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edited by

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EDITORIAL NOTE

The present volume collects together accounts of some new departures in the area of logic. Each is, or was at its time a novelty, each required inventive *initiative* to come to fruition, each meant an advance in logic. The samples range over a wide span of time, from the Classical Greek period to modern computer technique.

Together the selection illustrates the fact that not only developing logic as such, as distinct from merely utilizing already perfected logical techniques, is an art, but that is a typical art. By that we mean that it requires inventive intelligence and ingenuity, qualities that cannot be taught, only perfected and developed under guidance. This contrasts with skills that can in principle be taught to anybody of sufficient ability. A person who cannot be taught skills is dense, but a person who cannot be an artist is not dense, sometimes he could even be abler than the artistic, creative individual. Of course, logic is not the kind of art that can be practised by the less than able, ability is a prerequisite, as it is for every type of art, but that prerequisite is not enough. Like every other science, creative ingenuity is a *conditio sine qua non* for the work that really extends the domain of logic.

To illustrate this is a modest aim of the present volume. It is of course impossible to present a history of creativity in logic at all thoroughly in a small collection of articles, yet the point is made without it, and it is hoped that each and everyone of the contributions carries its own direct interest in a way that gives the whole a strong appeal.

* * *

The arrangement of contributions is organized as follows.

The volume starts with two essays concerning the great initiator in logic - Leibniz. Wolfgang Lenzen's essay ***Leibniz's calculus of strict implication*** shows how Leibniz anticipated Lewis' modal calculi; the comparison gets possible due to the suitable axiomatizations proposed by the author. A formally alternative approach, that making use of algebraic means, is suggested by Maciej Juniewicz's paper ***Leibniz's modal calculus of concepts***. These historical reconstructions are followed by a new approach to conditionals, obviously related to modal logics, as adopted by Ingemund Gullvåg in his paper entitled ***The logic of conditions.***

The next group of papers is devoted to foundational studies. Among the pioneering thinkers in this field was Henri Poincaré. His original views on the relations between logics and mathematics, esp. set theory, are analyzed in Gerhard Heinzmann's paper ***Philosophical pragmatism in Poincaré.*** The following ***A note on Zeno B3*** by Nicholas Denyer recalls the Greek troubles with infinity which lie at the beginning of historical chain ending in set theory. Should the reader be interested in further fate of Gödel's theorem up to early 1980's, he will find it in Roman Murawski's paper ***Generalization and strengthenings of Gödel's incompleteness theorem.***

Some of the papers deal with less known Polish logical initiatives, both belonging to pure mathematical logic and to application of logic to philosophy. Some of Mordchaj Wajsberg's results in mathematical logic are presented by Stanisław Surma in the context of Lvov-Warsaw School in the essay ***The logical work of Mordchaj Wajsberg***, while a specific Wajsberg result is critically discussed by M. N. Bezhanishvili in his ***Notes on Wajsberg's proof of separation theorem.*** In the Polish climate of 1930's, favourable both to logic and philosophy there appeared attempts similar to those discussed by Edward Nieznański in his ***Logical analysis of Thomism – the Polish programme that originated in 1930's;*** this covers also recent developments. In the same period Kazimierz Ajdukiewicz initiated a logical theory of questions; this field is the subject of Leon Koj's paper ***On justification of questions***, where he suggests a pragmatic approach.

The contributions closing this volume have in common what may be called an alogorithmic-oriented approach. Wojciech Buszkowski's article ***The logic of types*** belongs to the chain of inquiries initiated by K. Ajdukiewicz' algorithm for checking syntactic connexion, and developed essentially by Lambek's results. Witold Marciszewski under the title ***System of computer-aided reasoning for mathematics and natural language*** reports on recent research in this field, especially in Poland; this discussion is complemented by two **technical reports**, by Lesław W. Szczerba, and by Anna Zalewska, concerning the use of such a system in teaching logic and mathematics.

J. S. - W. M.

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WOLFGANG LENZEN

LEIBNIZ'S CALCULUS OF STRICT IMPLICATION

At first the title of this paper will provoke both logicians and Leibnitians to shake their heads: while talking about calculi of strict implication one rather associates it with the names of C. I. Lewis, W. Ackermann, or some other contemporary logician, but hardly with the 17th century philosopher G. W. Leibniz[1]. Yet already some 30 years ago in an essay on "Leibniz's interpretation of his logical calculi" the wellknown Leibniz scholar, Nicholas Rescher **(1954)**, had put forward the then bold thesis: in an interpretation that Leibniz himself had suggested, one of these calculi would become "a precursor of C. I. Lewis' systems of strict implication" (p.10). Unfortunately, Rescher did not give a factual justification of this prophetic view. Had it been done, it would have thrown some light on the real significance of Leibniz's logic. One reason for this omission consists, perhaps, in the fact that Rescher incorrectly interpreted the important logical constant 'est Ens', or, synonymously, 'est Res' or 'est Possibile' as logical necessity instead of logical possibility. Furthermore, he viewed his term 'Ens', taken in itself, as a conceptual constant, on a par with the "normal" terms A, B, C, ..., such that it may enter into fundamental sentence-schema '...est ---' as a subject- and a predicate-concept[2]. This misinterpretation of the Leibnitian intentions, however, leads to a series of inconsistencies that shall be discussed briefly below.

The task of this paper will be

- i to reconstruct and to axiomatise consistently (and completely) the concerned Leibnitian logic calculus;
- ii to describe in some detail the relevant Leibnitian interpretation of this calculus as a sentential logic

and

- iii to prove that this interpretation of the calculus does in fact yield a system of strict implication - to be exact, the so-called Lewis-modal system $S2^{o}$.

The Leibnitian logic-calculus L1

In the domain of logic, Leibniz's main aim was to create a comprehensive calculus that would allow him, in particular, to prove the entire laws of the then "accepted" logic, i.e. the syllogism. The generally accepted view is that Leibniz failed to attain this goal. Thus, for example, W. & M. Kneale **(1962)** assert "... although he worked on the subject in 1679, in 1689 [?] and in 1690, he never succeeded in producing a calculus which covered even the whole theory of the syllogism" (p.377).

Of course it is undeniable that Leibniz never came to a definitive formulation or even publication of one such calculus. Also, it can hardly be disputed that in his manifold efforts to embed syllogism in his diverse drafts of a general calculus, Leibniz repeatedly failed to derive either certain concrete syllogisms or some such general syllogistic principle as the law of opposition, of conversion, or of subalternation. And it appears that the pessimistic dictum "post tot logicas nondum logica qualem desidero scripta est"[3] reflects the sincere opinion of even the later Leibniz. Yet it has to be noted that already in the **Generales Inquisitiones (GI)** of 1686 Leibniz developed a calculus which is substantially more powerful than he himself (and his 20th century critics) suspected. At any rate, it easily attains the above mentioned goal. That is, in the form of his basic calculus L1 (without "indefinite concepts")[4] Leibniz had laid out a *complete axiomatization of Boole-Schroeder set-algebra* which trivially includes the whole syllogistic logic. The "problems" that arose for Leibniz while attempting to prove the syllogism in L1 are mainly rooted in his somewhat uncertain and partly erroneous theory of negation. A discussion of these two topics transcends the framework of the present paper; the lacunae are compensated for by two other papers, "'Non est' non est 'est non'" and "Zur Einbettung der Syllogistic in Leibnizens allgemainen Kalkül"[5].

The calculus in question, L1, can be characterized syntactically as follows. Starting from a set of *conceptual-constants* A, B, C,... the general concepts or *terms* are generated by means of the operations of concept-*negation* and concept-*conjunction*. Leibniz expresses the latter through concatenation and juxtaposition of the terms involved (e.g. AB, BCD, ...), while the former operation is expressed by means of "non" (with or without hyphenation) in the form of "non A", "non-B"..., where the scope of the negation operator is sometimes indicated by a bar placed over the terms (to distinguish e.g. non-$\overline{BC}$ from non-$\overline{B}C$. For the sake of simplicity we like to use what is customary in set-theory – the raised-bar itself as the negation symbol and thus take, for example, '$\overline{BC}$' as short for the negation of the conjunction 'non-BC'. By contrast, the conjunction of the individual negations, 'non-B non-C' is formalized as '$\overline{B}$ $\overline{C}$'. Leibniz's idea of how to construct the set of terms of L1 is captured by the following inductive definition

Def.1
- **i** every conceptual-constant is a term of L1,
- **ii** if τ is a term of L1, so too is $\overline{\tau}$,
- **iii** if σ and τ are terms of L1, so too is $\sigma\ \tau$,
- **iv** only the expressions in accordance with **i** – **iii** are terms of L1.

These terms are then connected into sentences especially by means of the relations 'est' (or 'continet') and '=' (coincidunt), whereby each of these operators can be defined according to one's choice. In what follows we take 'est' as the basic relation and symbolise it – as Leibniz himself once suggested[6] – by 'e'; accordingly we adopt his definition of identity "A esse B et B esse A idem est quod A et B coincidere" (**GI**, §30), formally,

Def.2 $$A = B := AeB \wedge BeA$$

Leibniz normally preferred to define conversely 'e' by means of '=': "Generaliter A esse B idem est quod A = AB" (**GI**, §83). In our approach this becomes a provable theorem:

K6 $$AeB \equiv A = AB$$

As both these formulae show, we represent the informal Leibnitian sentential-operators 'et' and 'idem est quod' by the modern symbols '$\wedge$' and '$\equiv$' of conjunction and equivalence, respectively. Similarly, we will use in the following the symbol '$\neg$' for the sentence negation 'non' and '$\vee$' for disjunction 'vel'; finally, the if-then relation, 'si tunc' (or, likewise 'infert', 'si ..., sequitur ---' shall be symbolized, for the time being, by the operator of implication, '$\supset$'. This should not, however, prejudge the question whether or not in each individual case Leibniz used his particles of propositional logic precisely in the modern truth-functional sense. Indeed, the main aim of this paper is to establish that for Leibniz 'si tunc' is primarily a *strict* implication and not merely a *material* implication. However, for a preliminary, provisional description of *conceptual*-logic, L1, the use of the "critical" symbols '$\supset$' and '$\equiv$' is not problematic; moreover, when in the next section L1 is reinterpreted as a *sentential*-logic these symbols will be formally replaced by other ones.

The construction of the set of sentences of L1 obtainable by the term operators 'e' or '=' and the propositional connectives is, of course, not complete. One decisive element is still missing: the operator 'est Ens' or, likewise, 'est Res' or 'est Possibile'. That the expression 'possibile' (in the Leibnitian sense of being free from contradiction) is not itself an ordinary concept denoting a property of things, but rather a 2nd order concept denoting a property of concepts, is evident from the definition: "A non-A contradictorium est. *Possibile* est quod non est: Y non-Y" (**GI**, line 330-1).

Thus one could define in the framework of the stronger logic L2 with "indefinite concepts", that B is possible if and only if (iff) there is no Y such that $\mathrm{BeY}\overline{\mathrm{Y}}$. Alternatively, one could postulate in L1, that B be possible iff $\neg\mathrm{BeA}\overline{\mathrm{A}}$. In fact, however, this definition can be further simplified, for on the basis of some Leibnitian basic principles, the equivalence

P3 $$\mathbf{P}(\mathrm{A}) \equiv \mathrm{A}\not{e}\overline{\mathrm{A}}$$

is valid, where '**P**(A)' stands for 'A is possible' and 'A$\not{e}$B' is shorthand for the negation $\neg$(AeB).

The decisive logical law needed here to establish a relation between a categorical proposition of the form AeB and a corresponding (im-)possibility assertion about the complex concept $\mathrm{A}\overline{\mathrm{B}}$, is casually formulated in **GI** (cf. the 'seu'):

"A non-B est impossibilis seu A continet B" (§128). More explicitly, in the first sketch of **Illatio, Veritas, Probatio Duplex** (**AV**, 2, 403): "vera propositio est *A continet B* si *A non-B* infert contradictionem". I.e., the following law holds:

P1 $$\mathrm{AeB} \equiv \neg\mathbf{P}(\mathrm{A}\overline{\mathrm{B}}).$$

In view of the trivial principle of double negation,
"Non-non-A = A" (**GI**, §96),

N1 $$\overline{\overline{\mathrm{A}}} = \mathrm{A},$$

and the equally trivial law

K5 $$\mathrm{AA} = \mathrm{A}$$

(**GI**, §171), P3 follows from P1 even though apparently Leibniz himself didn't notice this simplified definition of the possibility operator[7].

Now, there is abundant textual evidence to show that at least as applied to terms, i.e. to concepts, Leibniz always uses 'est Ens' as synonymous with 'est Possibile': the phrase 'Ens seu possibile' is almost a standard idiom in many of his fragments[8]. Accordingly 'est non-Ens' means the same as 'non est Ens' or 'est impossibilis', as the following passage from C.261-4 shows: "Impossibile est terminus, vel Non Ens, qui si ponitur esse, sequitur esse contradictorium". Therefore, in view of the conjunction principle K6 stated above, Leibniz can also express the fundamental law in **GP** VII, 212 alternatively as follows: "Universalis Affirmativa: A est B, id est aequivalent AB et A seu A non B est non-Ens".

A third paraphrase of the possibility-condition is provided by the words 'est Res', where occasionally 'Res' is even dropped. Thus §151 of **GI** provides a reduction of the four categorical sentence-forms to the corresponding 'Res' propositions in line with P1, especially "Omne A est B dat: *A non-B non est res*". And the final paragraphs 199, 200 contain the shortened versions, especially: "*Universalis affirmativa* A non-B non est" and "... si dicam A non-B non est, idem est ac si dicam A continet ... B".

Thus, on the whole, there is overwhelming evidence showing that Leibniz expresses the possibility-operator 'A est possibile' equally by means of 'A est Res', 'A est Ens' or even 'A est'. In *one* place of his writings, however, as a trial 'Ens' is assumed as a conceptual-constant, and correspondingly the proposition 'A est Ens' is interpreted as a predication of the form 'AeEns'. In the fragment on **Difficultates quaedam Logicae** (**GP** VII, 211-7) - which otherwise contains very valuable ideas - Leibniz felt the temptation to combine both ways of reducing the categorical sentence-forms to identities à la K6 and to possibility propositions a la P1:

"Caeterum venit in mentem, etiam propositiones, Universalem Negativam et ei oppositam particularem affirmativam, reduci posse ad aequipollentiam hoc modo: Nullum A est B, id est AB est non Ens, etiam sic exprimi poterit: non aequivalent AB et AB Ens. ... Ita omnes propositiones logicas categoricas reduximus ad calculum aequipollentiarum" (o.c. 213/4).

Surely *if* it were legitimate to interpret 'Ens' as a predicate, i.e. as a concept on a par with the ordinary concepts A, B, C, ... (and not otherwise as Leibniz always presumes - as a concept of the second order, i.e. as a conceptual operator), then one could represent ‘AB est Ens’ qua ‘ABeEns’, and, in accordance with K6, transform this into the equation ‘AB = AB Ens’. But this attempt leads to inconsistencies and must therefore be considered as one of the very few impasses that Leibniz reached in his search for a suitable "calculus aequipollentiarum".

The "predicative" interpretation of ‘Ens’ would allow us, for example, to derive ‘ABeEns’ immediately from the possibility of A – in the sense of ‘AeEns’ due to the trivial conjunction law,

K2 ABeA

(cf. C.263, #(15)). I.e. for any self-compatible A, every conjunction AB would also automatically be free from contradiction, a fortiori even the conjunction $A\overline{A}$! Even larger absurdities result if the operator of concept-negation is applied to the "constant" ‘Ens’ as Rescher has attempted to do. As Leibniz remarked, both a concept A and its negation, $\overline{A}$, may simultaneously be self-compatible: "Etsi AB esset Ens, tamen etiam Non $\overline{AB}$ potest esse Ens"[9]. Now, if A is some concept such that $P(A) \wedge P(A)$, then the representation of these premises as AeEns∧$\overline{A}$ens in conjunction with the law of contraposition ("Generaliter A esse B, idem est quod non-B esse non-A", **GI**, §77),

N2 $AeB \equiv \overline{B}e\overline{A}$,

would allow us to derive $\overline{Ens}e\overline{A}$ and hence, by transitivity, $\overline{Ens}$eEns. This, however, as Dummett rightly remarked, is an "absurd theorem"[10].

All these difficulties vanish if ‘est Ens’ is interpreted correctly as a possibility-*operator* which is anyway the interpretation that Leibniz elsewhere always had in mind. One gets a unified "calculus of (non-)equations" for the four categorical sentence-forms either in line with K6:

U.A.	$A = AB$
P.A.	$A \neq A\overline{B}$
U.N.	$A = A\overline{B}$
P.N.	$A \neq AB$

or in line with 1:

U.A.	$\neg P(AB)$
P.A.	$P(AB)$
U.N.	$\neg P(AB)$
P.N.	$P(A\overline{B})$

By contrast one has to reject the hybrid forms of combination of these "calculi" that are probed in the **Difficultates quaedam**

Logicae. In order to round off the syntactic construction of L1, one should not therefore accept 'Ens' as a new conceptual-constant, but merely postulate that for every concept B "B est Ens", i.e. **P**(B), be a sentence of L1. This condition would be redundant if one were to introduce the possibility-operator by definition à la P3. We are, however, presupposing it here as a primitive concept and hence have to define the set of sentences of L1 inductively as follows:

Def.3 **i** if σ and τ are terms of L1, then $(\sigma e \tau)$ is a sentence;
ii if τ is a term of L1, then $\mathbf{P}(\tau)$ is a sentence;
iii if α is a sentence of L1, then so is $\neg\alpha$;
iv if α and β are sentences of L1, then so are $(\alpha \wedge \beta)$ and $(\alpha \supset \beta)$;
v only the expressions in accordance with **i** – **iv** are sentences of L2.

The remaining two sentence-operators can be introduced – according to Leibniz – by definition:

Def.4
$$(\alpha \vee \beta) := \neg(\neg\alpha \wedge \neg\beta)$$
$$(\alpha \equiv \beta) := (\alpha \supset \beta) \wedge (\alpha \supset \beta)$$ [11]

The extensional semantics of L1, as intended by Leibniz, can be characterized as follows: the conceptual-constants shall be interpreted as the extensions of the corresponding concepts, i.e. as sets of (possible) objects that fall under the respective concepts; concept negation and conjunction are to be interpreted as set-theoretical complement and intersection, respectively; the basic operator 'e' accordingly represents the inclusion among sets, and the possibility proposition **P**(B) has to be interpreted as true iff the extension of the concept B is not empty, i.e. if at least one *possible* object exists that has the property expressed by B. As shown by the work cited in reference 11, one can transform such an extensional semantics – in line with Leibniz's ideas – equivalently into an "intensional" one: in the intensional semantics, a conceptual constant B has to be interpreted as *the set of those concepts* which are contained in B; and (AeB) becomes true under an "intensional" interpretation if the "intension" of A, i.e. the set of concepts contained in A, comprises the "intension" of B. In what follows, however, semantic consideration do not play any role. We will first present an axiomatisation of L1 (as a conceptual-logic) the adequacy – i.e. consistency and completeness – of which follows *purely syntactically* from proving L1 to be deductively equivalent to the Boolean set-algebra.

Regarding the relevant laws of logic put forward by Leibniz mainly in the **GI** one may mention the principle of transitivity and reflexivity of the 'e'-relation: "...si A sit B et B sit C, A erit C" (§19); i.e.

E1 $$AeB \wedge BeC \supset AeC;$$

or (§37) "B est B", i.e.

E2 $$AeA.$$

The fundamental principle of concept-conjunction says: "A continere B et A continere C idem est quod A continere BC" (§35), formally:

K1 $$AeBC \equiv AeB \wedge AeC.$$

From this, one can easily derive the already cited conjunction principle K2 and its symmetric counterpart (cf. §38):

K3 $$ABeB;$$

furthermore one easily obtains the previously mentioned K6 and K5; the law of symmetry (cf. C.235):

K4 $$AB = BA$$

and finally the "praeclarum theorema"

K7 $$AeB \wedge CeD \supset ACeBD$$

from the earlier 'Ad specimen calculi universalis addenda' (cf. **GP** VII, 223).

The most important negation principles N1 and N2 have already been stated above together with their formulation by Leibniz. The last operator **P** can be axiomatized by P1, P3, the trivial law "A non-A non est Res" (**GI** §171),

P4 $$\neg\mathbf{P}(A\bar{A}),$$

which follows from E2 with P1 plus the following principle:

P2 $$AeB \wedge \mathbf{P}(A) \supset \mathbf{P}(B).$$

Leibniz rather incidentally formulated it in the remarkable §55 of **GI**: "Si A continet B, et A est vera, etiam B est vera. er falsam literam intelligo vel terminum falsum (seu impossibilem, seu qui est non Ens) vel propositionem falsam. Et per veram eodem modo intelligi possit terminus possibilis vel propositio vera".

That is, Leibniz envisages here *the simultaneous interpretation of the terms both as concepts and as sentences* which will be analysed in detail in the next section. It allows him to formulate *at the same time* the law P2 and the inference rule of *modus ponens*. For, in the case of sentences, A, B, 'A continet B' or 'A est B' is taken to mean that A *implies* B: "...cum dico A est B, et A et B sunt propositiones, intelligo ex A sequi B" (C.260). For *concepts* A, B, in contrast, AeB asserts that from the "truth", i.e., from the possibility of A the possibility of B always follows[12].

To complete the axiomatic construction of L1, let us consider briefly some principles for the identity relation (introduced in accordance with Def.2). The law of reflexivity,

I1 $$A = A,$$

("A et A sunt prima coincidentia", **GI**, 284) follows trivially from E2; the law of transitivity,

I2 $$A = B \wedge B = C \supset A = C$$

("si A coincidit ipsi B etiam B coincidit ipsi C, etiam A coincidit ipsi C", **GI**, 8) follows analogously; and the law of symmetry,

I3 $$A = B \supset B = A$$

("si A coincidit ipsi B etiam B coincidit ipsi A", **GI**, 269–70) is an elementary consequence of the symmetry of propositional conjunction. Further, one obtains a "weak" law of contraposition for the identity relation directly from the "strong" contraposition law N2 for the e-relation: "Si A coincidit ipsi B; non-A coincidit ipsi non-B" (**GI**, 9). Thus

I4 $$A = B \supset \overline{B} = \overline{A}.$$

Since in view of the conjunction laws K1 – K6 "Si A = B erit AC = BC" (**GI**, §171), i.e.

I5 $$A = B \supset AC = BC,$$

also is valid, one can prove by induction that for any *term* τ

I6 $$A = B \vdash \tau[A] = \tau[B],$$

and for the sentence α

RI $$A = B \vdash \alpha[A] = \alpha[B]$$

I6 and RI are formalized explications of the famous "Leibniz law of identity": "*Coincidunt A ipsi B*, si alterum in alterius locum substitui potest salva veritate" (**GI**, 257–8), or succintly: "coincidentia sibi substitui possunt" (**GI**, §198).

As already mentioned, the conceptual-logic L1 as axiomatised by these principles is deductively equivalent to the axiom system of Boole-Schroeder set-algebra, provided that in both systems the required laws of propositional logic are presupposed. In this sense, therefore, the well-known "Boolean" algebra was not invented only in 1847, but actually put forward already in 1686 in the GI[13] as a Leibnitian algebra. In the following, however, it is not the conceptual-logic L1 itself that is under discussion, but just the (prima facie missing) propositional logical foundation for it.

The interpretation of L1 as a propositional logic

With the exception of the early investigations of legal logic[14], Leibniz was generally little concerned with working out specific principles of propositional logic. He used to make the requisite propositional inferences and transformations rather implicitly. Inference rules such as the cited *modus ponens* or the related *modus tollens* were mentioned by him only in passing. E.g., the latter was first put forward in §55 of **GI** in the form "Si A continet B et B est falsa, etiam A est falsa" but then it was dropped in favour of the former. Leibniz did not do so, however, because he considered this rule as false; probably he just thought it to be redundant. Anyway, in a marginal note to **De Formis Syllogismorum Mathematice Definiendis** (C.410-6) he formulated the related inference of so called regression as follows: "In Regressu utimur hoc principio, quod conclusione existente falsa ... et una praemissarum existente vera, altera praemissarum necessario debeat esse falsa" (o.c.,p.412)[15]. Significantly, even the important ("de Morgan") laws concerning the reduction of (nonexclusive and exclusive) disjunction to negation and conjunction appear only in the margin of a text in the **Analysis Didactica** (o.c., ref.11.) which is quite alien to propositional logic. In view of this peripheral treatment of the laws of propositional logic our thesis that Leibniz had access to a full blown calculus of strict implication may appear somewhat implausible. This prima facie implausibility disappears, however, when one considers that in the course of the development of his logical calculus Leibniz became more and more conscious that all the principles of sentential-logic are *virtually* contained in the laws of concept-logic that he had invented. Already in 1678, when his concept-logic existed only in a very rudimentary form, he called attention to the parallel between the "analysis" of concepts on the one hand and the "analysis" of sentences on the other. In **Analysis Linguaram** (C.351-4) he wrote:

"Porro cum scientiae omnes, quae demonstrationibus constant, nihil aliud tradant, quam cogitationum aequipollentias seu substitutiones, ostendunt enim in propositione aliqua necessaria tuto substitui praedicatum in locum subjecti ... posse, et inter demonstrandum in locum quarundum veritatum quas praemissas vocant, tuto substituti aliam quae conclusio appelabatur; hinc manifestum est, illas ipsas veritates in charta ordine exhibitum iri sola characterum *analysis*, seu substitutione ordinata continuata" (o.c., p.352).

This idea that the conclusion K of an inference from premises $P_1,...,P_N$ may be substituted for the premiss(es) P_i (or the conjunction of P_i) in the same manner as one can substitute the predicate P for the subject S in a categorical proposition SeP, is, of course somewhat inaccurate. For, from the truth of SeP it follows that one may substitute P for S in those propositions where S takes a *predicate position*: if, e.g., S=PQ, it follows that SeP∧SeQ, but one may not deduce therefrom that PeQ or that QeP. An analogous restriction applies to the substitutability of K for the premiss(es) P_i.

In the **Notationes Generales**, which probably was written between 1683 and 1686[16], the parallel in question is expressed more clearly. Just as the "propositio simplex: *A est B*" - in which A is called the "subjectum", B the "praedicatum" - is true, "si praedicatum in subjecto continetur", similarly a "propositio conditionalis: *Si A est B, C est D*", - in which now 'A est B' is designated as 'antecedens', 'C est D' as 'consequens' - is true, "si consequens continetur in antecedente" (c.f. o.c., p.184). In works obviously written later, Leibniz compressed this idea into formulations such as "vera autem propositio est cujus praedicatum continetur in subjecto, *vel generalius* cujus consequens continetur in antecedente" (C.401, emphasis is ours) and "Semper igitur praedicatur seu consequens inest subjecto seu antecedenti" (**Primae Veritates**, C.518).

On the basis of these parallels, the hunch dawned on Leibniz that the logical laws for the "hypothetical propositions" could be developed in complete analogy to the laws for the relation 'est' (or 'continet'). In the **GI**, he expressed this hope prophetically as follows: "Si ut spero, possim concipere omnes propositiones instar terminorum et omnes Hypotheticas instar categoricarum, et universaliter tractare omnes, miram ea Res in mea characteristica ... promittit facilitatem, *eritque inventum maximi momenti*" (§75, emphasis ours).

