .$$

It follows from (2.14) that

$$\|x_m\| \le \|Q_m x_m\| + \|(1-Q_m)x_m\| < 2\|(1-Q_m)x_m\| .$$

Therefore,

$$N_{m+1} > (\lambda^{-1} - 2K\lambda - 4\delta)\,\|(1-Q_m)x_m\| .$$

Hence, if conditions (2.12) hold, then

$$\|(1-Q_{m+1})x_{m+1}\| - \|Q_{m+1}x_{m+1}\| > 7\|(1-Q_m)x_m\| > 0 . \tag{2.18}$$

When proving (2.18) we use the fact that

$$\lambda^{-1} \ge 8K \ge 8\|Q_n\| \ge 8 .$$

Thus, equation (2.18) follows from (2.14), i.e. $N_m > 0$ implies $N_{m+1} > 0$. Hence,

$$\|(1-Q_n)x_n\| > \|Q_n x_n\| \quad \text{for all} \quad n \ge m .$$

Moreover, (2.18) gives us that

$$K\cdot\|x_n\| \ge \|(1-Q_n)x_n\| \ge 7^{n-m}\|(1-Q_m)x_m\| , \qquad n \ge m .$$

Therefore, $\|x_n\| \to +\infty$ as $n \to +\infty$. This contradicts the assumption $\boldsymbol{x} = \{x_n\} \in l^\infty_X$. Thus, for all $n \in \mathbb{Z}$ estimate (2.13) is valid. In particular it leads us to the inequality

$$\|x_n\| \le \|(1-Q_n)x_n\| + \|Q_n x_n\| \le 2\|Q_n x_n\| . \tag{2.19}$$

Therefore, it follows from (2.17) that

$$\|Q_{n+1}x_{n+1}\| = \|Q_{n+1}A_n x_n\| \le 2(K\lambda+\delta)\|Q_n x_n\|$$

for all $n \in \mathbb{Z}$. We use conditions (2.12) to find that

$$\|Q_{n+1}x_{n+1}\| \le \frac{1}{2}\|Q_n x_n\| , \qquad n \in \mathbb{Z}. \tag{2.20}$$

If $\boldsymbol{x} = \{x_n\} \neq 0$, then inequality (2.19) gives us that there exists $m \in \mathbb{Z}$ such that $\|Q_m x_m\| \neq 0$. Therefore, it follows from (2.20) that

$$\|Q_n x_n\| \geq 2^{m-n} \|Q_m x_m\| > 0$$

for all $n \leq m$. We tend $n \to -\infty$ to obtain that $\|Q_n x_n\| \to +\infty$ which is impossible due to (2.9) and the boundedness of the sequence $\{x_n\}$. Therefore, there does not exist a nonzero $\boldsymbol{x} \in l_X^\infty$ such that $L\boldsymbol{x} = 0$. Thus, the mapping L is injective.

Let us now prove the surjectivity of L. Let us consider an operator R in the space l_X^∞ acting according to the formula

$$(R\boldsymbol{y})_n = Q_n y_n - B_n(1-Q_{n+1}) y_{n+1}, \qquad \boldsymbol{y} = \{y_n\} \in l_X^\infty,$$

where the operator $B_n = [(1-Q_{n+1})A_n]^{-1}$ acts from $(1-Q_{n+1})X$ into $(1-Q_n)X$ and is inverse to $(1-Q_{n+1})A_n\big|_{(1-Q_n)X}$. It follows from (2.9) and (2.11) that

$$\|R\boldsymbol{y}\|_{l^\infty} \leq (K+\lambda)\|\boldsymbol{y}\|_{l^\infty}, \qquad \boldsymbol{y} \in l_X^\infty. \tag{2.21}$$

It is evident that

$$(LR\boldsymbol{y})_n - y_n = -(1-Q_n)y_n - B_n(1-Q_{n+1})y_{n+1} -$$
$$- A_{n-1}Q_{n-1}y_{n-1} + A_{n-1}B_{n-1}(1-Q_n)y_n .$$

Since

$$(1-Q_n)A_{n-1}B_{n-1}(1-Q_n) = 1-Q_n ,$$

we have that

$$(LR\boldsymbol{y})_n - y_n = -B_n(1-Q_{n+1})y_{n+1} - A_{n-1}Q_{n-1}y_{n-1} +$$
$$+ Q_n A_{n-1}(1-Q_{n-1})B_{n-1}(1-Q_n)y_n .$$

Consequently,

$$\|(LR\boldsymbol{y})_n - y_n\| \leq \|B_n(1-Q_{n+1})\| \cdot \|y_{n+1}\| + \|A_{n-1}Q_{n-1}\| \cdot \|y_{n-1}\| +$$
$$+ \|Q_n A_{n-1}(1-Q_{n-1})\| \cdot \|B_{n-1}(1-Q_n)\| \cdot \|y_n\| .$$

Therefore, inequalities (2.10), (2.11), and (2.12) give us that

$$\|LR\boldsymbol{y} - \boldsymbol{y}\|_{l^\infty} \leq \lambda(2+\delta)\|\boldsymbol{y}\|_{l^\infty} \leq \frac{1}{2}\|\boldsymbol{y}\|_{l^\infty},$$

i.e. $\|LR - 1\| \leq 1/2$. That means that the operator LR is invertible and

$$\|(LR)^{-1}\| \leq (1 - \|LR - 1\|)^{-1} \leq 2 . \tag{2.22}$$

Let $\boldsymbol{h} = \{h_n\}$ be an arbitrary element of l_X^∞. Then it is evident that the element $\boldsymbol{y} = R(LR)^{-1}\boldsymbol{h}$ is a solution to equation $L\boldsymbol{y} = \boldsymbol{h}$. Moreover, it follows from (2.21) and (2.22) that

$$\|\boldsymbol{y}\|_{l^\infty} \leq 2(K+\lambda)\|\boldsymbol{h}\|_{l^\infty}.$$

Hence, L is surjective and $\|L^{-1}\| \leq 2K+1$. **Theorem 2.2 is proved**.

§ 3 Hyperbolicity of Invariant Sets for Differentiable Mappings

Let us remind the definition of the differentiable mapping. Let X and Y be Banach spaces and let $\mathscr{U}$ be an open set in X. The mapping f from $\mathscr{U}$ into Y is called (Frechét) ***differentiable*** at the point $x \in \mathscr{U}$ if there exists a linear bounded operator $Df(x)$ from X into Y such that

$$\lim_{\|v\| \to 0} \frac{1}{\|v\|} \|f(x+v) - f(x) - Df(x)v\| = 0 .$$

If the mapping f is differentiable at every point $x \in \mathscr{U}$, then the mapping Df: $x \to Df(x)$ acts from $\mathscr{U}$ into the Banach space $\mathscr{L}(X, Y)$ of all linear bounded operators from X into Y. If $Df: \mathscr{U} \to \mathscr{L}(X, Y)$ is continuous, then the mapping f is said to be ***continuously differentiable*** (or C^1-mapping) on $\mathscr{U}$. The notion of the derivative of any order can also be introduced by means of induction. For example, $D^2 f(x)$ is the Frechét derivative of the mapping $Df: \mathscr{U} \to \mathscr{L}(X, Y)$.

— Exercise *3.1* Let g and f be continuously differentiable mappings from $\mathscr{U} \subset X$ into Y and from $\mathscr{W} \subset Y$ into Z, respectively. Moreover, let $\mathscr{U}$ and $\mathscr{W}$ be open sets such that $g(\mathscr{U}) \subset \mathscr{W}$. Prove that $(f \circ g)(x) = f(g(x))$ is a C^1-mapping on $\mathscr{U}$ and obtain a chain rule for the differentiation of a composed function

$$D(f \circ g)(x) = Df(g(x))Dg(x), \qquad x \in \mathscr{U}.$$

— Exercise *3.2* Let f be a continuously differentiable mapping from X into X and let f^n be the n-th degree of the mapping f, i.e. $f^n(x) = f(f^{n-1}(x))$, $n \ge 1$, $f^1(x) \equiv f(x)$. Prove that f^n is a C^1-mapping on X and

$$(Df^n)(x) = Df(f^{n-1}(x)) \cdot Df(f^{n-2}(x)) \cdot \ldots \cdot Df(x) . \qquad (3.1)$$

Now we give the definition of a hyperbolic set. Assume that f is a continuously differentiable mapping from a Banach space X into itself and Λ is a subset in X which is invariant with respect to f $(f(\Lambda) \subset \Lambda)$. The set Λ is called ***hyperbolic*** (with respect to f) if there exists a collection of projectors $\{P(x): x \in \Lambda\}$ such that

a) $P(x)$ continuously depends on $x \in \Lambda$ with respect to the operator norm;

b) for every $x \in \Lambda$

$$Df(x) \cdot P(x) = P(f(x)) \cdot Df(x); \qquad (3.2)$$

c) the mappings $Df(x)$ are invertible for every $x \in \Lambda$ as linear operators from $(1 - P(x))X$ into $(I - P(f(x)))X$;

d) for every $x \in \Lambda$ the following equations hold:

$$\|(Df^n)(x)P(x)\| \leq Kq^n, \qquad n \geq 0, \tag{3.3}$$

$$\|[(Df^n)(x)]^{-1}(1-P(f^n(x)))\| \leq Kq^n, \qquad n \geq 0, \tag{3.4}$$

with the constants $K > 0$ and $0 < q < 1$ independent of $x \in \Lambda$. Here f^n is the n-th degree of the mapping f ($f^n(x) = f(f^{n-1}(x))$ for $n \geq 1$ and $f^1(x) \equiv f(x)$).

It should be noted that properties (b) and (c) as well as formula (3.1) enable us to state that $(Df^n)(x)$ maps $(1-P(x))X$ into $(1-P(f^n(x)))$ and is an invertible operator. Therefore, the value in the left-hand side of inequality (3.4) exists.

Exercise *3.3* Let $\Lambda = \{x_0\}$, where x_0 is a fixed point of a C^1-mapping f, i.e. $f(x_0) = x_0$. Then for the set Λ to be hyperbolic it is necessary and sufficient that the spectrum of the linear operator $Df(x_0)$ does not intersect the unit circumference (*Hint*: see Example 2.1).

Let Λ be an invariant hyperbolic set of a C^1-mapping f and let $\gamma = \{x_n : n \in \mathbb{Z}\}$ be a complete trajectory (in Λ) for f, i.e. $\gamma = \{x_n\}$ is a sequence of points from Λ such that $f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$. Let us consider a difference equation obtained as a result of linearization of the mapping f along γ:

$$u_{n+1} = Df(x_n)u_n, \qquad n \in \mathbb{Z}. \tag{3.5}$$

Exercise *3.4* Prove that the evolutionary operator $\Phi(m, n)$ of difference equation (3.5) has the form

$$\Phi(m, n) = (Df^{m-n})(x_n), \qquad m > n, \qquad m, n \in \mathbb{Z}.$$

Exercise *3.5* Prove that difference equation (3.5) admits an exponential dichotomy over $\mathbb{Z}$ with (i) the constants K and q given by equations (3.3) and (3.4) and (ii) the projectors $P_n = P(x_n)$ involved in the definition of the hyperbolicity.

It should be noted that property (a) of uniform continuity implies that the projectors $P(x)$ are similar to one another, provided the values of x are close enough. The proof of this fact is based on the following assertion.

Lemma 3.1.

Let P and Q be projectors in a Banach space X. Assume that

$$\|P\| \leq K, \qquad \|1-P\| \leq K, \qquad \|P-Q\| < \frac{1}{2K} \tag{3.6}$$

for some constant $K \geq 1$. Then the operator

$$J = PQ + (1-P)(1-Q) \tag{3.7}$$

possesses the property $PJ = JQ$ *and is invertible, therewith*

$$\|J^{-1}\| \le (1-2K\cdot\|P-Q\|)^{-1} . \tag{3.8}$$

Proof.

Since $P^2 + (1-P)^2 = 1$, we have

$$J-1 = J-P^2-(1-P)^2 = P(Q-P)+(1-P)(P-Q) .$$

It follows from (3.6) that

$$\|J-1\| \le 2K\|P-Q\| < 1 .$$

Hence, the operator J^{-1} can be defined as the following absolutely convergent series

$$J^{-1} = \sum_{n=0}^{\infty} (1-J)^n .$$

This implies estimate (3.8). The permutability property $PJ = JQ$ is evident. Lemma 3.1 is proved.

Exercise *3.6* Let Λ be a connected compact set and let $\{P(x):\ x \in \Lambda\}$ be a family of projectors for which condition (a) of the hyperbolicity definition holds. Then all operators $P(x)$ are similar to one another, i.e. for any $x,\ y \in \Lambda$ there exists an invertible operator $J = J_{x,\ y}$ such that $P(x) = JP(y)J^{-1}$.

The following assertion contains a description of a situation when the hyperbolicity of the invariant set is equivalent to the existence of an exponential dichotomy for difference equation (3.5) (cf. Exercise 3.5).

Theorem 3.1.

Let $f(x)$ ***be a continuously differentiable mapping of the space*** X ***into itself. Let*** x_0 ***be a hyperbolic fixed point of*** f ***(***$f(x_0)=x_0$***) and let*** $\{y_n\colon n \in \mathbb{Z}\}$ ***be a homoclinic trajectory (not equal to*** x_0***) of the mapping*** f***, i.e.***

$$f(y_n) = y_{n+1}, \quad n \in \mathbb{Z}, \quad y_n \to x_0, \quad n \to \pm\infty . \tag{3.9}$$

Then the set $\Lambda = \{x_0\} \cup \{y_n\colon n \in \mathbb{Z}\}$ ***is hyperbolic if and only if the difference equation***

$$u_{n+1} = Df(y_n)u_n, \quad n \in \mathbb{Z}, \tag{3.10}$$

possesses an exponential dichotomy over $\mathbb{Z}$.

Proof.