Even though this "invention of highest significance" was not systematically exploited by him in the sense that he did not *explicitly* transpose or translate the conceptual-logical principles such as E1, E2, etc. into corresponding sentential-logical principles, yet he believed to be entitled to note in a midresumee (§137): "Multa ergo arcana deteximus magni momenti ad analysin omnium nostrarum cogitationum, invetionemque et demonstrationem veritatum. Nempe ... quomodo veritates absolutae et hypotheticae *unas easdemque habeant leges, iisdemque generalibus theorematibus contineantur*, ita ut omnes syllogismi fiant categorici" (emphasis ours). And at the end of the **GI** he formulated as a general principle (§189, *sexto*): "quaencunque dicuntur de termino continente terminum, etiam dici possunt de propositione ex qua sequitur alia propositio".

The decisive step, that was only hinted at in the **GI**, *viz.*, the step of *formally identifying* the propositional connective 'si, tunc' with the conceptual connective 'est', was taken by Leibniz in a series of – apparently later – fragments. Thus he says in the draft of a calculus C.259-61, §16:

"Si a sit propositio vel enuntiatio, per non-A intelligo propositionem A esse falsam. Et cum dico A est B, et A et B sunt propositiones, intelligo ex A sequi B. Sed demonstrandus erit harum substitutionum successus. Utile etiam hoc ad compendiose demonstrandum, ut si pro L est A dixissemus C et pro L est B dixissemus D pro ista si L est A sequitur quod L est B, substitui potuisset C est D."

Naturally, this substitution of 'est' for 'si ..., sequitur ---' should be valid not merely for the special case where antecedent and consequent of the if-then sequence have the same concept as its subject (L above), but quite generally: "Itaque cum dicimus Ex A est B sequitur E est F, *idem est ac si diceremus* A esse B est E esse F" (ibid., emphasis ours). In the same vein, only more briefly, Leibniz says in the thematically related fragment C.261-4: "Hypothetica nihil aliud est quam categorica, vertendo antecedens in subjectum et consequens in praedicatum. Ex.gr. ... A est B, ergo C est D. A esse B sit L, et C esse D sit M, dicemus est M."

These passages clearly show that Leibniz considered the if-then relation between sentences as logically absolute equivalent with the 'est'-relation between concepts, and that he was therefore convinced that he could derive sentential-logic from conceptual-logic L1 through a plain identification of the sentence-operators with the corresponding concept-operators. In line with this, in the next passage, he transfers the important law P1 to the realm of propositions:

"*Vera propositio* est A continet B, si A non-B infert contradictionem. *Comprehenduntur et categoricae et hypotheticae propositiones*, v.g. si A continet B, C continet D, potest sic formari: A continere B continet C continere D; itaque A continere B, et simul C non continere D infert contradictionem" (C.407; second emphasis is ours).

To formalize such sentential-logical principles, let us replace both "implications", '$\supset$' and 'e', uniformly by a new symbol '$\Rightarrow$'. Also, the two conjunctions – of concepts and of sentences – shall be formalized uniformly by '$\wedge$'. Finally, both negations – for which Leibniz anyway always used one and the same particle 'non' – shall be represented uniformly by '$\neg$'. In consequence, the sentential-logical counterpart of P1, formulated above, takes the following shape:

P1S $\quad (A \Rightarrow B) \Leftrightarrow \neg P(A \wedge \neg B),$

where '$\Leftrightarrow$' is to be understood (in analogy to the previous definitions 2 and 4) as a mutual '$\Rightarrow$' relation:

Def.5 $\quad (A \Leftrightarrow B) := (A \Rightarrow B) \wedge (B \Rightarrow A).$

Similarly, one gets the sentential-logical principle

K6S $\quad (A \Rightarrow B) \Leftrightarrow (A \Leftrightarrow A \wedge B)$

formalizing Leibniz's definition: "*Vera propositio hypothetica* primi gradus est si A est B, et inde sequitur C est D ... Status quo A est B vocetur L, et status quo C est D vocetur M. Erit L ∞ LM *ita reducitur hypothetica ad categoricam*" (C.408; second emphasis is ours).

As already noted by N. Rescher, Leibniz's P1S represents a definition of *strict implication* or of "entailment in terms of negation, conjunction, and the notion of possibility" (o.c., p.10). C.I. Lewis's definition, formulated almost a quarter of a millenium after the **GI**: "Thus ... 'p strictly implies q' is to mean 'it is false that it is possible that p should be true and q false' or 'The statement 'p is true and q false'' is not self-consistent"[17], reads like a literal translation of Leibniz's definition given in another fragment: "Itaque si dico si L est vera sequitur quod M est vera, sensus est, non simul suponi potest quod L est vera, et quod M est falsa"[18].

The idea of reducing sentential-logic to conceptual-logic was already described by L. Couturat rightly as a "idée capitale ... peut-être sa plus belle découverte"[19]. Similarly, other authors such as Kauppi, Burkhardt and Schupp all have drawn attention to it[20]. But, apparently, to date, the exact extent of this 'inventio maximi momenti' has not yet been explored in a systematic way, perhaps because Leibniz himself exemplified it only in the case of the above cited principles, K6(S) and P1(S). With that much at hand, however, it does not require much of a logical genius to derive further 'S'-principles from the above principles of the concept logic L1. The "translation scheme" clearly outlined by Leibniz yields, e.g., transitivity and reflexivity of strict implication:

E1S $\qquad (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

E2S $\qquad A \Rightarrow A.$

The different conjunction principles will be transformed analogously into

K1S $\qquad (A \Rightarrow B \wedge C) \Leftrightarrow (A \Rightarrow B) \wedge (A \Rightarrow C)$

K2S $\qquad A \wedge B \Rightarrow A$

K3S $\qquad A \wedge B \Rightarrow B$

K4S $\qquad A \wedge B \Leftrightarrow B \wedge A$

K5S $\qquad A \wedge A \Leftrightarrow A$

K7S $\qquad (A \Rightarrow B) \wedge (C \Rightarrow D) \Rightarrow (A \wedge C \Rightarrow B \wedge D).$

The sentential-logical counterparts of the negation-principles are as follows:

N1S $\qquad \neg\neg A \Leftrightarrow A$

N2S $\qquad (A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A).$

And for the possibility operator, one obtains besides P1S:

P2S $\qquad (A \Rightarrow B) \wedge \mathbf{P}(A) \Rightarrow \mathbf{P}(B)$

P3S $\qquad \mathbf{P}(A) \Leftrightarrow \neg(A \Rightarrow \neg A)$

P4S $\qquad \neg\mathbf{P}(A \wedge \neg A).$

We can forego an account of the sentential-logical counterparts of I1 - I5 (which now express the properties of the relation of strict equivalence); suffice it to mention that the earlier substitution principles I6 and RI transform themselves into the rule of substitutability of strict equivalent expressions:

RIS $A \Leftrightarrow B \vdash \alpha[A] \Leftrightarrow \alpha[B]$.

Undoubtedly all these principles may be viewed as authentic Leibnitian principles of a logic of strict implication.

In spite of his tendency to neglect propositional logic, Leibniz obtained the just now listed laws not only indirectly through the general transition from conceptual- to sentential-logic, but also put forward at least some of them independently and explicitly in scattered fragments. For instance, in **De Conditionibus** I (o.c. ref.14) he characterizes the transitivity of the inference relation, E1S, as follows: "C[onditi]o C[onditio]nis est C[onditi]o C[onditiona]ti. Si posito A positur B, et posito B positur C; etiam posito A positur C"[21]. Concerning the reflexivity of '$\Rightarrow$', i.e. E2S, one may point to the fragment **De Calculo Enutiationum Absolutarum et Conditionalium** (AV,1,123-7) where Leibniz adds to the law of concept logic "Veritas primitiva absoluta *A est A*" (i.e. E2) the sentential counterpart "Prima consequentia" "*A est B ergo A est B*" or the "*axiom*" "3) *Si A est B, etiam est B*" (o.c., p.126). Moreover, he puts forward the sentential conjunction principle K1S in **De Veritatibus Enuntiationum** (AV, 1, 80-5) for the special case A = (a est b), B = (e est d) and C = (l est m) by asserting that the proposition "*Si a est b sequitur quod e est d et l est m*" is (equivalently) resolvable into conjunction of the propositions "*Si a est b sequitur quod e est d*" and "*Si a est b sequitur quod l est m*"[22]. Further sentential versions of the principle of double negation, N1S, are to be found in the form "Coincidunt L ... et L esse falsam est falsa" (GI, §4) or (in the special case where L = (A = B) or L = (AeB)) more formally: "Idem sunt A∞B ... et A non non ∞B" (C.235) or: "A non non est B, idem est quod A est B" (C.262). Finally, the **Analysis Particularum** (o.c., ref.18) contains along the with the sentential principle of contraposition N2S: "Si ex propositione L ... sequitur propositio M ... tunc contra ex falsitate propositionis M sequitur falsitas propositionis L" (o.c., p.145), also the above cited paraphrase of P1S, according to which M follows from L, if it is impossible ("non supponi potest") that L is true and at the same time M is false.

On the basis of this textual evidence it may be taken for granted that the 'S'-principles set out above were all viewed by Leibniz as valid laws of sentential logic. In the next section we want to axiomatise the corresponding logic L1S and compare it with the modern systems of strict implication.

L2S and the Lewis systems of strict implication

In accordance with the informal explanations of the last section, we can give the following precise definition of the syntax of the sentential-logical system L1S, generated via the Leibnitian "translation rule" by the conceptual-logic L1:

Def.6
- **i** every constant (A,B,C, ...) is a term of L1S;
- **ii** if τ is a term of L1S, so is $\neg\tau$;
- **iii** if σ and τ are terms of L1S, so are $(\sigma\wedge\tau)$ and $(\sigma \Rightarrow \tau)$;
- **iv** only the expressions in accordance with **i** – **iii** are terms of L1S.

In this we assume – which, however, is inessential – the possibility operator as a defined concept (in line with the previous principle P3)[23]:

Def.7 $$P(A) := \neg(A \Rightarrow \neg A)$$

The "translation rule" in question thus has to be interpreted as mapping L1 into L1S:

Def.8 The Leibnitian "translation" function ϕ (from L1 in L1S) associates to every expression (i.e. every term and every sentence) of L1 a term of L1S in the following manner:

i	$\phi(A)$	$= A$	for every conceptual constant A of (L1)
ii	$\phi(\overline{\tau})$	$= \neg\phi(\tau)$	for arbitrary terms σ, τ (of L1)
iii	$\phi(\sigma\tau)$	$= \phi(\sigma) \wedge \phi(\tau)$	
iv	$\phi(\sigma e \tau)$	$= \phi(\sigma) \Rightarrow \phi(\tau)$	
v	$\phi(\neg\alpha)$	$= \neg\phi(\alpha)$	for arbitrary sentences α,β (of L1)
vi	$\phi(\alpha\wedge\beta)$	$= \phi(\alpha) \wedge \phi(\beta)$	
vii	$\phi(\alpha\supset\beta)$	$= \phi(\alpha) \Rightarrow \phi(\beta)$	

Roughly speaking, the axioms of L1S shall be the ϕ-images of the axioms of the Leibnitian algebra L1. The necessary modifications of this idea will be discussed below. First, however, we want to deal with the rules of deduction of L1S, which have altogether been neglected so far. For the purpose of comparison we will refer to C.I. Lewis' (and H.G. Langford's) **Symbolic Logic (1959)**. There it is postulated: "Either of two equivalent expressions may be substituted for the other" (p.125). But this is nothing else but our rule RIS that was obtained by applying the function ϕ to Leibniz's law of identity, RI.

Secondly, Lewis has a general substitution rule saying "Any proposition, or any expression ... may be substituted for [A], or [B], or [C], etc., in any assumption or established theorem" (ibid.). In the same sense, Leibniz explains in §26 of **GI**:

"Admonenda adhuc quaedam circa hunc calculum quae praemittere debueramus. Nenpe quod de quibuslibet literis usurpatis asseritur generaliter, vel concluditur non tanquam Hypothesis, id de quotlibet aliis literis intelligi"[24].

Thirdly, Lewis needs the conjunction rule: "Any two expressions which have been separately asserted may be jointly asserted. That is, if [A] has been asserted, and [B] has been asserted, then [A∧B] may be asserted" (p.126). Almost literally the same rule is formulated by Leibniz in the short fragment C.326-7 "Generalis transitus est, et positus A et B dicere liceat AB". Since obviously a "transitus ab enuntiatione ad enuntiationem seu consequentia" is envisaged here, the 'et' (like the corresponding 'and' of Lewis, too) has to be taken as a meta-linguistic expression, so that the deducibility of the proposition 'AB' – i.e. in our standardized terminology, 'A ∧ B' – from the two premises gets asserted.

The last one of Lewis' deduction rules is (strict) *modus ponens*: "If [A] has been asserted, and [A ⇒ B] is asserted, then [B] may be asserted" (ibid.). While presenting the conceptual-logic L1, we have already referred to §55 of **GI** where Leibniz writes analogously: "Si A continet B et A est vera, etiam B est vera". Now, Lewis has drawn attention to the fact that from a proof-theoretical point of view the rule of deduction:

MPS $(A \Rightarrow B), A \vdash B$

has to be distinguished from the corresponding sentence (which in **Symbolic Logic** carries the designation 11.7),

11.7 $(A \Rightarrow B) \wedge A \Rightarrow B.$

"Contentwise" both say obviously the same thing; yet whereas for an axiomatic calculus of strict implication the rule MPS is indispensable, the principle 11.7 as an axiom can indeed be abandoned[25]. And since Leibniz's formulation speaks in favour of 11.7 rather than MPS, one may suspect here a gap in the rule-network of the calculus L1S.

A definite clarification of the question (often discussed in literature) to what extent did Leibniz know the *modus ponens* (in common or strict form), is rendered a bit difficult by the fact that Leibniz often interprets the inferences quasi as propositions;[26] at any rate he did not attach much importance to the distinction between the inference: '$A_1,\dots,A_n$, therefore B' – and the corresponding proposition: 'if A_1 and ... and A_n, then B'. Only in the context of grammatical investigations and linguistic analyses he refers explicitly to the distinction 'si, tunc' and 'ergo'. Thus, for example, one finds in the **Analysis Particularum** the explanation: "Cum dico Sapiens est Rex, ergo felix est civitas, non tantum dico si sapiens est rex, ergo felix est civitas, sed etiam affirmo sapientem regem et civitatem felicem esse, ac proinde totus syllogismus hypotheticus in his absolvitur ..." (o.c., p.147). In the **Notae Grammaticae** (AV, 1, 102–6; C.243–4), this "syllogismus theoreticus" is described more precisely as follows: "*Ergo*, significat: *Si A est B, tunc C est D. Atqui A est B. Ergo C est D.*"[27] This, then, is an unmistakable version of (strict) *modus ponens* or the special case of this deduction-rule, in which the propositions α and β are categorical propositions of the form AeB and CeD, and where from α plus $\alpha \Rightarrow \beta$ the conclusion β is derived.

Thus all the four relevant deduction rules may safely be regarded as genuine Leibnitian principles, and we can now turn to an axiomatic comparison of L1S with the Lewis' systems of strict implication. In another place[28] we have asserted, (somewhat hastily), that L1S coincides *exactly* with the Lewis' calculus S2. This assertion, however, is somewhat problematic, because "the" calculus L1S is, in a way, undetermined. No doubt the syntax of L1S has been determined once and for ever by Def.6; in view of the preceding discussion the set of deduction rules of L1S may be taken to be determined once and for all; but it is not at all clear whether those and only those 'S'-principles that were listed in the previous paragraph – i.e. the ϕ-images of the former axiomatization of L1, – axiomatise the sentential-logic L1S. More precisely, the problem consists in that there exist other (logically equivalent) axiomatizations of the Leibnitian algebra L1 which will generate by means of the translation-function ϕ variants of L1S which themselves are *not* necessarily equivalent to each other. For example there is no reason to doubt that Leibniz would have agreed to the following version of his law P2:

P2** $(AeB) \supset (\mathbf{P}(A) \supset \mathbf{P}(B))$.

Its "translation" under ϕ is:

P2S** $(A \Rightarrow B) \Rightarrow (\mathbf{P}(A) \Rightarrow \mathbf{P}(B))$.

On the other hand, in the framework of the conceptual-logic L1, the principle P2 could also have been replaced equivalently by the special case

P2* $\mathbf{P}(AB) \supset \mathbf{P}(B)$

that follows directly from P2** (or from P2), and which, conversely, implies P2 or P2**. For, according to K6, if AeB then A = AB is true, so that **P**(A) entails **P**(AB), - hence P2* allows one to infer the desired **P**(B). The ϕ-image of P2* takes the shape

P2S* $$\mathbf{P}(A \wedge B) \Rightarrow \mathbf{P}(B).$$

Whereas Leibniz surely would have accepted the conceptual laws P2, P2* and P2** as equivalent, the sentential counterparts P2S, P2S* and P2S** of L1S are not equivalent at all (e.g. on the basis of Lewis' S1): while P2S is a theorem of S1, P2S* is the characteristic axiom of the new stronger system S2, and replacing by P2S** leads to the even stronger calculus S3! Therefore our provisional determination of the axiom of L1S as the set of the ϕ-images of the axioms of L1 has to be taken with a pinch of salt: this stipulation is not invariable with respect to equivalent transformations of the set of axioms of L1.

In order to arrive at statements precise to some degree at least, in spite of this tricky situation of the undetermined nature of Leibniz's sentential-logic, let us consider three axiomatic variants of L1S. The primary axiomatization should contain precisely those 'S'-principles put together in the previous section, i.e. {E1S, E2S, K1S-K7S, N1S, N2S, P1S-P4S} - they are the ϕ-images of those symbolizations of the laws of concept logic that do best conform to Leibniz's semi-formalized versions. The secondary axiomatization of L1S differs from the first only in that it contains P2S* instead of P2S; finally, the tertiary axiomatization contains analogously P2S**. Of course it is possible to imagine giving variants in the case of certain other conceptual-logical principles which might be proved as authentically Leibnitian by citing appropriate texts; and this should be taken care of for quartiary etc. axiomatizations of L1S. But such considerations would undermine the framework of this paper. With regard to the most important three axiomatizations of L1S, it may be now shown:

Proposition 1 The primary axiomatization of L1S is deductively equivalent with the Lewis-system S2°.

In accordance with Zeman (p.96) by S2° we refer to the calculus S2 minus the possibility-axiom A ⇒ **P**(A) or its equivalent 11.7. As axioms of S2° the following laws from ch.6 of **Symbolic Logic** are available:[29]

11.1 $$A \wedge B \Rightarrow B \wedge A$$
11.2 $$A \wedge B \Rightarrow A$$
11.3 $$A \Rightarrow A \wedge A$$
11.4 $$(A \wedge B) \wedge C \Rightarrow A \wedge (B \wedge C)$$

11.5	$A \Rightarrow \neg\neg A$
11.6	$(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
19.6	$P(A \wedge B) \Rightarrow P(A)$.

By the way, the first 6 principles axiomatize the weaker system $S1^o$, and S1 is equal to $S1^o$ plus 11.7.

The proof of *Proposition 1* is given in the Appendix. Suffice it to mention here that the "lack" of the law of associativity like Lewis's 11.4, which has often been criticised by other commentators, is not a real deficiency in the Leibnitian system L1(S), since 11.4 can be proved by means of the conjunction principle K1S. Furthermore it can easily be shown that the alternative axiomatizations of L1S diverge from $S2^o$ – if at all – only slightly[30].

For one thing we have:

Proposition 2 The substitution of P2S through P2S* does not alter the logical strength of L1S (i.e. the secondary axiomatization of L1S also is deductively equivalentwith $S2^o$).

Only when P2S** replaces P2S(*) does one ascend one level higher in the hierarchy of Lewis's system:

Proposition 3 The tertiary axiomatization of L1S (P2S**) is deductively equivalent with with the Lewis-system $S3^o$.

Again, proofs are supplied in the Appendix. The room for vagueness surrounding the axiomatization of L1S is thus not very great. No matter whether $S2^o$ or $S3^o$ results, in the context of a discussion of the *Leibnitian logic* only the following is important: by way of his conceptual-logic L1, Leibniz only provided a complete axiomatization of the Boolean algebra more than 150 years before Boole, but by explaining in detail the "translation" of conceptual-logical into sentential-logical principles he also developed a complete calculus of strict implication that was to be "discovered" only more than 230 years after the GI. Indeed Leibniz has even provided an answer – though, of course, only in directly – to the question that occupied C.I.Lewis throughout his life and for which he did not find a (definitive) answer, *viz.* the question which modal logical system corresponds in a natural manner to the Boolean algebra? The answer à la Leibniz is: $S2^o$ or $S3^o$.

Leibniz's modal logic

The outcome of *Proposition 1* through *Proposition 3*, however, does not mean that S2° or S3° represent the modal logic favoured by Leibniz in the sense that these systems would exactly mirror his ideas about the logic of alethic modalities. S2° or, likewise, S3° is merely the faithful ϕ-image of his concept logic L1. As we noted above in connection with the law of *modus ponens*, 11.7, he presumably accepted some further laws for the concept of logical possibility and necessity. Although, e.g., 11.7 is unquestionably valid for Leibniz, unlike the coresponding deduction rule this axiom does not appear in L1S, because, roughly speaking, its "original" with respect to the mapping ϕ has no place in L1. More precisely: corresponding expressions like (AeB∧A)eB or (AeB∧A) ⊃ B (which would be mapped by ϕ onto 11.7) are not well-formed formulae of L1.

As J.J. Zeman (**1973**) has shown (p.94), even in the framework of the weakest system of strict implication 11.7 is equivalent with the more familiar principle (the designation which we borrow from Lewis/Langford):

18.4 $$A \Rightarrow \mathbf{P}(A).$$

This is also missing in L1S because its "original" is not a syntactically well-formed sentence of the concept logic L1: AeP(A) is not well-formed, because according to Def.2 terms and not sentences must occur on both sides of 'e'; similarly A ⊃ P(A) is not well-formed because, according to Def.3, sentences but not terms must stand on both sides of '⊃'. But **P**(A) is a *sentence* iff A is a *term*!

On the other hand, it is certainly unquestionable that Leibniz accepted this principle as valid. Onto-logical versions appear, for example, in **De Veritatis Realitate** (AV, 1, 65-6) in the form "Quicquid existit, est possibile" or in **De Veritatibus Primis** (AV, 1, 115-5) as a marginal note: "Quod omne existens est possibile, debet demonstrari ex definitione existentiae". It is somewhat difficult to find unobjectionable logical variants of 18.4, because it is often unclear whether Leibniz speaks about concepts or about sentences. Thus his remark in **De Libertate et Necessitate** (AV, 2, 272-8; **Grua** 287-91) "quod contradictionem implicat esse falsum" (o.c. 275) may be well understood to say that an impossible, contradictory *concept* is designated as false; indeed Leibniz did this in the earlier cited §55 of **GI**[31].

However, the following passage from **De principiis** (C.183–4) speaks clearly for a sentential-logical interpretation: "Duo illa prima principia: unum rationis: *Identica sunt vera, et contradictionem implicantia sunt falsa*, alterum expenentiae ...". Since "identica" always have to be identical *propositions*, *viz.* propositions of the form 'A = A', accordingly therefore, by "contradictionem implicantia" one has to understand contradictory *propositions* and not concepts. And if, as Leibniz asserts, every impossible proposition is *a fortiori* false, then by contraposition it follows that every true proposition must be possible, as claimed in 18.4.

Thus Leibniz's general theory of modal propositions goes beyond L1S since it contains either 18.4 or 11.7 and is hence at least as strong as the Lewis system S2 or S3, probably, however, still considerably stronger. First it should be pointed out that both the latter systems show two unpleasant characteristics that Leibniz certainly would not have accepted. First they are "unreasonable" (in the sense of Hallden), i.e. they contain certain theorems of the form $\alpha \vee \beta$ although neither α nor β itself is provable and although α and β are contentwise independent of each other in the sense that they contain completely different terms[32]. Such a presumption clearly contradicts the basic rationalist principle of Leibniz, according to which there must be a sufficient reason for every true proposition and even more so for every provable proposition. But, if there is no sufficient ground either for α or β to be true, what reason could there be for the disjunction to be true if both constituents α and β are independent of each other?

A second undesirable characteristic of S2 and S3 consists in the fact that in these systems the deduction rule – for every theorem α, the necessity proposition $\neg\mathbf{P}\neg\alpha$ also is provable – holds only within limits[33]. But, as noted already by Hans Poser, for Leibniz the validity of this "rule of necessitation" results "as the direct conclusion from his definition of necessity of a proposition as the contradictoriness of the negation of the proposition"[34], – or somewhat more precisely: from his view formulated in many places, that every (finitarily) provable proposition is necessary. Unfortunately we could not discover in Leibniz's scripts a definitive version of this view – in the sense of a logical inference rule[35]. If one adds it to the principles of S2 or S3 that are defended so far, then one obtains either the so called calculus T or the more familiar (and more attractive) system S4 as the extended Leibnitian modal theory[36].

That Leibniz would have favoured the latter is discernible, e.g. from the following fact. As is shown in Zeman (**(1973)**,ch.11) S4 can be axiomatized alternatively by adding to S2 or to S3 or even to the weakest system S1^0 only one more axiom, *viz.*, the iteration law (cf. Lewis/Langford **(1959)**, p.497):

C.10.1 $\quad$ **PP(A) $\Rightarrow$ P(A).**

S4 is no longer "Hallden unreasonable", and the "rule of necessitation" holds there without qualification. Poser has drawn attention to a passage from **De Affectibus** (**Grua**, p.512-37), which can be interpreted as a formulation of this central modal-logical law: "Nam quod impossibile est esse actu, id impossibile est esse possibile" (o.c.534). Provided that Leibniz here refers to the possibility and the possible possibility of *propositions*, and, furthermore, that he means that a proposition "actu est" only when it "est", i.e. when it is true, then this quote would really represent a formulation of (the contraposition) of C.10.1.

Unfortunately this assumption is not fully corroborated by the context. Besides, this (relatively early) fragment from 10 April 1679 contains in the sequel a series of statements and definitions that are partly untenable and which do not quite fit together with Leibniz's later views on modalities. Thus, for example, the concept of possibility is defined rather in a confused manner: "Possibile est quod esse aliquid et non-non esse aliquid est idem" and this definition is taken to entail the no less confused statement: "Possibile quod est, id non-non est. Possibile quod non-non est id est". Only the further conclusion "Unde sequitur impossibile est simul esse et non esse" corresponds to some extent with the ripe view which underlies the account given so far; and according to it a concept (or sentence) is possible, if it does not contain (or imply) any concept (or sentence) A and its negation at one and the same time.

Even though we have no firm proof that Leibniz adopted C.10.1 as a *logical* law, still a plausibility case can be made for the fact that he accepted it - at any rate as a *meta*-logical principle. In the §§132-5 of **GI** Leibniz unfolds his theory of necessary and contingent truths:

"Propositio vera necessaria, probari potest reductione ad identicas, vel oppositae reductione ad contradictorias; unde opposita dicitur impossibilis. Propositio vera contingens non potest reduci ad identicas, probatur tamen, ostendendo continuata magis magisque resolutione, accedi quidem perpetuo ad identicas, nunquam tamen ad eas perveniri ... Hinc veritatum necessarium a contingentibus idem discrimen est, quod Linearum occurentium, et Asymptotarum, vel Numerorum commensurabilium et incommensurabilium."

But now C.10.1 says per contraposition, that every necessary proposition is necessarily necessary. If one considers any necessary proposition A, it can be traced back, according to Leibniz, in finitely many steps to an "identity" (i.e. to an axiom). But, if this is the case, then a related and equally finite analysis of the concepts that occur in A and of concept of necessity shows that 'necessarily A' can also be traced back to an "identity". That is, the proposition 'necessarily A' is not simply true, but itself necessary!

Therefore, the "material" validity of C.10.1 as well as the "validity" of the related "rule of necessitation" result from Leibniz's view of necessity as finitary probability. But, of course, it does not follow from this that these principles would be derivable as logical laws or rules from the laws of S2 or S3 that have been formulated explicitely by Leibniz. Even if one would add the definition of necessity as finitary provability to Leibniz's calculus of strict implication, still C.10.1 would become provable only if one were to have the "rule of necessitation" at one's disposal. If α is a theorem, i.e. finitarily provable, then indeed $\neg P \neg \alpha$ is true by "definition". But what has to be shown is that this sentence is itself a theorem, i.e. (finitarily) provable, and this result is obtainable only by introducing the corresponding inference-rule:

$$\alpha \vdash \neg P \neg \alpha.$$

To put it in a nutshell: whereas Leibniz's explicitly given modal logic results in $S2^0$ or $S3^0$ depending on the choice of the axiomatization, his meta-logical modal *theory* is at least as strong as S4.

H. Poser (1969) has gone still a step further and has asserted that the characteristic axiom of S5,

C11 $$P(A) \Rightarrow \neg P \neg P(A),$$

– according to which every possible proposition is necessarily possible – should also be valid in Leibniz's modal theory. However, unlike the case of C.10.1 he did not even cite a single quote as evidence for C11; and the following argumentation is neither conclusive nor does it really express Leibnitian thinking: "... alles, was möglich ist, besitzt die Eigenschaft, möglich zu sein, absolut und im Bereich der Ideen. Die Annahme, ein Element A dieses Bereiches, ein Ens, sei nicht möglich, führt deshalb auf einen Widerspruch; folglich ist A notwendigerweise möglich" (Poser **(1969)**, p.59).

Of course a plain contradiction would be involved if one would assume of "an Ens" A, i.e. of some A for which 'A est Ens', i.e. P(A) is true, that is A is not possible, ¬P(A). But this interpretation of Poser's argument only proves the all too trivial law that the conjunction P(A)∧¬P(A) is impossible. If, in contrast, Poser's argument is understood in such a way that if A is an "Ens" then this very proposition, P(A), may not be assumed to be impossible, even then, granted for the time being that the latter, ¬PP(A), be incompatible with the former, P(A), one would have proved too little - *viz.* only ¬P(P(A))∧¬PP(A). But this is just the converse of C.10.1, which as a special case of 18.4 is at any rate valid for Leibniz. In order to accomplish the intended justification of C11, one would have to show instead that for an "Ens" A the assumption that it is possible that A is not an "Ens", P¬P(A), contradicts P(A).

But, according to Leibniz's view of necessity and contingency, the proposition P(A) says that A can never be refuted in finitely many steps, i.e. it cannot be led back to a contradiction. So in case C11 would be valid, it would follow that the meta-proposition P(A) - as a necessary proposition - is finitarily provable. But, how is one to show in finitely many steps - through analysis of the concept of provability as well as of the constituent concepts that appear in A - that no finite analysis of A does lead to a contradiction? This itself appears to be a sheer impossibility.

In contrast to Poser, therefore, one ought not subsume C11 in the Leibnitian modal theory. Leibniz's conception of necessity as finitary provability (and consequently, of possibility as non-finitary refutability) not only does not speak in favor of C11, but - as already noted by R. M. Adams[37] - really against it. Leibniz's modal theory, then, certainly may be taken to be definitely weaker than S5 but at least as strong as S4. We have to forgo the attempt at a closer demarcation.. For one thing, the spectrum of the modal calculi between S4 and S5 contains an endless number of systems[38]. For the other, Leibniz's meta-logical (or meta-physical) writings on necessity and impossibility are too imprecise to allow us to answer the question whether, e.g., the characteristic axiom of S4.2: What is possibly necessary, is necessarily possible - or the S4.4 - axiom: Every proposition that is at the same time true and possibly necessary, is necessary – whether these hold under Leibniz's interpretation of the logical modalities[39] or not. To the best of our knowledge, even amongst the most competent contemporary logicians there is no agreement on the question, what is the precise structure of the concept 'logically possible'? This may be surprising, for the alethic modalities have been investigated in the philosophical and logical literature since Aristotle, thus essentially longer than, for example, the deontic or epistemic modalities the structure of which has been determined quite precisely[40]. It is presumptuous to expect an answer from a 17-th century philosopher to a question the solution of which today still causes one to rack one's brains. Anyway, it is surprising enough that Leibniz's informal model theory fairly agrees with the results of modern logical research. That is, according to the views of A. R. Anderson and N. D. Belnap, only the S4 laws hold for the notion of logical necessity, whereas diverse objections (independent of Leibniz's considerations) have been brought forward against the S5 principle[41].

Surely it is idle to speculate, what modern-logic would look like if Leibniz had lived 300 years later. It appears to make sense, however, to suggest to the contemporary logicians that they take account of the Leibnitian work in their own researches. Someone who – centuries ago – invented in one stroke both the Boolean algebra and, by means of an ingenuously simple translation, its sentential-logical derivate in the form of a calculus of strict implication has at any rate graded himself as an extremely competent dialogue-partner. He will have much to say to every logician, even though, unfortunately, he cannot speak to us anymore.

Appendix

Proof of *Proposition 1*.

As we have seen, both calculi have the same deduction rules; therefore it is sufficient to deduce all axioms of the primary axiomatization of L1S in S2°, and vice versa.

(A) In one direction, we can make use of the work of Lewis and Langford; in ch.6 they have proved all the relevant Leibnitian principles within their system S2. It only has to be observed that the principle 11.7 or its equivalent 18.4 can wholly be dispensed with in these deductions, so that the proofs also hold in the weaker calculus S2°. First, Leibniz's E1S coincides with Lewis's axiom 11.6; E2S is Lewis's theorem 12.1, an immediate consequence of 11.2 and 11.6 along with 11.3; K2S is an axiom for Lewis, namely 11.2; K4S which constitutes the strengthening of Lewis's axiom 11.1 into a strict equivalence, is proved as theorem 12.15; K5S, the analogous strengthening of 11.3, appears as theorem 12.7; K3S is proved as a counterpart to 11.2 in theorem 12.17; K6S arises relatively late as theorem 16.33[42]. And in contrast to the foregoing principles, all of which are also theorems of the weaker system S1°, K1S, which is Lewis's theorem 19.63, is provable only in S2° with the help of 19.01; equally, K7S may be obtained only in S2° as theorem 19.68. N1S is deduced by Lewis by means of 11.5 as theorem 12.3; N2S similarly as theorem 12.44; for Lewis P1S is simply the definition 11.02 of strict implication; P3S coincides with Lewis' theorem 18.1; similarly P4S is theorem 18.8; and finally P2S appears as theorem 18.51.

(B) Conversely, one obtains the definition 11.2 within the Leibnitian system L1S as principle P1S; further, the axiom 11.1 is obtained from K4S with the help of K2S and the definition of strict equivalence; 11.2 is identical with K2S; 11.3 follows readily from K5S; 11.5 similarly from N1S; 11.6 coincides with E1S; thus only 19.01 - the characteristic axiom of S2° - as well as the law of associativity, 11.4, remain to be proved in L1S.

The latter is obtained as follows:

i	$(A \wedge B) \wedge C \Rightarrow C$	K3S
ii	$(A \wedge B) \wedge C \Rightarrow (A \wedge B)$	K2S
iii	$(A \wedge B) \Rightarrow A$	K2S
iv	$(A \wedge B) \Rightarrow B$	K3S

v	$(A\wedge B)\wedge C \Rightarrow A$	from **ii** and **iii** by means of the rule of conjunction, E1S and MPS
vi	$(A\wedge B)\wedge C \Rightarrow B$	analogously from **ii** and **iv**
vii	$((A\wedge B)\wedge C \Rightarrow B)\wedge((A\wedge B)\wedge C \Rightarrow C)$	by the rule of conjunction from **i** and **vi**
viii	$(A\wedge B)\wedge C \Rightarrow (B\wedge C)$	from **vii** with K1S
ix	$((A\wedge B)\wedge C \Rightarrow A)\wedge((A\wedge B)\wedge C \Rightarrow (B\wedge C))$	rule of conjunction applied to **v** and **viii**
x	$(A\wedge B)\wedge C \Rightarrow A\wedge(B\wedge C)$	from **ix** with K1S

Thus it is already proved that L1S contains at least the system $S1^{\circ}$. The proof of 19.01 that still remains to be given may be simplified by using Zeman's meta-theorem according to which for every truth-functional tautology α, the sentence $\neg\mathbf{P}\neg\alpha$ is provable in $S1^{\circ}$, hence also in L1S[43]. We will designate the applications of this meta-theorem in the commentary by 'MTZ' and thus we can show

xi	$B \Rightarrow B\wedge\neg(A\wedge\neg A)$	MTZ
xii	$\neg\mathbf{P}(B) \Rightarrow \neg\mathbf{P}(B\wedge\neg(A\wedge\neg A))$	RIS (**xi**)
xiii	$(B \Rightarrow A\wedge\neg A) \Rightarrow \neg\mathbf{P}(B\wedge\neg(A\wedge\neg A))$	P1S
xiv	$(B \Rightarrow A\wedge\neg A) \Rightarrow (B \Rightarrow \neg A)$	K1S, K3S
xv	$(B \Rightarrow \neg A) \Rightarrow \neg\mathbf{P}(B\wedge\neg\neg A)$	P1S
xvi	$(B\wedge\neg\neg A) \Leftrightarrow (A\wedge B)$	MTZ
xvii	$(B \Rightarrow \neg A) \Rightarrow \neg\mathbf{P}(A\wedge B)$	RIS (**xv**, **xvi**)
xviii	$\neg\mathbf{P}(B) \Rightarrow \neg\mathbf{P}(A\wedge B)$	from **xii**, **xiii**, **xiv** and **xvii** by means of E1S
xix	$\mathbf{P}(A\wedge B) \Rightarrow \mathbf{P}(B)$	from **xviii** with N2S

With this proof of 19.01 in L1S, not only *Proposition 1* but at the same time also *Proposition 2* is verified. The remaining

Proof of *Proposition 3*

however, is also a rather trivial consequence of the former proof of *Proposition 1*. For, according to Zeman (p.161) $S1^{\circ}$ extended by the axiom P2S** just yields the system $S3^{\circ}$, i.e. the tertiary axiomatization of L1S contains $S3^{\circ}$, and conversely, all the principles of L1S including P2S** are provable in $S3^{\circ}$.

Notes

[1] We will use the following abbreviations for the Leibnitian works on logic: **A** = G. W. Leibniz: **Sämtliche Schriften und Briefe**. Ed. by the Preußische (later: Deutsche) Akademie der Wissenschaften; cited by series, volume and pages; **AV** = **Voraus-edition** to the **VI** series of **A**, edited by the Leibniz Forschung-stelle der Universität Münster; cited by fascicle and pages; **C** = L. Couturat: G. W. Leibniz **Opuscules et fragments inedits**, Paris 1903, Reprint Hildesheim 1961; **GI** = **Generales Inquisitiones de Analysi Notionum et Veritatum**, ed. and translated by F. Schupp, Hamburg 1982, cited by § or by number of the line; **GP** = **Die Philosophischen Schriften von G. W. Leibniz**, ed. by C. I. Gerhardt, Berlin 1875 – 1890 (reprint Hildesheim 1960–1), cited by vol. and pages; **Grua** = G. Grua: G. W. Leibniz **Textes inedits**, Paris 1948.

[2] Cf. Rescher **(1954)**, where the author speaks of "the 'term'-constant *Ens* or *Res*" and where, especially in the formulae 21–24, he subsumes 'Ens' as conceptual-constant under the transformation rules of the calculus. A related approach – that, however, leads to even worse absurdities and misinterpretations of Leibniz's ideas – may be found in H.-N. Castaneda's paper "Leibniz's Syllo-gistico-Propositional Calculus", **Notre Dame Journal of Formal Logic** 17 (1976), 481–500.

[3] This is the title of the fragment – unfortunately undated – Nr. 56 in **AV**, 1, 176–79. A similar opinion was held by Leibniz at least at the age of 50. In a famous letter sent to Gabriel Wagner at the end of 1696, he defends the till then familiar logic with the confession "... so muss ich zwar bekennen, daß alle unsre bisherigen Logicken kaum ein schatten deßen seyn, so ich wünd-sche, und so ich gleichsam von ferne sehe" (**GP**, VII, 516); how big the gap between what was achieved till then and what was prophetically intuited as a hunch by Leibniz becomes evident by the fact that for him the syllogistic "diese arbeit des *Aristote-lis*" represents "nur ein Angfang und gleichsam das A, B, C" (o.c. p.519), and that "aber diese Vernunfft Kunst noch unvergleichlich höher zu bringen, halte ich vor gewiß, und glaube es zu sehen, auch einigen Vorschmack davon zu haben ... Was nun meines ermeßens darinn zu leisten müglich, ist von solchen begriff, daß ich mir nicht getraue ohne wirkliche Proben genungsamen glauben zu finden" (p.522).

[4] The "indefinita" function as disguised conceptual-quantifiers for Leibniz; for a detailed account of the Leibnitian logic of the quantifiers, cf. our paper "'Unbestimmte Begriffe bei Leibniz" **Studia Leibnitiana**, XVI (1984), 1-26.

[5] The first one appeared in **Studia Leibnitiana** XVIII (1986), 1-37, the second is forthcoming in the contributions to the symposium **Leibniz: Questions de Logique**, Brüssel, Louvain-la-Neuve 1985.

[6] In the **Definitiones** from around 1679 (**AV**, 1, 146-7) one can find a marginal sketch of a characteristic in which especially 'est' is abbreviated by 'e'. Leibniz did not use this symbol, however, in any of the known drafts of a calculus.

[7] At least one half of the equivalence P3, namely $\mathbf{P}(B) \Rightarrow (B \not\in \overline{B})$ was, however, recognised by Leibniz as valid. Cf. §43 **GI**:
"B continere non-B est falsa ... Patet et ex aliter. B continet B (per [E2]). Ergo non continet non-B alioqui foret impossibilis."

[8] Cf. **GI**, line 168, §73, §146, §148, §190; **C**.259, Principle (2), **C**.261, Principles (3) and (4), **C**.271, **C**.421, Principles (6) and (9); as well as **Grua**, pp.324 and 325.

[9] **C**.262; the bar of course does not mean a second negation, but only serves Leibniz as a bracket.

[10] Compare M. Dummet, Review of Rescher **(1954)** in **Journal of Symbolic Logic** 21 (1956), p.198; also cf. Castaneda **(1976)**, ref.3, p.484, where Leibniz is blamed for the following: "... having analyzed 'some A's are B's' as 'AB exists' [he] does not go on to interpret this as '*AB* contains existence' which would be symbolized as 'AB ≡ AB (Existence)'. He takes this step [in **GP** VII, 213] but he leaves the concept *existence* or *entity* somewhat isolated." Fortunately Leibniz did not further pursue this mistaken approach otherwise, and the "serious troubles" that Castaneda deduces later on (especially pp.489 ff) do not refute the system of Leibnitian logic but only the miscarried reconstruction of it by Castaneda.
That 'Ens' should not be viewed as a conceptual-constant has first been noticed by L. Couturat: cf. his **La Logique de Leibniz**, Paris 1901 (Reprint Hildesheim 1966), p.353, ref.2. A very detailed discussion of this point may also be found in R. Kauppi, **Über die Leibnizche Logik**, Helsinki 1960, especially pp.215-222.
Finally, it should be pointed out that analogous interpretation of the truth concept as a 1st order conceptual-*constant* as probed in §108 **GI** leads to the same difficulties. With 'V' abbre-

viating 'verum' Leibniz attempts to interpret the proposition 'A est verum' – in which 'A' stands for a concept such as 'Homo' – predicatively and thus obtains in accordance with K6 the equation A = AV: "A = A verum seu A est verum". However, because of ABeA, this interpretation would entail that if A is true, then every conjunction AB is true as well, especially $A\bar{A}$ would be true. But that is absurd.

[11] The definition of the disjunction is given (with a slip of the pen) in the **Analysis Didactica** (C.424–6); as to the question of the discovery of this "de Morganian" principle, cf. our earlier paper "Zur extensionalen und 'intensionalen' Interpretation der Leibnizchen Logik", **Studia Leibnitiana** 15 (1983), and the further literature mentioned there, especially the papers of H. Schepers and of Ph. Boehner. The definition of equivalence ('idem est quod' or 'aequivalent') as mutual implication is always presupposed by Leibniz but seldom formulated explicitly. Thus he defines the (strict) equivalence of sentences by means of the condition: "Coincidere dico enuntiationes, si una alteri substitui potest salva veritate", only to add just casually: "seu quae se reciproce inferunt" (**GI**, lines 311–312).

[12] Cf. also R. Kauppi's **(1960)** similar interpretation of these principles; however, she expresses a qualification: "In dieser Form sind sie nicht ausdrücklich von LEIBNIZ aufgestellt worden" (p.182, footnote 2).

[13] A proof of this assertion is given in Lenzen, "Leibniz und die Boolesche Algebra", **Studia Leibnitiana** XVI (1984), 187–203. L. Kruger hit this fact very nearly when he said in **Rationalismus und Entwurf einer universalen Logik bei Leibniz**, Frankfurt 1969, pp.17–8, "daß Leibniz die Boolesche Algebra sozusagen um Haaresbreite verfehlt [hat]".

[14] Those are the two parts of the juristic disputation **De Conditionibus** (A VI, 1, 101–24 and 129–50) from 1665 and the **Specimen Certitudinis seu Demonstrationum in Jure** (ibid. 169–430) from 1667. For the discussion of the laws of propositional logic developed here cf. H. Schepers: "Leibniz' Disputationen 'De Conditionibus' Ansätze zu einer juristischen Aussagenlogik", in **Akten des II Internationalen Leibniz-Kongresses**, vol.V (1975),1–17.

[15] In **De Vero et Falso, Affirmatione et Negatione, et de Contradictoriis** (AV, 1, 86–8) it is said analogously "12) Si positis enuntiationibus sequatur nova et haec sit falsa, etiam aliqua ex illis erit falsa". By the way, Leibniz remarks: "Hoc est axioma" instead of "Hoc est regula"!

[16] AV, 1, 184–90; the following quotation does not appear in the partial edition in **Grua**, 322–4, but is found in F. Schmidt (ed.), **G. W. Leibniz – Fragmente zur Logik** (Berlin-East 1960), 474–8. Later references to this fragment refer to the Edition in AV.

[17] C. I. Lewis & H. G. Langford, **Symbolic Logic**, New York[2] 1959, S.124.

[18] **Analysis Particularum**, ed. by F. Schupp in **Studia Leibnitiana** Sonderheft 8 (1979), 138–53; quotation p.145.

[19] Couturat **(1901)**, p.354; Couturat also points out that this idea was rediscovered by Boole; naturally he could not know that it might be rediscovered by C. I. Lewis in this century for a second time. If Lewis had viewed the Leibnitian logic with somewhat less scepticism, then perhaps he would have noticed that the question he left open: which calculus of strict implication corresponds to the Boolean Algebra? – has been answered by Leibniz in an interesting and rather obvious way.

[20] Cf. Kauppi **(1960)**, especially ch.IV, §3; also Kauppi, "Zur Analyse der hypothetischen Aussage bei Leibniz", in A. Heinekamp & F. Schupp (ed.), **Die Intensionale Logik bei Leibniz und in der Gegenwart**, Wiesbaden 1979, 1–9; H. Burkhardt, **Logik und Semiotik in der Philosophie von Leibniz**, München 1980, passim (cf. under the heading 'hypothetisch'); H. Ishiguro in M. Hooker (ed.), **Leibniz: Critical and Interpretative Essays**, Minneapolis 1982, 90–102 is indeed concerned with the theme "Leibniz on hypothetical truth" but she ignores the reduction of hypothetical to categorical propositions that is advocated by Leibniz with verve; cf. finally F. Schupp's **(1982)** commentary to **GI**, o.c., especially 164–5, where further literature is given.

[21] O.c., p.110; cf. similarly the versions in **Specimen Certitudinis**, o.c., p.372.

[22] O.c., p.82; Leibniz says that the former proposition "resolvi potesti in hac duas" – i.e., into the latter propositions.

[23] As to the legitimacy of this definition cf. the passage of GI quoted in ref.7. That one might take over P3 from the domain of concepts to that of sentences, is evidenced by the remark: "Falsum esse B continere non-B, intelligendum est et de propositione B, quae non continet contradictionem" (§43). Thus, according to Leibniz, at least the implication $P(B) \Rightarrow \neg(B \Rightarrow \neg B)$ is true. If one would yet want to reject Def.7 as non-Leibnitian,

because the converse implication, $\neg P(B) \Rightarrow (B \Rightarrow \neg B)$, had not been put forward by him explicitly, then one might treat **P** as an undefined constant by adding the condition: $\phi(P\tau) = P\phi(\tau)$ to Def.8 of the "translation function".

[24] Cf. also the earlier version in **Ad Specimen Calculi Universalis Addenda** (**GP**, VII,221 ff): "P r i n c i p i a C a l c u l i. 1) Quicquid conclusum est in literis quibusdam indefinitis, idem intelligi debet conclusum in aliis quibusscunque easdem conditiones habentibus ..." (o.c., p.224).

[25] This need not mean that 11.7 were derivable from MPS and the other laws of strict implication. If one drops 11.7 without substitution, weaker systems result that are designated as respective "'nought systems'" (S1°, S2°, S3°, ...) in the terminology of J. J. Zeman, **Modal Logic – The Lewis Modal Systems** (Oxford 1973).

[26] Cf., e.g. the already quoted §137 of **GI**: "... Omnes syllogismi fiant Categoricae". Furthermore in **Ad Specimen Calculi Universalis Addenda** (**AV**, 1, 107 ff; cf. **GP** VII, 221-7 and C.249) axiom K1 is stated as the rule "*a est bcd*, Ergo *a est b*, et *a est c* et *a est d*" (p.111) and similarly E1 is described as a "consequentia per se vera": "*a est b et b est c. Ergo a est c*" (p.110). Finally, one can also find a rule-version of the "praeclarum theorema" K7: "Generaliter si sint quotcunque propositiones: *a est b*, *c est d*, *e est f*, inde fieri poterit una: *ace est bdf*, per additionem illinc subjectorum, hinc praedicatorum" (p.109).

[27] A further version of the *modus ponens* is provided e.g. in **De Calculo Enuntiationum** (**AV**, o.c., p.125): "Significatio particulae *ergo* talis a me accipitur: Esto: *si A est B tunc C est D* item: *A est B* tunc poni poterit: *Ergo C est D.*" Finally, cf. also the **Notationes Generales**, o.c., where the "C o n s e q u e n t i a p r i m a H y p o t h e t i a" is analogously formulated as follows: "*Si A est B, C est D. Jam A est B. Ergo C est D.*" Therefore the misgivings of Rescher (**1954**): "Leibniz cannot ... give a wholly adequate statement of this rule *modus ponens*" (p.3, ref.8) are without foundation.

[28] In our contribution to the IVth International Leibniz Congress, Hannover, 1983: "Leibniz und die Entwicklung der Modernen Logik", p.423.

[29] For the sake of uniformity Lewis's sentential constants *p*, *q*, *r*, ... have been replaced by A, B, C, ...; similarly, his

different logical operators have been replaced by the symbols used in this paper.

[30] As to the "lack" of the law of associativity cf. K. Durr, **Neue Beleuchtung einer Theorie von Leibniz** (Darmstadt 1930), pp.53 ff; Rescher **(1954)** p.11; Kauppi **(1960)**, p.173; and H. Burkhardt **(1980)**, p.353 with suggestions to further readings (ref.399).

[31] Cf. also §194: "Terminus falsus est qui continet oppositos A non-A. terminus verus est non-falsus."

[32] Cf. Zeman **(1973)**, pp.176-7.

[33] There the "rule of necessitation" is valid only for the truth-functional tautologies and for the sentences of the form $\neg P \neg \beta \Rightarrow \gamma$; cf. Zeman **(1973)**, pp.104 ff. and 184 ff.

[34] H. Poser, **Zur Theorie der Modalbegriffe bei G. W. Leibniz**, Wiesbaden 1969, p.60.

[35] Poser refers to a passage from **Quod Ens Perfectissimum existit** (GP, VII, 261-2), where Leibniz attempts to justify the compatibility of any "perfections", i.e., simple, positive and absolute qualities. But the explanation "omnes autem propositiones necessario verae sunt aut demonstrabiles, aut per se notae" (p.261) shows at best that every (finitary) provable proposition α is *ipso facto* necessary. In order to obtain an informal version of the "rule of necessitation", however, one would have to have in addition that the necessity-statement $\neg P \neg \alpha$ itself is finitarily provable or necessary.

[36] S4, unlike T, contains only finitely many non-equivalent "modalities"; cf. Zeman **(1973)**, 179-81.

[37] Cf. his paper "Leibniz's Theories of Contingency", reprinted in M. Hooker (ed.), **Leibniz: Critical and Interpretative Essays,** 243-83. On p.275 Adams remarks: "... the characteristic axiom of S5 ... is not valid on the demonstrability conception of necessity. For a proposition may be indemonstrable without being demonstrably indemonstrable."

[38] This follows from K. Fine's investigation: "An ascending chain of S4 logics", **Theoria** 40 (1974), 110-6. In a letter dated 13.03.78 Steven Schmidt pointed out that there are even infinitely many systems between S4 and S5, that can be characterised by means of "one-variable axioms" alone, *viz.* through the axioms $\neg P \neg P(A) \wedge \alpha_{n+1} \Rightarrow \alpha_n$ with $\alpha_1 = \neg P \neg A$; $\alpha_{2n} = A \Rightarrow \alpha_{2n-1}$; and $\alpha_{2n+1} = \neg A \Rightarrow \alpha_{2n}$.

[39] Particularly in "Epistemologische Betrachtungen zu [S4, S5]" **Erkenntnis** 14 (1979), 33–56, we have attempted to show, that whereas the S4.2 axiom, in *epistemic* interpretation, should be viewed as valid of a general concept of knowledge, the characteristic axiom of S4.4 only holds of the specific concept of knowledge as true conviction. The characteristic axioms of other S4-extensions (cf. ibid., p.35) are, as a rule, so complex that even a paraphrasing in ordinary language is hardly possible, let alone a justified decision on its "intuitive" acceptability.

[40] For a short outline of deontic-logical systems cf., for instance, F. von Kutschera, **Einführung in die Logik der Normen, Werte und Entscheidungen** (Freiburg 1973), ch.1. Systems of epistemic logic are described in detail in our **Glauben, Wissen und Wahrscheinlichkeit** (Wien 1980).

[41] Cf. A. R. Anderson & N. D. Belnap, **Entailment I** (Princeton 1975), 12 and 22.3.

[42] To be more precise, Lewis/Langford prove only the following weakened version of K6S: $(A \Rightarrow B) \Leftrightarrow (A \Rightarrow A \wedge B)$, in which one may, however, strengthen the implication on the right side to an equivalence.

[43] Cf. Zeman **(1973)** p.86; as a corollary of this proposition one obtains the further meta-theorem, that $S1^{0}$ – thus also Leibniz's L1S – covers the whole of propositional-logic.

MACIEJ JUNIEWICZ

LEIBNIZ'S MODAL CALCULUS OF CONCEPTS

The aim of this paper is to uncover Leibniz's modal calculus of concepts. Here we employ the word "uncover" quite seriously, because his ideas of modal logic, however sketchy in character and obscured by the dust of occasional mistakes, are developed to such an extent that bringing them to completion is almost automatic for a modern "underlabourer".

Our point of departure is the observation that what we are after here is not a sentential modal logic (at least not primarily), since for Leibniz a sentence to which modal operators are applied has quite a definite structure: in the simplest case it is of the form A est B or A = B, and therefore his system is a modal extension of his calculus of concepts – in other words, a theory of Boolean algebras with underlying modal logic.