If Λ is hyperbolic, then (see Exercise 3.5) equation (3.10) possesses an exponential dichotomy over $\mathbb{Z}$. Let us prove the converse assertion. Assume that equation (3.10) possesses an exponential dichotomy over $\mathbb{Z}$ with projectors $\{P_n\colon\ n \in \mathbb{Z}\}$

and constants K and q. Let us denote the spectral projector of the operator $Df(x_0)$ corresponding to the part of the spectrum inside the unit disc by P. Without loss of generality we can assume that

$$\left\| [Df(x_0)]^n P \right\| \le Kq^n , \qquad n \ge 0 ,$$

$$\left\| [Df(x_0)]^{-n} (1-P) \right\| \le Kq^{-n} , \qquad n \ge 0 .$$

Thus, for every $x \in \Lambda$ the projector $P(x)$ is defined: $P(x_0) = P$, $P(y_k) = P_k$. The structure of the evolutionary operator of difference equation (3.10) (see Exercise 3.4) enables us to verify properties (b)–(d) of the definition of a hyperbolic set. In order to prove property (a) it is sufficient to verify that

$$\left\| P_k - P \right\| \to 0 \quad \text{as} \quad k \to \pm\infty . \tag{3.11}$$

Since Λ is a compact set, then

$$M = \sup\{\|Df(x)\| : \; x \in \Lambda\} < \infty . \tag{3.12}$$

Let us consider the following difference equations

$$v_{n+1} = Df(x_0) v_n , \qquad n \in \mathbb{Z} , \tag{3.13}$$

and

$$w_{n+1}^{(k)} = Df(y_{n+k}) w_n^{(k)} , \qquad n \in \mathbb{Z} , \tag{3.14}$$

where k is an integer. It is evident that equation (3.14) admits an exponential dichotomy over $\mathbb{Z}$ with constants K and q and projectors $P_n^{(k)} = P_{n+k}$. Let $G(n, m)$ and $G^{(k)}(n, m)$ be the Green functions (see Section 2) of difference equations (3.13) and (3.14). We consider the sequence

$$x_n = G(n, 0)z - G^{(k)}(n, 0)z , \qquad z \in X .$$

Since (see Exercise 2.8)

$$\|G(n, 0)\| \le Kq^{|n|} , \qquad \|G^{(k)}(n, 0)\| \le Kq^{|n|} , \tag{3.15}$$

we have that the sequence $\{x_n\}$ is bounded. Moreover, it is easy to prove (see Exercise 2.9) that $\{x_n\}$ is a solution to the difference equation

$$x_{n+1} - Df(x_0) x_n = h_n \equiv [Df(x_0) - Df(y_{n+k})] G^{(k)}(n, 0)z .$$

It follows from (3.12) and (3.15) that the sequence $\{h_n\}$ is bounded. Therefore, (see Exercise 2.9),

$$G(n, 0)z - G^{(k)}(n, 0)z \equiv x_n = \sum_{m \in \mathbb{Z}} G(n, m+1) h_m .$$

If we take $n = 0$ in this formula, then from the definition of the Green function we obtain that

$$(P - P_k)z = \sum_{m \in \mathbb{Z}} G(0, m+1)(Df(x_0) - Df(y_{m+k}))\, G^{(k)}(m, 0)z .$$

Therefore, equation (3.15) implies that

$$\|(P-P_k)z\| \le K^2 \sum_{m \in \mathbb{Z}} \|Df(x_0) - Df(y_{m+k})\| \cdot q^{|m|+|m+1|} \cdot \|z\| .$$

Consequently,

$$\|P-P_k\| \le K^2 \sum_{m \in \mathbb{Z}} \|Df(x_0) - Df(y_{m+k})\| \cdot q^{2|m|-1} \le$$

$$\le K^2 \left\{ \max_{|m| \le N} \|Df(x_0) - Df(y_{m+k})\| \cdot \sum_{|m| \le N} q^{2|m|-1} + 2M \sum_{|m| > N} q^{2|m|-1} \right\} ,$$

where N is an arbitrary natural number. Upon simple calculations we find that

$$\|P-P_k\| \le \frac{2K^2}{q(1-q^2)} \left(\max_{|m| \le N} \|Df(x_0) - Df(y_{m+k})\| + 2Mq^{2N+2} \right)$$

for every $N \ge 1$. It follows that

$$\overline{\lim_{k \to \pm\infty}} \|P-P_k\| \le \frac{4K^2M}{1-q^2} q^{2N+1} , \qquad N = 1,\ 2,\ \dots .$$

We assume that $N \to +\infty$ to obtain that

$$\overline{\lim_{k \to \pm\infty}} \|P-P_k\| \le 0 .$$

This implies equation (3.11). Therefore, **Theorem 3.1 is proved**.

It should be noted that in the case when the set $\Lambda = \{x_0\} \cup \{y_n\colon\ n \in \mathbb{Z}\}$ from Theorem 3.1 is hyperbolic the elements y_n of the homoclinic trajectory $\gamma = \{y_n\colon\ n \in \mathbb{Z}\}$ are called ***transversal homoclinic points***. The point is that in some cases (see, e.g., [4]) it can be proved that the hyperbolicity of Λ is equivalent to the transversality property at every point y_n of the stable $W^s(x_0)$ and unstable $W^u(x_0)$ manifolds of a fixed point x_0 (roughly speaking, transversality means that the surfaces $W^s(x_0)$ and $W^u(x_0)$ intersect at the point y_n at a nonzero angle). In this case the trajectory γ is often called a ***transversal homoclinic trajectory***.

§ 4 *Anosov's Lemma on ε-trajectories*

Let f be a C^1-mapping of a Banach space X into itself. A sequence $\{y_n\colon\ n \in \mathbb{Z}\}$ in X is called a δ***-pseudotrajectory*** (or δ-pseudoorbit) of the mapping f if for all $n \in \mathbb{Z}$ the equation

$$\|y_{n+1} - f(y_n)\| \le \delta$$

is valid. A sequence $\{x_n\colon\ n \in \mathbb{Z}\}$ is called an ε***-trajectory*** of the mapping f corresponding to a δ-pseudotrajectory $\{y_n\colon\ n \in \mathbb{Z}\}$ if

(a) $f(x_n) = x_{n+1}$ for any $n \in \mathbb{Z}$;

(b) $\|x_n - y_n\| \le \varepsilon$ for all $n \in \mathbb{Z}$.

It should be noted that condition (a) means that $\{x_n\}$ is an orbit (complete trajectory) of the mapping f. Moreover, if a pair of C^1-mappings f and g is given, then the notion of the ε-trajectory of the mapping g corresponding to a δ-pseudoorbit of the mapping f can be introduced.

The following assertion is the main result of this section.

Theorem 4.1.

Let f be a C^1-mapping of a Banach space X into itself and let Λ be a hyperbolic invariant $(f(\Lambda) \subset \Lambda)$ set. Assume that there exists a Δ-vicinity $\mathcal{O}$ of the set Λ such that $f(x)$ and $Df(x)$ are bounded and uniformly continuous on the closure $\overline{\mathcal{O}}$ of the set $\mathcal{O}$. Then there exists $\varepsilon_0 > 0$ possessing the property that for every $0 < \varepsilon \le \varepsilon_0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any δ-pseudoorbit $\{y_n\colon\ n \in \mathbb{Z}\}$ lying in Λ has a unique ε-trajectory $\{x_n\colon\ n \in \mathbb{Z}\}$ corresponding to $\{y_n\}$.

As the following theorem shows, the property of the mapping f to have an ε-trajectory is rough, i.e. this property also remains true for mappings that are close to f.

Theorem 4.2.

Assume that the hypotheses of Theorem 4.1 hold for the mapping f. Let $\mathcal{W}_\eta(f)$ be a set of continuously differentiable mappings g of the space X into itself such that the following estimates hold on the closure $\overline{\mathcal{O}}$ of the Δ-vicinity $\mathcal{O}$ of the set Λ:

$$\|f(x) - g(x)\| < \eta, \qquad \|Df(x) - Dg(x)\| < \eta. \tag{4.1}$$

Then $\varepsilon_0 > 0$ can be chosen to possess the property that for every $\varepsilon \in (0, \varepsilon_0]$ there exist $\delta = \delta(\varepsilon) > 0$ and $\eta = \eta(\varepsilon) > 0$ such that for any δ-pseudotrajectory $\{y_n\colon\ n \in \mathbb{Z}\}$ (lying in Λ) of the mapping f and for any $g \in \mathcal{W}_\eta(f)$ there exists a unique trajectory $\{x_n\colon\ n \in \mathbb{Z}\}$ of the mapping g with the property

$$\|y_n - x_n\| \le \varepsilon \quad \textit{for all} \quad n \in \mathbb{Z}.$$

It is clear that Theorem 4.1 is a corollary of Theorem 4.2 the proof of which is based on the lemmata below.

Lemma 4.1.

Let $\mathcal{U}$ be an open set in a Banach space X and let $\mathcal{F}: \mathcal{U} \to X$ be a continuously differentiable mapping. Assume that for some point $y \in \mathcal{U}$ there exist an operator $[D\mathcal{F}(y)]^{-1}$ and a number $\varepsilon_0 > 0$ such that

$$\|D\mathcal{F}(x) - D\mathcal{F}(y)\| \le \left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.2}$$

for all x with the property $\|x-y\| \le \varepsilon_0$. Assume that for some $\varepsilon \in (0, \varepsilon_0]$ the inequality

$$\|\mathcal{F}(y)\| \le \bar{q} \cdot \varepsilon\left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.3}$$

is valid with $0 < \bar{q} < 1$. Then for any C^1-mapping $\mathcal{G}: \mathcal{U} \to X$ such that

$$\|\mathcal{G}(x) - \mathcal{F}(x)\| \le \varepsilon(1-\bar{q})\left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.4}$$

and

$$\|D\mathcal{G}(x) - D\mathcal{F}(x)\| \le \frac{1}{2}\left(2\|[D\mathcal{F}(y)]^{-1}\|\right)^{-1} \tag{4.5}$$

for $\|x-y\| \le \varepsilon_0$, the equation $\mathcal{G}(x) = 0$ has a unique solution x with the property $\|x-y\| \le \varepsilon$.

Proof.

Let $\Gamma = (2\|[D\mathcal{F}(y)]^{-1}\|)^{-1}$ and let

$$\eta(x) = \mathcal{F}(x) - \mathcal{F}(y) - D\mathcal{F}(y)(x-y) .$$

For $x_1, x_2 \in B_{\varepsilon_0} \equiv \{z: \|z-y\| \le \varepsilon_0\}$ we have that

$$\eta(x_1) - \eta(x_2) = \mathcal{F}(x_1) - \mathcal{F}(x_2) - D\mathcal{F}(y)(x_1 - x_2) =$$

$$= \int_0^1 (D\mathcal{F}(x_1 + \xi(x_2 - x_1)) - D\mathcal{F}(y))(x_1 - x_2)\, d\xi .$$

Since

$$\|\eta(x_1) - \eta(x_2)\| \le \int_0^1 \|D\mathcal{F}(x_1 + \xi(x_2 - x_1)) - D\mathcal{F}(y)\|\, d\xi\, \|x_1 - x_2\| ,$$

it follows from (4.2) that

$$\|\eta(x_1) - \eta(x_2)\| \le \Gamma\|x_1 - x_2\| \tag{4.6}$$

for all x_1 and x_2 from B_{ε_0}. Now we rewrite the equation $\mathcal{G}(x) = 0$ in the form

$$x = T(x) \equiv y - [D\mathcal{F}(y)]^{-1}(\mathcal{G}(x) - \mathcal{F}(x) + \mathcal{F}(y) + \eta(x)) .$$

Let us show that the mapping T has a unique fixed point in the ball $B_\varepsilon = \{x : \|x-y\| \le \varepsilon\}$. It is evident that

$$\|T(x)-y\| \le \|[D\mathcal{F}(y)]^{-1}\| \Big(\|\mathcal{G}(x)-\mathcal{F}(x)\| + \|\mathcal{F}(y)\| + \|\eta(x)\|\Big)$$

for any $x \in B_\varepsilon$. Since $\eta(y) = 0$, we obtain from (4.6) that

$$\|\eta(x)\| = \|\eta(x)-\eta(y)\| \le \Gamma\|x-y\| \le \Gamma\varepsilon .$$

Therefore, estimates (4.3) and (4.4) imply that

$$\|T(x)-y\| \le \varepsilon \quad \text{for } x \in B_\varepsilon ,$$

i.e. T maps the ball B_ε into itself. This mapping is contractive in B_ε. Indeed,

$$\|T(x_1)-T(x_2)\| \le \frac{1}{2\Gamma}\Big(\|\mathcal{H}(x_1, x_2)\| + \|\eta(x_1)-\eta(x_2)\|\Big),$$

where

$$\mathcal{H}(x_1, x_2) \equiv \mathcal{G}(x_1)-\mathcal{G}(x_2)-\mathcal{F}(x_1)+\mathcal{F}(x_2) =$$

$$= \int_0^1 [D\mathcal{G}(x_1+\xi(x_1-x_2)) - D\mathcal{F}(x_1+\xi(x_1-x_2))]\, d\xi \; (x_1-x_2) .$$

It follows from (4.5) that

$$\|\mathcal{H}(x_1, x_2)\| \le \frac{1}{2}\Gamma\|x_1-x_2\| .$$

This equation and inequality (4.6) imply the estimate

$$\|T(x_1)-T(x_2)\| \le \frac{3}{4}\|x_1-x_2\| .$$

Therefore, the mapping T has a unique fixed point in the ball $B_\varepsilon = \{x : \|x-y\| \le \varepsilon\}$. The lemma is proved.

Let the hypotheses of Theorems 4.1 and 4.2 hold. We assume that $\eta < 1$ in (4.1). Then for any element $g \in \mathcal{W}_\eta(f)$ the following estimates hold:

$$\|g(x)\| \le M, \qquad \|Dg(x)\| \le M, \qquad x \in \overline{\mathcal{O}}, \tag{4.7}$$

where $M > 0$ is a constant. In particular, these estimates are valid for the mapping f.

Lemma 4.2.

Let $\{y_n : n \in \mathbb{Z}\}$ be a δ-pseudotrajectory of the mapping f lying in Λ. Then for any $k \ge 1$ the sequence $\{z_n \equiv y_{nk} : n \in \mathbb{Z}\}$ is a $\delta \cdot M_{k-1}$-pseudotrajectory of the mapping f^k. Here M_k has the form

$$M_k = 1 + M + \ldots + M^k, \qquad k \ge 1, \quad M_0 = 1, \tag{4.8}$$

and M is a constant from (4.7).