Leibniz does not provide explicitly any axiomatization of the modal calculus of concepts. In this situation we investigate the part of his thought dealing with strict correspondence between sentences and concepts, which enables us to notice in the theorem on the reduction of hypothetical to categorical sentences a whole series of modal formulas. These formulas, connecting intimately modal and Boolean structures, we admit as the most fundamental axioms.

In the resulting system, necessity can be interpreted as derivability. For reasons of space, we are not able to discuss this interesting fact that reminds of Leibniz's definition of necessity as reducibility to identity. Instead, we concentrate upon a group of formulas (which we would like to call "the calculus of strokes") never, to our knowledge, commented on in the literature of the subject in question. The acquired technique turns out to be a useful tool for explaining these strange at first sight expressions, confirming the correctness of our axiomatization of Leibniz's calculus of concepts.

I

Leibniz's calculus of concepts, presented, e.g. in **Generales Inquisitiones de Analysi Notionum et Veritatum** (GI), can be described rather adequately as a formalized theory of Boolean algebras, using a quantifier-free language which has an infinite set of concept constants (cf. the paper by Professor Lenzen in this volume). More precisely, we assume that the language L of the calculus has an alphabet consisting of the following symbols: $\cup$, $^{-}$ (functional symbols), $\leqslant$, $=$ (predicate symbols), 1, A, B, ... (concept constants forming an infinite set Cn), $\vee$, $\wedge$, $\sim$, $\rightarrow$, $\longleftrightarrow$ (logical constants). The sets of terms and formulas, denoted respectively by T and F, are constructed in the standard way. We shall write $t_1 \cap t_2$ instead of $\overline{\overline{t}_1 \cup \overline{t}_2}$, $t_1 \neq t_2$ instead of $\sim t_1 = t_2$ ($t_1, t_2 \in T$), and $\overline{0}$ instead of 1.

The Leibnitian expressions AB (the composition of concepts A, B), non A, A est B, A est Ens (equivalently: A est possibile, A est res, A est verum, A est), A non est Ens (equivalently: A est impossibile, A non est res, A est falsum, A non est) are rewritten in L as: $A \cup B$, $\overline{A}$, $B \leqslant A$, $A \neq 1$. The fact that we read A est B as $B \leqslant A$, which accords with Leibniz's generally synonymous use of "est" and "continet", and that we denote the unique impossible concept by 1 shows that the intended interpretation of the language is the so-called intentional one.

The calculus of concepts is now a theory $B_0 = (L, Ax, Cons)$, where Ax is a set of axioms, Cons is a consequence operation. Ax contains any set of Boolean axioms, formed with respect to $\cup$, $^{-}$, 1 and closed under the substitution of terms for constants. The last property agrees with Leibniz's statement that there are infinitely many axioms ("propositions identiques"), because there are infinitely many names for concepts ("termes" – C.186). Furthermore, beside the substitutional instances of axioms of sentential logic, to Ax belong all formulas of the type $t \neq \overline{t}$, where $t \in T$ (*"Propositio per se falsa* est A coincidit ipsi non A" – C.365). B_0 is thus a theory of nondegenerated Boolean algebras – an important fact in our subsequent constructions. Cons is based on modus ponens as a sole rule of inference. $\vdash \alpha$ means that the formula α is a theorem of B_0.

Let C be the Boolean algebra of concepts. By an interpretation of the language L in C we shall mean any function $v\colon Cn \longrightarrow C$ such that $v(1) = 1$. The interpretation v standardly assigns truth-values to every formula of L. We write $C \vDash_v \alpha$, if the formula α is true in C under the interpretation v.

II

Commenting in GI on theorems of his calculus, Leibniz writes often: "A autem B significare possunt terminos, vel propositiones alias" (C.365). In the latter case, when, moreover, L, M, N are sentences, non B is interpreted as the negation of B, MN denotes the conjunction of M and N ("Scilicet sit $\overline{\text{A esse B}}$ = L, et sit B = CD et $\overline{\text{A esse C}}$ = M, et $\overline{\text{A esse D}}$ = N, utique fiet: L = MN" – C.372), and the expression A est B means that the sentence A contains, i.e. implicates, the sentence B: "Propositionem ex propositione sequi nihil aliud est quam consequens in antecendenti contineri ut terminum in termino" (C.398).

Throughout the rest of the present paper we shall meditate upon this idea of Leibniz's, showing that what he has in mind is surprisingly far more complicated than the "ordinary" Lindenbaum algebra of the theory B_0. The point is that Leibniz maintains not only that the algebraic structure in the set of sentences is simply analogous to that in the set of concepts, but claims, moreover, that the former originates from (is isomorphic to) the latter via some operators associating terms with sentences (concepts with interpreted sentences).

The Leibnitian reduction of "propositiones tertii adjecti" (categorical sentences) to "propositiones secundi adjecti" is the set of the following four equivalences (C.393):

(R1) Quoddam A est B dat: *AB est res*;
(R2) Quoddam A non est B dat: *A non B est res*;
(R3) Omne A est dat: *A non B non est res*;
(R4) Nullum A est B dat: *AB non est res.*

The form of sentences on the right-hand side suggests that the reduction can be pushed on still further – to objects "primi adjecti". In this way there arise two operators which are more or less explicitly present in Leibniz's texts:

1. V: T $\longrightarrow$ F, V(t) is the formula $t \neq \mathbf{1}$;
2. Z: T $\longrightarrow$ F, Z(t) is the formula $t = \mathbf{0}$ (or t est necessarium in Leibniz's notation – C 259).

Let us introduce at one stroke two remaining analogous operators:

3. $\overline{V}$: T $\longrightarrow$ F, $\overline{V}$(t) is the formula $t = \mathbf{1}$;
4. $\overline{Z}$: T $\longrightarrow$ F, $\overline{Z}$(t) is the formula $t \neq \mathbf{0}$.

In assigning terms to formulas, Leibniz employs, first of all, the operator V^{-1}. The following passage illustrates the procedure in question: "Ipsa propositio concipi potest instar termini, sic qu. A esse B, seu AB esse terminum verum, est terminus, nempe AB verum" (C.397). AB verum means here just AB: "verum facit hoc loco officium quod unitas in Arithmetica (...) A = A verum" (C.381, 382).

Let us now see how Leibniz fares with sentences which have reduction of the form $t = 1$ or $t = 0$: "Sic omne A esse B, seu A non B esse falsum, seu A non B falsum est terminus verus" (C.397). The sentence A non B est falsum can be transformed into the term A non B by means of V^{-1}. However, if the initial sentence is true, then the term denotes an impossible concept, which is for Leibniz a perplexing situation. For this reason he chooses the expression A non B falsum. If we interpret it as non(AnonB), we obtain the transformation:

Omne A est B $\longmapsto$ non(AnonB),

which consists in applying the operator Z^{-1}, because $\vdash B \leqslant A \longleftrightarrow A \cup B = 0$. Perhaps our seeing in C.397 Leibniz's allusion to Z is a little simple-minded, but it should be stressed that the theory under consideration just calls for the operator Z, which can therefore be introduced in our model a priori.

On the whole, Leibniz is convinced that one can associate a term with any sentence by means of the operator V^{-1}. This is, of course, not so, but the very idea remains clear, and his formulations can easily be corrected. He says, for instance: "Et revera omnis propositio seu omne complexum potest vicissim reduci ad incomplexum per *est* primi adjecti ut vocant. Et si loco propositionis homo est rationalis, dicam τό hominem esse rationalem, est" (GP II, 472). Unfortunately, the sentence *homo est rationalis* cannot be written in the form $t \neq 1$, where $t \in T$. Let us paraphrase the above statement, substituting the sentence *quidam homo est rationalis* for the original one. The new sentence is equivalent, on account of (R1), to: *homo rationalis est (res)*, and therefore we obtain

τό quendam hominem esse rationalem = homo rationalis.

If we want to retain Leibniz's original example, we should, of course, use the operator Z^{-1} with the result:

τό hominem esse rationalem = non(homo non rationalis).

One observes that the term Leibniz assigns to a sentence α is, as a rule, denoted by *τό* α, or solely by the acc. cum inf.-form of α.

The converse transformation of terms into sentences can also take place: "Propositio ipsa fit terminus si termino ipsi adjiciatur verum aut falsum; ut sit *A* terminus, et *A est* vel *A verum est*, sit propositio, *A verum*, seu *A verum esse*, seu *A esse* erit terminus novus, de quo rursus fieri potest propositio" (C.398), or

$$A \overset{V}{\longmapsto} A \text{ est verum} \overset{V^{-1}}{\longmapsto} A \text{ verum},$$

where, as we noted earlier, A verum = A.

III

The discovery of strict correspondence between terms and sentences enables Leibniz to elucidate the nature of implication, and, in particular, to "reduce" hypothetical sentences to categorical ones. Now we pass on to a detailed analysis of an example of such a reduction (C.260):

(1) Ex A est B sequitur E est F, idem est ac si diceremus A esse B est E esse F.

We begin by establishing the sense of the implication on the left-hand side. In Leibniz's texts "ex α sequitur β" expresses not material implication, but a much stronger connection between the sentences α and β, which consists in α's "containing" β, and which Leibniz describes also in a way which amounts to saying that β is a logical consequence of α and of a set of the definitions of concepts (C.408). "Ex α sequitur β" is therefore a kind of strict implication: $N(\alpha \to \beta)$, where N is a sign of necessity.

Let us now consider alternative readings of the expressions A esse B, E esse F in (1), which Leibniz might as well have written *τό* A esse B, *τό* E esse F (C.389). According to what was said earlier, A esse B, E esse F are just terms K(A est B), K(E est F) where K is potentially one of the following operators: V^{-1}, Z^{-1}, $\overline{V}^{-1}$, $\overline{Z}^{-1}$.

As we have remarked, Leibniz assumes in the first place, that $K = V^{-1}$. But in this case the operator K is erroneously applied to sentences which cannot be written in the form $t \neq 1$, and therefore we paraphrase (1) as follows:

(2) Ex quoddam A est B sequitur quoddam E est F, idem est ac si diceremus quoddam A esse B est quoddam E esse F.

In view of (R1), the sentences A quoddam A est B, quoddam E est F lead to the following formulas of the language L: $A \cup B \neq 1$, $E \cup F \neq 1$, and consequently K(q. A est B), K(q. E est F) are, respectively, the terms $A \cup B$, $E \cup F$. We thus obtain

(3) $$N(A \cup B \neq 1 \rightarrow E \cup F \neq 1) \longleftrightarrow E \cup F \leqslant A \cup B.$$

Assume next that $K = Z^{-1}$ in (1). Then, analogously, we get

(4) $$N(B \leqslant A \rightarrow F \leqslant E) \longleftrightarrow \overline{E} \cap F \leqslant \overline{A} \cap B.$$

Finally, when $K = \widetilde{V}^{-1}$ or $K = \overline{Z}^{-1}$, we obtain, respectively (in the latter case we consider (2) instead of (1))

(5) $$N(B \leqslant A \rightarrow F \leqslant E) \longleftrightarrow E \cup \overline{F} \leqslant A \cup \overline{B},$$

(6) $$N(A \cup B \neq 1 \rightarrow E \cup F \neq 1) \longleftrightarrow \overline{E} \cap \overline{F} \leqslant \overline{A} \cap \overline{B}.$$

To sum up, we see that the operators $\overline{V}$ and $\overline{Z}$ cannot be used for associating terms with sentences in Leibniz's sense (at least as long as est=continet), because from (5) (or from (6)) by putting $A = 0$, $B = E = F = 1$, even under very general syntactic assumptions, one deduces the formula $0 = 1$, which contradicts the axiom $t \neq \overline{t}$. On the other hand, Leibniz's idea of reducing hypothetical to categorical sentences, realized by means of the operators V and Z, leads immediately to formulas (3) and (4), which are particular instances of

(7) $$N(t_1 = 1 \rightarrow t_2 = 1) \longleftrightarrow t_1 \leqslant t_2,$$

where $t_1, t_2 \in T$.

We intend to regard formulas of the type (7) as structural axioms of Leibniz's modal calculus of concepts. It is easy to perceive their Boolean sense. To this aim, denote the formula $t_1 = 1 \rightarrow t_2 = 1$ by α. Firstly, we obviously have $\vdash t_1 \leqslant t_2 \rightarrow \alpha$. Secondly, it is not difficult to prove that if $\vdash \beta \rightarrow \alpha$, where β is any conjunction of equalities (of the form $p = q$, $p,q \in T$), then $\vdash \beta \rightarrow t_1 \leqslant t_2$. The formula $t_1 \leqslant t_2$ can thus be called the weakest, with respect to conjunction of equalities, condition implying the truth of α. If in place of $t_1 = 1 \rightarrow t_2 = 1$ we consider the more general formula: $t = 1 \rightarrow (t_1 = 1 \vee \ldots \vee t_n = 1)$ (denoted also by α), we see that the role of $t_1 \leqslant t_2$ is now played by the set $\{t \leqslant t_1, \ldots, t \leqslant t_n\}$, because $\vdash t \leqslant t_i \rightarrow \alpha$, and if $\vdash \beta \rightarrow \alpha$, where β is a conjunction of equalities, then $\vdash \beta \rightarrow t \leqslant t_i$ for some $i = 1, \ldots n$.

IV

The above investigations suggest that all formulas of the type

(8) $N(t = 1 \rightarrow (t_1=1 \vee \dots \vee t_n=1)) \longleftrightarrow t \leqslant t_1 \vee \dots \vee t \leqslant t_n$,

where $t, t_i \in T$, n is a natural number, should be candidates for axioms of Leibniz's modal calculus of concepts.

We shall now extend the language L by adding to its alphabet the symbol for necessity N. The set F_1 of formulas of the new language L_1 is thus closed under the operation $\alpha \longmapsto N\alpha$, where $\alpha \in F_1$. $M\alpha$ will be an abbreviation for $\sim N \sim \alpha$. Leibniz's modal calculus of concepts is a formal system $B_1 = (L_1, Ax_1, Cons_1)$. The set Ax_1 of axioms of the theory B_1 contains all axioms of the theory B and all formulas of the type (8). Furthermore, Ax_1 contains formulas of the form $N\alpha \rightarrow \alpha$, $N(\alpha \rightarrow \beta) \rightarrow (N\alpha \rightarrow N\beta)$, $N\alpha \rightarrow NN\alpha$, where $\alpha \in F_1$, and substitutional instances of axioms of sentential logic. The consequence operation $Cons_1$ is based on modus ponens and on the following rule of inference: if $\vdash_1 \alpha$ then $\vdash_1 N\alpha$ (for $\alpha \in F_1$, $\vdash_1 \alpha$ means that α is a theorem of B_1).

For reasons of space, we are not able to explain in this place why exactly S4 should be chosen as the logic of Leibniz's modal theory. For the argument, based on his definition of necessity as provability, we refer the reader to Juniewicz **(1986)** or to Professor Lenzen's paper in this volume.

We shall now present the main metatheoretical properties of B_1 (for proofs see Juniewicz **(1986)**). These will not be of use in the sequel, but should enable the reader to get a feeling of what our formalization of Leibniz's modal calculus is.

Proposition 1. B_1 is a consistent nonessential extension of the theory of Boolean algebras B_0 (i.e. if $\vdash_1 \alpha$ then $\vdash \alpha$, for $\alpha \in F$).

Proposition 2. The underlying logic of B_1 is in fact McKinsey's system S4.1, i.e. $\vdash_1 NM\alpha \rightarrow MN\alpha$ for $\alpha \in F_1$.

Let Id denote the family of all proper ideals in C (the Boolean algebra of concepts). Let R be the relation on Id defined as follows: $I_1 R I_2$ iff the ideal I_1 is contained in the ideal I_2. We shall treat the partially ordered set of quotient algebras $(\{C/I\}_{I \in Id}, R)$ as a structure of possible worlds in the sense of Kripke. Given an interpretation $v: Cn \longrightarrow C$ of the language L in

C, a formula α is false or true in each possible world C/I, namely, under the interpretation v_I: Cn $\longrightarrow$ C/I, $v_I = k_I \circ v$, where k_I: C $\longrightarrow$ C/I is the canonical epimorphism. Thus we obtain a Kripke semantics for the modal language L_1. C/I $\models_v \beta$ means that the formula β is true in the possible world C/I within this semantics.

Proposition 3. (Completeness theorem). Let α be any formula of the modal language L_1. Then $\vdash_1 \alpha$ iff C/I $\models_v \alpha$ for every ideal I $\in$ Id, and every interpretation v: Cn $\longrightarrow$ C.

Let A_I^v stand for the set of the atomic formulas (of L) which are true in C/I under the interpretation v.

Proposition 4. For any $\alpha \in F_1$ we have C/I $\models_v$ Nα iff $A_I^v \vdash_1 \alpha$.

Thus, unsurprisingly, B_1 belongs to the family of those formal theories with modal logic in which necessity can be interpreted as derivability. S. Kripke **(1963)** describes semantically a similar system which is a modal extension of arithmetic. Kripke does not provide any complete axiomatisation, but observes that the logic is in fact S4.1.

The above-mentioned fundamental property of B_1 accords neatly with Leibniz's explanation of necessity in terms of provability. This suggests that it is possible to arrive at the theory B_1 in a completely different way, namely, starting from his famous definition stating that a sentence is necessarily true if it is "reducible" to identity. For this we again refer the reader to Juniewicz **(1986)**, where it is also shown how Leibnitian possible worlds can be naturally described as the algebras C/I.

V

Leibniz's constructions using the operators which set up the correspondence between terms and sentences point to the fact that he regards the algebra of sentences not only as analogous but, more strongly, as "isomorphic" to the algebra of concepts. Let us return once again to the statement (1), which we interpreted earlier as a formula of the modal language L_1. Now we can look at (1) from a slightly different point of view. Let v: Cn $\longrightarrow$ C be an interpretation. The equivalence (1) may be written as

(9) $$C \models_v N(\alpha \rightarrow \beta) \text{ iff } C \models_v K(\beta) \leqslant K(\alpha),$$

where $K = V^{-1}$ or $K = Z^{-1}$. But the fact that the sentence α

implies the sentence β means, according to Leibniz, that α contains β. We thus obtain

(10) $$[\alpha]_v \leqslant [\beta]_v \text{ iff } C \models_v K(\beta) \leqslant K(\alpha),$$

where $[\alpha]_v$, $[\beta]_v$ are equivalence classes of α and β, i.e. elements of some Boolean algebra of sentences. (1) states therefore that the operator K preserves (at least) the ordering in that algebra and in the algebra of concepts.

The technique of the theory B_1 enables us to formulate more precisely the above observations.

***Proposition** 5*. Let $t, t_1, t_2 \in T$. Formulas of the following form are theorems of B_1:

(a) $$M{\sim}t \neq 1 \longleftrightarrow \overline{t} \neq 1;$$

(b) $$MN(t_1 \neq 1 \wedge t_2 \neq 1) \longleftrightarrow t_1 \cup t_2 \neq 1;$$

(c) $$(t_1 \neq 1 \wedge t_2 \neq 1) \longleftrightarrow t_1 \cap t_2 \neq 1;$$

(d) $$N{\sim}t = 0 \longleftrightarrow \overline{t} = 0;$$

(e) $$NM(t_1 = 0 \vee t_2 = 0) \longleftrightarrow t_1 \cap t_2 = 0;$$

(f) $$(t_1 = 0 \wedge t_2 = 0) \longleftrightarrow t_1 \cup t_2 = 0.$$

Proof.

(a) and (d). Substituting in (8) (where n=1) the constant **0** for t_1 we obtain $\vdash_1 Nt \neq 1 \longleftrightarrow t = \mathbf{0}$, and dually $\vdash_1 Mt = 1 \longleftrightarrow \overline{t} \neq 1$.

(b) and (e). The following formulas are equivalent within B_1 $MN(t_1 \neq 1 \wedge t_2 \neq 1)$, $M(Nt_1 \neq 1 \wedge Nt_2 \neq 1)$, $M(\overline{t}_1 = 1 \wedge \overline{t}_2 = 1)$, $M\overline{t}_1 \cap \overline{t}_2 = 1$, $t_1 \cup t_2 \neq 1$. Finally we have (b). We obtain (e) by duality.

(c) and (f) are theorems of B_0 and thereby theorems of B_1.

Let $[\alpha]$ be the equivalence class of a formula $\alpha \in F_1$ with respect to the equivalence relation: $\alpha \approx \beta$ iff $\vdash_1 \alpha \longleftrightarrow \beta$. Denote by S_1 the set of all elements of the form $[t \neq 1]$, where $t \in T$. From *Proposition 5* it follows that S_1 is a Boolean algebra with operations:

(i) $\overline{[\alpha]} = [M{\sim}\alpha]$;

(ii) $[\alpha] \cup_* [\beta] = [MN(\alpha \wedge \beta)]$;

(iii) $[\alpha] \cap_* [\beta] = [\alpha \vee \beta]$.

The class $[1 \neq 1]$ is the greatest element, the class $[0 \neq 1]$ is the least (S_1 is in fact the Boolean algebra of regularly closed elements in the topological Boolean algebra $F_1/{\approx}$).

Let $[t]$ be the equivalence class of a term $t \in T$ with respect to the equivalence relation: $t_1 \approx t_2$ iff $\vdash t_1 = t_2$. From *Proposition 5* and from the fact that if $\vdash t_1 \neq 1 \longleftrightarrow t_2 \neq 1$ then $\vdash t_1 = t_2$, it follows that the function $V: T/\approx \longrightarrow S_1$, defined by the formula $V([t]) = [t \neq 1]$ is an isomorphism of Boolean algebras.

Analogously, let S_2 be the Boolean algebra of sentences which is the set of elements of the form $[t = 0]$, where $t \in T$, endowed with the following operations:

(iv) $[\overset{*}{\bar{\alpha}}] = [N{\sim}\alpha]$;

(v) $[\alpha] \cup^{*} [\beta] = [\alpha \wedge \beta]$;

(vi) $[\alpha] \cap^{*} [\beta] = [NM(\alpha \vee \beta)]$.

The class $[1 = 0]$ is the greatest element, the class $[0 = 0]$ is the least element (S_2 is in fact the Boolean algebra of regularly open elements in the topological Boolean algebra $F_1/\approx$). *Proposition 5* and the fact that if $\vdash t_1 = 0 \longleftrightarrow t_2 = 0$ then $\vdash t_1 = t_2$ show that the function $Z: T/\approx \longrightarrow S_2$, defined by the formula $Z([t]) = [t = 0]$ is an isomorphism of Boolean algebras.

It is now clear how to obtain the genuine Leibnitian algebras of sentences, namely those which are isomorphic to the algebra of concepts (and not of terms). To this aim, it suffices to modify the above constructions, considering in F_1 the equivalence relation: $\alpha \approx_v \beta$ iff $C \vdash_v N(\alpha \longleftrightarrow \beta)$ instead of the formerly used one : $\alpha \approx \beta$ iff $\vdash_1 \alpha \longleftrightarrow \beta$. Here we omit the details, because the simpler algebras S_1 and S_2 equally well explain the thought of Leibniz. We only point out that in these algebras indeed $[\beta]_v \leqslant [\alpha]_v$ iff $C \vDash_v N(\alpha \rightarrow \beta)$, in accordance with Leibniz's interpretation of implication as a relation of containing (the proof in Juniewicz (1986)).

We conclude that Leibniz's ideas concerning the nature of the algebra of sentences are quite reasonable, although we have had to correct him at several points. Firstly, Leibniz seems to be convinced that the algebra of all sentences is isomorphic to the algebra of terms, whereas in reality this property is possessed by the algebras S_1 and S_2, which are proper subsets of $F_1/\approx$. Secondly, Leibniz proposes the classical negation as the operation on sentences, parallel to the operation *non* on concepts. We take instead the strict negation $N{\sim}$ (in the case of S_2). "Composition" of sentences in S_2 agrees with the Leibnitian operation (which is conjunction), but in S_1 the situation is not so neat in view of the defining formula (ii).

These divergencies could raise objections to our interpretation, were it not for the pleasant fact that the apparatus can be employed to illuminate some of Leibniz's ideas which are hardly accessible within any other framework.

VI

Our analysis of Leibniz's "calculus of strokes" from §107-108 of GI is based on the following obvious consequence of *Proposition 5*:

Proposition 6. (Leibniz's principle of the duality sentence-concept). Let $t(A_1,\dots,A_n)$ be a term. Then:

(a) $\vdash_1 t(A_1,\dots,A_n) \neq 1 \longleftrightarrow t_*(A_1\neq 1,\dots,A_n\neq 1)$

(b) $\vdash_1 t(A_1,\dots,A_n) = 0 \longleftrightarrow t^*(A_1=0,\dots,A_n=0)$,

where $t_*(\alpha_1,\dots,\alpha_n)$ (resp. $t^*(\alpha_1,\dots,\alpha_n)$) denotes the modal operator that arises from the term t by indexing with the lower (resp. upper) asterisk each symbol of Boolean operation on t.

In the above-mentioned fragment of GI Leibniz introduces a very interesting method of notation of formulas of the calculus of concepts. In this notation the meaning of an expression depends on the context in which it occurs.

1. An expression of the type $\frac{1\;2\;3}{A\quad B}$ is in principle a sentence in which A is the subject, B is the predicate. The numbers mark empty places in which some symbols should be written, indicating the kind of the judgement (in this role we shall use just the sequences of natural numbers): "locus 1 designabit quantitatem vel qualitatem, etc. secundum quam hic adhibetur terminus A (...) et locus 2 naturam propositionis AB, locus 3 modum termini B" (C.381). In this case we shall write $\frac{1\;2\;3}{A\quad B}$ as $t(A,B) \neq 1$ or as $t(A,B) = 0$, where t is a term.

2. The same expression $\frac{1\;2\;3}{A\quad B}$ can appear in a context like

$$\begin{array}{ccc} 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \cline{1-3} A & B & C \end{array}$$

Our previous considerations suggest that if C is a name of a concept, and the sequence (4,5,6) codes the kind of judgement then $\frac{1\;2\;3}{A\quad B}$ denotes not a sentence (this would be nonsensical), but the corresponding (with respect to the operators V or Z) term, i.e.

$t(A,B)$. Leibniz writes about it, using in a characteristic way the Greek article: "Locus 4 (designes) modum adhibendi $\tau\acute{o}$ AB seu L "(ibid.).

3. Although, as we have said, $\dfrac{1\ 2\ 3}{A\quad B}$ is in the first place a sentence with the subject A and the predicate B, it is syntactically also possible to substitute for A and B expressions which are clearly sentences. In this case we shall interpret (1,2,3) as the modal operator in two arguments $t_{*}(\alpha,\beta)$ or $t^{*}(\alpha,\beta)$ accordingly, as the original interpretation of $\dfrac{1\ 2\ 3}{A\quad B}$ is $t(A,B) \neq 1$ or $t(A,B) = 0$. For instance

$$\cfrac{28\quad 29\qquad 30}{\cfrac{7\ 8\ 9}{B\quad V}\ \cfrac{10\ 11\ 12}{C\qquad D}}$$

(C.382; a subformula of Leibniz's formula) is a modal formula of this type, whose subformulas are the sentences B est verum and $\dfrac{10\ 11\ 12}{C\qquad D}$.

For the sake of illustration we shall now concentrate on a formula of Leibniz's "calculus of strokes", showing at the same time what modal machinery is involved in his synonymous treatment of "ex ... sequitur" and "continet" or "est".

Let (1,2,3) denote the Boolean relation est. The expression $\dfrac{1\ 2\ 3}{A\quad B}$ in the position "categorical sentence" means thus: A est B. Let us define A est_1 B = $\bar{A}\cap B$. Since $\vdash \bar{A}\cap B = 0 \longleftrightarrow B \leqslant A$, the expression under consideration in the position "term" is just A est_1 B. One verifies easily that $\vdash_1 N(\alpha \rightarrow \beta) \longleftrightarrow \overset{*}{\bar{\alpha}}\cap^{*}\beta$, when α and β are formulas equivalent to $t_1 = 0$ and $t_2 = 0$, respectively ($t_1,t_2 \in T$). $\dfrac{1\quad 2\quad 3}{A=0\ \ B=0}$ is therefore the formula $N(A = 0 \rightarrow B = 0)$, or, introducing a new symbol est_2, the formula $(A = 0)\ \mathrm{est}_2\ (B = 0)$.

Consider now the expression

$$\cfrac{4\quad 5\quad 6}{\cfrac{1\ 2\ 3}{A\quad B}\quad C}$$

(C.381). Let us assume that also (4,5,6) designates initially the relation est. If C is the name of a concept, then we obtain

(11) $\qquad (A\ \mathrm{est}_1\ B)\ \mathrm{est}\ C.$

According to the Leibnitian idea of associating terms with sentences by means of an appropriate operator, or even of identifying terms with sentences, C may be regarded as the sentence C = 0. Leaving without change the status of A and B we get

(12) $$(A \text{ est } B) \text{ est}_2 (C = 0).$$

Finally, treating A, B, C as sentences via the operator Z we have

(13) $$((A = 0) \text{ est}_2 (B = 0)) \text{ est}_2 (C = 0).$$

The sense of this procedure consists in the fact that the formulas (11) – (13) are equivalent within the modal calculus of concepts by virtue of Leibniz's duality principle.

Now we pass on to investigate Leibniz's original example of a transformation in which he makes use of the operator V. In the fragment of GI under consideration we find the following formula (C.381):

(14) $$\begin{array}{cccccc} 13 & & 14 & & & 15 \\ \hline 10 & 11 & & 12 & & \\ \cline{1-4} & 7 & 8 & 9 & & \\ \cline{2-4} & & \multicolumn{2}{c}{1\ \ 2\ \ 3} & \multicolumn{2}{c}{4\ \ 5\ \ 6} \\ \cline{3-4}\cline{5-6} A & B & C & D & E & F \end{array}$$

It is more or less clear that the letters are here the names of concepts. This means that

$$\frac{1\ \ 2\ \ 3}{C\quad D} \quad \text{and} \quad \frac{7\quad 8\quad 9}{B\quad \frac{1\ 2\ 3}{C\quad D}}$$

are in the same semantic category. For definiteness, we assume that $\frac{4\ \ 5\ \ 6}{E\quad F}$ and the expression under (10,11,12) are sentences (these may be treated as terms as well). Consequently, the sequence (13,14,15) codes a modal operator in two variables. In our notation the formula (14) takes on the form:

(15) $$p_*(q(A,r(B,s(C,D))) \neq 1,\ t(E,F) \neq 1),$$

where $p,q,r,s,t \in T$. (We choose the variant that is most convenient for applying the operator V).

After recalling that "Omnis terminus etiam incomplexus potest haberi pro propositione", Leibniz forms a new expression, which is to be equivalent to the first (C.382):

(16) $$\begin{array}{cccccccccccccccccccccccc} 43 & & & & & & & & & & & 44 & & & & & & & & & & & & 45 \\ \hline 37 & & & & & 38 & & & & & & 39 & 40 & & & & & 41 & & & & & & 42 \\ \cline{1-12}\cline{13-24} 25 & & & 26 & & 27 & 28 & & 29 & & & 30 & 31 & & & 32 & & 33 & 34 & & & 35 & & 36 \\ \cline{1-6}\cline{7-12}\cline{13-18}\cline{19-24} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ \cline{1-3}\cline{4-6}\cline{7-9}\cline{10-12}\cline{13-15}\cline{16-18}\cline{19-21}\cline{22-24} A & & V & V & & V & B & & V & C & & D & E & & V & V & & V & F & & V & V & & V. \end{array}$$

The point of the transformation is clear. The letter B appears as the sentence B est verum, which automatically makes $\dfrac{10\ 11\ 12}{C \qquad D}$ a sentence. The symbol A (and similarly E and F) has been replaced by a more complicated sentence, containing three occurrences of V. If Leibniz was not intent on obtaining the symmetry of the graph, he could as well have written:

(17)
$$
\begin{array}{ccccccccccccccc}
25 & & & & & & & 26 & & & & & & & 27 \\ \hline
22 & & & 23 & & & & & 24 & & & & & & \\ \cline{1-9}
 & & & 16 & & 17 & & & 18 & 19 & & & 20 & & 21 \\ \cline{4-9} \cline{10-15}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ \cline{1-3} \cline{4-6} \cline{7-9} \cline{10-12} \cline{13-15}
A & & V & B & & V & C & & D & E & & V & F & & V
\end{array}
$$

or, in our notation:

(18) $p_*(q_*(A \neq 1, r_*(B \neq 1, s(C,D) \neq 1)), t_*(E \neq 1, F \neq 1))$.

The equivalence of (14) and (17) is thus again an application of Leibniz's duality principle.

There remains to explain the meaning of the formulas under (25,26,27), (31,32,33), (34,35,36) in (16). The transformation of (14) into (17) consists in applying several times the equivalence $t(A,B) \neq 1 \longleftrightarrow t_*(A \neq 1, B \neq 1)$, where t is any term. But we obviously do not have to stop at $\dfrac{1\ 2\ 3}{A \quad V}$ because the operation of transferring "$\neq 1$" to the inside of an expression can be carried out once more just with respect to that seemingly closing formula. If t is the term $A \cup \overline{B}$, we have:

$$\vdash_1 A \cup \overline{B} \neq 1 \longleftrightarrow (A \neq 1) \cup_* (B \neq 1),$$

and putting 1 in the place of B:

$$\vdash_1 A \neq 1 \longleftrightarrow (A \neq 1) \cup_* (1 \neq 1).$$

Instead of $\dfrac{1\ 2\ 3}{A \quad V}$ we can therefore write

$$
\begin{array}{cccccc}
25 & & & 26 & & 27 \\ \hline
1 & 2 & 3 & 4 & 5 & 6 \\ \cline{1-3} \cline{4-6}
A & & V & 1 & & V
\end{array}
$$

(and analogously for E, F), where (25,26,27) denotes the modal operator $\alpha \cup_* \overline{\beta}$, or, equivalently $MN(\alpha \wedge M{\sim}\beta)$. Leibniz's mistake, i.e. writing the letter V instead of 1 under the places 4, 16, 22 in (16) originates from the fact that he treats sentences of the type A est verum as formally analogous to those of the type A est B, where B is a concept constant.

VII

We conclude our exposition with several remarks.

Let $t \in T$. Each of the following expression-forming operators in two variables: $t(A,B) \neq 1$, $t(A,B)$, $t_*(\alpha,\beta)$ will be called parallel to any of the remaining ones. We shall also say that the expression A is parallel to the expression $A \neq 1$ (and vice versa; this can be regarded as a particular instance of the first parallelism). Let in an expression of the type (14) or (17) every sequence of natural numbers designate an expression-forming operator of the form $p(A,B) \neq 1$, $q(A,B)$, $r_*(\alpha,\beta)$, where $p,q,r \in T$ (the assumption does not concern the sequences occurring in an expression of the form $\frac{1\;2\;3}{A\quad V}$ which is always interpreted as $A \neq 1$). An expression obtained in this way is a well formed formula of "the calculus of strokes", while it is not, in general, a well formed formula of the language L_1. However, by substituting, if necessary, expression-forming operators or expression for parallel ones, we can get a well formed formula of L_1, and Leibniz's duality principle guarantees that the resulting formulas are equivalent to each other.

Analogous remarks can be made with respect to the context of the operator Z. In that case, as noticed earlier, for example A est B, A est_1 B, α est_2 β ($B \leqslant A$, $\bar{A} \cap B$, $N(\alpha \rightarrow \beta)$), are parallel expression-forming operators.

Let us now return to Leibniz's synonymous use of "antecedens" and "subjectum", "ex ... sequitur" and "est", "consequens" and "praedicatum", as formulated e.g. in C.518: "Semper igitur praedicatum seu consequens inest subjecto seu antecendenti et in hoc ipso consistit natura veritatis in universum". According to what has just been said, the strange expressions $(A = 0)$ est $(B = 0)$ or A est_2 B are well formed formulas of "the calculus of strokes", the corresponding equivalent formulas of L_1 being $(A = 0)$ est_2 $(B = 0)$ and A est B. In brief: Leibniz's synonymy is a particular case of the phenomenon of parallelism (as to the explicit use of an expression of the form A est_2 B, where A, B are concept constants, cf. C.259: "*Sequitur* (vel infertur) A ex B (...). Per A (aut B) hic intelligo vel terminum vel enuntiationem"). In view of the fact that $\vdash_1 N(A \neq 1 \rightarrow B \neq 1) \longleftrightarrow B \leqslant A$, the synonymy can also be explained in terms of the operator V, but we shall not dwell on this point.

As remarked earlier, in GI Leibniz emphasizes that "A autem B significare possunt terminos, vel propositiones alias" (C.365).

By virtue of the considerations developed in the preceding sections, it is clear that the sentences which can be substituted for the concept constants A, B, ..., occurring in the formulas of the calculus of concepts, are by no means arbitrary, but they are exactly $A \neq 1$, $B \neq 1$, ... (or $A = 0$, $B = 0$, ...). We also know that, as a result of such a substitution, the symbol *est* acquires the meaning of strict implication. In this way there arise many functions from the set F to the set F_1. We shall precisely describe one of them, based on the operator V.

To begin with, we define an auxiliary function $f_0\colon T \longrightarrow F_1$ as follows: Let $t(A_1,\ldots,A_n)$ be any term. Then $f_0(t)$ is the formula $t_*(A_1 \neq 1,\ldots,A_n \neq 1)$. The Leibnitian transformation of Boolean formulas into modal formulas is the function $f\colon F \longrightarrow F_1$ given by the following rules (for simplicity we assume that all atomic formulas of L are of the form $t_1 \leqslant t_2$, $t_1,t_2 \in T$):

1. If α is of the form $t_1 \leqslant t_2$, then $f(\alpha)$ is the formula

$$N(f_0(t_2) \rightarrow f_0(t_1)).$$

2. If α is of the form $\sim\beta$, then $f(\alpha)$ is the formula $\sim f(\beta)$, and similarly for the rest of logical constants.

For example, in Leibniz' notation, f assigns to A est B the formula ex A est verum sequitur B est verum.

From Leibniz's duality principle and from the fact that

$$\vdash_1 N(A \neq 1 \rightarrow B \neq 1) \longleftrightarrow B \leqslant A,$$

it follows that $\vdash_1 \alpha \longleftrightarrow f(\alpha)$ for any $\alpha \in F$. In particular, theorems of the calculus of concepts are mapped into theorems of the modal calculus of concepts.

INGEMUND GULLVÅG

THE LOGIC OF CONDITIONS

1. Introduction

The logic of necessary or sufficient conditions has been investigated by a number of authors, including C.D. Broad, G.H. von Wright, K.E. Tranøy and A. Pasch. The first systematic discussion of it was Broad's "The Principles of Demonstrative Induction", I, **Mind** (1930).

In **A Treatise on Induction and Probability** (1952), von Wright gives the following definitions:

> That (the property denoted by) A is a Sufficient Condition of (the property denoted by) B means that whenever A is present, then B is also present....

and

> That (the property denoted by) A is a Necessary Condition of (the property denoted by) B means that whenever B is present, then A is also present....

Let "ASB" mean that (the property denoted by) A is a sufficient condition of (the property denoted by) B; and correspondingly for "ANB", mutatis mutandis. We may render von Wright's definitions formally as follows:

(D1) $$ASB =_D (x)\,(Ax \supset Bx)$$

and

(D2) $$ANB =_D (x)\,(Bx \supset Ax).$$

These definitions have the result that denials of statements about sufficient or necessary conditions involve existential commitments. Let "$A\bar{S}B$" ("$A\bar{N}B$") mean that (the property denoted by) A is not a sufficient (necessary) condition of (the property denoted by) B.

From (D1) we get

$$A\bar{S}B = -(x)\,(Ax \supset Bx) = (\exists x)\,(Ax \;\&\; \bar{B}x)$$

and from (D2)

$$A\bar{N}B = -(x)\,(Bx \supset Ax) = (\exists x)\,(Bx \;\&\; \bar{A}x).$$

In **Logic and the English Language**, Alan Pasch takes the notions of necessary and sufficient conditions as basic. Let P, Q, R, ... stand for nominalizations such as "being a man", "being a mammal", "being mortal". "PSQ" ("PNQ") may be read as "being a P is a sufficient (necessary) condition for being a Q". Pasch says that negations of these expressions, "$P\bar{S}Q$" and "$P\bar{N}Q$", involve no existential commitments. But he does not analyze the notions of necessary and sufficient conditions further. An obvious way of avoiding existential commitments here would be to regard PSQ and PNQ as modal statements.

2. Conditions and modalities: a first attempt

The first and most obvious attempt to relate the notions of necessary and sufficient conditions to modal logic is simply to put an L in front of von Wright's definitions:

$$PSQ =_D L(x)\,(Px \supset Qx)$$

$$PNQ =_D L(x)\,(Qx \supset Px).$$

This gives us interdefinability of S and N:

$$PSQ = QNP.$$

And it makes S and N reflexive, as von Wright requires, since, in standard modal systems,

$$L(x)\,(Px \supset Px)$$

is a thesis, and this is equivalent with PSP as well as PNP. As here defined, S and N are also transitive, as von Wright has them; for, in standard modal systems the following arguments are valid:

$$\begin{array}{l} PSQ\text{: } L(x)\,(Px \supset Qx) \\ QSR\text{: } L(x)\,(Qx \supset Rx) \\ \hline PSR\text{: } L(x)\,(Px \supset Rx) \end{array} \qquad \begin{array}{l} PNQ\text{: } L(x)\,(Qx \supset Px) \\ QNR\text{: } L(x)\,(Rx \supset Qx) \\ \hline PNR\text{: } L(x)\,(Rx \supset Px) \end{array}$$

These definitions of S and N also avoid existential commitments of negative condition-statements:

$$P\bar{S}Q = -L(x)\ (Px \supset Qx)$$
$$= M(\exists x)\ (Px \ \&\ -Qx).$$

This merely says that it is possible for something to be a P without being a Q. Analogously for N:

$$P\bar{N}Q = -L(x)\ (Qx \supset Px)$$
$$= M(\exists x)\ (Qx \ \&\ -Px)$$

says that it is possible for something to be a Q without being a P.

If we assume that the definitions of S and N are grafted on to a modal system at least as strong as T, we get other theses:

$$PSQ = (x)\ (Px \supset Qx) \qquad \text{(from A5 in T),}$$
$$PSQ = \bar{Q}S\bar{P}$$
$$PNQ = \bar{Q}N\bar{P}.$$

The inference-rules called (by Pasch) "S-denial" and "N-denial" become valid:

<u>S-denial</u>:

$P\bar{S}Q$: $-L(x)\ (Px \supset Qx) = M(\exists x)\ (Px \ \&\ -Qx)$
QNR: $L(x)\ (Rx \supset Qx) = L(x)\ (-Qx \supset -Rx)$

$P\bar{S}R$: $-L(x)\ (Px \supset Rx) = M(\exists x)\ (Px \ \&\ -Rx)$

Proof:

{1}	1. $M(\exists x)\ (Px \ \&\ -Qx)$	P
{2}	2. $L(x)\ (-Qx \supset -Rx)$	P
{1,2}	3. $M((\exists x)\ (Px \ \&\ -Qx) \ \&\ (x)\ (-Qx \supset -Rx))$	1,2 by T
{1,2}	4. $M((\exists x)\ (Px \ \&\ -Rx))$	3, by PC and T

q.e.d.

<u>N-denial</u>:

$P\bar{N}Q$: $-L(x)\ (-Px \supset -Qx) = M(\exists x)\ (-Px \ \&\ Qx)$
QSR: $L(x)\ (Qx \supset Rx) = L(x)\ (Qx \supset Rx)$

$P\bar{N}R$: $-L(x)\ (-Px \supset -Rx) = M(\exists x)\ (-Px \ \&\ Rx)$

This is proved in the same way as S-denial.

Further, the rules called (by Pasch) "Sufficient-Condition Satisfied" (SCS) and "Necessary Condition Not Satisfied" (NCN) become valid:

<u>SCS</u>:

$$\begin{array}{ll} \text{PSQ:} & L(x)\ (Px \supset Qx) \\ \text{Pa:} & Pa \\ \hline \text{Qa:} & Qa \end{array}$$

<u>NCN</u>:

$$\begin{array}{ll} \text{PNQ:} & L(x)\ (Px \supset Qx) \\ -\text{Pa:} & -Pa \\ \hline -\text{Qa:} & -Qa \end{array}$$

But we do *not* get the rules that Pasch calls "Premiss Introduced and Eliminated" (PIE). This rule is

(PIE) (P & Q)SR ⊢ If P is the case, QSR.

If we reconstruct this in terms of propositions, we get

$$L((p \,\&\, q) \supset r) \vdash (p \supset L(q \supset r))$$

which is not valid in T. And if we analyze it in terms of modal predicate logic, we get

$$\begin{array}{l} L(x)\ ((Px \,\&\, Qx) \supset Rx) \\ \quad Pa \\ \hline L(x)\ (Px \supset Qx) \end{array}$$

which is not valid either. Hence our first and most obvious attempt at a modal analysis of the theory of necessary and sufficient conditions in Pasch's version begins to seem inadequate.

And we encounter other problems when we look at Pasch's "Principles of Non-Triviality". Let us see what these principles amount to in our reconstruction:

$$PSQ \vdash P\overline{SQ}$$

becomes

$$L(x)\ (Px \supset Qx) \vdash -L(x)\ (Px \supset -Qx).$$

This is to say that something is a sufficient condition of something only if it is consistent or satisfiable, for what this principle rules out is

$$L(x)\ (Px \supset Qx)\ \&\ L(x)\ (Px \supset -Qx),$$

i.e.,

$$L(x)\ (Px \supset (Qx\ \&\ -Qx))$$

or

$$L(x)\ -Px$$

or

$$-M(\exists x)\ Px.$$

$$PSQ \vdash \overline{PSQ}$$

becomes

$$L(x)\ (Px \supset Qx) \vdash -L(x)\ (-Px \supset Qx).$$

This is to say that something *has* a sufficient condition only if it is not a condition that is necessarily satisfied by everything; for this principle rules out

$$L(x)\ (Px \supset Qx)\ \&\ L(x)\ (-Px \supset Qx),$$

i.e.,

$$L(x)\ Qx.$$

These requirements restrict the conditions or properties denoted by P, Q etc. in S- and N-contexts, to *contingent* conditions or properties: neither tautologous nor contradictory. Hence properties necessarily satisfied by everything, as well as properties necessarily satisfied by nothing, are excluded from such contexts, for they would violate the requirements of non-triviality.

$$PNQ \vdash P\overline{NQ}$$

and

$$PNQ \vdash \overline{PN}Q$$

are easily seen to be equivalent each with one of the previous principles of non-triviality.

These principles, then, amount to requirements of contingency for the conditions or properties which may occur in S- and N-contexts. Assuming that expressions like "Pa", "Qb" etc. are substituted for variables in theses in T, the principles of non-triviality amount to a restriction on the rule TR 1, of substitution for variables, in T. And we cannot use the rule of Necessitation, TR 3, without restriction. For we have, for example, $(x)\ (Px \supset (Qx \vee -Qx))$ as a truth in predicate logic, and from this we get by TR 3

$$L(x)\ (Px \supset (Qx \vee -Qx))$$

i.e.,

$$PS(Q \vee \bar{Q}).$$

But we also have (x) (−Px ⊃ (Qx v −Qx)) as a logical truth; hence by TR 3 we get

$$L(x)\ (-Px \supset (Qx \vee -Qx)),$$

i.e.,

$$\bar{P}S(Q \vee \bar{Q}).$$

But these two theses conflict with one of the principles of non-triviality.

So our attempted reconstruction breaks down. We shall make another attempt which will give us a rule corresponding to (PIE), but the principles of non-triviality only in weakened versions.

3. Conditions and modalities; a second attempt

A. Introduction

In the following, I shall first develop a system where the conditions considered are states of affairs or propositions.

Let "$N_q p$" mean "(the state of affairs) p is a necessary condition for (the state of affairs) q", or the "necessary conditions of q require that p". "$N_q{-}p$" will mean that the necessary conditions of q exclude p, or: not-p is a necessary condition for q. "$-N_q{-}p$" will mean that the necessary conditions of q do not exclude p, or, in other words, p is compatible with the necessary conditions of q. Of course, p is compatible with all the necessary conditions of q if and only if p is compatible with q. Let "$M_q p$" mean that p is compatible with the necessary conditions of q, or, briefly, p is compatible with q. We have

(Def M) $$M_q p =_D -N_q{-}p.$$

Assuming substitution for variables in theses, and that logically equivalent expressions can be exchanged in N- and M-contexts, we have

(1) $$N_q p = -M_q{-}p,$$

that is, p is a necessary condition of q if and only if not-p is incompatible with the necessary conditions of q.

"$N_p{-}p$" will mean that not-p is a necessary condition for p, hence in order for p to be the case, not-p must be the case; hence the necessary conditions of p are inconsistent: p is necessarily false.

"$N_{-p}p$" will mean that p is a necessary condition for not-p; hence in order for not-p to be the case, p must be the case. Hence the necessary conditions of not-p are inconsistent: p is necessarily true.

"$M_p p$" or, equivalently, "$-N_p-p$", means that p is not necessarily false, i.e., p is consistent.

It seems reasonable to assume

(2) $$M_q p = M_p q,$$

that is, p is compatible with the necessary conditions of q if and only if q is compatible with the necessary conditions of p; or, briefly, p is compatible with q if and only if q is compatible with p.

By (Def M), substitution and PC, we get from (2),

(3) $$N_q p = N_{-p}-q,$$

that is, p is a necessary condition of q if and only if not-q is a necessary condition of not-p.

p is a sufficient condition of q if and only if q is a necessary condition of p:

(Def S) $$S_q p =_D N_p q.$$

The following seem plausible as theses for N and M:

(4) $$N_p(q \supset r) \supset (N_p q \supset N_p r)$$

(5) $$N_p q \supset (M_p r \supset M_p(q \& r))$$

(6) $$N_p(q \& r) = (N_p q \& N_p r)$$

(7) $$M_p(q \vee r) = (M_p q \vee M_p r)$$

(8) $$(N_p q \vee N_p r) \supset N_p(q \vee r)$$

(9) $$M_p(q \& r) \supset (M_p q \& M_p r).$$

From (2) and (7) we get

(10) $$M_{(q \vee r)} p = (M_q p \vee M_r p).$$

Whatever is compatible with something is compatible with its necessary conditions:

(11) $$M_p q \supset (N_p r \supset M_r q).$$

From (11) we get, by substitution,

(12) $$M_q q \supset (N_q p \supset M_q p),$$

i.e., the necessary conditions of a consistent proposition are compatible with it, that is, consistent. From (12) follows

(13) $$M_q q \supset (N_q p \supset -N_q -p).$$

From (13) we get, by (3) and substitution,

(14) $$M_{-q} -q \supset (N_p q \supset -N_{-p} q),$$

and, by PC and (Def M),

(15) $$(N_p q \ \&\ N_{-p} q) \supset N_{-q} q,$$

i.e., a proposition is a necessary condition of contradictory propositions only if it is necessarily true. (13) and (14) give us the following rules, reminiscent of the Principles of Non-Triviality in Pasch's system:

$$(M_q q \ \&\ N_q p) \vdash -N_q -p$$

$$(M_{-q} -q \ \&\ N_p q) \vdash -N_{-p} q.$$

By (Def S) we get from these

$$(M_q q \ \&\ S_p q) \vdash -S_{-p} q$$

$$(M_{-q} -q \ \&\ S_q p) \vdash -S_{-q} p.$$

From (11) we get

(16) $$(N_p q \ \&\ N_q r) \supset N_p r,$$

that is, N is transitive. From (16) we get

(17) $$(S_p q \ \&\ S_q r) \supset S_p r,$$

i.e., S is transitive. (16) and (17) give us transitivity rules for S and N.

N and S are also reflexive:

(18) $$N_p p$$

(19) $$S_p p.$$

The necessary conditions of a state of affairs are satisfied if that state of affairs is realized:

(20) $$N_p q \supset (p \supset q).$$

From (20) we get, by (Def S) and PC,

(21) $$S_p q \supset (q \supset p).$$

From (20) we get, by PC,

(22) $$N_p q \supset (-q \supset -p).$$

From (21) and (22) we get rules corresponding to Pasch's Sufficient Condition Satisfied and Necessary Condition Not Satisfied:

$$\frac{N_p q \quad -q}{-p} \qquad\qquad \frac{S_p q \quad q}{p}$$

If p is the case and r is compatible with (the necessary conditions of) q, then (p & r) is compatible with q:

(23) $(p \,\&\, M_q r) \supset M_q(p \,\&\, r).$

(23) is equivalent with

(24) $N_{p\&q} r \supset (p \supset N_q r)$

and

(25) $S_p(q \,\&\, r) \supset (q \supset S_p r).$

(25) gives us a derived rule of inference corresponding to Pasch's Premiss Introduced and Eliminated:

(PIE) $S_p(q \,\&\, r) \vdash (q \supset S_p r).$

In this system, we do not need to avoid a Rule of Necessitation (corresponding to TR 3 of T). We have, for any p,

(N) $\vdash \alpha \rightarrow \vdash N_p\alpha,$

i.e., a logical truth is a necessary condition of anything. This is consistent with (13) and (14) and hence with the weakened versions of Pasch's Principles of Non-Triviality. By (N) we get

$$\vdash \alpha \rightarrow \vdash (N_p\alpha \,\&\, N_{-p}\alpha)$$

and by (15)

$$\vdash \alpha \rightarrow \vdash N_{-\alpha}\alpha,$$

which simply says that if it is a thesis then it is also a thesis that is necessarily true.

From (N) and (4) we get a derived rule,

(RN) $\vdash (\alpha \supset \beta) \rightarrow \vdash (N_p\alpha \supset N_p\beta),$

i.e., whatever is entailed by a necessary condition of something, is itself a necessary condition of it. From (RN) we get, by (Def M) and PC, another derived rule,

(RM) $\vdash (\alpha \supset \beta) \rightarrow \vdash (M_p\alpha \supset M_p\beta).$

Further, by (2) we get

(RM') $$\vdash (\alpha \supset \beta) \rightarrow \vdash (M_{\alpha}p \supset M_{\beta}p),$$

that is, whatever is compatible with (the necessary conditions of) some state of affairs, is compatible with (the necessary conditions of) its logical consequences.

By (Def S), we get from (RN),

(RS) $$\vdash (\alpha \supset \beta) \rightarrow \vdash (S_{\alpha}p \supset S_{\beta}p),$$

i.e., a sufficient condition of some state of affairs is a sufficient condition of its logical consequences as well.

Another possible candidate for a thesis is

(26) $$N_p(N_pq \supset q),$$

i.e., it is a necessary condition of any state of affairs p that its necessary conditions be satisfied (true).

A state of affairs q is compatible with the necessary conditions of a state of affairs p if and only if (p & q) is consistent:

(27) $$M_pq \equiv M_{p\&q}p \,\&\, q$$

If p is necessary and q is consistent, then (p & q) is consistent:

(28) $$(N_{-p}p \,\&\, M_qq) \supset M_{p\&q}p \,\&\, q.$$

From this follows

(29) $$N_{p\supset q}p \supset q \supset (N_{-p}p \supset N_{-q}q),$$

i.e., if necessarily p implies q, then if p is necessary, so is q.