Proof.

Let us use induction to prove that

$$\left\|y_{nk+i} - f^i(y_{nk})\right\| \le \delta \cdot M_{i-1}\,, \qquad 1 \le i \le k\,. \tag{4.9}$$

Since $\{y_n\}$ is a δ-pseudotrajectory, then it is evident that for $i = 1$ inequality (4.9) is valid. Assume that equation (4.9) is valid for some $i \ge 1$ and prove estimate (4.9) for $i+1$:

$$\left\|y_{nk+i+1} - f^{i+1}(y_{nk})\right\| \le \left\|y_{nk+i+1} - f(y_{nk+i})\right\| + \left\|f(y_{nk+i}) - f(f^i(y_{nk}))\right\|.$$

With the help of (4.7) we obtain that

$$\left\|y_{nk+i+1} - f^{i+1}(y_{nk})\right\| \le \delta + M\left\|y_{nk+i} - f^i(y_{nk})\right\| \le \delta + M \cdot \delta M_{i-1} = \delta \cdot M_i\,.$$

Thus, Lemma 4.2 is proved.

Lemma 4.3.

Let $\{y_n\}$ be a δ-pseudoorbit of the mapping f lying in Λ. Let $\{x_n\}$ be a trajectory of the mapping $g \in \mathcal{W}_\eta(f)$ such that

$$\left\|y_{nk} - x_{nk}\right\| \le \varepsilon, \qquad n \in \mathbb{Z}, \tag{4.10}$$

for some $k \ge 1$. If

$$\max(\varepsilon,\ \delta + \eta) M_k \le \Delta\,, \tag{4.11}$$

then

$$\left\|y_n - x_n\right\| \le \max(\varepsilon,\ \delta + \eta) \cdot M_k\,, \tag{4.12}$$

where M_k has the form (4.8).

Proof.

We first note that

$$\left\|y_{nk+1} - x_{nk+1}\right\| = \left\|y_{nk+1} - g(x_{nk})\right\| \le$$

$$\le \left\|y_{nk+1} - f(y_{nk})\right\| + \left\|f(y_{nk}) - g(y_{nk})\right\| + \left\|g(y_{nk}) - g(x_{nk})\right\|.$$

Therefore, it is evident that

$$\left\|y_{nk+1} - x_{nk+1}\right\| \le \delta + \eta + M\left\|y_{nk} - x_{nk}\right\| \le \max(\varepsilon,\ \delta + \eta)(1 + M)\,. \tag{4.13}$$

Here we use the estimate

$$\|g(y) - g(x)\| \le \int_0^1 \|Dg(y + \xi(x - y))\|\,\mathrm{d}\xi \cdot \|x - y\| \le M\|x - y\|$$

which follows from (4.7) and holds when the segment connecting the points x and y lies in $\overline{\mathcal{O}}$. Condition (4.11) guarantees the fulfillment of this property at each stage of reasoning. If we repeat the arguments from the proof of (4.13), then it is easy to complete the proof of (4.12) using induction as in Lemma 4.2. Lemma 4.3 is proved.

Lemma 4.4.

Let $y \in \Lambda$ *and* $\|x-y\| \le \varepsilon$. *Assume that*

$$\max(\varepsilon, \eta)(1 + \ldots + M^{k-1}) < \Delta. \tag{4.14}$$

Then the estimates

$$\|f^j(y) - f^j(x)\| \le M^j \|x-y\|, \qquad \|Df^j(x)\| \le M^j, \tag{4.15}$$

$$\|f^j(y) - g^j(x)\| \le \max(\varepsilon, \eta)(1 + \ldots + M^j), \tag{4.16}$$

$$\|f^j(x) - g^j(x)\| \le \eta(1 + \ldots + M^{j-1}) \tag{4.17}$$

are valid for $j = 1, 2, \ldots, k$ *and for every mapping* $g \in \mathcal{W}_\eta(f)$.

Proof.

As above, let us use induction. If $j = 1$, it is evident that equations (4.15)–(4.17) hold. The transition from j to $j+1$ in (4.15) is evident. Let us consider estimate (4.16):

$$f^{j+1}(y) - g^{j+1}(x) = \Big(f(f^j(y)) - f(g^j(x))\Big) + f(g^j(x)) - g(g^j(x)). \tag{4.18}$$

Condition (4.14) and the induction assumption give us that $g^j(x)$ lies in the ball with the centre at the point $f^j(y) \in \Lambda$ lying in $\mathcal{O}$. Therefore, it follows from (4.18) that

$$\|f^{j+1}(y) - g^{j+1}(x)\| \le M\|f^j(y) - g^j(x)\| + \eta \le \max(\varepsilon, \eta)(1 + \ldots + M^{j+1}).$$

The transition from j to $j+1$ in (4.17) can be made in a similar way. Lemma 4.4 is proved.

Lemma 4.5.

There exists $\Delta' \le \Delta$ *such that the equations*

$$\sup\left\{\|f^k(x) - g^k(x)\| : \ x \in \mathcal{O}'\right\} \le \rho_k(\eta) \tag{4.19}$$

and

$$\sup\left\{\|(Df^k)(x) - (Dg^k)(x)\| : \ x \in \mathcal{O}'\right\} \le \rho_k(\eta) \tag{4.20}$$

are valid in the Δ'*-vicinity* $\mathcal{O}'$ *of the set* Λ *for any function* $g \in \mathcal{W}_\eta(f)$. *Here* $\rho_k(\eta) \to 0$ *as* $\eta \to 0$.

The proof follows from the definition of the class of functions $\mathcal{W}_\eta(f)$ and estimates (4.7) and (4.17).

Let us also introduce the values

$$\omega_k(\varkappa) = \sup\left\{\|Df^k(y) - Df^k(x)\| : \ y \in \Lambda, \ \|x-y\| \le \varkappa\right\} \tag{4.21}$$

and

$$\omega(\varkappa) = \sup\left\{\|P(y)-P(x)\|, \quad x, y \in \Lambda, \ \|x-y\| \le \varkappa\right\}. \tag{4.22}$$

The requirement of the uniform continuity of the derivative $Df(x)$ (see the hypotheses of Theorem 4.1) and the projectors $P(x)$ (see the hyperbolicity definition) enables us to state that

$$\omega_k(\varkappa) \to 0, \qquad \omega(\varkappa) \to 0 \ \text{ as } \ \varkappa \to 0. \tag{4.23}$$

Let $\{y_n\}$ be a δ-pseudotrajectory of the mapping f lying in Λ. Then due to Lemma 4.2 the sequence $\{\bar{y}_n = y_{nk}: \ n \in \mathbb{Z}\}$ is a δM_{k-1}-pseudotrajectory of the mapping f^k. Let us consider the mappings $\mathcal{F}(\boldsymbol{x})$ and $\mathcal{G}(\boldsymbol{x})$ in the space $l_X^\infty \equiv l^\infty(\mathbb{Z}, X)$ (for the definition see Section 2) given by the equalities

$$[\mathcal{F}(\boldsymbol{x})]_n = \bar{y}_n + x_n - f^k(\bar{y}_{n-1} + x_{n-1}), \tag{4.24}$$

$$[\mathcal{G}(\boldsymbol{x})]_n = \bar{y}_n + x_n - g^k(\bar{y}_{n-1} + x_{n-1}), \tag{4.25}$$

where $\boldsymbol{x} = \{x_n: \ n \in \mathbb{Z}\}$ is an element from l_X^∞. Thus, the construction of ε-trajectories of the mapping f^k and g^k corresponding to the sequence $\{\bar{y}_n\}$ is reduced to solving of the equations

$$\mathcal{F}(\boldsymbol{x}) = 0 \ \text{ and } \ \mathcal{G}(\boldsymbol{x}) = 0$$

in the ball $\{\boldsymbol{x}: \ \|\boldsymbol{x}\|_{l_X^\infty} \le \varepsilon\}$. Let us show that for k large enough Lemma 4.1 can be applied to the mappings $\mathcal{F}$ and $\mathcal{G}$. Let us start with the mapping $\mathcal{F}$.

Lemma 4.6.

The function $\mathcal{F}$ is a C^1-smooth mapping in l_X^∞ with the properties

$$\|\mathcal{F}(0)\| \le \delta M_{k-1}, \tag{4.26}$$

$$\|D\mathcal{F}(\boldsymbol{x}) - D\mathcal{F}(0)\| \le \omega_k(\varepsilon), \qquad \|\boldsymbol{x}\|_{l_X^\infty} \le \varepsilon. \tag{4.27}$$

Proof.

Estimate (4.26) follows from the fact that $\{\bar{y}_n\}$ is a δM_{k-1}-pseudotrajectory. Then it is evident that

$$[D\mathcal{F}(\boldsymbol{x})\boldsymbol{h}]_n = h_n - Df^k(\bar{y}_{n-1} + x_{n-1})h_{n-1}, \tag{4.28}$$

where $\boldsymbol{x} = \{x_n: \ n \in \mathbb{Z}\}$ and $\boldsymbol{h} = \{h_n: \ n \in \mathbb{Z}\}$ lie in l_X^∞. Therefore, simple calculations and equation (4.21) give us (4.27).

In order to deduce relations (4.2) and (4.3) from inequalities (4.26) and (4.27) for $\boldsymbol{y} = 0$, we use Theorem 2.2. Consider the operator $L = D\mathcal{F}(0)$. It is clear that

$$[L\boldsymbol{h}]_n = h_n - Df^k(\overline{y}_{n-1})h_{n-1}, \qquad \boldsymbol{h} = \{h_n\}.$$

Let us show that equations (2.9)–(2.11) are valid for $A_n = Df^k(\overline{y}_{n-1})$ and $Q_n = P(\overline{y}_{n-1})$ and then estimate the corresponding constants. Property (2.9) follows from the hyperbolicity definition. Equations (3.3) and (4.15) imply that

$$\|A_n Q_n\| \le Kq^k \quad \text{and} \quad \|A_n\| \le M^k.$$

Further, the permutability property (3.2) gives us that

$$Q_{n+1}A_n(1-Q_n) = [Q_{n+1}A_n - A_nQ_n](1-Q_n) =$$

$$= \left[Q_{n+1} - P(f^k(\overline{y}_{n-1}))\right]A_n(1-Q_n)\ .$$

Hence (see (4.22)),

$$\|Q_{n+1}A_n(1-Q_n)\| \le \omega(\delta M_{k-1})\cdot\|A_n\|\cdot\|1-Q_n\| \le \omega(\delta M_{k-1})M^k\cdot K\ .$$

Similarly, we find that

$$\|(1-Q_{n+1})A_nQ_n\| \le \omega(\delta M_{k-1})M^k\cdot K\ .$$

The operator

$$J_n = Q_{n+1}P(f^k(\overline{y}_{n-1})) - (1-Q_{n+1})(1-P(f^k(\overline{y}_{n-1})))$$

is invertible if (see Lemma 3.1)

$$\|Q_{n+1} - P(f^k(\overline{y}_{n-1}))\| \le \omega(\delta M_{k-1}) < \frac{1}{2K}\ .$$

Moreover,

$$\|J_n^{-1}\| \le (1-2K\omega(\delta M_{k-1}))^{-1} < 2\ ,$$

provided $2K\omega(\delta M_{k-1}) < 1/2$. Due to the hyperbolicity of the set Λ, the operator A_n is an invertible mapping from $(1-Q_n)X$ into $(1-P(f^k(\overline{y}_{n-1})))X$. Therefore, since

$$(1-Q_{n+1})A_n(1-Q_n) = J_nA_n(1-Q_n)\ ,$$

the operator $(1-Q_{n+1})A_n(1-Q_n)$ is invertible as a mapping from $(1-Q_n)X$ into $(1-Q_{n+1})X$. Moreover, by virtue of (3.4) we have that

$$\|[(1-Q_{n+1})A_n]^{-1}(1-Q_{n+1})\| = \|[A_n(1-Q_n)]^{-1}\cdot J_n^{-1}(1-Q_{n+1})\| \le 2K^2q^k\ .$$

Thus, under the conditions

$$4K\omega(\delta M_{k-1}) < 1\ , \qquad 8KM^k\omega(\delta M_{k-1}) \le 1\ , \qquad 16K^3q^k \le 1\ , \tag{4.29}$$

Theorem 2.2 implies that the operator $L = D\mathcal{F}(0)$ is invertible and $\|L^{-1}\| \le 2K+1$. Let us fix some $\overline{\varepsilon} > 0$. If

$$\delta M_{k-1} \le \frac{1}{2}(4K+2)^{-1}\cdot\bar{\varepsilon}\,, \qquad \omega_k(\bar{\varepsilon}) \le (4K+2)^{-1}, \tag{4.30}$$

then by Lemma 4.6 relations (4.2) and (4.3) hold with $y = 0$, $\varepsilon = \bar{\varepsilon}$, and $\bar{q} = 1/2$. If

$$\rho_k(\eta) \le \frac{1}{2}\min(\bar{\varepsilon},\ 1)(4K+2)^{-1}\,, \tag{4.31}$$

then equations (4.4) and (4.5) also hold with $\boldsymbol{y} = 0$, $\varepsilon = \bar{\varepsilon}$, and $\bar{q} = 1/2$. Hence, under conditions (4.29)–(4.31) there exists a unique solution to equation $\mathcal{G}(\boldsymbol{x}) = 0$ possessing the property $\|\boldsymbol{x}\|_{l^\infty} \le \bar{\varepsilon}$. This means that for any δ-pseudoorbit $\{y_n\colon n \in \mathbb{Z}\}$ (lying in Λ) of the mapping f there exists a unique trajectory $\{z_n\colon n \in \mathbb{Z}\}$ of the mapping $g \in \mathcal{W}_\eta(f)$ such that

$$\|z_{nk} - y_{nk}\| \le \bar{\varepsilon}\,, \qquad n \in \mathbb{Z}\,,$$

provided conditions (4.29)–(4.31) hold. Therefore, under the additional condition

$$(\bar{\varepsilon} + \delta + \eta)M_k \le \Delta$$

and due to Lemma 4.3 we get

$$\|y_n - z_n\| \le (\bar{\varepsilon} + \delta + \eta)M_k\,, \qquad n \in \mathbb{Z}\,.$$

These properties are sufficient for the completion of the proof of Theorem 4.2.