If p is necessary, then it is necessarily necessary:

(30) $$N_{-p}p \supset N_{-N_{-p}p}N_{-p}p.$$

If p is consistent, then it is necessarily consistent:

(31) $$M_pp \supset N_{-M_pp}M_pp.$$

Finally, it is compatible with a state of affairs q that a state of affairs p is compatible with a state of affairs r, if and only if p is compatible with (q & r):

(32) $$M_qM_rp = M_{q\&r}p.$$

B. Basis

A sufficient basis for this set of theses and rules is PC and the following:

(Def M)	$M_p q =_D -N_p -q$
(Def S)	$S_p q =_D N_q p$
(N)	$\vdash \alpha \rightarrow \vdash N_p \alpha$

<u>Axioms</u>:

A1	$N_p p$
A2	$N_{-p} p \supset p$
A3	$M_p q \equiv M_{p\&q} p \& q$
A4	$N_p(q \supset r) \supset (N_p q \supset N_p r)$
A5	$(N_p q \& N_q r) \supset N_p r$
A6	$N_p q \supset (p \supset q)$
A7	$(N_{-p} p \& M_q q) \supset M_{p\&q} p \& q$
A8	$M_p q \equiv M_q p$
A9	$N_p(N_p q \supset q)$
A10	$N_{-p} p \supset N_{-N_{-p}p} N_{-p} p$
A11	$M_p p \supset N_{-M_p p} M_p p$
A12	$S_p(q \& r) \supset (q \supset S_p r)$
A13	$M_q M_r p \equiv M_{q\&r} p$

C. Semantics

We interpret N and M in terms of a possible world semantics. Let Ω be a set of possible worlds, and let λ, μ, ν be variables ranging over Ω. Let "$(p)\lambda$" be short for "the state of affairs represented by p is present in the possible world λ" (or: "the proposition p is true in λ"). Let H be a triadic relation between a proposition and a pair of possible worlds. "$H_p\lambda\mu$" may be understood as "the necessary conditions of the proposition p in the possible world λ are satisfied in the possible world μ", or, in other words, "μ is compatible with the necessary conditions of p in λ". We interpret $(N_p q)\lambda$ as truth of q in every world compatible with the necessary conditions of p in λ: $(\forall\mu)\ (H_p\lambda\mu \supset (q)\mu)$, and $(M_p q)\lambda$ as truth of q in some world compatible with the necessary conditions of p in λ: $(\exists\mu)\ (H_p\lambda\mu \ \& \ (q)\mu)$. $(S_p q)\lambda$ is interpreted as truth of p in every world compatible with the necessary conditions of q in λ: $(\forall\mu)$ $(H_q\lambda\mu \supset (p)\mu)$. We assume the following conditions on the relation H:

I. (Tertium non datur). For any proposition (state of affairs) p and worlds λ, μ, μ is compatible either with the necessary conditions of p in λ or with the necessary conditions of $-$p in λ:

$$(\forall\lambda)\ (\forall\mu)\ (H_p\lambda\mu \vee H_{-p}\lambda\mu).$$

II. (Satisfaction). A possible world μ is compatible with the necessary conditions of a state of affairs p in a world λ only if p is present (p is true) in μ:

$$(\forall\lambda)\ (\forall\mu)\ (H_p\lambda\mu \supset (p)\mu).$$

By substitution we get

$$(\forall\lambda)\ (\forall\mu)\ (H_{-p}\lambda\mu \supset (-p)\mu);$$

and from this and I we get

$$(\forall\lambda)\ (\forall\mu)\ ((p)\mu \supset H_p\lambda\mu).$$

Hence from I and II follows

$$(\forall\lambda)\ (\forall\mu)\ (H_p\lambda\mu \equiv (p)\mu).$$

I and II also entail a principle of Limited Reflexivity:

$$(\forall\lambda)\ (\forall\mu)\ (H_p\lambda\mu \supset H_p\mu\mu).$$

III. (Compatibility with subset of conditions). A possible world is compatible with a set (conjunction) of necessary conditions only if it is compatible with each subset of the conditions:

$$(\forall\lambda)\ (\forall\mu)\ (H_{p\&q}\lambda\mu \supset H_p\lambda\mu).$$

IV. (H-Transitivity). If a world ν is compatible with the necessary conditions of a state of affairs q in a world μ compatible with the necessary conditions of a state of affairs p in a world λ, then ν is compatible with the necessary conditions of (p & q) in λ:

$$(\forall\lambda)\ (\forall\mu)\ (\forall\nu)\ (H_p\lambda\mu\ \&\ H_q\mu\nu) \supset H_{p\&q}\lambda\nu).$$

A4 is made valid by the interpretation of N. Principle II (Satisfaction) makes A1, A3, A5, A6 and A7 valid. I (Tertium non datur) and II make A2, A8, A9, A10 and A11 valid. II and IV (H-Transitivity) make A12 valid. And I, II, III (Compatibility with subset of conditions) and IV make A13 valid.

4. Constraints

A. Introduction

Let us consider Barwise & Perry's notion of *constraint* (in **Situations and Attitudes** (1983)). Constraints are said to be systematic relations of a special sort between different types of situations; e.g., every woman is a human (there is a constraint to the effect that anything that is a woman is a human); no smoke without fire (there is a constraint to the effect that if there is smoke, there is fire); when the bell rings, the class ends (there is a constraint to the effect that the class ends if the bell rings).

Constraints can be analyzed by means of the notions of necessary and sufficient conditions. Let "Con p,q" mean that there is a constraint relating the state of affairs p to the state of affairs q. We may construe this as meaning that p must be a sufficient condition of q, in some sense of "must". But we may distinguish between various senses of "must" here, and between corresponding kinds of constraints. We may take "p must be the case" to mean that p is logically or conceptually necessary, or that p is causally necessary or a natural law, or that there is a convention or rule to the effect that p. Following Barwise and Perry, we may distinguish between (logically or conceptually) necessary constraints, nomic constraints, and conventional constraints - without necessarily regarding this as an exhaustive classification of constraints.

Let "$Con_{\Box}p,q$" mean that there is a necessary constraint relating the state of affairs p to the state of affairs q; let "$Con_{L}p,q$" mean that there is a nomic constraint L relating p to q; and let "$Con_{C}p,q$" mean that there is a conventional constraint C relating p to q. In the following sections, we shall consider how these concepts may be analyzed by means of the notions of necessary and sufficient conditions.

B. Necessary constraints

Let "$\Box p$" mean that p is necessary, and "$\Diamond p$" that p is possible.

We assume the system of §3 as a basis, with the following additions:

(Def $\Box$) $\Box p =_{D} N_{-p}p.$

(Def $\Diamond$) $\Diamond p =_{D} -\Box -p.$

From (Def $\Box$) follows

(33) $$\Box-p = N_p-p.$$

From (Def $\Diamond$) follows

(34) $$\Diamond p = -N_p-p$$

and

(35) $$\Diamond p = M_p p.$$

TR 3 of T,

$$\vdash \alpha \ \rightarrow \ \vdash \Box\alpha,$$

may be derived from (N) by (Def $\Box$). Hence all tautologies and all the theses of §3, in necessitated form, are theses. Theorems include

(36) $$\Box p \supset p,,$$

(37) $$\Box(p \supset q) \supset (\Box p \supset \Box q),$$

(38) $$\Box p \supset \Box\Box p,$$

(39) $$\Diamond p \supset \Box\Diamond p.$$

Hence this system includes S5 as a part.

Due to the definitions of $\Box$ and $\Diamond$, their interpretations are:

$$(\Box p)\lambda \quad \text{iff} \quad (\forall\mu)\ (H_{-p}\lambda\mu \supset (p)\mu)$$

and

$$(\Diamond p)\lambda \quad \text{iff} \quad (\exists\mu)\ (H_p\lambda\mu \ \&\ (p)\mu).$$

But due to the principles I and II, these are equivalent with

$$(\Box p)\lambda \quad \text{iff} \quad (\forall\mu)\ H_p\lambda\mu$$

and

$$(\Diamond p)\lambda \quad \text{iff} \quad (\exists\mu)\ H_p\lambda\mu,$$

and these are equivalent, respectively, with

$$(\Box p)\lambda \quad \text{iff} \quad (\forall\mu)\ (p)\mu$$

and

$$(\Diamond p)\lambda \quad \text{iff} \quad (\exists\mu)\ (p)\mu.$$

Due to the definition

(Def $Con_\Box$) $$Con_\Box p,q =_D \Box S_q p$$

and principle I, we have the interpretation

$$(Con_\Box p,q)\lambda \quad \text{iff} \quad (\forall\mu)\ (\forall\nu)\ (H_p\mu\nu \supset (q)\nu),$$

i.e., there is a necessary constraint relating a state of affairs p to a state of affairs q in some possible world if and only if q is present in any world compatible with the necessary conditions of p in any world.

C. Nomic constraints

We assume the system of §4B as a basis (including the one of §3), with the addition of the letter "L" as a subscript to "N", "M" and "S", to denote any set of nomic constraints, or natural laws. "$N_L p$" will mean that the state of affairs represented by p is required by (is a necessary condition of) the actual nomic constraints (natural laws) L; or, briefly, there is an actual nomic constraint to the effect that p. "$N_L -p$" will mean that the state of affairs p is excluded by the actual natural laws L, i.e., the occurrence of p would be "against nature", or a miracle. "$M_L p$" will mean that the state of affairs represented by p is compatible with the actual nomic constraints (natural laws) L, hence the occurrence of p would not be miraculous.

The definitions, rules and axioms of §3 apply to N_L/M_L. Hence we have

(40) $$M_L p = -N_L -p$$

and

(41) $$N_L p = -M_L -p$$

as theorems.

As suggested, the occurrence of a state of affairs excluded by a set of actual nomic constraints L, i.e., the truth of '$N_L -p$ & p', would be a miracle. If we assume, as we shall, that genuine miracles do not occur, we reject '$N_L -p$ & p', or '$N_L p$ & $-p$', as possibilities. This amounts to accepting a further axiom in addition to those of §3:

A14 $$N_L p \supset p,$$

i.e., actual nomic constraints are inviolable. This does not preclude that what is at a certain time *regarded* as a miracle may occur, or that what are believed or postulated to be natural laws may be violated. The actual occurrence of some state of affairs p believed to be ruled out by a set of nomic constraints L will simply show that $N_L -p$ is false. Either the constraints in L are not actual, or they do not really exclude p.

We assume that actual nomic constraints do not violate necessary constrains, hence that

(42) $$\Box p \supset -M_L-p$$

holds for any set L of actual nomic constraints. We shall adopt an equivalent formula as a further axiom:

A15 $$\Box p \supset N_Lp.$$

The converse of A15 obviously does not hold: lawlikeness is a weaker notion than logical necessity. $\Box$ as well as N_L constitute squares of opposition, but the one of N_L is "inside" the one of $\Box$.

From A15 and the rule TR 3 of §4B, we derive a rule

(L) $$\vdash \alpha \rightarrow \vdash N_L\alpha,$$

i.e., necessary states of affairs are required by any set L of nomic constraints.

S4, applied to N_L/M_L, is a part of this system. '$N_Lp \supset p$' is an axiom; '$N_L(p \supset q) \supset (N_Lp \supset N_Lq)$', '$N_L(p\&q) \equiv (N_Lp \& N_Lq)$', '$M_L(p \vee q) \equiv (M_Lp \vee M_Lq)$' etc. are theorems; and so is '$N_Lp \supset N_LN_Lp$'.

Another theorem is

(43) $$N_{(I\&L)}E \supset (I \supset N_LE),$$

i.e., if a set L of actual nomic constraints in conjunction with a state of affairs I (the "initial condition"), require that the state of affairs E be the case, then if I is the case, the nomic constraints L require that E occur.

In the semantics, $H_L\lambda\mu$ will mean that the possible world μ respects (is compatible with) the nomic constraints L actual in λ, hence that μ is a causally possible world from the point of view of the natural laws L actual in λ. We assume a principle in addition to the ones of §3C:

V. (L-Reflexivity). Any possible world respects the nomic constraints actual in it:

$$(\forall\lambda)\ H_L\lambda\lambda.$$

We may now define the notion that a nomic constraint L relates two states of affairs p and q, as meaning that p is a sufficient causal condition of q, or: there is an actual nomic constraint L to the effect that p a is sufficient condition for q:

(Def Con_L) $$Con_Lp,q =_D N_LS_qp.$$

D. Conventional constraints

We assume the system of §3B as a basis, with the addition of the letter "C" as a subscript to "N", "M", and "S", to denote any set of rules or tacit expectations of some group. "$N_C p$" will mean that there is a (set of) actual conventional constraint(s) requiring or presuming that p be the case. "$M_C p$" will mean that the occurrence of p does not (would not) violate the actual nomic constraints C.

The definitions, axioms and rules of §3 apply to N_C/M_C. The axioms of §4C, A14 and A15, do not apply. Conventional constraints are not necessarily consistent, and not inviolable. Nor do they necessarily respect nomic constraints. Conventional presumptions may be (probably often are) in conflict with actual nomic constraints (cf. e.g. beliefs in magic or supernatural phenomena).

Theorems for N_C/M_C include '$N_C(N_C p \supset p)$' (i.e. there is an actual conventional constraint requiring or presuming that actual conventional constraints be respected), '$N_C(p \supset q) \supset (N_C p \supset N_C q)$', '$N_C(p \& q) \equiv (N_C p \& N_C q)$', '$M_C(p \vee q) \equiv (M_C p \vee M_C q)$', '$N_C p \supset N_C N_C p$'.

In the semantics, "$H_C \lambda \mu$" will mean that the possible world μ respects (is compatible with) the conventional constraints C actual in the possible world λ; or, in other words, μ is an ideal world with respect to the conventional presumptions C actual in λ. No principles beyond those of §3C are required.

We may define a conventional constraint relating a state of affairs p to another one q, as consisting of conventional requirement or presumption that p be a sufficient condition for q:

(Def Con_C) $$Con_C p,q =_D N_C S_q p.$$

5. Universals and Involvement

As noted, in the previous systems the conditions considered are states of affairs or propositions. But the notions of necessary and sufficient conditions are commonly applied to properties, or universals more generally. E.g., Pasch's "P" and "Q" in "PNQ", "PSQ", stand for nominalizations denoting possesion of properties, such as being a man, being mortal, being old. "PNQ" may be read, "being (a) P is a necessary condition for being (a) Q", or "having the property P is a necessary condition for having the property Q".

We may be tempted to render "PNQ", as here understood, as $(x)N_{Qx}Px$, i.e., for any x, it is a necessary condition for its being (a) Q that it is (a) P; and correspondingly for "PSQ". But this will lead to existential commitments for denials of PNQ and PSQ, which we want to avoid. We appeal to the notions of necessary, nomic and conventional constraints and define as follows:

$$\mathrm{Con}_{\Box}P,Q =_{D} \Box(x)N_{Qx}Px$$

$$\mathrm{Con}_{L}P,Q =_{D} N_{L}(x)N_{Qx}Px$$

$$\mathrm{Con}_{C}P,Q =_{D} N_{C}(x)N_{Qx}Px.$$

Obviously, denials of these do not involve existential commitments.

If we let the variables x, y, ... range over situations or events, we may take "P" and "Q" as denoting types of events or situations. We may then introduce another notion, the one of involvement. Barwise & Perry construe involvement as a relationship between types of events (situations), e.g., kissing involves touching. One type of event P involves another type of event Q if every actual event x of type P is *part* of an actual event (situation) y of type Q. This requires the notion of one event's or situation's being part of, or extending, another event or situation. Let "Ext x,y" mean that y is an extension of x, or x is part of y. Let "Inv P,Q" mean that (the type of event) P involves (the type of event) Q. We may distinguish between necessary, nomic and conventional involvement, and define these as follows:

(Def $\mathrm{Inv}_{\Box}$) $$\mathrm{Inv}_{\Box}P,Q =_{D} \Box(x)\ S_{(\exists x)(\mathrm{Ext}\ x,y\ \&\ Qy)}\,Px$$
$$= \Box(x)\ N_{Px}\,(\exists x)(\mathrm{Ext}\ x,y\ \&\ Qy),$$

i.e., that the type of event P necessarily involves the type Q means that, necessarily, it is a necessary condition for an event's being of the type P that it is part of an event of the type Q. Analogous definitions apply to Inv_{L} and Inv_{C}, mutatis mutandis.

GERHARD HEINZMANN

PHILOSOPHICAL PRAGMATISM IN POINCARE*

At the beginning Poincaré is using the terms 'intuition' and 'analysis' in order to describe two psychological attitudes involved in the logic of invention: Riemann and Klein represent the attitude of intuition, Hermite and Weierstrass the attitude of analysis. Later on, but usually without explicit indication, these two terms stand likewise for two theories about the nature of mathematical activity: on the one hand you concentrate on investigations into the conditions governing the construction (intuition) of mathematical objects, on the other hand you try to describe (analyze) domains of already existing objects.

In arithmetic and in foundational studies - in this paper I exclude his approach to geometry - Poincaré is almost always an intuitionist. Yet, even though the philosophy of Poincaré remains on the whole an intuitionistic one, he displays analytical features, too. Hence, you find a solution of the problem of predicativity not only by providing limitations to the domains under discussion (these are procedures on the level of construction) but also by providing restrictive clauses with respect to the rules of quantification (these are procedures on the level of description), i.e. prima facie without limiting the domains of quantification. Certainly, here, these two levels correspond and you even arrive at logically equivalent presentations, if you translate the underlying ideas into a formal language. In the first case you end up with an 'exclusive' theory of types, in the second case with a cumulative one.

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* This paper, written originally in French, was delivered as a lecture at the University of Aix-en-Provence in October 1985; it derives from an attempt to improve on chapter II of my book: **Entre intuition et analyse. Poincaré et le concept de prédicativité**, Paris 1985.

However, I want to argue for the claim that the modern idea of a unification of the approach on the level of construction and of the approach on the level of description, these levels being two inseparable aspects of the common ground of actions, can find one of its roots in the philosophical pragmatism of Poincaré. Nevertheless, an attempt of reducing Poincaré's pragmatism to its role as a forerunner of a philosophy conducted in a pragmatist framework, would be a biased approach. Poincaré was first of all an anti-logicist (and anti-formalist), afterwards, as a reaction, he became an intuitionist, and only by retrospection he is found as being placed between 'the methodological fronts'.

In two consecutive papers under the common title 'Les mathématiques et la logique', published in 1905 and 1906[1], Poincaré argued against the logicist claim of being able to "démontrer toutes les vérités mathématiques ... une fois admis les principes de la logique"[2]. For this would mean to give up either the analytical nature of logic[3] or the synthetical nature of mathematics, i.e. to advocate a solution to the problem of defining the relation between logic and mathematics which rests basically on a Leibnitian tradition. Poincaré suspects that during the centennials in honour of Kant's *death* an equivocation of the term 'logic' is introduced such that 'logic' does not exclusively refer to traditional logic but to a 'new logic' which comprises both synthetic principles of demonstration and the formation of non-logical concepts.

Poincaré sees very clearly here. Predicate logic is not only richer than traditional logic which Kant referred to but in order to deal with reductionism one should also take into account that in the 'new logic' more freedom with respect to certain (set theoretic) existence postulates prevails. For example, is it really an analytical procedure (of the second order) to turn predicates into names and afterwards affirm their existence, i.e. the existence of entities signified? It is obvious that the 'conditional' solution of Russell would not satisfy the constructivist, nor had Poincaré been satisfied.

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[1] Cf. Poincaré **(1905/1906)** and Poincaré **(1906)**.

[2] Poincaré **(1905/1906)** p.817.

[3] Poincaré follows Kant in calling propositions analytic when the subject-concept is contained in the predicate-concept.

But Poincare's criticism of logicism extends even further. If logicism pretends to derive all of mathematics from indefinable 'logic' by means of deduction rules and direct definitions, "il faudrait que l'on eût le moyen de démontrer qu'ils n'impliquent pas contradiction"[4]. And such a proof would have to make use of a principle of induction which is not yet available. And if it were, you would be in a vicious circle. According to Poincaré this difficulty of using complete induction in a justified manner connects Russell's logicism with Hilbert's formalism. Hence, Poincaré's reservation towards dealing in this way can be understood as an anticipation of the constructivist attitude of the intuitionists and of the descriptive attitude of the analysts: on the one hand he is in accord with Brouwer's refusal to distinguish mathematics from metamathematics, on the other hand he acts in foresight - taken in a very large sense - of difficulties articulated precisely and confirmed by Gödel in the thirties.

Of course, it is not necessary to look at his criticism on the methodological level only. For, the reductionist programme of logicism seems to waver already by the antinomies occurring in the new logic. If, Poincaré says, "la logistique n'est plus stérile, elle engendre l'antinomie"[5], this has become possible because you have tacitly relied on a false intuition. A 'true' intuition can be distinguished from simple evidence by the fact that it refers to what can be done instead of merely do something that is. So, the certainty with respect to complete induction taken as a synthetic judgement a priori, derives from the fact that it is the affirmation of a direct intuition into the capacity of the mind to comprehend the indefinite repetition of one and the same act.

We would say today that such an intuition obtains with respect to a schema (of an action) which is 'pure' (or a 'forme'), because it is not generated but only represented by indefinite repetition, and that it is called 'intuitive' because it cannot be determined conceptually, but only by singular actualization. It is possible to trace a platonist feature within this interpretation of Poincaré's inasmuch as the intuitive schema of construction counts as a 'forma intelligibilis' which, like the corresponding 'forma sensibilis' in Kant, precedes its actua-

[4] Poincaré (1905/1906) p.829.

[5] Poincaré (1906) p.316.

lization rather than having been derived from them by 'purification'.

But, disregarding this aspect of pure intuition, Poincaré is anti-platonist from about 1909 onwards. The antinomies are for him in the last resort a necessary consequence of the erroneous method of conceptual realism to invoke intuition with respect to abstract entities. For explication I want to examine Poincaré's conceptual interpretation of explicit definition of a set. The existence conditions contained in it allow - according to Poincaré - conclusions concerning the formation of classes, or pseudo-definitions, which are impredicative and, therefore, responsible for antinomies.

According to Poincaré explicit definitions of a set follow two procedures: "soit par *genus proximum et differentiam specificam* soit par construction"[6]. These two methods mirror the dispute between realists and nominalists taken up again by Poincaré with the terms 'cantorians' and 'pragmatists', the former acting from the point of view of intension, the latter from the point of view of extension:

> "Si on se place au point de vue de l'extension, une collection se constitue par l'adjonction successive de nouveaux membres; nous pouvons en combinant les objets anciens construire les objets nouveaux, puis avec ceux-ci des objets encore plus nouveaux ... Au point de vue de la compréhension au contraire, nous partons de la collection où se trouvent les objets préexistants, qui nous apparaissent d'abord comme indistincts, mais nous finissons par reconnaître quelques-uns d'entre eux parce que nous y collons des étiquettes et que nous les rangeons dans les tiroirs; mais les objets sont antérieurs aux étiquettes, et la collection existerait quand même il n'y aurait pas le conservateur pour les classer."[7]

Poincaré places the logicists - they are in his eyes followers of Peano or Russell - together with the adherents of Cantor and treats himself as belonging to the pragmatists.

Though the term 'pragmatist' seems to have been chosen by Poincaré rather accidentally - he introduces it with the words

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[6] Poincaré (1912) p.5.

[7] Ibid., p.4.

"il faut bien ... donner un nom"[8] – it does characterize the interpretation of nominalism by Poincaré very luckily; in refusing to start with an analysis of domains considered to be already existing, he is not any more content with merely a synthesis of elements to be constructed. To have useful constructions one should rather follow them with a descriptive analysis of the constructions themselves. In this sense we have here a reconciliation of the two methods: they follow one another as aspects of an ordered sequence and thus characterize the pragmatic spirit as started by Poincaré:

> "On a attaché, et à juste titre, une grande importance à ce procédé de la 'construction' et on a voulu y voir la condition nécessaire et suffissante de progrès des sciences exactes. Nécessaire, sans doute, mais suffissante, non. Pour qu'une construction puisse être utile, ... il faut d'abord qu'elle possède une sorte d'unité, qui permette d'y voir autre chose que la juxtaposition de ses eléments. Ou plus exactement, il faut qu'on trouve quelque avantage à considérer la construction plutôt que ses éléments eux-mêmes."[9]

Only a *statement* expressing an 'analogy' among constructions will lead to a level of abstraction afterwards where the analogous objects can be identified:

> "Une construction ne devient donc intéressante que quand on peut la ranger à côté d'autres constructions analogues formant les espèces d'un même genre."[10]

The 'analytic' feature here pertains to the means of construction. Hence, analysis is not any more the traditional inverse of synthesis or "une marche du général au particulier", because the constructions are obviously not regarded as something more special than their elements. In this sense "la mathématique est l'art de donner le même nom à des choses differentes",[11] and not by their form but by their content. Yet, Poincaré did not

[8] Ibid, p.2.

[9] Poincaré (1902), p.44.

[10] Loc. cit., ibid.

[11] Poincaré (1908), p.29.

know how to use the means now at his disposal in order to formulate a principle of abstraction for the genus. By "gravissant un ou plusieurs échelons" he proceeds directly to propositions which express a property of the genus: the statement of analogy is now complete induction which serves to "démontrer les propriétés du genre sans être forcé de les établir successivement pour chacune des espèces"[12].

Pragmatists, let us repeat, are not realists. They forbid, so to speak, to read the arbor porphyriana from top to bottom, i.e. to consider the "genre ... antérieur à l'espèce"[13] and to stop at an abstract level. Hence, a definition which does not define an individual but a whole genus is incomplete,[14] because individuation cannot be derived logically from the abstract unit (of a genus):

> "La connaissance du genre ne ... fait pas connaître tous ses individus, elle ... donne seulement la possibilité de les construire tous, ou plutôt d'en construire autant que vous voudrez. Ils n'existeront qu'après qu'ils auront été construits, c'est-à-dire après qu'ils auront été définis."[15]

Even though the terms used by Poincare are traditional, he is giving them a non-traditional meaning: extension and intension, e.g., appear only as metapredicates (on predicates). For, to define a genus by means of a predicate is eo ipso an intensional procedure, whereas extension is not connected with a predicate but with a way of construction.

If you restrict definition of a set to its being an abstract entity, you are deprived of the constructive aspect of definition, which - in the eyes of the followers of Cantor - counts as an artificial restriction. Following this way the guaranty of individual existence of the elements of a set is substituted by a proof of consistency. For the pragmatist, on the other hand, a direct definition which you get by following the inverse method of the Cantorians, can be 'corrected': you supplement it by a second part which replaces the postulation of

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[12] Poincaré (1902), p.44/45.

[13] Poincaré (1906), p.317.

[14] Cf. Poincaré (1912), p.5.

[15] Poincaré (1912), p.7.

an abstract entity as the reference of the genus. You have to "sous-entendre l'ensemble des individus qui satisfont à la définition"[16]. Since generality is - from the point of view of extension - individual or numerical universality, in that second part of the definition the *differentia specifica* has to lead directly to elements of first level; without such a supplement a proposition about all elements of a set "n'aurait aucun sens" and the object of the proposition would be unthinkable[17]. In a paper of 1912 the expression 'aucun sens' acquires even a philosophical value. A pragmatist uses 'having a sense' with respect to a definition, and that means existence of veridical instantiations, together with consistency as a criterion for the admissibility of a definition. He uses, so to speak, a restriction 'from below' to the effect that the means employed for the transition from finite to infinite do not go beyond the 'legal' one: the complete (non-transfinite) induction.

Only against the background of such ideas it is possible to understand Poincaré's refusal of impredicative definitions which is a logical consequence already within his system; for, if classes were considered to be real objects existing independently the definition of their members, an impredicative definition would not be circular. But it is exactly this attitude of Platonism which Poincaré attacks, and which he considers to be responsible for the antinomies; here he reacts already like Herman Weyl who wrote some twenty years later:

> "Als Wurzel der Antinomien vermag man aber nur die schon von Anfang an in der Mathematik begangene Kühnheit aufzudecken: dass ein Feld konstruktiver Möglichkeiten als geschlossener Inbegriff an sich seiender Gegenstände behandeldt werde."[18]

This is the essential difference between the pragmatists and the followers of Cantor. The axiom-schema of comprehension (Zermelo's 'Aussonderungsaxiom')

$$\bigvee y \bigwedge x \; (x \in y \leftrightarrow x \in a \wedge \varphi(x))$$

is not always representable, and, hence, not admitted by Poincaré, even though it might be consistent. In assuming a set *a*

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[16] Ibid., p.5.

[17] Cf. Poincaré (1909b), p.479.

[18] Weyl (1976), p.71.

beforehand, Poincaré says, Zermelo "a élevé un mur de clôture qui arrête les gêneurs qui pourraient venir du dehors. Mais il ne se demande pas s'il ne peut pas y avoir des gêneurs du dedans qu'il a enfermés avec lui dans son mur"[19]. Whenever Poincaré's pragmatism has been associated with an utilitarian philosophy or with a pragmatic philosophy in the sense that a technician uses results provided their consistency will eventually be proved, it finally developed by using the formulation of 1912, into an intuitionism of principle. Considerations of consistency, be they conducted logically (analytically) or not, are not any more sufficient unless they are supplemented by processes of verification, i.e. by a concrete model. hence, in this context, the verification is, using Kantian terminology, a construction of concepts through instantiation in senuous intention.

Adopting the pragmatic reinterpretation of the Cantorian way of definition the terms 'analysis' and 'construction' are not only related to the strictly Cantorian way of definition on the one hand and its pragmatic complement on the other hand, but must be understood as referring to two aspects even within the level of determining the individuals. Poincaré's identification of 'construction' with 'definition'[20] signifies that the reduction of the *definiens* to a *definiendum* is not effected by means of language alone. In constructing the individuals of a genus, language is in the beginning only an aspect of an action governed by pragmatic norms. Language is its symbolic (analytic) aspect which alone permits to understand an actual construction of individuals as actualizations of an (intuitive) schema of construction, i.e. of a rule. In this semiotic sense language and construction are two inseparable features for the pragmatist. We suddenly find ourselves in the tradition of the philosophy of the later Wittgenstein, where language has lost its role of being something available on the metalevel with respect to the level of objects.

Of course, to look at Poincaré in this manner transcends the given texts, but it confirms and explains two further aspects in Poincaré's writings.

At first a remark about the theory of types of Bertrand Russell, which refers to the hypothetical admissibility of transfinite ordinals as indices of types in order to distinguish them.

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[19] Poincaré (**1909b**), p.477.

[20] Cf. quotation above at footnote 15.

Such a theory of types, Poincaré says, remains incomprehensible as long as as the theory of ordinals is not yet set up[21]. Poincaré is demanding a simultaneous reflection both about the constructed objects and about the linguistic tools used. Much later, the logicians G. Kreisel and S. Feferman tried to fulfil this demand by proceeding from predicative well-orderings - those which are of order type $< \omega_1$ - to orderings predicatively well-ordered, i.e. orderings of order-type $< \Gamma_0$ which can predicatively be ascertained as being a well-ordering[22].

A second corroboration of the proposed interpretation can be found in the linguistic turn given by Poincaré to the pragmatic identification of ontology with epistemology: to a pragmatist an individual "n'existe que quand il est pensé ... d'un sujet pensant" and only when it can be defined by "un nombre fini de mots". An a concept which cannot be defined in such a finite way is inadmissible because it cannot be conceived[23]. It, therefore, seems to be adequate to say with A. Heyting that definability with finitely many words signifies (in the linguistic sense: 'is a sign for') finite constructibility[24]. All elements of a genus have to participate in this finite constructibility. This point of view finds another confirmation when Poincaré takes issue with a claim by Schoenflies that finite definability and constructibility should be made independent from each other. The set of constant functions serves as a neat example:

> "Quand on dit 'une fonction constante', on a une formule d'un nombre fini de mots et qui s'applique à une infinité de fonctions; mais qui ne les définit pas ... Il n'est donc pas exact de dire que cette formule définit en un nombre fini de mots un ensemble de fonctions."[25]

Since definition of a set enforces 'knowledge' of all its members, the definition of an infinite set - infinite not in the sense of potentially infinite but as actually infinite - by one

[21] Cf. Poincaré (1909b), p.469.

[22] Cf. Kreisel (1960) and Feferman (1964).

[23] Cf. Poincaré (1909b), p.482 and Poincaré (1912), p. 9/10.

[24] Cf. Heyting (1934), p.4.

[25] Poincaré (1909a), p.195/196.

and the same formula containing only finitely many words is impossible:

> "Et en effet ce qui caractérise précisément une définition, c'est qu'elle permet de distinguer l'objet defini de tous les autres objets; si elle s'applique à une infinité d'objects, elle ne permet pas de les discerner les uns des autres; elle n'en définit aucun; elle n'est plus une définition."[26]

From the point of view of extension, the infinite is something in development and never a closed totality. It was Borel who made the first precisification of Poincaré's vague notion 'one could enumerate by distinguishing between a denumerable set and its effectively enumerable subsets': Borel takes a set to be admissible only when it is effectively enumerable, i.e. when it is possible to indicate "au moyen d'un nombre fini de mots, un procédé sûr pour attribuer sans ambiguité un rang déterminé à chacun de ses éléments"[27]. Today we know that relying alone on the concept of general recursiveness - a concept doubtlessly envisaged by Borel - unfortunately does not lead very far: a predicate defined on the natural numbers using unrestricted quantification will not, in general, even belong to the class of recursive predicates. This and the other fact that a constructive interpretation of elementary classical arithmetic is available, suggest a broadening of what Poincaré understood by the term 'pragmatism': it seems desirable to admit the totality of natural numbers and to distinguish the recursively undecidable predicates on that domain by measuring the complexity of their undecidability. The demands of pragmatism will then appear as a predicativism of second order to be applied to the denumerable infinite.

Whatever one decides, the pragmatic idea of Poincaré always satisfies what Vuillemin has called the principle of intuitionism. It refuses "à la disjonction de l'infini et du fini une validité universelle, c'est-à-dire indepéndante des conditions de l'intuition et de la construction"[28]. Thus, Poincaré belongs to the great intuitionistic stream which has started in antiquity and passed through Descartes and Kant.

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[26] Poincaré (1909a), p.195.

[27] Borel (1908), p.446/447.

[28] Vuillemin (1981), p.27.

Yet, with respect to Brouwer and his difference to Poincaré, one could equally well extend the term 'semi-intuitionism' to cover also the philosophy of Poincaré. The addition of 'semi' then marks - not taking into account the problem of justifying the principle of excluded middle - the special manner in which he conceives of the relation between intuition and analysis (or language), i.e. between construction and description; an object does not exist without being designated. Brouwer has never assigned to language such an essential function of control; for him a language remained a mere auxiliary means.

NICHOLAS DENYER

A NOTE ON ZENO B3

Εἰ πολλά ἐστιν, ἀνάγκη τοσαῦτα εἶναι ὅσα ἐστί καί οὔτε πλείονα αὐτῶν οὔτε ἐλάττονα. Εἰ δέ τοσαῦτά ἐστιν ὅσα ἐστί, πεπερασμένα ἄν εἴη.

If there are many things, it is necessary that they are just as many as they are, neither more nor less. And if they are just as many they are, they would be finite in number.

Let V be the set of things that there are. Suppose that V contains infinitely many members. We will now introduce the premiss:

(1) V contains just as many members as V, neither more nor less.

This is the only premiss to be deployed in my reconstruction that represents something explicit in Zeno's Greek. But it has at least the advantage of representing Zeno's words quite literally. Contrast the interpretation whereby Zeno contends in these crucial words that 'any plurality of things must consist of a *definite number* of things and so be finite in number'[1]. The only sense in which it follows from V's infinity that it contains no definite number of members would be that for no natural number *n* does V contain only *n* members. But this is not a thought happily framed as a denial that the members of V 'are as many as they are'; and in any case to infer this thought is merely to restate V's infinity, not to draw from it an evidently absurd conclusion. A similar contrast may be drawn also with the paraphrase of

[1] H.D.P.Lee, **Zeno of Elea** (Cambridge, 1936) p.31. Lee's interpretation seems to be shared by practically all other commentators.

Barnes, whereby Zeno reasoned thus: 'If there are many *As*, then there is some true proposition of the form: "There are as many *As* as *Bs*"'[2]. If Zeno's thought was that any plurality must contain just as many members as *another* plurality, then he was quite inept in framing it as a thought that any plurality contains just as many members as *itself*. Furthermore, so to reconstruct Zeno's argument is to leave him with what is, as Barnes himself puts it, 'an uninstructive sophism'. My more literal interpretation of these words will in the end also produce a sophism: for how could a 'proof' of falsehood be otherwise? Nevertheless I trust that it will not be entirely uninstructive.

Our premiss (1) is not of itself enough to reduce to absurdity our supposition that V contains infinitely many members. My reconstruction must therefore, like others, supply further premisses to give Zeno the semblance of a sound argument. Let me therefore supply:

(2) An infinite set is one that contains the same number of members as some proper subset of itself.

(2) may seem to have a suspiciously anachronistic air. After all, it first achieved a fully clear articulation and proof in the nineteenth century. Nevertheless, intimations of it can be found closer to Zeno's time. Some Stoics held that 'The man is not composed of more parts then the finger, nor the universe than the man; for division produces bodies to infinity, and of infinities none is greater or lesser'[3]. And it is not altogether implausible to see such ideas as present in Zeno himself. For, as the second horn of the B3 dilemma indicates, it was points on a line that above all Zeno has in mind here. And, however hard it may be to grasp (2) as a general definition of infinities, its particular application to infinities of points on a line is evident enough. For it takes but little reflection to see that if a line can be divided at an infinite number of points then the same holds of any part into which it is divided.

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[2] Jonathan Barnes, **The Presocratic Philosophers** (London, 1979) Vol.1, p.252.

[3] Plutarch **De communibus notitiis adversus Stoicos** 1079a (=SVF 2.484).

There is a third premiss we will use in our reconstruction.

(3) Each set contains more members than any proper subset of itself.

(3) is undeniably true if we restrict its application to sets of finite size, and it takes some sophistication to see that it is false when applied to infinities. If I am to spend every day from now for ever onwards in the Isles of the Blessed, and you are to spend only every other day there, then you have evident cause to envy me; and it would be only too natural for you to give the reason that I am to have more days in the Isles of the Blessed than you are. Again, it is only too natural to suppose that the line ABC can be divided at more points than can its segment AB; for the line ABC can be divided at all those points at which the segment AB can, and also at those further points at which the segment BC can be divided. The sophistication required to see the error here was in all likelihood not possessed by Zeno's contemporaries. For it seems to have been lacked by Aristotle, who in his argument against the view that infinity is a substance treats as evidently absurd its alleged consequence that the infinite has a part which is itself infinite[4]. Nor indeed is it entirely stupid to suppose that (3) is in general true. For the chief reason that we have to doubt (3) is simply that (1) and (2) are true, that V is infinite, and that (3) in conjunction with (1) and (2) entails that all sets are finite in size.

How then does it entail this? How did Zeno's argument from (1), (2) and (3) proceed? V, we are supposing, contains infinitely many members. Hence there is, by (2), a proper subset of V, containing just as many members as V itself. Call such a subset S. The number of members of V is now the same as the number of members of S. But by (3) the number of members of S is *less* than the number of members of V. So the number of members of V is less than itself. Similarly, the number of members of V is, by (3), *greater* than the number of members of S. But this is, by (2), the same as the number of members of V. So the number of members of V is greater than itself. But all this contradicts (1). We have thus reduced to absurdity our assumption that V contains infinitely many members; and if there are many things they are in consequence only finite in number.

[4] **Physics** 204a20ff (= **Metaphysics** 1066b11ff.).

ROMAN MURAWSKI

GENERALIZATIONS AND STRENGTHENINGS OF GÖDEL'S INCOMPLETENESS THEOREM

1. Historical background

In 1931 in the journal **Monatshefte für Mathematik und Physik** a short paper (a bit more than 20 pages) of an Austrian mathematician and logician Kurt Gödel was published - paper which has turned out to be one of the greatest and most important papers in mathematical logic and foundations of mathematics. Its title was "Über formal unentscheidbare Sätze der 'Principia Mathematica' und verwandter Systeme. I". In it Gödel proved that arithmetic of natural numbers and all systems containing it are essentially incomplete provided they are consistent. It means that there are sentences which are undecidable in them, i.e. sentences φ such that neither φ, nor $\neg\varphi$ are theorems. What's more, we know which sentence of the pair $\varphi, \neg\varphi$ is true in the basic model of the theory, i.e. in the model to the description of which the theory was formulated. This incompleteness is essential, i.e. it cannot be removed by adding the undecidable sentences as a new axioms because new undecidable sentences will appear (undecidable in the new, richer theory). This theorem (so called 1^{st} Gödel theorem) indicates the cognitive limitations of the deductive method. It shows that one cannot include whole mathematics in a consistent formalized system based on the first order predicate calculus - what's more in such a system one cannot even include all truths about natural numbers! There will always be undecidable sentences of the form $\forall x\, \varphi(x)$ such that all substitutions of φ, i.e. sentences $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, ... are theorems.

Gödel's results struck the program of Hilbert's formalism. Namely, Hilbert proposed a program of justifying classical mathematics. His proposal was Kantian in character. Hilbert (1925) wrote as follows:

> Kant taught - and it is an integral part of his doctrine - that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure.
>
> As a further precondition for using logical deduction and carrying out logical operations, something must be given in conception, *viz.*, certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else and which requires no reduction. This is the basic philosophy which I find necessary not just for mathematics, but for all scientific thinking, understanding and communicating. The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable"

Such concrete objects are just natural numbers considered as numerals (certain systems of symbols): 1, 11, 111, ... One can exactly describe them and relations between them. The part of mathematics talking about those objects is certainly consistent (because facts cannot contradict themselves). But in mathematics, beside such finitistic, real theorems describing concrete objects, we have also infinitistic, ideal ones talking about the actual infinity (to which no real objects correspond). And therefore mathematics needs a justification and foundations. The convincing proof of the consistency of mathematical theory ought to be a finitistic one (i.e. a proof using no ideal assumptions). Hilbert thought that such a proof was possible and proposed a program of providing it. It consisted of two steps. The first step was just the formalization of mathematics (Hilbert, first of all, thought here about arithmetics, analysis and set theory). It ought to be carried out by fixing an artifical symbolic language and rules of building in it well-formed formulas. Further axioms and rules of inference ought to be fixed (the rules could refer only to the form, to the shape of formulas and not to their sense or meaning). In such a way theorems of mathematics become those formulas of our formal language which have a formal proof based on a given set of axioms and given rules of inference. There was one condition put on the set of axioms: they ought to be chosen in such a way that they suffice to solve any problem formulated in the language of considered theory, i.e. they ought to form a complete set of axioms. The second step in Hilbert's program was now to give a proof of the consistency of mathematics. Such a proof could be carried out by finitistic methods because it was enough to consider formal proofs, i.e. sequences of symbols, and to verify if there were two sequences such that one of them finishes with formula φ and the other with formula $\neg\varphi$. If there were such proofs then mathematics would be inconsistent, if not, then it would be consistent. But the study of formal proofs deals with finite, concrete objects (namely sequences of symbols formed according to some rules) and hence is finitistic.

Gödel's theorem showed that it is impossible to build a consistent and complete system of mathematics. In particular it showed that no formal system for arithmetic of natural numbers is adequate with respect to the set of all arithmetical truths (provided it is consistent). Such systems and their methods of proving theorems are not (and cannot be) adequate with respect to the usual mathematical practice. In other words: provability it is not the same as truth - every theorem of formalized arithmetic is true but there are true sentences which are not theorems, i.e. there are sentences which are undecidable.

Gödel had shown even more in the paper mentioned above. He had announced (but not proved, promising to give the proof in the second part of the paper which was never written) a theorem stating that there cannot exist a proof of the consistency of a formalized system of arithemetic which uses only the methods of that system. This showed the unrealizability of the second step of Hilbert's program - namely of finitistic proof of the consistency.

Both Gödel's theorems were obtained with the help of a new sophisticated method of arithmetization (or gödelization, as we often call it today) of syntax. Gödel observed, namely, that one can fix a one-one correspondence between formulas of a given formalized theory (such formulas are simply sequences of basic symbols) and natural numbers. In such a way to a formula φ of the language of the arithmetic of natural numbers corresponds a natural number which we denote by $\ulcorner\varphi\urcorner$. Moreover, the correspondence can be defined in such a way that to natural syntactic relations between formulas φ, ψ etc. (e.g. to the relation of being a subformula or being a consequence) correspond some natural arithmetical relations between numbers $\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner$ etc. Hence instead of talking about formulas we can talk about numbers. In this way, if a language of a considered theory contains the language of arithmetic then one can talk in it about itself!

The method of arithmetization together with some ideas known from old self-referential paradoxes (e.g. the ancient paradox of a liar) enabled Gödel to construct a sentence which, as he proved, is true in the standard model of arithmetic (i.e. in the structure $N_0 = \langle N, 0, S, +, \cdot \rangle$ where N is the set of natural numbers 0, 1, 2, ..., S is the natural succesor function, + is the usual addition, and $\cdot$ is the usual multiplication) but is undecidable in the formal system of arithmetic he considered. This sentence, though talking about natural numbers, had in fact a metamathematical contents. It was stating: 'I am not a theorem' - hence it stated its own unprovability[1].

This metamathematical and not mathematical contents of Gödel's sentence belittled the philosophical significance of his results. It was known that arithmetic is incomplete but all known examples of undecidable sentences were artifical from the mathematical point of view (after Gödel's results some other undecidable sentences were obtained; cf. sentences of Rosser, Kreisel and Levy, Kent, Mostowski, Shepherdson - see Smoryński (1981)) because they all had metamathematical contents. Hence there was an open problem (interesting also from the point of view of the philosophy of mathematics): is it possible to indicate examples of undecidable sentences of mathematical contents, in particular of number-theoretical contents? The question was even more interesting because after Gödel's results it was still possible to cherish hopes that all sentences which are interesting and reasonable from the mathematical point of view are decidable.

On the other hand there was a methodological problem connected with Gödel's results. It was asked if, instead of using the arithmetization of syntax, it was possible to indicate a sentence φ and two models of arithmetic M_1, M_2 such that $M_1 \models \varphi$, $M_2 \models \neg\varphi$ (cf. Mostowski (1955) where among problems to be solved he mentioned the following one: 'To prove the incompleteness of the axiomatic arithmetic without applying the method of arithmetization by giving suitable models showing the consistency and independence of an appropriately chosen number-theoretical axioms.'). Observe that such a method was succesfully applied in the foundations of set theory (result of Gödel from 1938, showing the consistency of the axiom of choice and of the continuum hypothesis, and results of Cohen from 1963, showing their independence).

From now on we shall fix one particular formal system of arithmetic, namely the so called Peano arithmetic PA (its axioms are based on axioms for natural numbers given in 1889 by G. Peano). This system is a standard system used in studies of the foundations of arithmetic. It is formalized in the first order predicate calculus and based on the following nonlogical axioms:

$$Sx = Sy \rightarrow x = y,$$

$$Sx \neq 0,$$

$$x + 0 = x,$$

$$x + Sy = S(x + y),$$

$$x \cdot 0 = 0,$$

$$x \cdot Sy = x \cdot y + x,$$

$$\varphi(0) \;\&\; \forall x\,(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x\,\varphi(x).$$

The problem mentioned above was solved only in 1977 (i.e. 46 years after Gödel's results). It was done by J. Paris (1978). Working on nonstandard models[2] of PA he had invented a new method of constructing sentences which are independent of PA, but true in the standard model. The sentences of Paris were simplified by L. Harrington and at the end a new elegant undecidable sentence of a combinatorial contents was obtained (cf. Paris, Harrington (1977)). Soon a lot of new such sentences were found (by McAloon, Clote, Pudlak, Friedman, Mills, Murawski, Ratajczyk, Kirby, Simpson, Tverskoj). But the solution was still not completely satisfying. There was still no number-theoretical sentence. Only in 1982 J. Paris and L. Kirby found such a sentence (cf. Paris, Kirby (1982)) (it is interesting that the construction of their sentence uses some ideas of R. L. Goodstein (1944)). Hence, only 51 years after Gödel's results there was found an arithmetical sentence proving (by its existence) the incompleteness of arithmetic.

2. New undecidable sentences

Let us now describe the Paris-Harrington and Goodstein-Kirby-Paris sentences. We need some notation. If X is a set of natural numbers then $[X]^n$ denotes the family of all n-element subsets of X. A function $C\colon [X]^n \longrightarrow c$ (c being a natural number which we identify with the set of its predecessors, i.e. $c = \{0,1,\dots,c-1\}$) is said to be a colouring function. It may be interpreted as a colouring of n-element subsets of X by colours 0, 1, 2, ..., c−1. The English mathematician F. P. Ramsey (1929) proved that if C is a function colouring $[X]^n$ and X is big with respect to c and n then there exists a big set Y such that all its n-element subsets are coloured by one colour. Such a set $Y \subseteq X$ we call homogenous with respect to C. In fact Ramsey proved the following two theorems:

THEOREM 1 (Infinite Ramsey Theorem). Let n, c be positive natural numbers. For any colouring function $C: [N]^n \longrightarrow c$ there is an infinite set $Y \subseteq N$ such that Y is homogenous with respect to C, i.e. $C \restriction [Y]^n$ is constant.

THEOREM 2 (Finite Ramsey Theorem). Let s, n, c be positive natural numbers such that $s \geqslant n + 1$. Then there is a number R(s,n,c) such that for every $r \geqslant R(s,n,c)$, for any set X having r elements and any colouring function $C: [X]^n \longrightarrow c$ there exists a set homogenous with respect to C having s elements.

These theorems are not intuitively obvious and need proofs. They can be treated as generalizations of Dirichlet's Scubfach-prinzip. For n = 1 Theorem 1 says that if one divides an infinite set into a finite number of disjoint parts then one of these parts must be infinite. Theorem 2 for n = 1, s = 2 and R(2,1,c) = = c+1 is exactly the Dirichlet's principle: if one divides a set containing c + 1 elements (or more) into c parts, then one of them must contain at least 2 elements.

It turns out that Finite Ramsey Theorem can be proved in PA[3]. Harrington observed that modifying it a bit we obtain a sentence independent of PA. Call a set $X \subseteq N$ relatively large iff $card(X) \geqslant min(X)$. Then for example the set {2,3,89,92} is rela-vely large but the set {10,13,7,9} is not relatively large. The Paris-Harrington sentence φ_0 says now:

> for any natural numbers s, n, c there exists a natural number H(s,n,c) such that for any $h \geqslant H(s,n,c)$, any set X of cardinality h, any $C: [X]^n \longrightarrow c$ there is a set Y homogenous with respect to the function C and such that $card(Y) \geqslant s$ and Y is relatively large.

It can be proved[4] that $N_0 \models \varphi_0$ (in fact φ_0 is a consequence of Infinite Ramsey Theorem) but PA non $\vdash \varphi_0$. Hence φ_0 is an undeci-dable sentence of a combinatorial contents.

Now to describe the Goodstein-Kirby-Paris sentence let m, n be natural numbers and define a representation of m by the basis n: we write m as a sum of powers of n (e.g. if $m = 266$, $n = 2$ then $266 = 2^8 + 2^3 + 2^1$). We do the same with all exponents and at the end we get:

$$266 = 2^{2^{2+1}} + 2^{2+1} + 2^1.$$

We define now a number $G_n(m)$ as follows:

if $m = 0$ then $G_n(m) = 0$,
if $m \neq 0$ then $G_n(m)$ is a number obtained by replacing everywhere in the representation of m (by the basis n) the number n by n+1 and subtracting 1.

For example: $G_2(266) = 3^{3^{3+1}} + 3^{3+1} + 2 \approx 10^{38}$.

Goodstein's sequence for m is now defined in the following way:

$$m_0 = m,$$
$$m_1 = G_2(m_0),$$
$$m_2 = G_3(m_1),$$
$$\vdots$$

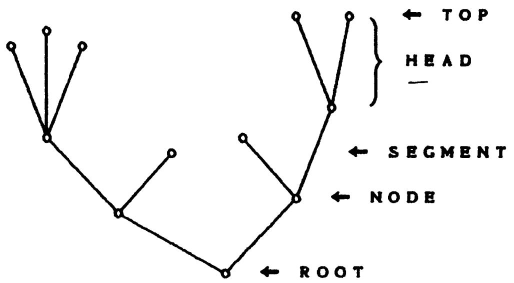
For example:

$$m_0 = 266_0 = 2^{2^{2+1}} + 2^{2+1} + 2,$$
$$m_1 = 266_1 = G_2(m_0) = 3^{3^{3+1}} + 3^{3+1} + 2 \approx 10^{38},$$
$$m_2 = 266_2 = G_3(m_1) = 4^{4^{4+1}} + 4^{4+1} + 1 \approx 10^{616},$$
$$m_3 = 266_3 = G_4(m_2) = 5^{5^{5+1}} + 5^{5+1} \approx 10^{10000} \quad \text{etc.}$$

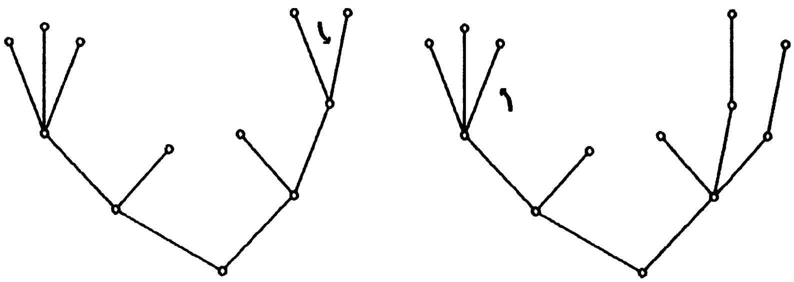
Observe that this procedure of constructing the sequence m_k can be described in the language of PA. Consider now the following sentence φ_1 of L(PA): $\forall m \exists k\, (m_k = 0)$. It can be proved that $N_0 \models \varphi_1$ but PA non $\vdash \varphi_1$. The unprovability of φ_1 has its source roughly speaking, in the fact that $m_k = 0$ only for very big k, e.g. if $m = 4$, $m_k = 0$ for $k = 3 \cdot 2^{402653211} - 3 \approx 10^{121000000}$. Observe that the whole number of atoms in the Universe is estimated as 10^{80}.