Let us fix k such that $16K^3q^k \le 1$. We choose $\varepsilon_0 \le \Delta' \le \Delta$ (Δ' is defined in Lemma 4.5) such that

$$\omega_k(\bar{\varepsilon}) \le (4K+2)^{-1} \quad \text{for all} \quad \bar{\varepsilon} \le \frac{\varepsilon_0}{2M_k}\,.$$

Let us fix an arbitrary $\varepsilon \in (0,\ \varepsilon_0]$ and take $\bar{\varepsilon} = \varepsilon[2M_k]^{-1}$. Now we choose $\delta = \delta(\varepsilon)$ and $\eta = \eta(\varepsilon)$ such that the following conditions hold:

$$4K\omega(\delta M_{k-1}) < 1\,, \qquad 8KM^k\omega(\delta M_{k-1}) \le 1\,,$$

$$\delta M_{k-1} \le \frac{1}{4}\bar{\varepsilon}(2K+1)^{-1}\,, \quad \rho_k(\eta) \le \frac{1}{4}\min(\bar{\varepsilon},\ 1)(2K+1)^{-1}\,, \quad 2(\delta+\eta)M_k \le \varepsilon\,.$$

It is clear that under such a choice of δ and η any δ-pseudoorbit (from Λ) of the mapping f has a unique ε-trajectory of the mapping g. Thus, **Theorem 4.2 is proved**.

Exercise *4.1* Let the hypotheses of Theorem 4.1 hold. Show that there exist $\Delta > 0$ and $\delta > 0$ such that for any two trajectories $\{x_n\colon n \in \mathbb{Z}\}$ and $\{y_n\colon n \in \mathbb{Z}\}$ of a dynamical system (X, f) the conditions

$$\operatorname{dist}(x_n,\ \Lambda) < \Delta\,, \qquad \operatorname{dist}(y_n,\ \Lambda) < \Delta\,, \qquad \sup_n \|x_n - y_n\| \le \delta$$

imply that $x_n \equiv y_n$, $n \in \mathbb{Z}$. In other words, any two trajectories of the system (X, f) that are close to a hyperbolic invariant set cannot remain arbitrarily close to each other all the time.

Exercise *4.2* Show that Theorem 4.1 admits the following strengthening: if the hypotheses of Theorem 4.1 hold, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exists $\delta = \delta(\varepsilon)$ with the property that for any δ-pseudoorbit $\{y_n\}$ such that $\operatorname{dist}(y_n, \Lambda) < \delta$ there exists a unique ε-trajectory.

Exercise *4.3* Prove the analogue of the assertion of Exercise 4.2 for Theorem 4.2.

Exercise *4.4* Let $\Lambda = \{x_n\colon\ n \in \mathbb{Z}\}$ be a periodic orbit of the mapping f, i.e. $f^k(x_n) \equiv x_{n+k} = x_n$ for all $n \in \mathbb{Z}$ and for some $k \geq 1$. Assume that the hypotheses of Theorem 4.2 hold. Then for $\eta > 0$ small enough every mapping $g \in \mathscr{W}_\eta(f)$ possesses a periodic trajectory of the period k.

§ 5 Birkhoff-Smale Theorem

One of the most interesting corollaries of Anosov's lemma is the Birkhoff-Smale theorem that provides conditions under which the chaotic dynamics is observed in a discrete dynamical system (X, f). We remind (see Section 1) that by definition the possibility of chaotic dynamics means that there exists an invariant set Y in the space X such that the restriction of some degree f^k of the mapping f on Y is topologically equivalent to the Bernoulli shift S in the space Σ_m of two-sided infinite sequences of m symbols.

Theorem 5.1.

Let f be a continuously differentiable mapping of a Banach space X into itself. Let $x_0 \in X$ be a hyperbolic fixed point of f and let $\{y_n\colon\ n \in \mathbb{Z}\}$ be a homoclinic trajectory of the mapping f that does not coincide with x_0, i.e.

$$f(x_0) = x_0; \qquad f(y_n) = y_{n+1}, \quad y_n \neq x_0, \quad n \in \mathbb{Z}; \qquad y_n \to x_0,\ n \to \pm\infty.$$

Assume that the trajectory $\{y_n\colon\ n \in \mathbb{Z}\}$ is transversal, i.e. the set

$$\Lambda = \{x_0\} \cup \{y_n\colon n \in \mathbb{Z}\}$$

is hyperbolic with respect to f and there exists a vicinity $\mathscr{O}$ of the set Λ such that $f(x)$ and $Df(x)$ are bounded and uniformly continuous on the closure $\overline{\mathscr{O}}$. By $\mathscr{W}_\eta(f)$ we denote a set of continuously differentiable mappings g of the space X into itself such that

$$\|f(x) - g(x)\| \leq \eta, \qquad \|Df(x) - Dg(x)\| \leq \eta, \qquad x \in \overline{\mathscr{O}}.$$

Then there exists $\eta > 0$ ***such that for any mapping*** $g \in \mathscr{W}_\eta(f)$ ***and for any*** $m \ge 2$ ***there exist a natural number*** l ***and a continuous mapping*** φ ***of the space*** Σ_m ***into a compact subset*** $Y \equiv \varphi(\Sigma_m)$ ***in*** X ***such that***

a) $Y = \varphi(\Sigma_m)$ ***is strictly invariant with respect to*** g^l***, i.e.*** $g^l(Y) = Y$***;***

b) ***if*** $a = (\dots a_{-1}, a_0, a_1, \dots)$ ***and*** $a' = (\dots a'_{-1}, a'_0, a'_1, \dots)$ ***are elements of*** Σ_m ***such that*** $a_i \ne a'_i$ ***for some*** $i \ge 0$***, then*** $\varphi(a) \ne \varphi(a')$***;***

c) ***the restriction of*** g^l ***on*** Y ***is topologically conjugate to the Bernoulli shift*** S ***in*** Σ_m***, i.e.***

$$g^l(\varphi(a)) = \varphi(Sa), \qquad a \in \Sigma_m .$$

Moreover, if in addition we assume that for the mapping g ***there exists*** $\varepsilon_0 > 0$ ***such that for any two trajectories*** $\{x_n:\ n \in \mathbb{Z}\}$ ***and*** $\{\bar{x}_n:\ n \in \mathbb{Z}\}$ ***(of the mapping*** g***) lying in the*** ε_0***-vicinity of the set*** Λ ***the condition*** $x_{n_0} = \bar{x}_{n_0}$ ***for some*** $n_0 \in \mathbb{Z}$ ***implies that*** $x_n = \bar{x}_n$ ***for all*** $n \in \mathbb{Z}$***, then the mapping*** φ ***is a homeomorphism.***

The proof of this theorem is based on Anosov's lemma and mostly follows the standard scheme (see, e.g., [4]) used in the finite-dimensional case. The only difficulty arising in the infinite-dimensional case is the proof of the continuity of the mapping φ. It can be overcome with the help of the lemma presented below which is borrowed from the thesis by Jürgen Kalkbrenner (Augsburg, 1994) in fact.

It should also be noted that the condition under which φ is a homeomorphism holds if the mapping g does not "glue" the points in some vicinity of the set Λ, i.e. the equality $g(x) = g(\bar{x})$ implies $x = \bar{x}$.

Lemma 5.1.

Let the hypotheses of Theorem 5.1 *hold. Let us introduce the notation*

$$J_\nu \equiv J_\nu(k_0, \mu) = \{k \in \mathbb{Z}:\ |k - k_0| \le \mu\nu\},$$

where $k_0 \in \mathbb{Z}$ *and* $\mu, \nu \in \mathbb{N}$. *Let* $z = \{z_n:\ n \in J_\nu\}$ *be a segment (lying in* Λ*) of a* δ*-pseudoorbit of the mapping* f*:*

$$z_n \in \Lambda, \qquad \|z_{n+1} - f(z_n)\| \le \delta, \qquad n,\ n+1 \in J_\nu . \tag{5.1}$$

Assume that $x = \{x_n:\ n \in J_\nu\}$ *and* $\bar{x} = \{\bar{x}_n:\ n \in J_\nu\}$ *are segments of orbits of the mapping* $g \in \mathscr{W}_\eta(f)$*:*

$$g(x_n) = x_{n+1}, \qquad g(\bar{x}_n) = \bar{x}_{n+1}, \qquad n,\ n+1 \in J_\nu , \tag{5.2}$$

such that

$$\|z_n - x_n\| \le \varepsilon, \qquad \|z_n - \bar{x}_n\| \le \varepsilon . \tag{5.3}$$

Then there exist $\delta, \eta, \varepsilon > 0$*, and* $\mu \in \mathbb{N}$ *such that conditions* (5.1)–(5.3) *imply the inequality*

$$\|x_{k_0} - \bar{x}_{k_0}\| \le 2^{1-\nu}\varepsilon . \tag{5.4}$$

Proof.

It follows from (5.2) that

$$x_{k_0+\mu} - \bar{x}_{k_0+\mu} = (Df^{\mu})(z_{k_0})(x_{k_0} - \bar{x}_{k_0}) + R_{k_0}, \tag{5.5}$$

where

$$R_{k_0} = g^{\mu}(x_{k_0}) - g^{\mu}(\bar{x}_{k_0}) - (Df^{\mu})(z_{k_0})(x_{k_0} - \bar{x}_{k_0}).$$

Since the set Λ is hyperbolic with respect to f, there exists a family of projectors $\{P(x)\colon\ x \in \Lambda\}$ for which equations (3.2)–(3.4) are valid. Therefore,

$$(1 - P(f^{\mu}(z_{k_0})))(x_{k_0+\mu} - \bar{x}_{k_0+\mu}) =$$
$$= (Df^{\mu})(z_{k_0})(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0}) + (1 - P(f^{\mu}(z_{k_0})))R_{k_0}\ .$$

It means that

$$(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0}) =$$
$$= [(Df^{\mu})(z_{k_0})]^{-1}[1 - P(f^{\mu}(z_{k_0}))][x_{k_0+\mu} - \bar{x}_{k_0+\mu} - R_{k_0}]\ .$$

Consequently, equation (3.4) implies that

$$\|(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0})\| \ \le\ Kq^{\mu}(\|x_{k_0+\mu} - \bar{x}_{k_0+\mu}\| + \|R_{k_0}\|)\ .$$

Let us estimate the value R_{k_0}. It can be rewritten in the form

$$R_{k_0} = \int_0^1 [(Dg^{\mu})(\xi x_{k_0} + (1-\xi)\bar{x}_{k_0}) - Df^{\mu}(z_{k_0})]\,\mathrm{d}\xi \cdot (x_{k_0} - \bar{x}_{k_0}).$$

It follows from (5.3) that $\|\xi x_{k_0} + (1-\xi)\bar{x}_{k_0} - z_{k_0}\| \le \varepsilon$. Hence, using (4.20) and (4.21), for $\varepsilon > 0$ small enough we obtain that

$$\|R_{k_0}\| \ \le\ (\rho_{\mu}(\eta) + \omega_{\mu}(\varepsilon))\|x_{k_0} - \bar{x}_{k_0}\|, \tag{5.6}$$

where $\rho_{\mu}(\eta) \to 0$ as $\eta \to 0$ and $\omega_{\mu}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore,

$$\|(1 - P(z_{k_0}))(x_{k_0} - \bar{x}_{k_0})\| \ \le\ Kq^{\mu}(\|x_{k_0+\mu} - \bar{x}_{k_0+\mu}\| + \|x_{k_0} - \bar{x}_{k_0}\|)\ , \tag{5.7}$$

provided $\rho_{\mu}(\eta) + \omega_{\mu}(\varepsilon) \le 1$. Further, we substitute the value $k_0 - \mu$ for k_0 in (5.5) to obtain that

$$x_{k_0} - \bar{x}_{k_0} = Df^{\mu}(z_{k_0-\mu})(x_{k_0-\mu} - \bar{x}_{k_0-\mu}) + R_{k_0-\mu}.$$

Therefore, using (3.2) we find that

$$P(f^{\mu}(z_{k_0-\mu}))(x_{k_0} - \bar{x}_{k_0}) =$$
$$= Df^{\mu}(z_{k_0-\mu})P(z_{k_0-\mu})(x_{k_0-\mu} - \bar{x}_{k_0-\mu}) + P(f^{\mu}(z_{k_0-\mu}))R_{k_0-\mu}.$$

Hence, equations (3.3) and (5.6) with $k_0 - \mu$ instead of k_0 give us that

$$\left\| P(f^{\mu}(z_{k_0-\mu}))(x_{k_0} - \bar{x}_{k_0}) \right\| \leq$$

$$\leq K\{q^{\mu} + (\rho_{\mu}(\eta) + \omega_{\mu}(\varepsilon))\} \left\| x_{k_0-\mu} - \bar{x}_{k_0-\mu} \right\| . \tag{5.8}$$

Since

$$z_{k_0} - f^{\mu}(z_{k_0-\mu}) = \sum_{j=0}^{\mu-1} \left\{ f^j(z_{k_0-j}) - f^j(f(z_{k_0-j-1})) \right\} ,$$

it follows from (4.15) and (5.1) that

$$\left\| z_{k_0} - f^{\mu}(z_{k_0-m}) \right\| \leq \delta \sum_{j=0}^{\mu-1} M^j \leq \delta \cdot \mu(1 + M^{\mu})$$

for δ small enough. Therefore,

$$\left\| P(z_{k_0}) - P(f^{\mu}(z_{k_0-\mu})) \right\| \leq \omega(\delta \cdot \mu(1 + M^{\mu})) \equiv \omega(\delta,\ \mu) ,$$

where $\omega(\xi) \to 0$ as $\xi \to 0$ (cf. (4.22)). Consequently, estimate (5.8) implies that

$$\left\| P(z_{k_0})(x_{k_0} - \bar{x}_{k_0}) \right\| \leq$$

$$\leq K\{q^{\mu} + \rho_{\mu}(\eta) + \omega_{\mu}(\varepsilon) + K^{-1} \cdot \omega(\delta,\ \mu)\} \left\| x_{k_0-\mu} - \bar{x}_{k_0-\mu} \right\| . \tag{5.9}$$

It is evident that estimates (5.7) and (5.9) enable us to choose the parameters $\mu,\ \eta,\ \varepsilon$, and δ such that

$$\left\| x_{k_0} - \bar{x}_{k_0} \right\| \leq \frac{1}{2} \max\{ \left\| x_k - \bar{x}_k \right\| :\ k \in J_1(k_0,\ \mu)\} .$$

Using this inequality with k instead of k_0 we obtain that

$$\left\| x_k - \bar{x}_k \right\| \leq \frac{1}{2} \max\{ \left\| x_n - \bar{x}_n \right\| :\ n \in J_2(k_0,\ \mu)\}$$

for all $k \in J_1(k_0,\ \mu)$. Therefore,

$$\left\| x_{k_0} - \bar{x}_{k_0} \right\| \leq \frac{1}{4} \max\{ \left\| x_n - \bar{x}_n \right\| :\ n \in J_2(k_0,\ \mu)\} .$$

If we continue to argue like that, then we find that

$$\left\| x_{k_0} - \bar{x}_{k_0} \right\| \leq 2^{-\nu} \max\{ \left\| x_n - \bar{x}_n \right\| :\ n \in J_{\nu}(k_0,\ \mu)\} .$$

Since $\left\| x_n - \bar{x}_n \right\| \leq \left\| x_n - z_n \right\| + \left\| \bar{x}_n - z_n \right\|$, this and estimate (5.3) imply (5.4). Lemma 5.1 is proved.

Proof of Theorem 5.1.

Let $p_1,\ p_2,\ \ldots,\ p_{m-1}$ be distinct integers. Let us choose and fix the parameters $\varepsilon,\ \eta,\ \delta > 0$ and the integer $\mu > 0$ such that (i) Theorem 4.2 and Lemma 5.1 can be applied to the hyperbolic set $\Lambda = \{x_0\} \cup \{y_n :\ n \in \mathbb{Z}\}$ and (ii)

$$\varepsilon < \frac{1}{2}\min\left\{\left\|y_{p_i}-y_{p_j}\right\|,\ \left\|y_{p_i}-x_0\right\|:\quad i, j = 1, \ldots, m-1,\quad i \neq j\right\}. \tag{5.10}$$

Assume that N is such that

$$\|y_n - x_0\| < \frac{\delta}{2} \quad \text{for} \quad n = N + p_i + 1 \quad \text{and} \quad n = p_i - N \tag{5.11}$$

for all $i = 1, 2, \ldots, m-1$. Let us consider the segments C_i of the orbits of the mapping f of the form

$$C_i = (y_{p_i - N}, \ldots, y_{p_i}, \ldots, y_{p_i + N}), \qquad i = 1, 2, \ldots, m-1,$$

$$C_m = (x_0, x_0, \ldots, x_0).$$

The length of every such segment is $2N+1$. Let $a = (\ldots a_{-1}\, a_0\, a_1 \ldots) \in \Sigma_m$. Let us consider a sequence of elements γ_a made up of the segments C_i by the formula

$$\gamma_a = (\ldots C_{a_{-1}} C_{a_0} C_{a_1} \ldots). \tag{5.12}$$

It is clear that $\gamma_a \in \Lambda$ and by virtue of (5.11) γ_a is a δ-pseudoorbit of the mapping f. Therefore, due to Theorem 4.2 there exists a unique trajectory $\{w_n \equiv w_n(a): n \in \mathbb{Z}\}$ of the mapping g such that

$$\|w_{n(N,\, i,\, j)} - z_{ij}(a)\| \le \varepsilon, \tag{5.13}$$

where $n(N, i, j) = N + (i-1)(2N+1) + j$ and $z_{ij}(a)$ is the j-th element of the segment C_{a_i}, $i \in \mathbb{Z}$, $j = 1, 2, \ldots, 2N+1$. Let us define the mapping φ from Σ_m into X by the formula

$$\varphi(a) = w_0, \tag{5.14}$$

where w_0 is the zeroth element of the trajectory $\{w_n\}$. Since the trajectory $\{w_n\}$ possessing the property (5.13) is uniquely defined, equation (5.14) defines a mapping from Σ_m into X.

If we substitute $i+1$ for i in (5.13) and use the equations

$$w_{n(N,\, i+1,\, j)} = w_{n(N,\, i,\, j)+2N+1} = g^{2N+1}\big(w_{n(N,\, i,\, j)}\big),$$

we obtain that

$$\left\|g^{2N+1}(w_{n(N,\, i,\, j)}) - z_{i+1,\, j}(a)\right\| \le \varepsilon$$

for all $i \in \mathbb{Z}$ and $j = 1, 2, \ldots, 2N+1$. Therefore, the equality $z_{i+1,\, j}(a) = z_{ij}(Sa)$ with S being the Bernoulli shift in Σ_m leads us to the equation

$$\left\|g^{2N+1}(w_{n(N,\, i,\, j)}) - z_{i+1,\, j}(Sa)\right\| \le \varepsilon.$$

Consequently, the uniqueness property of the ε-trajectory in Theorem 4.2 gives us the equation

$$w_n(Sa) = g^{2N+1}(w_n(a)), \qquad n \in \mathbb{Z}.$$

This implies that

$$\varphi(Sa) = g^{2N+1}(\varphi(a)), \qquad a \in \Sigma_m, \tag{5.15}$$

i.e. property (c) is valid for $l = 2N+1$. It follows from (5.15) that

$$g^{2N+1}(\varphi(S^{-1}a)) = \varphi(a).$$

Therefore, the set $Y = \varphi(\Sigma_m)$ is strictly invariant with respect to g^{2N+1}. Thus, assertion (a) is proved.

Let us prove the continuity of the mapping φ. Assume that the sequence of elements $a^{(s)} = (\ldots, a_{-1}^{(s)}, a_0^{(s)}, a_1^{(s)}, \ldots)$ of Σ_m tends to $a = (\ldots, a_{-1}, a_0, a_1, \ldots) \in \Sigma_m$ as $s \to +\infty$. This means (see Exercise 1.4) that for any $M \in \mathbb{N}$ there exists $s_0 = s_0(M)$ such that

$$a_i^{(s)} = a_i \quad \text{for} \quad |i| \le M, \; s \ge s_0. \tag{5.16}$$

Assume that $\gamma^{(s)} = \{z_k^{(s)}\}$ and $\gamma = \{z_k\}$ are δ-pseudoorbits in Λ constructed according to (5.12) for the symbols $a^{(s)}$ and a, respectively. Equation (5.16) implies that $z_k^{(s)} = z_k$ for $|k| \le M(2N+1)$. Let $\{w_n\}$ and $\{w_n^{(s)}\}$ be ε-trajectories corresponding to γ and $\gamma^{(s)}$, respectively. Lemma 5.1 gives us that

$$\left\|w_0^{(s)} - w_0\right\| \le 2^{1-\nu}\varepsilon, \tag{5.17}$$

provided $M(2N+1) \ge \nu\mu$, i.e. for any $\nu \in \mathbb{N}$ equations (5.17) is valid for $s \ge s_0(\nu, M)$. This means that

$$\|\varphi(a^{(s)}) - \varphi(a)\| \equiv \left\|w_0^{(s)} - w_0\right\| \to 0$$

as $s \to +\infty$. Thus, the mapping φ is continuous and $Y = \varphi(\Sigma_m)$ is a compact strictly invariant set with respect to g^{2N+1}.

Let us now prove nontriviality property (b) of the mapping φ. Let $a, a' \in \Sigma_m$ be such that $a_i \ne a'_i$ for some $i \ge 0$. Let $\{w_n\}$ and $\{w'_n\}$ be ε-trajectories corresponding to the symbols a and a', respectively. Then

$$\left\|w_{n(N,\,i,\,N+1)} - w'_{n(N,\,i,\,N+1)}\right\| \ge \|z_{i,\,N+1}(a) - z_{i,\,N+1}(a')\| -$$

$$-\left\|w_{n(N,\,i,\,N+1)} - z_{i,\,N+1}(a)\right\| - \left\|w'_{n(N,\,i,\,N+1)} - z_{i,\,N+1}(a')\right\|.$$

Therefore, it follows both from (5.13) and the definition of the elements $z_{ij}(a)$ that

$$\left\|w_{(2N+1)i} - w'_{(2N+1)i}\right\| \ge \left\|y_{q_i} - y_{q'_i}\right\| - 2\varepsilon,$$

where $q_i = p_{a_i}$ and $q'_i = p_{a'_i}$. We apply (5.10) to obtain that

$$\left\|w_{(2N+1)i} - w'_{(2N+1)i}\right\| > 0. \tag{5.18}$$

Therefore, if $i \ge 0$, then

$$g^{(2N+1)i}(w_0) \ne g^{(2N+1)i}(w'_0).$$

Hence, $\varphi(a) \equiv w_0 \ne w'_0 \equiv \varphi(a')$. This completes the proof of assertion (b).

If the trajectories of the mapping g cannot be "glued" (see the hypotheses of Theorem 5.1), then for some $i \in \mathbb{Z}$ equation (5.18) gives us that $w_0 \neq w_0'$, i.e. $\varphi(a) \neq \varphi(a')$ if $a \neq a'$. Thus, the mapping φ is injective in this case. Since Σ_m is a compact metric space, then the injectivity and continuity of φ imply that φ is a homeomorphism from Σ_m onto $\varphi(\Sigma_m)$. **Theorem 5.1 is proved**.

It should be noted that equations (5.13) and (5.14) imply that the set $Y = \varphi(\Sigma_m)$ lies in the ε-vicinity of the hyperbolic set Λ. Therewith, the values η and l involved in the statement of the theorem depend on ε and one can state that for any vicinity $\mathcal{U}$ of the set Λ there exist η and l such that the conclusions of Theorem 5.1 are valid and $\varphi(\Sigma_m) \subset \mathcal{U}$. It is also clear that the set $Y = \varphi(\Sigma_m)$ is not uniquely determined.

Exercise *5.1* Assume that $g = f$ in Theorem 5.1. Prove that the mapping φ can be constructed such that $\varphi(\Sigma_m) \supset \{x_0\} \cup \{y_{nl}:\ n \geq 0\}$, where $\{y_n:\ n \in \mathbb{Z}\}$ is a homoclinic orbit of the mapping f.

Exercise *5.2* Prove the Birkhoff theorem: if the hypotheses of Theorem 5.1 hold, then for any $\varepsilon > 0$ small enough there exist $\eta > 0$ and $l \in \mathbb{N}$ such that for every mapping $g \in \mathcal{W}_\eta(f)$ there exist periodic trajectories of the mapping g^l of any minimal period in the ε-vicinity of the set Λ.

Exercise *5.3* Use Theorem 1.1 to describe all the possible types of behaviour of the trajectories of the mapping g on a set

$$W = \bigcup_{k=1}^{l} g^k(\varphi(\Sigma_m)).$$

In conclusion, it should be noted that different infinite-dimensional versions of Anosov's lemma and the Birkhoff-Smale theorem have been considered by many authors (see, e.g., [6], [10], [11], [12], and the references therein).

§ 6 Possibility of Chaos in the Problem of Nonlinear Oscillations of a Plate

In this section the Birkhoff-Smale theorem is applied to prove the existence of chaotic regimes in the problem of nonlinear plate oscillations subjected to a periodic load. The results presented here are close to the assertions proved in [13]. However, the methods used differ from those in [13].

Let us remind the statement of the problem. We consider its abstract version as in Chapter 4. Let H be a separable Hilbert space and let A be a positive operator with discrete spectrum in H, i.e. there exists an orthonormalized basis $\{e_k\}$ in H such that

$$Ae_k = \lambda_k e_k , \qquad 0 < \lambda_1 \le \lambda_2 \le \ldots , \qquad \lim_{n \to \infty} \lambda_n = \infty .$$

The following problem is considered:

$$\begin{cases} \ddot{u} + \gamma \dot{u} + A^2 u + (\varkappa \|A^{1/2} u\|^2 - \Gamma) A u + L u = h \cos \omega t , & (6.1) \\ u\big|_{t=0} = u_0 \in F_1 = D(A), \quad \dot{u}\big|_{t=0} = u_1 \in H . & (6.2) \end{cases}$$

Here γ, $\varkappa$, Γ, and ω are positive parameters, h is an element of the space H, L is a linear operator in H subordinate to A, i.e.

$$\|Lu\| \le K \|Au\| , \tag{6.3}$$

where K is a constant. The problem of the form (6.1) and (6.2) was studied in Chapter 4 in details (nonlinearity of a more general type was considered there). The results of Section 4.3 imply that problem (6.1) and (6.2) is uniquely solvable in the class of functions

$$\mathcal{W}_+ = C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, H) . \tag{6.4}$$

Moreover, one can prove (cf. Exercise 4.3.9) that Cauchy problem (6.1) and (6.2) is uniquely solvable on the whole time axis, i.e. in the class

$$\mathcal{W} = C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, H) .$$

This fact as well as the continuous dependence of solutions on the initial conditions (see (4.3.20)) enables us to state that the monodromy operator G acting in $\mathcal{H} = D(A) \times H$ according to the formula

$$G(u_0;\ u_1) = \left(u\left(\frac{2\pi}{\omega}\right),\ \dot{u}\left(\frac{2\pi}{\omega}\right) \right) \tag{6.5}$$

is a homeomorphism of the space $\mathcal{H}$ (see Exercise 4.3.11). Here $u(t)$ is a solution to problem (6.1) and (6.2)

The aim of this section is to prove the fact that under some conditions on L and h chaotic dynamics is observed in the discrete dynamical system $(\mathcal{H}, G)$ for some set of parameters γ, $\varkappa$, Γ, and ω.

Lemma 6.1.

The mapping G defined by equality (6.5) *is a diffeomorphism of the space $\mathcal{H} = D(A) \times H$.*

Proof.

We use the method applied to prove Lemma 4.7.3. Let $u_1(t)$ be a solution to problem (6.1) and (6.2) with the initial conditions $\bar{y} = (u_0, u_1) \in \mathcal{H}$ and let

$u_2(t)$ be a solution to it with the initial condions $\bar{y}+z \equiv (u_0+z_0,\ u_1+z_1) \in \mathcal{H}$. Let us consider a linearization of problem (6.1) and (6.2) along the solution $u_1(t)$:

$$\begin{cases} \ddot{w}(t)+\gamma\dot{w}(t)+A^2 w+(\varkappa\left\|A^{1/2}u_1(t)\right\|^2-\Gamma)Aw+ \\ \qquad +2\varkappa(A^{1/2}u_1(t),\ A^{1/2}w(t))Au_1(t)+Lw = 0\ , & (6.6) \\ w\big|_{t=0}=z_0, \quad \dot{w}\big|_{t=0}=z_1\ . & (6.7) \end{cases}$$

As in the proof of Theorem 4.2.1, it is easy to find that problem (6.6) and (6.7) is uniquely solvable in the class of functions (6.4). Let $v(t) = u_2(t)-u_1(t)-w(t)$. It is evident that $v(t)$ is a weak solution to problem

$$\begin{cases} \ddot{v}+\gamma\dot{v}(t)+A^2 v(t) = F(t) \equiv F(u_1(t),\ u_2(t),\ w(t))\ , & (6.8) \\ v\big|_{t=0}=0\,, \quad \dot{v}\big|_{t=0}=0\ , & (6.9) \end{cases}$$

where

$$F(u_1,\ u_2,\ w) = -\left(\varkappa\left\|A^{1/2}u_2\right\|^2-\Gamma\right)Au_2+\left(\varkappa\left\|A^{1/2}u_1\right\|^2-\Gamma\right)Au_1+$$

$$+\left(\varkappa\left\|A^{1/2}u_1\right\|^2-\Gamma\right)Aw+2\varkappa(Au_1,\ w)Au_1-L(u_2-u_1-w)\ .$$

A simple calculation shows that

$$F(u_1,\ u_2,\ w) = F_1(u_1,\ u_2,\ w)+F_2(u_1,\ u_2,\ w) \equiv F_1+F_2\,,$$

where

$$F_1 = -\left(\varkappa\left\|A^{1/2}u_2\right\|^2-\Gamma\right)Av-Lv-2\varkappa(Au_1,\ v)A(u_1+w)\,,$$

$$F_2 = -\varkappa\left\|A^{1/2}(u_1-u_2)\right\|^2A(u_1+w)-2\varkappa(Au_1,\ w)Aw\,.$$

We assume that $\|\bar{y}\|_{\mathcal{H}} \le R$ and $\|z\|_{\mathcal{H}} \le 1$. In this case (see Section 4.3) the estimates

$$\begin{aligned} \left\|Au_j(t)\right\| &\le C_{R,\,T}\,, \\ \left\|A(u_1(t)-u_2(t))\right\| &\le C_{R,\,T}\|z\|_{\mathcal{H}}\,, \\ \|Aw(t)\| &\le C_{R,\,T}\|z\|_{\mathcal{H}}\,, \end{aligned} \qquad (6.10)$$

are valid on any segment $[0,\ T]$. Here $C_{R,\,T}$ is a constant. Therefore,

$$\left\|F_1\right\| \le C_1\|Av\|\,, \qquad \left\|F_2\right\| \le C_2\|z\|^2_{\mathcal{H}}\,, \qquad t \in [0,\ T]\,,$$

where C_1 and C_2 are constants depending on R and T. Hence,

$$\|F(t)\|^2 \le C_1\|Av\|^2+C_2\|z\|^4_{\mathcal{H}}\,, \qquad t \in [0,\ T]\,.$$

Therefore, the energy equation

$$\frac{1}{2}\left(\|\dot{v}(t)\|^2 + \|Av(t)\|^2\right) + \gamma \int_0^t \|\dot{v}(\tau)\|^2 \, d\tau = \int_0^t (F(\tau), \dot{v}(\tau)) \, d\tau \tag{6.11}$$

for problem (6.8) and (6.9) leads us to the estimate

$$\|\dot{v}(t)\|^2 + \|Av(t)\|^2 \le \int_0^t \left(C_1\|Av(\tau)\|^2 + C_2\|z\|^4\right) d\tau \,, \qquad t \in [0, T] .$$

Using Gronwall's lemma we find that

$$\|\dot{v}(t)\|^2 + \|Av(t)\|^2 \le C\|z\|^4 , \qquad t \in [0, T] ,$$

where the constant C depends on T and R. This estimate implies that the mapping

$$G' : z \equiv (z_0; z_1) \to \left(w\left(\frac{2\pi}{\omega}\right); \dot{w}\left(\frac{2\pi}{\omega}\right)\right) \tag{6.12}$$

is a Frechét derivative of the mapping G defined by equality (6.5). Here $w(t)$ is a solution to problem (6.6) and (6.7). It follows from (6.10) and (6.6) that G' is a continuous linear mapping of $\mathcal{H}$ into itself. Using (6.10) it is also easy to see that $G' = G'[u_0, u_1]$ continuously depends on $\bar{y} = (u_0; u_1)$ with respect to the operator norm. Lemma 6.1 is proved.

Further we will also need the following assertion.

Lemma 6.2.

Let G_j *be the monodromy operator of problem* (6.1) *and* (6.2) *with* $L = L_j$ *and* $h = h_j$, $j = 1, 2$. *Assume that for* $L = L_j$ *equation* (6.3) *is valid and* $h_j \in H$, $j = 1, 2$. *Moreover, assume that*

$$\|L_j A^{-1}\| \le \rho, \qquad \|h_j\| \le \rho, \qquad j = 1, 2 .$$

Then the estimates

$$\sup_{y \in B_R} \|G_1(y) - G_2(y)\| \le C\left(\|(L_1 - L_2) A^{-1}\| + \|h_1 - h_2\|\right) \tag{6.13}$$

and

$$\sup_{y \in B_R} \|G_1'(y) - G_2'(y)\| \le C\left(\|(L_1 - L_2) A^{-1}\| + \|h_1 - h_2\|\right) \tag{6.14}$$

are valid. Here B_R *is a ball of the radius* R *in* $\mathcal{H} = D(A) \times H$, $R > 0$ *is an arbitrary number while the constant* C *depends on* R *and* ρ *but does not depend on the parameters* $\omega, \gamma, \varkappa \ge 0$, *and* Γ *provided they vary in bounded sets.*

Proof.

Let $u_j(t)$ be a solution to problem (6.1) and (6.2) with $L = L_j$ and $h = h_j$, $j = 1, 2$. It is evident (see Section 4.3) that

$$\|Au_j(t)\| \le C \equiv C(R, \rho, T), \qquad t \in [0, T]. \tag{6.15}$$

Therefore, it is easy to find that the difference $u(t) = u_1(t) - u_2(t)$ satisfies the equation

$$\begin{cases} \ddot{u} + \gamma u + A^2 u = F(t, u_1, u_2), \\ u|_{t=0} = 0, \quad \dot{u}|_{t=0} = 0, \end{cases}$$

where the function $F(t, u_1, u_2)$ can be estimated as follows:

$$\|F(t, u_1, u_2)\| \le C_1 \|Au\| + C_2 \left(\|(L_1 - L_2)A^{-1}\| + \|h_1 - h_2\| \right).$$

As in the proof of Lemma 6.1, we now use energy equality (6.11) and Gronwall's lemma to obtain the estimate

$$\|\dot{u}(t)\|^2 + \|Au(t)\|^2 \le C\left(\|(L_1 - L_2)A^{-1}\|^2 + \|h_1 - h_2\|^2 \right). \tag{6.16}$$

This implies inequality (6.13). Estimate (6.14) can be obtained in a similar way. In its proof equations (6.12), (6.15), and (6.16) are used. We suggest the reader to carry out the corresponding reasonings himself/herself. Lemma 6.2 is proved.

Let us now prove that there exist an operator L and a vector h such that the corresponding mapping G possesses a hyperbolic homoclinic trajectory. To do that, we use the following well-known result (see, e.g., [1], [13], as well as Section 7) related to the Duffing equation.

Theorem 6.1.

Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ ***be a monodromy operator corresponding to the Duffing equation***

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(f \cdot \cos \omega t - \delta \dot{x}), \tag{6.17}$$

i.e. the mapping of the plane $\mathbb{R}^2$ ***into itself acting according to the formula***

$$g(x_0, x_1) = \left(x\left(\frac{2\pi}{\omega}\right);\ \dot{x}\left(\frac{2\pi}{\omega}\right) \right), \tag{6.18}$$

where $x(t)$ ***is a solution to equation*** (6.17) ***such that*** $x(0) = x_0$ ***and*** $\dot{x}(0) = x_1$. ***All the parameters contained in*** (6.17) ***are assumed to be positive. Let us also assume that***

$$f > f_{cr} \equiv \delta \cdot \frac{2\beta^{3/2}}{3\omega\sqrt{2\alpha}} \cdot \cosh\left(\frac{\pi\omega}{2\sqrt{\beta}} \right). \tag{6.19}$$

Then there exists $\varepsilon_0 > 0$ ***such that for every*** $\varepsilon \in (0, \varepsilon_0)$ ***the mapping*** g ***possesses a fixed point*** z ***and a homoclinic trajectory*** $\{y_n : n \in \mathbb{Z}\}$ ***to it,*** $y_n \neq z$***, therewith the set*** $\{z\} \cup \{y_n : n \in \mathbb{Z}\}$ ***is hyperbolic.***

Let P_k be the orthoprojector onto the one-dimensional subspace generated by the eigenvector e_k in H. We consider problem (6.1) and (6.2) with $L_k = L(1-P_k)$ instead of L and $h = h_k \cdot e_k$, where h_k is a positive number. Then it is evident that every solution to problem (6.1) and (6.2) with the initial conditions $u_0 = c_0 e_k$ and $u_1 = c_1 e_k$ has the form

$$u(t) = x(t)\, e_k ,$$

where $x(t)$ is a solution to the Duffing equation

$$\ddot{x} + \gamma\dot{x} - \lambda_k(\Gamma - \lambda_k)x + \varkappa\lambda_k^2 x^3 = h_k \cos\omega t \tag{6.20}$$

with the initial conditions $x(0) = c_0$ and $\dot{x}(0) = c_1$. In particular, this means that the two-dimensional subspace $\mathscr{L}_k = \mathrm{Lin}\{(e_k;\ 0),\ (0;\ e_k)\}$ of the space $\mathscr{H}$ is strictly invariant with respect to the corresponding monodromy operator G_k while the restriction of G_k to $\mathscr{L}_k$ coincides with the monodromy operator corresponding to the Duffing equation (6.20). Therefore, if h_k is small enough and the conditions

$$0 < \lambda_k < \Gamma , \qquad \frac{h_k}{\gamma} > \frac{2\lambda_k(\Gamma-\lambda_k)}{3\omega\sqrt{2\varkappa}}\left(\frac{\Gamma}{\lambda_k} - 1\right)^{1/2} \cosh\frac{\pi\omega}{2\sqrt{\lambda_k(\Gamma-\lambda_k)}}$$

hold, then the mapping G_k possesses a hyperbolic invariant set

$$\Lambda_k = \{z e_k\} \cup \{y_n \cdot e_k :\ n \in \mathbb{Z}\}$$

consisting of the fixed point $(z_0 e_k;\ z_1 e_k)$ and its homoclinic trajectory

$$\{y_n e_k :\ n \in \mathbb{Z}\}, \quad \text{where } y_n = (y_n^0;\ y_n^1) \in \mathbb{R}^2 .$$

Thus, if $\omega > 0$ and for some k the condition $0 < \lambda_k < \Gamma$ holds, then there exists an open set $\mathscr{P}$ in the space of parameters $\{\gamma, h_k\}$ such that for every $(\gamma, h_k) \in \mathscr{P}$ the monodromy operator G_k corresponding to problem (6.1) and (6.2) with $L_k = L(1-P_k)$ instead of L and $h = h_k e_k$ possesses a hyperbolic set consisting of a fixed point and a homoclinic trajectory. This fact as well as Lemmata 6.1 and 6.2 enables us to apply the Birkhoff-Smale theorem and prove the following assertion.

Theorem 6.2.

Let $\omega > 0$ ***and let the condition*** $0 < \lambda_k < \Gamma$ ***hold for some*** k***. Then there exist*** $\mu > 0$ ***and an open set*** $\mathscr{P}$ ***in the metric space*** $\mathbb{R}_+ \times H$ ***such that if***

$$\|Le_k\|\lambda_k^{-1} < \mu, \qquad (\gamma, h) \in \mathscr{P},$$

then some degree G^l ***of the monodromy operator*** G ***of problem*** (6.1) ***and*** (6.2) ***possesses a compact strictly invariant set*** Y $(G^l Y = Y)$ ***in the space*** $\mathscr{H}$ ***in which the mapping*** G^l ***is topologically conjugate to the Bernoulli shift of***

sequences of m ***symbols, i.e. there exists a homeomorphism*** $\varphi\colon \Sigma_m \to Y$ ***such that***

$$G^l(\varphi(a)) = \varphi(Sa), \qquad a \in \Sigma_m.$$

— Exercise *6.1* Prove that if the hypotheses of Theorem 6.2 hold, then equation (6.1) possesses an infinite number of periodic solutions with periods multiple to ω.

— Exercise *6.2* Apply Theorem 6.2 to the Berger approximation of the problem of nonlinear plate oscillations:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \gamma \dfrac{\partial u}{\partial t} + \Delta^2 u - \left(\varkappa \displaystyle\int_\Omega |\nabla u(x,t)|^2 \, dx - \Gamma \right) \Delta u \; + \\ \qquad + \rho \dfrac{\partial u}{\partial x_1} = h(x) \cos \omega t, \qquad x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2, \quad t > 0, \\ u\big|_{\partial\Omega} = \Delta u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0(x), \quad \dfrac{\partial u}{\partial t}\Big|_{t=0} = u_1(x). \end{cases}$$

§ 7 *On the Existence of Transversal Homoclinic Trajectories*

Undoubtedly, Theorem 6.1 on the existence of a transversal (hyperbolic) homoclinic trajectory of the monodromy operator for the periodic perturbation of the Duffing equation is the main fact which makes it possible to apply the Birkhoff-Smale theorem and to prove the possibility of chaotic dynamics in the problem of plate oscillations. In this connection, the question as to what kind of generic condition guarantees the existence of a transversal homoclinic orbit of monodromy operators generated by ordinary differential equations gains importance. Extensive literature is devoted to this question (see, e.g., [1], [2] and the references therein). There are several approaches to this problem. All of them enable us to construct systems with transversal homoclinic trajectories as small perturbations of "simple" systems with homoclinic (not transversal!) orbits. In some cases the corresponding conditions on perturbations can be formulated in terms of the Melnikov function.

This section is devoted to the exposition and discussion of the results obtained by K. Palmer [14]. These results help us to describe some classes of systems of ordinary differential equations which generate dynamical systems with transversal homoclinic orbits. Such differential equations are obtained as periodic perturbations of autonomous equations with homoclinic trajectories.

In the space $\mathbb{R}^n$ let us consider a system of equations

$$\dot{x}(t) = g(x(t)), \qquad x(t) \in \mathbb{R}^n, \tag{7.1}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is a twice continuously differentiable mapping. Assume that the Cauchy problem for equation (7.1) is uniquely solvable for any initial condition $x(0) = x_0$. Let us also assume that there exist a fixed point z_0 $(g(z_0)=0)$ and a trajectory $z(t) \neq z_0$ homoclinic to z_0, i.e. a solution to equation (7.1) such that $z(t) \to z_0$ as $t \to \pm\infty$. Exercises 7.1 and 7.2 given below give us the examples of the cases when these conditions hold. We remind that every second order equation $\ddot{x} + V(x) = 0$ can be rewritten as a system of the form (7.1) if we take $x_1 = x$ and $x_2 = \dot{x}$.

— Exercise *7.1* Consider the Duffing equation

$$\ddot{x} - \beta x + \alpha x^3 = 0, \qquad \alpha, \beta > 0.$$

Prove that the curve $z(t) = (\eta(t), \dot{\eta}(t)) \in \mathbb{R}^2$ is an orbit of the corresponding system (7.1) homoclinic to 0. Here $\eta(t) = \sqrt{2\beta/\alpha}\ \mathrm{sech}\sqrt{\beta}\,t$.

— Exercise *7.2* Assume that for a function $U(x) \in C^3(\mathbb{R})$ there exist a number E and a pair of points $a < b$ such that

$$U(a) = U(b) = E; \qquad U(x) < E, \quad x \in (a, b);$$
$$U'(a) = 0; \qquad U''(a) < 0; \qquad U'(b) > 0.$$

Then system (7.1) corresponding to $\ddot{x} + U'(x) = 0$ possesses an orbit homoclinic to $(a, 0)$ that passes through the point $(b, 0)$.

Unfortunately, as the cycle of Exercises 7.3–7.5 shows, the homoclinic orbit of autonomous equation (7.1) cannot be used directly to construct a discrete dynamical system with a transversal homoclinic trajectory.

— Exercise *7.3* For every $\tau > 0$ define the mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by the formula $f(x_0) = x(\tau; x_0)$, where $x(t; x_0)$ is a solution to equation (7.1) with the initial condition x_0. Show that f is a diffeomorphism in $\mathbb{R}^n$ with a fixed point z_0 and a family of homoclinic orbits $\{y_n^s\colon n \in \mathbb{Z}\}$, where $y_n = z(s + n\tau)$, $s \in [0, \tau)$.

— Exercise *7.4* Prove that the derivative f' of the mapping f constructed in Exercise 7.3 can be evaluated using the formula $f'(x_0)w = y(\tau, w)$, where $y(t, w)$ is a solution to problem

$$\dot{y}(t) = g'(x(t))\,y(t), \qquad y(0) = w_0.$$

Here $x(t) = x(t; x_0)$ is a solution to equation (7.1) with the initial condition x_0.

Exercise *7.5* Let f be the mapping constructed in Exercise 7.3 and let $\{z(t)\colon\ t \in \mathbb{Z}\}$ be a homoclinic orbit of equation (7.1). Show that $\{w_n = \dot{z}(n\tau)\colon\ n \in \mathbb{Z}\}$ is a bounded solution to the difference equation $w_{n+1} = f'(y_n)w_n$, where $y_n = z(n\tau)$ (*Hint*: the function $w(t) = \dot{z}(t)$ satisfies the equation $\dot{w} = g'(z(t))w$).

Thus, due to Theorems 2.1 and 3.1 the result of Exercise 7.5 implies that the set

$$\Lambda = \{z_0\} \cup \{z(n\tau)\colon\ n \in \mathbb{Z}\}$$

cannot be hyperbolic with respect to the mapping f defined by the formula $f(x_0) = x(\tau;\ x_0)$, where $x(t;\ x_0)$ is a solution to equation (7.1) with the initial condition x_0. Nevertheless we can indicate some quite simple conditions on the class of perturbations $\{h(t,\ x,\ \mu)\}$ periodic with respect to t under which the monodromy operator of the problem

$$\dot{x}(t) = g(x(t)) + \mu h(t,\ x(t),\ \mu), \qquad x(t) \in \mathbb{R}^n, \tag{7.2}$$

possesses a transversal (hyperbolic) homoclinic trajectory for μ small enough.

Further we will use the notion of exponential dichotomy for ordinary differential equations (see [15], [16] as well as [5] and the references therein)

Let $A(t)$ be a continuous and bounded $n \times n$ matrix function on the real axis. We consider the problem

$$\dot{x}(t) = A(t)x(t), \qquad t \in \mathbb{R}, \quad x|_{t=s} = x_0, \tag{7.3}$$

in the space $X = \mathbb{R}^n$. It is easy to see that it is solvable for every initial condition. Therefore, we can define the evolutionary operator $\Phi(t,\ s)$, $t,\ s \in \mathbb{R}$, by the formula

$$\Phi(t,\ s)x_0 = x(t) \equiv x(t,\ s;\ x_0), \qquad t,\ s \in \mathbb{R},$$

where $x(t)$ is a solution to problem (7.3).

Exercise *7.6* Prove that

$$\Phi(t,\ s) = \Phi(t,\ \tau)\Phi(\tau,\ s), \qquad \Phi(t,\ t) = I$$

for all $t,\ s,\ \tau \in \mathbb{R}$ and the following matrix equations hold:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\ s) = A(t)\,\Phi(t,\ s), \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Phi(t,\ s) = -\Phi(t,\ s)\,A(s). \tag{7.4}$$

Exercise *7.7* Prove the inequality

$$\|\Phi(t,\ s)\| \le \exp\left\{\int_s^t \|A(\tau)\|\,\mathrm{d}\tau\right\}, \qquad t \ge s.$$

Let $\mathcal{J}$ be some interval of the real axis. We say that equation (7.3) admits an ***exponential dichotomy*** over the interval $\mathcal{J}$ if there exist constants $K, \alpha > 0$ and a family of projectors $\{P(t):\ t \in \mathcal{J}\}$ continuously depending on t and such that

$$P(t)\,\Phi(t,\,s) = \Phi(t,\,s)\,P(s)\,, \qquad t \ge s\,; \tag{7.5}$$

$$\|\Phi(t,\,s)P(s)\| \le Ke^{-\alpha(t-s)}\,, \qquad t \ge s\,, \tag{7.6}$$

$$\|\Phi(t,\,s)\,(I-P(s))\| \le Ke^{-\alpha(s-t)}\,, \qquad t \le s\,, \tag{7.7}$$

for $t,\, s \in \mathcal{J}$.

— Exercise *7.8* Let $A(t) \equiv A$ be a constant matrix. Prove that equation (7.3) admits an exponential dichotomy over $\mathbb{R}$ if and only if the eigenvalues on A do not lie on the imaginary axis.

The assertion contained in Exercise 7.8 as well as the following theorem on the roughness enables us to construct examples of equations possessing an exponential dichotomy.

Theorem 7.1.

Assume that problem (7.3) ***possesses an exponential dichotomy over an interval*** $\mathcal{J}$. ***Then there exists*** $\varepsilon > 0$ ***such that equation***

$$\dot{x}(t) = (A(t)+B(t))\,x(t) \tag{7.8}$$

possesses an exponential dichotomy over $\mathcal{J}$, ***provided*** $\|B(t)\| \le \varepsilon$ ***for*** $t \in \mathcal{J}$. ***Moreover, the dimensions of the corresponding projectors for*** (7.3) ***and*** (7.8) ***are the same.***

The proof of this theorem can be found in [15] or [16], for example.

The exercises given below contain some simple facts on systems possessing an exponential dichotomy. We will use them in our further considerations.

— Exercise *7.9* Prove that equations (7.5)–(7.7) imply the estimates

$$\|\Phi(t,\,s)\xi\| \ge K^{-1}e^{\alpha(t-s)}\|(1-P(s))\,\xi\|\,, \qquad t \ge s\,,$$

$$\|\Phi(t,\,s)\,\xi\| \ge K^{-1}e^{\alpha(s-t)}\|P(s)\,\xi\|\,, \qquad t \le s\,,$$

for any $\xi \in X \equiv \mathbb{R}^n$.

— Exercise *7.10* Assume that equation (7.3) admits an exponential dichotomy over $\mathbb{R}_+ = [0,\ +\infty)$. Prove that $P(0)X = V_+$, where

$$V_+ \equiv \left\{\xi \in X \equiv \mathbb{R}^n:\ \ \sup_{t>0}\|\Phi(t,\,0)\,\xi\| < \infty\right\} \tag{7.9}$$

(*Hint*: $\|\Phi(t,\,0)\,\xi\| \ge K^{-1}e^{\alpha t}\|(1-P(0))\,\xi\|\,,\quad t \ge 0$).

Exercise *7.11* If equation (7.3) possesses an exponential dichotomy over $\mathbb{R}_- = (-\infty, 0]$, then $(I - P(0))X = V_-$, where

$$V_- = \left\{ \xi \in X \equiv \mathbb{R}^n : \sup_{t<0} \|\Phi(t, 0)\xi\| < \infty \right\} \tag{7.10}$$

(*Hint*: $\|\Phi(t, 0)\xi\| \geq K^{-1} e^{\alpha t} \|P(0)\xi\|, \quad t \leq 0$).

Exercise *7.12* Assume that equation (7.3) possesses an exponential dichotomy over $\mathbb{R}_+$ (over $\mathbb{R}_-$, respectively). Show that any solution to problem (7.3) bounded on $\mathbb{R}_+$ (on $\mathbb{R}_-$, respectively) decreases at exponential velocity as $t \to +\infty$ (as $t \to -\infty$, respectively).

Exercise *7.13* Assume that equation (7.3) possesses an exponential dichotomy over the half-interval $[a, +\infty)$, where a is a real number. Prove that equation (7.3) possesses an exponential dichotomy over any semiaxis of the form $[b, +\infty)$.
(*Hint*: $P(t) = \Phi(t, a) P(a) \Phi(a, t)$).

Exercise *7.14* Prove the analogue of the assertion of Exercise 7.13 for the semiaxis $(-\infty, a]$.

Exercise *7.15* Prove that for problem (7.3) to possess an exponential dichotomy over $\mathbb{R}$ it is necessary and sufficient that equation (7.3) possesses an exponential dichotomy both over $\mathbb{R}_+$ and $\mathbb{R}_-$ and has no nontrivial solutions bounded on the whole axis $\mathbb{R}$.

Exercise *7.16* Prove that the spaces V_+ and V_- (see (7.9) and (7.10)) possess the properties

$$V_+ \cap V_- = \{0\}, \qquad V_+ + V_- = X \equiv \mathbb{R}^n,$$

provided problem (7.3) possesses an exponential dichotomy over $\mathbb{R}$.

Exercise *7.17* Consider the following equation adjoint to (7.3):

$$\dot{y}(t) = -A^*(t) y(t), \tag{7.11}$$

where $A^*(t)$ is the transposed matrix. Prove that the evolutionary operator $\Psi(t, s)$ of problem (7.11) has the form $\Psi(t, s) = [\Phi(s, t)]^*$.

Exercise *7.18* Assume that problem (7.3) possesses an exponential dichotomy over an interval $\mathcal{I}$. Then equation (7.11) possesses exponential dichotomy over $\mathcal{I}$ with the same constants K, $\alpha > 0$ and projectors $Q(t) = I - P(t)^*$.

Exercise *7.19* Assume that problem (7.3) possesses an exponential dichotomy both over $\mathbb{R}_+$ and $\mathbb{R}_-$. Let $\dim V_+ + \dim V_- = n$, where $V_\pm$ are defined by equalities (7.9) and (7.10). Show that the dimensions of the spaces of solutions to problems (7.3) and (7.11) bounded on the whole axis are finite and coincide.

Exercise *7.20* Assume that problem (7.3) possesses an exponential dichotomy over $\mathbb{R}$. Then for any $s \in \mathbb{R}$ and $\tau > 0$ the difference equation $x_n = \Phi(s + \tau n, s) x_{n-1}$ possesses an exponential dichotomy over $\mathbb{Z}$ (for the definition see Section 2).

Let us now return to problem (7.1). Assume that z_0 is a hyperbolic fixed point for (7.1), i.e. the matrix $g'(z_0)$ does not have any eigenvalues on the imaginary axis. Let $z(t)$ be a trajectory homoclinic to z_0. Using Theorem 7.1 on the roughness and the results of Exercises 7.13 and 7.14 we can prove that the equation

$$\dot{y} = g'(z(t))\, y \tag{7.12}$$

possesses an exponential dichotomy over both semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$. Moreover, the dimensions of the corresponding projectors are the same and coincide with the dimension of the spectral subspace of the matrix $g'(z_0)$ corresponding to the spectrum in the left semiplane. Therewith it is easy to prove that $\dim V_+ + \dim V_- = n$, where $V_\pm$ have form (7.9) and (7.10). The result of Exercise 7.15 implies that equation (7.12) cannot possess an exponential dichotomy over $\mathbb{R}$ ($y(t) = \dot{z}(t)$ is a solution to (7.12) bounded on $\mathbb{R}$) while Exercise 7.19 gives that the number of linearly independent bounded (on $\mathbb{R}$) solutions to (7.12) and to the adjoint equation

$$\dot{y} = -[g'(z(t))]^* y \tag{7.13}$$

is the same. These facts enable us to formulate Palmer's theorem (see [14]) as follows.

Theorem 7.2.

Assume that $g(x)$ ***is a twice continuously differentiable function from*** $\mathbb{R}^n$ ***into*** $\mathbb{R}^n$ ***and equation***

$$\dot{x} = g(x)$$

possesses a fixed hyperbolic point z_0 ***and a trajectory*** $\{z(t)\colon\ t \in \mathbb{R}\}$ ***homoclinic to*** z_0. ***We also assume that*** $y(t) = \dot{z}(t)$ ***is a unique (up to a scalar factor) solution to equation***

$$\dot{y} = g'(z(t))\, y \tag{7.14}$$

bounded on $\mathbb{R}$. ***Let*** $h(t, x, \mu)$ ***be a continuously differentiable vector function*** T***-periodic with respect to*** t ***and defined for*** $t \in \mathbb{R}$, $|x - z(t)| < \Delta_0$, $|\mu| < \sigma_0$, $\mu \in \mathbb{R}$. ***If***

$$\int_{-\infty}^{\infty} (\psi(t),\ h(t, z(t), 0))_{\mathbb{R}^n} \mathrm{d}t = 0, \qquad \int_{-\infty}^{\infty} (\psi(t),\ h_t(t, z(t), 0))_{\mathbb{R}^n} \mathrm{d}t \neq 0, \tag{7.15}$$

where $\psi(t)$ ***is a bounded (unique up to a constant factor) solution to the equation adjoint to*** (7.14), ***then there exist*** Δ ***and*** σ ***such that for*** $0 < |\mu| < \sigma$ ***the perturbed equation***

$$\dot{x} = g(x) + \mu h(t, x, \mu) \tag{7.16}$$

possesses the following properties:

(a) ***there exists a unique*** T***-periodic solution*** $\xi_0(t, \mu)$ ***such that***

$$|z_0 - \xi_0(t, \mu)| < \Delta, \quad t \in \mathbb{R},$$

and

$$\sup_{t \in \mathbb{R}} |z_0 - \xi_0(t, \mu)| \to 0, \quad \mu \to 0 ;$$

(b) ***there exists a solution*** $\xi(t, \mu)$ ***bounded on*** $\mathbb{R}$ ***and such that***

$$|\xi(t, \mu) - z(t)| < \Delta, \quad t \in \mathbb{R},$$

$$\sup_{t \in \mathbb{R}} |\xi(t, \mu) - z(t)| = 0(\mu),$$

and

$$\lim_{t \to \pm\infty} |\xi(t, \mu) - \xi_0(t, \mu)| = 0 ;$$

(c) ***the linearized equation***

$$\dot{y} = \left\{ g'(\eta(t, \mu)) + \mu h'_x(t, \eta(t, \mu), \mu) \right\} y, \tag{7.17}$$

where $\eta(t, \mu)$ ***is equal to either*** $\xi(t, \mu)$ ***or*** $\xi_0(t, \mu)$, ***possesses an exponential dichotomy over*** $\mathbb{R}$.

This theorem immediately implies (see Exercise 7.20 and Theorem 3.1) that under conditions (7.15) the monodromy operator for problem (7.16) has a hyperbolic fixed point in a vicinity of the orbit $\{z(t): t \in \mathbb{R}\}$ and a transversal trajectory homoclinic to it.

We will not prove Theorem 7.2 here. Its proof can be found in paper [14]. We only outline the scheme of reasoning which enables us to construct a homoclinic trajectory $\xi(t, \omega)$. Here we pay the main attention to the role of conditions (7.15). If we change the variable $x = z(t) + \zeta$ in equation (7.16), then we obtain the equation

$$\dot{\zeta} = g(z(t) + \zeta) - g(z(t)) + \mu h(t, z(t) + \zeta, \mu).$$

We use this equation to construct a mapping $\mathcal{F}$ from $C_b^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}$ into $C_b^0(\mathbb{R}, \mathbb{R}^n)$ acting according to the formula

$$(\zeta, \mu) \to \mathcal{F}(\zeta, \mu) = \dot{\zeta} - \{g(z(t) + \zeta) - g(z(t)) + \mu h(t, z(t) + \zeta, \mu)\}. \tag{7.18}$$

We remind that $C_b^k(\mathbb{R}, X)$ is the space of k times continuously differentiable bounded functions from $\mathbb{R}$ into X with bounded derivatives with respect to t up to the k-th order, inclusive.

Thus, the existence of bounded solutions to problem (7.16) is equivalent to the solvability of the equation $\mathcal{F}(\zeta, \mu) = 0$. It is clear that $\mathcal{F}(0, 0) = 0$. Therefore, in order to construct solutions to equation $\mathcal{F}(\zeta, \mu) = 0$ we should apply an appropriate version of the theorem on implicit functions. Its standard statement requires that the operator $L = D_\zeta \mathcal{F}(0, 0)$ be invertible. However, it is easy to check that the operator L has the form

$$(Ly)(t) = \dot{y}(t) - g'(z(t))\, y(t) \;.$$

Therefore, it possesses a nonzero kernel $(L\dot{z} = 0)$. Hence, we should use the modified (nonstandard) theorem on implicit functions (see Theorem 4.1 in [14]). Roughly speaking, we should make one more change $\zeta = \mu w$ and consider the equation

$$\mathscr{F}(\mu w,\ \mu) = 0\,. \tag{7.19}$$

If this equation is solvable and the solution w depends on μ smoothly, then w satisfies the equation

$$D_\zeta \mathscr{F}(\mu w,\ \mu)[w + \mu w_\mu] + D_\mu \mathscr{F}(\mu w,\ \mu) = 0\,, \tag{7.20}$$

where w_μ is the derivative of w with respect to the parameter μ. This equation can be obtained by differentiation of the identity $\mathscr{F}(\mu w(\mu),\ \mu) = 0$ with respect to μ. Due to the smoothness properties of the mapping $\mathscr{F}$, it follows from (7.20) the solvability of the problem

$$D_\zeta \mathscr{F}(0,\ 0) w_0 + D_\mu \mathscr{F}(0,\ 0) = 0\ , \tag{7.21}$$

which is equivalent to the differential equation

$$\dot{w}_0 = g'(z(t)) w_0 + h(t,\ z(t),\ 0) \tag{7.22}$$

in the class of bounded solutions. It is easy to prove that the first condition in (7.15) is necessary for the solvability of (7.22) (it is also sufficient, as it is shown in [14]).

Further, the necessary condition of the dichotomicity of (7.17) for $\eta(t,\ \mu) = \xi(t,\ \mu)$ on $\mathbb{R}$ is the condition of the absence of nonzero solutions to equation (7.17) bounded on $\mathbb{R}$. However, this equation can be rewritten in the form

$$D_\zeta \mathscr{F}(\mu w,\ \mu)\, y(\mu) = 0\,, \tag{7.23}$$

where $w = w(\mu)$ is determined with (7.19). If we assume that equation (7.23) has nonzero solutions, then we differentiate equation (7.23) with respect to μ at zero, as above, to obtain that

$$\Big[D_{\zeta\zeta} \mathscr{F}(0,\ 0) w_0 + D_{\zeta\mu} \mathscr{F}(0,\ 0)\Big] y_0 + D_\zeta \mathscr{F}(0,\ 0) y_1 = 0\,, \tag{7.24}$$

where w_0 is a solution to equation (7.21), $y_0 = y(0)$, $y_1 = D_\mu y(0)$, and $y(\mu)$ is a solution to equation (7.23). Equation (7.17) transforms into (7.14) when $\mu = 0$. Therefore, the condition of uniqueness of bounded solutions to (7.14) gives us that $y_0 = c_0 \dot{z}(t)$, therewith we can assume that $c_0 = 1$. Hence, equation (7.24) transforms into an equation for y_1 of the form

$$\dot{y}_1 - g'(z(t)) y_1 = a(t)\,, \tag{7.25}$$

where

$$a(t) = -\Big(\Big[D_{\zeta\zeta} \mathscr{F}(0,\ 0) w_0 + D_{\zeta\mu} \mathscr{F}(0,\ 0)\Big]\dot{z}\Big)(t) =$$
$$= \Big[D_{xx}\, g(z(t))\, w_0(t) + D_x h(t,\ z(t),\ 0)\Big]\dot{z}(t)\ .$$

Here $w_0(t)$ is a solution to (7.22). A simple calculation shows that equation (7.25) can be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(y_1(t)+\dot{w}_0(t)) - g'(z(t))(y_1(t)+\dot{w}_0(t)) = h_t(t,\ z(t),\ 0)\,. \tag{7.26}$$

The second condition in (7.15) means that equation (7.26) cannot have solutions bounded on the whole axis. It follows that equation (7.23) has no nonzero solutions, i.e. equation (7.17) is dichotomous for $\eta(t,\ \mu) = \xi(t,\ \mu)$.

Thus, the first condition in (7.15) guarantees the existence of a homoclinic trajectory $\xi(t,\ \mu)$ while the second one guarantees the exponential dichotomicity of the linearization of the equation along this trajectory.

As to the existence and properties of the periodic solution $\xi_0(t,\ \mu)$, this situation is much easier since the point z_0 is hyperbolic. The standard theorem on implicit functions works here.

It should be noted that condition (7.15) can be modified a little. If we consider a "shifted" homoclinic trajectory $z_s(t) = z(t-s)$ for $s \in \mathbb{R}$ instead of $z(t)$ in Theorem 7.2, then the first condition in (7.15) can be rewritten in the form

$$\Delta(s) \equiv \int_{-\infty}^{\infty} \Big(\psi(t-s),\ h(t,\ z(t-s),\ 0)\Big)_{\mathbb{R}^n} \mathrm{d}t = 0\,.$$

If we change the variable $t \to t+s$, then we obtain that

$$\Delta(s) = \int_{-\infty}^{\infty} \Big(\psi(t),\ h(t+s,\ z(t),\ 0)\Big)_{\mathbb{R}^n} \mathrm{d}t\,. \tag{7.27}$$

It is evident that

$$\Delta'(s) = \int_{-\infty}^{\infty} \Big(\psi(t-s),\ h_t(t,\ z(t-s),\ 0)\Big)_{\mathbb{R}^n} \mathrm{d}t\,.$$

Therefore, the second condition in (7.15) leads us to the requirement $\Delta'(s) \neq 0$. Thus, if the function $\Delta(s)$ has a simple root s_0 ($\Delta(s_0) = 0$, $\Delta'(s_0) \neq 0$), then the assertions of Theorem 7.2 hold if we substitute the value $z(t-s_0)$ for $z(t)$ in (b). Performing the corresponding shift in the function $\xi(t,\ \mu)$, we obtain the assertions of the theorem in the original form. Thus, condition (7.15) is equivalent to the requirement

$$\Delta(s_0) = 0\,, \quad \Delta'(s_0) \neq 0 \quad \text{for some } s_0 \in \mathbb{R}\,, \tag{7.28}$$

where $\Delta(s)$ has form (7.17).

In conclusion we apply Theorem 7.2 to prove Theorem 6.1. The unperturbed Duffing equation can be rewritten in the form

$$\begin{cases} \dot{x}_1 = x_2\,, \\ \dot{x}_2 = \beta x_1 - \alpha x_1^3\,. \end{cases} \tag{7.29}$$

The equation linearized along the homoclinic orbit $z(t) = (\eta(t), \dot{\eta}(t))$ (see Exercise 7.1) has the form

$$\begin{cases} \dot{y}_1 = y_2 , \\ \dot{y}_2 = \beta y_1 - 3\alpha\eta(t)^2 y_1 . \end{cases} \tag{7.30}$$

Let us show that system (7.30) has no solutions which are bounded on the axis and not proportional to $\dot{z}(t)$. Indeed, if $w(t) = (v(t), \dot{v}(t))$ is another bounded solution, then due to the fact that $|\dot{\eta}(t)| + |\ddot{\eta}(t)| \to 0$ as $|t| \to \infty$, the Wronskian $W(t) = v(t)\ddot{\eta}(t) - \dot{v}(t)\dot{\eta}(t)$ possesses the properties

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t) = 0 \quad \text{and} \quad \lim_{|t| \to \infty} W(t) = 0 .$$

This implies that $W(t) \equiv 0$ and therefore $v(t)$ is proportional to $\dot{\eta}(t)$.

Evidently, the equation adjoint to (7.30) has the form

$$\begin{cases} \dot{y}_1 = -\beta y_2 + 3\alpha\eta(t)^2 y_2 , \\ \dot{y}_2 = -y_1 . \end{cases} \tag{7.31}$$

Since we have that

$$\ddot{y}_2 - \beta y_2 + 3\alpha\eta(t)^2 y_2 = 0 ,$$

a solution to (7.31) bounded on $\mathbb{R}$ has the form $\psi(t) = (-\ddot{\eta}(t); \dot{\eta}(t))$. Let us now consider the corresponding function $\Delta(s)$. Since in this case $h(t, x, \mu) = (0; f\cos\omega t - \delta x_2)$, we have

$$\Delta(s) = \int_{-\infty}^{\infty} \dot{\eta}(t)[f\cos\omega(t+s) - \delta\dot{\eta}(t)]\,\mathrm{d}t ,$$

where

$$\eta(t) = \sqrt{\frac{2\beta}{\alpha}}\,\mathrm{sech}\sqrt{\beta}\,t = \sqrt{\frac{8\beta}{\alpha}}\left(e^{\sqrt{\beta}t} + e^{-\sqrt{\beta}t}\right)^{-1} .$$

Calculations (try to do them yourself) give us that

$$\Delta(s) = 2f\omega\sqrt{\frac{2}{\alpha}} \cdot \frac{\sin\omega s}{\cosh\left(\frac{\pi\omega}{2\sqrt{\beta}}\right)} - \frac{4}{3\alpha}\delta\beta^{3/2} .$$

Therefore, equation $\Delta(s) = 0$ has simple roots under condition (6.19). Thus, the assertion of Theorem 6.1 follows from Theorem 7.2.

It should be noted that in this case the function $\Delta(s)$ coincides with the famous Melnikov function arising in the geometric approach to the study of the transversality (see, e.g., [1], [2] and the references therein). Therewith conditions (7.28) transform into the standard requirements on the Melnikov function which guarantee the appearance of homoclinic chaos.

Addition to the English translation:

The monographs by Piljugin [1*] and by Palmer [2*] have appeared after publication of the Russian version of the book. Both monographs contain an extensive bibliography and are closely related to the subject of Chapter 6.

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