There was also found (cf. Paris, Kirby (1982)) an interesting example of an undecidable sentence of a ... mythological contents! As we know Hercules after killing his wife and children in a fit of madness went, regaining consciousness, to the oracle to ask her how he could now expiate his crime. Pythia told him to go to Mycenae and to enlist into the service with the king Eurystheus. There he ought to follow his commands until he does

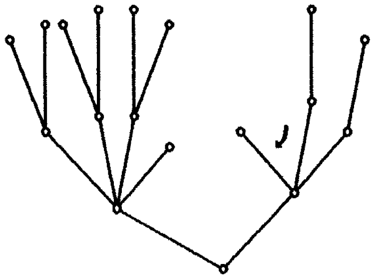
12 works. One of Eurystheus' commands was to kill the hydra of Lerna. What did that monster look like? Mathematics helps us to describe it. We can imagine it as something reminding by its shape what we call in mathematics a finite tree. Schematically we can represent it as follows:



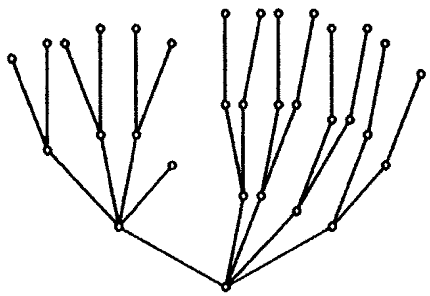
The battle between Hercules and hydra proceeds as follows: at stage n (n ⩾ 1) Hercules chops off one head from the hydra. The hydra then grows n "new heads" in the following manner: from the node that used to be attached to the head which was just chopped off, traverse one segment towards the root until the next node is reached. From this node sprout n replicas of that part of the hydra (after decapitation) which is "above" the segment just traversed, i.e. those nodes and segments from which, in order to reach the root, this segment would have to be traversed. If the head just chopped off had the root as one of its nodes, no new head is grown. For example (an arrow marks always the head which Hercules decides to chop off):



after stage 1



after stage 2



after stage 3

Hercules will win if after a finite number of stages nothing is left of the hydra but its root. Of course he may chop off the heads in any order. By a strategy we mean a function which determines for Hercules which head to chop off at each stage of the battle and by a winning strategy we mean a strategy which enables Hercules to win a battle with any hydra. It turns out that every strategy is a winning strategy, i.e. by chopping off the heads of any hydra in any order he always win!

Now consider the battle in a slightly different way. A hydra can be coded by a single natural number (hence a mathematician is more powerful than Hercules – he can reduce a hydra to a single not dangerous number!). This enables us to talk about the battle with a hydra in the language of Peano arithmetic. We cannot speak in this language about arbitrary strategies but we can speak about recursive (effective) ones. Consider now a sentence: 'Any recursive strategy is a winning strategy'. It is of course weaker than the sentence stating that every strategy is a winning strategy, therefore it is also true. But in turns out that even such a weak statement is not provable in PA. Peano arithmetic is to weak to prove some true sentences about the battle between Hercules and a hydra.

The methods and ideas of Paris, Kirby and Harrington were applied also to find sentences undecidable in sub- and supertheories of arithmetic. Let us mention here only the results of A. A. Tverskoj (1980) and H. Friedman (cf. Smoryński (1982)). With any recursive function f Tverskoj associates a formula KTP(f) of L(PA) (below we describe its sense) and proves that there is a sequence of recursive functions $f_{-1}, f_0, f_1, f_2, \ldots$ such that for any $n \geqslant 0$:

(1) $\qquad PA \vdash KTP(f_{n-1}) \rightarrow$ "f_n is total",

(2) $PA \vdash KTP(f_n) \to KTP(f_{n-1})$,

(3) PA non $\vdash KTP(f_{n-1}) \to KTP(f_n)$.

Hence we have a sequence of stronger and stronger sentences $KTP(f_n)$ such that $KTP(f_n)$ is independent of $PA + KTP(f_{n-1})$. Moreover, by (2), $KTP(f_n)$ is independent of $PA + \{KTP(f_k): k < n\}$. The function f_{-1} may be chosen in such a way that the sentence $KTP(f_{-1})$ is equivalent to the Paris-Harrington sentence φ_0.

Friedman has found sentences of combinatorial contents which are independent of some interesting (from the point of view of not only foundations of mathematics but also of the mathematical practice) fragments of the second order arithmetic called ATR_0 and Π^1_1-CA. (Second order arithmetic is a theory formalized in a two sorted language - individual variables representing natural numbers and set variables representing sets of natural numbers - which extends Peano arithmetic and is already so powerful that a big part of classical mathematics can be formalized within it.) Friedman's sentences talk about embeddability of finite trees. The precise description of those sentences needs the introduction of a lot of technical notions and details and we shall not give them here.

3. Some words about proofs

In general, there are two types of proofs of undecidability of those new sentences - call them syntactical and semantical. Semantical proof (e.g. Paris' (1977)) are based on the so called indicator theory founded by Paris and Kirby studying initial segments of nonstandard models of PA. To explain this method consider a countable nonstandard model M of PA and a certain property Q of initial segments of M (e.g. Q = being a model of PA). The indicator for Q is definable function which informs us if between any two elements $a,b \in M$ there exists an initial segment $I \subseteq_e M$ such that Q(I).

Let $M \models PA$ be countable and assume that there is an indicator $Y(x,y)$ for the family of initial segments I of M such that $I \models PA$. It is proved that there is an initial segment $I_0 \subseteq_e M$ such that $I_0 \models PA$ and I_0 non $\models \forall z \forall x \exists y\, Y(x,y) \geq z$. On the other hand $N_0 \models \forall z \forall x \exists y\, Y(x,y) \geq z$. Hence the sentence $\forall z \forall x \exists y\, Y(x,y) \geq z$ is undecidable in PA.

This is a general scheme. The whole problem is now to find a "good" (i.e. giving a sentence $\forall z \forall x \exists y\, Y(x,y) \geq z$ interesting from the mathematical point of view) indicator Y. The indicator found by Paris and Harrington and giving φ_0 was based on Ramsey theorem. (In the appendix we give a detailed, semantical type proof for another undecidable sentence, namely for Pudlak's sentence.)

The syntactical method does not use any models or other semantical notions. It was applied in the paper of Paris and Harrington (1977). They considered φ_0, constructed a theory (extending PA) and proved that

$$\mathrm{PA} \vdash \mathrm{Con}(T) \rightarrow \mathrm{Con}(\mathrm{PA}),$$

$$\mathrm{PA} \vdash \varphi_0 \rightarrow \mathrm{Con}(T),$$

where Con(PA) (resp. Con(T)) is a sentence of L(PA) expressing the fact that PA (resp. T) is consistent. Hence, by Gödel's second theorem, we get that PA non $\vdash \varphi_0$. On the other hand $N_0 \vDash \varphi_0$ and hence φ_0 is undecidable in PA.

4. Philosophical remarks

Consider now the new undecidable sentences from the philosophical point of view. First observe that they tell us more about the incompleteness of Peano arithmetic PA than Gödel's results, since they show some undecidable sentences of combinatorial and number-theoretical contents, i.e. sentences interesting from the usual mathematical point of view. Secondly, despite their mathematical contents they are strongly connected with some metamathematical sentences. McAloon (1979) has namely shown (and the same can be proved about almost all new sentences) that

$$\mathrm{PA} \vdash \varphi_0 \equiv \mathrm{Rfn}_{\Sigma_1}(\mathrm{PA}),$$

where $\mathrm{Rfn}_{\Sigma_1}(\mathrm{PA})$ is metamathematical sentence: for every sentence φ from L(PA) which contains only existential quantifiers appearing at the beginning, if φ is provable then φ is true. (By using the technique of arithmetization $\mathrm{Rfn}_{\Sigma_1}(\mathrm{PA})$ can be written as a sentence of L(PA).) Hence new sentences have in fact also some metamathematical contents. In the third place the new sentences are stronger then Gödel's sentence. Namely

$$\mathrm{PA} \vdash \text{Gödel's sentence} \equiv \mathrm{Con}(\mathrm{PA})$$

but

$$\mathrm{PA} + \mathrm{Con}(\mathrm{PA}) \text{ non} \vdash \mathrm{Rfn}_{\Sigma_1}(\mathrm{PA}).$$

Hence φ_0 is undecidable in PA + Con(PA).

To give some ideas how strong Paris-Harrington and Friedman sentences really are consider the following hierarchy of theories (α is here an ordinal):

$$T_0 = PA, \quad T_{\alpha+1} = T_\alpha + Con(T_\alpha), \ T_\lambda = \bigcup_{\alpha<\lambda} T_\alpha \ (\lambda \text{ limit}).$$

It can be shown that:

1° PA + φ_0 has the same proof-theoretic power as T_{ε_0} where ε_0 is the first ordinal ρ such that $\omega^\rho = \rho$, i.e. ε_0 is the limit of the following sequence of ordinal numbers $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ (Recall on this occasion that ε_0 is very well known in the proof theory – G. Gentzen showed that with the help of induction up to ε_0 we can prove the consistency of Peano arithmetic PA.)

2° PA + F, where F is Friedman's sentence undecidable in ATR_0 is stronger than any of the theories $T_{\varepsilon_1}, T_{\varepsilon_2}, \ldots, T_{\varepsilon_{\varepsilon_0}}, \ldots$ where ε_α is the α^{th} ordinal ρ such that $\omega^\rho = \rho$. In fact PA + F has the same proof-theoretical power as T_γ where γ is a countable ordinal which is strongly impredicative, i.e. such that it cannot be described without any reference to the first uncountable ordinal. The sentence F implies over a reasonable weak theory the consistency of predicative analysis, i.e. of the fragment of the second order arithmetic with comprehension scheme restricted to formulas having quantifiers over natural numbers and no set quantifiers.

What are in fact the reasons for the unprovability in PA of true arithmetical sentences considered above, i.e. sentences of Gödel, Paris-Harrington, Goodstein-Kirby-Paris, Friedman etc.? Recall that Gödel's sentence was of the form $\forall x\, \varphi(x)$ where $\varphi(x)$ was a formula containing only bounded quantifiers[5] and such that PA $\vdash \varphi(0)$, PA $\vdash \varphi(1)$, PA $\vdash \varphi(2)$, ..., PA $\vdash \varphi(n)$, ... for any natural number n. The source of the unprovability of $\forall x\, \varphi(x)$ is the fact that proofs (in PA) of $\varphi(0), \varphi(1), \varphi(2), \ldots$ are not uniform, i.e. there is no general method of proving $\varphi(n)$ for any given n. In other words for every $n \in N$ the proof of $\varphi(n)$ is different from a proof $\varphi(m)$, $m \neq n$.

What about the Paris-Harrington sentence φ_0? One can see that φ_0 may be written in the form $\forall x \exists y\, \psi(x,y)$ where ψ contains only bounded quantifiers. If now a formula of such type were provable in PA then there would be a provably recursive function f(x) such that PA would prove $\forall x\, \psi(x,f(x))$, i.e. a provably recursive function f giving examples of y's for given x's fulfilling the formula ψ – we call such a function a witness. Recall on this occasion that a function $f: N \longrightarrow N$ is said to be provably recursive iff f is recursive and there is a term F of language of Peano arithmetic PA such that F represents f in PA and

$$\mathrm{PA} \vdash \forall x \exists! y\, [F(x) = y],$$

where $\exists! y$ means that there exists exactly one y. Observe that if the term F represents a recursive function f in PA then for any natural number $n \in N$: $\mathrm{PA} \vdash \exists! y\, [F(n) = y]$ but it may happen that PA non $\vdash \forall x \exists! y\, [F(x) = y]$. On the other hand any recursive function can be represented by a term of PA and vice versa, any term of PA represents a recursive function (which may be but need not be provably recursive).

Coming back to our considerations, if PA non $\vdash \forall x \exists y\, \psi(x,y)$ then the source of this unprovability lies in the fact that for any provably recursive function f, if F is a term representing f, then PA non $\vdash \forall x\, \psi(x,F(x))$, i.e. no provably recursive function is a witness for ψ.

To explain this phenomenon better let us introduce the following hierarchy of functions of natural numbers. Let $f_0(x) = x + 1$, $f_{n+1}(x) = f_n^{x+1}(x)$, where $f_n^{x+1}(x) = f_n(f_n(\ldots(f_n(x))\ldots))$ $(x + 1)$-times. Hence, e.g. $f_1(x) = 2x + 1$, $f_2(x)$ is similar to 2^x and $f_3(x)$ is similar to

$$\left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\}\ x \text{ times}$$

We say that the function h_1 is something like the function h_2 iff $\mathcal{F}(h_1) = \mathcal{F}(h_2)$ where $\mathcal{F}(h)$ is the smallest class of functions containing h, S, $+$, $\cdot$ and closed under composition. It can be now shown that for any $n \in N$, f_{n+1} is not something like f_n and that f_{n+1} majorizes all functions from $\mathcal{F}(f_n)$. We can extend the hierarchy of functions f_n to ordinal numbers α (e.g. if α is a successor, i.e. $\alpha = \beta + 1$ then we put $f_\alpha(x) = f_{\beta+1}(x) = f_\beta^{x+1}(x)$ and if α is a limit ordinal then we diagonalize, e.g. $f_\omega(x) = f_x(x)$). Let now ε_0 be the smallest ordinal α such that $\omega^\alpha = \alpha$. The following fact now holds.

THEOREM 3. Functions belonging to the set $\bigcup_{\alpha<\varepsilon_0} \mathcal{F}(f_\alpha)$ are precisely those provably recursive in PA.

After that necessary (but maybe a bit tedious) explanation let us return back to the Paris-Harrington sentence φ_0. If φ_0 were provable in PA, i.e. if $\mathrm{PA} \vdash \forall x \exists y\, \psi(x,y)$ then there would be (by Theorem 3) an $\alpha<\varepsilon_0$ such that $\mathrm{PA} \vdash \forall x \exists y{<}f_\alpha(x)\, \psi(x,y)$. But $\mathrm{PA} \vdash \forall x \exists y\, \psi(x,y)$. Hence for any $\alpha<\varepsilon_0$ PA non $\vdash \forall x \exists y{<}f_\alpha(x)\, \psi(x,y)$.

Consequently, the function f which is the witness of φ_0 grows more rapidly than any f_α for $\alpha<\varepsilon_0$. And in fact it can be proved that the function H(x+1,x,x) (H is the function from the Paris-Harrington sentence φ_0) is something like f_{ε_0}, i.e. the function witnessing φ_0 majorizes all provably recursive functions and therefore PA is not able to handle it.

Observe that Paris-Harrington sentence φ_0 has a similar property as Gödel's sentence. Namely for any given $n\in N$, $PA \vdash \exists y\, \psi(n,y)$.

A similar situation to the one described above we have in the case of Friedman's sentences and reasons for their unprovability are also similar (recall that those sentences are also of the form $\forall x\exists y\, \chi(x,y)$ where χ contains only bounded quantifiers). But here the function witnessing the formula χ grows even more rapidly than it was in the case of Paris-Harrington sentence φ_0. And as before, for any given $n\in N$, $PA \vdash \exists y\, \chi(n,y)$. How long are proofs of those sentences? The answer is: they are very long. For example, in the case of Friedman's sentence $\forall x\exists y\, \chi(x,y)$ undecidable in ATR_0, the proof of the sentence $\exists y\, \chi(10,y)$ has at least

$$\left. 2^{2^{2^{\cdot^{\cdot^{2}}}}} \right\} \text{1000 times symbols!}$$

Do the new results fully satisfy logicians and "normal" mathematicians? Is this part of the foundations of mathematics already closed? We must answer these two questions negatively. The new results are not completely satisfying because the undecidability of new sentences is shown by proving that there exists I_0 such that I_0 does not satisfy the particular sentence. And this proof is not constructive. It is not convincing for a usual number-theorist. He does not study nonstandard numbers and the fact that some nonstandard model I_0 does not satisfy the considered sentence gives him no information. On the other hand he is not interested in the fact that in some system some sentence cannot be proved. He asks if these sentences are true among natural numbers. In practice he does not work in any formal system but uses any "proper" methods. We can assume, to fix our attention, that any of his proofs can be reconstructed in set theory (say in Zermelo-Fraenkel set theory with the axiom of choice ZFC). Hence there arises a problem of finding mathematical (number-theoretical) sentences undecidable by any "proper" method, i.e. independent of ZFC. The problem is still open (only metamathematical sentences about natural numbers which are independent of ZFC are known). Maybe Fermat's theorem or Goldbach's conjecture are examples of such sentences?

5. Appendix

To give the reader an idea of what could indicators generating undecidable sentences (cf. part 3) look like, we shall present in this appendix a detailed proof of unprovability of Pudlak's sentence. It was described in an unpublished paper 'Another combinatorial sentence independent of Peano's axioms'. We have chosen just this sentence because it has a clear and easily understandable contents and one can provide clear and readable proof of its independece. Describing Pudlak's sentence and its properties we shall follow Smoryński (1980).

DEFINITION 1. Let $f\colon N \longrightarrow N$. We say that a finite set $A \subseteq N$, $A = \{a_0, a_1, ..., a_n\}$ where $a_0 < a_1 < a_2 < ... < a_n$ is an approximation to f iff for any $i < n$:

$$\forall x \leqslant a_i\ [x \in \mathrm{dom}(f) \rightarrow f(x) \leqslant a_{i+1} \vee f(x) > a_n],$$

where dom(f) is the domain of f.

It can be seen that $A = \{10,12,13,14\}$ is an approximation of the function $f(x) = x^2$. Observe that any set having 2 elements is an approximation to every function.

DEFINITION 2. Let X be a finite set of natural numbers. We say that X is 0-dense iff $\mathrm{card}(X) \geqslant 3$. We say that X is $(n + 1)$-dense iff for any function $f\colon N \longrightarrow N$ there is an n-dense set Y such that $Y \subseteq X$ and Y is an approximation to f.

The set $\{0,1,2\}$ is 1-dense. It is not easy to give other examples of 1-dense sets. But the following theorem holds.

THEOREM 4. For any $a,n \in N$ there is $a,b \in N$ such that $a < b$ and the interval $[a, b]$ is n-dense.

This theorem cannot be proved in PA because its proof is not effective and uses facts and methods which cannot be formalized in PA. Therefore it is really difficult to give concrete, effective examples of n-dense sets though we know they do exist.

Consider the following sentence φ:

$$\forall v_0, v_1\ \exists v_2\ (v_2 > v_0\ \&\ [v_0, v_2] \text{ is } v_1\text{-dense}).$$

It turns out that φ is independent of PA. By Theorem 4 the sentence is true in the standard model N_0. Hence it is undecidable in PA.

To prove that PA non $\vdash \varphi$ it is enough to find a model M_0 of PA such that M_0 non $\models \varphi$. We shall do it now.

Let M be a nonstandard model of PA such that M is an elementary extension of N_0, i.e. the sentences of the language of Peano arithmetic which are true in M are exactly those sentences true in N_0. Hence by Theorem 4, $M \models \varphi$. Let a and c be any nonstandard elements of M. There is a $b \in M$ such that $a < b$ and $M \models$ "[a, b] is c-dense". The model M is model of PA, hence the scheme of induction, and consequently the principle of minimum hold in it. So let b_0 be the smallest element b from M such that $a < b$ and $M \models$ "[a, b] is c-dense".

The most important (and most difficult) part of the proof is to show that the function

$$Y(x,y) = \max(c\colon [x, y] \text{ is } c\text{-dense})$$

is the indicator for the family of initial segments of M being models of PA (cf. part 3). We omit this proof of course.

Now if M_0 is an initial segment of the model M such that $a, c \in M_0$ and $b_0 \notin M_0$ then in M_0 there is no b such that $a < b$ and [a, b] is c-dense. Hence M_0 is the needed model of $\neg\varphi$. Observe that here we have used the fact that the sentence "[a, b] is c-dense" for $a, b, c \in M_0$ holds in M_0 iff it holds in M.

Hence we have shown that $N_0 \models \varphi$ but there is a model $M_0 \models$ PA such that $M_0 \models \neg\varphi$. Consequently PA non $\vdash \varphi$ and PA non $\vdash \neg\varphi$. Hence φ is the sentence of mathematical contents undecidable in PA.

Notes

[1] Readers interested in details of Gödel's results may consult e.g. Nagel, Newman (1959) or Smoryński (1977). It is in order here to mention two contributions concerning the independent arithmetical propositions expected to appear:

R. Murawski, Appendix to paper 'On the incompleteness of arithmetic once more', in Essays on Logic and Philosophy, Proc. of the XXX International Conference on the History of Logic, ed. J. Perzanowski.

S. G. Simpson, Nichtbeweisbarkeit von gewissen kombinatorischen Eigenschaften endlicher Bäume, Archiv f. Math. Logik und Grundlagenforschung.

[2] One of the consequences of the compactness theorem is the existence of models of arithmetic of natural numbers different from (i.e. nonisomorphic to) the basic model $N_0 = \langle N,0,S,+,\cdot\rangle$. This basic model is called standard, models which are not isomorphic to it are called nonstandard. Nonstandard models contain so called nonstandard natural numbers, i.e. objects a such that they have properties described by axioms of the arithmetic of natural numbers but are bigger than all standard numbers ($a > 0$, $a > 1$, $a > 2$ etc.). Any nonstandard model of arithmetic is ordered (by the natural order relation $a \leqslant b$ iff $\exists c\,(b = a + c)$) in the type $\omega + (\omega^* + \omega)\,\rho$, where ρ is a dense order type, ω is the order type of the set of natural numbers and $\omega^* + \omega$ is the order type of the set of integers. In the case of countable models ρ is the order type of rationals. Hence any nonstandard model contains an initial segment isomorphic to N.

[3] Observe that in PA we can talk about finite sets of natural numbers. We can simply code them by single natural numbers. If we have a set $X = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \ldots < a_n$ then it can be coded by the number $p_1^{a_1}..p_n^{a_n}$ where p_i is the i-th prime.

[4] Talking about provability of some semantical facts we mean always the provability in the metatheory which is e.g. Zermelo-Fraenkel set theory with the axiom of choice ZFC.

[5] Bounded quantifiers are quantifiers of the form $\forall x{<}t$ and $\exists x{<}t$ where t is a term of L(PA) and we define

$$\forall x{<}t\ \varphi(x) \equiv \forall x\ (x < t \rightarrow \varphi(x)),$$

$$\exists x{<}t\ \varphi(x) \equiv \exists x\ (x < t\ \&\ \varphi(x)).$$

STANISŁAW J. SURMA

THE LOGICAL WORK OF MORDCHAJ WAJSBERG

i. Introductory Remarks

In this paper Mordchaj Wajsberg's life and research work in logic are described, and an attempt is made to situate the latter among the accomplishments of the rest of the Polish school of logic.

ii. Wajsberg and the Polish School of Logic

Wajsberg belonged to a research formation called in the course of time the Polish school of logic. The undisputed leaders of the school were S. Leśniewski, J. Łukasiewicz, and at a later stage, A. Tarski who since 1923 became responsible for many outstanding contributions and systematic studies in logic, metalogic and semantics.

Although the school members centered on modern logic and its applications where they promoted a number of new trends and opened many fresh fields of research, they also showed a lively interest in methodology of deductive and empirical sciences as they took up in modern logical form many of the traditional major philosophical questions at the same time putting outside the scope of philosophy some such philosophical problems which could be either clearly stated or investigated by the methods of science. In these efforts the school members were supported by the prominent philosophers T. Kotarbiński and K. Ajdukiewicz.

It is this school which emerged as the most dominant force in academic logic and philosophy of Poland as well as the Polish intellectual life between the two world wars. And it is this school which should be seen responsible for the spectacular rise to prominence of formal and philosophical logic.

The leaders of the school soon became surrounded by a large number of talented students, young assistants and followers. Among those who essentially contributed to the school's success there were A. Lindenbaum, B. Sobociński, S. Jaśkowski, J. Słupecki, and of course, M. Wajsberg.

A distinctive feature of the school and one of the secrets of its success was the spirit of teamwork. The mutual collaboration among the members was so close and intimate that it is often hard to decide who should be credited with which particular results. Another its feature is that its members seemed to care more about making research progress than about making the results actually published or otherwise documented. Consequently, many findings appeared in print only in the form of abstract with proofs and other essentials missing. Moreover, some important findings were never published during their authors' lifetime. They became more and more dependent on oral communication thus contributing to the growth of the school's 'oral tradition'. For a general school's background see KUZAWA 1968 and MOSTOWSKI 1957.

Many useful findings were summarized and systematized by Łukasiewicz and Tarski in their joint paper ŁUKASIEWICZ-TARSKI 1930. A great number of references to the school's results can be found in TARSKI 1956. See also JORDAN 1945, 1963, and 1967.

Wajsberg emerges as a prominent representative of the school. Many of his research results have profoundly influenced further studies in the field. Among other things, he became a pioneer in the axiomatization of many-valued logic. He was the first to provide an adequate semantics for one of Lewis's modal systems. He also worked out an original method for the separable axiomatization of intuitionistic propositional logic. Wajsberg made an impression on many things which he touched, perfected many results by others, particularly by Łukasiewicz, Leśniewski, Tarski, Lewis and Hilbert. His research work gave a new impetus to further studies. And, although, unlike his teachers, Wajsberg said directly nothing on philosophical subjects, his research work has borne unquestionable philosophical implications.

Wajsberg published twelve papers. For the availability of their English translations see SURMA 1977. See also McCALL 1967 which contains English translations of three of his papers, WAJSBERG 1931, 1937, and 1938b.

iii. Childhood

Mordchaj Wajsberg was born on May 10, 1902 at Łomża, Białystok district. The years of 1909 to 1912 he spent in a local primary school. Then he moved to an intermediary school but the school was closed two years later when the first world war had broken out. In 1920 a year of military service in the revived Polish army followed thus interrupting his preparations for final school certificate. After completion of the service he passed successfully his entrance examination to the last but one form of the local secondary school from which he graduated in June 1923.

iv. University Study

Wajsberg spent his formative years in Warsaw. In October, 1923 he enrolled as a mathematics student at the Philosophy Department of the Warsaw University. He specialized in mathematical logic which he studied under Łukasiewicz. Apart from those by Łukasiewicz he also attended lectures on logic given at that time by Leśniewski and Kotarbiński.

As a second year student he read two papers to the Philosophy of Mathematics Section of the Association of Philosophy Students, one on "Russell's Theory of Functions of Apparent Variable", the other one on "Invariants of Logistic Transformation".

v. Early Findings. Pure Implication

Already as a third-year student Wajsberg obtained some original results. He described a number of alternative axiomatic systems for various fragments of classical propositional logic. In particular, he found new axioms for the logic of pure implication and for that of pure equivalence. Among them there is his 25-letter single axiom for pure implication:

CCCpqCCrstCCuCCrstCCpuCst

(Explanation of the symbolism: the above formula is rendered using the so called Polish notation, due to Łukasiewicz (see ŁUKASIEWICZ 1929), where 'C' denotes the connective of implication, and where 'Cab' reads as 'If a, then b'). This axiom is organic in the sense that none of its proper subformulae is a tautology; the notion of organic formula was also introduced by Wajsberg (see ŁUKASIEWICZ-TARSKI 1930).

Unlike the above axiom, the 25-letter single axiom:

CCCpCqpCCCCCrstuCCsuCruvv

found by Łukasiewicz and also referred to in ŁUKASIEWICZ-TARSKI 1930, contains tautology CpCqp as a subformula, and so it is not organic.

It has been shown by Łukasiewicz later that the following 13-letter formula:

CCCpqrCCrpCsp

is the shortest single axiom for pure implication (see ŁUKASIEWICZ 1948). Still later Ivo Thomas, using the work of R. Tursman (see TURSMAN 1968), has finally shown that there are no more shortest single axioms for pure implication (see THOMAS 1970).

vi. Pure Equivalence

Investigations into the logic of pure equivalence were initiated in Poland by Leśniewski to whom we owe what is now called Leśniewski's decidability criterion to the effect that each purely equivalential formula is a tautology if and only if each propositional variable occurs in it an even number of times. Leśniewski was also the first to prove that all pure equivalential tautologies can be axiomatized with the help of substitution and ordinary detachment for equivalence:

$$Eab,\ a \vdash b$$

together with the following axioms:

$$EEEprEqpErq,\ EEpEqrEEpqr.$$

For reference see LEŚNIEWSKI 1929. (Explanation of the symbolism: the above formulae are rendered using the Polish notation, where 'E' stands for the connective of equivalence, and where 'Eab' reads as 'a if and only if b').

The subject of pure equivalence attracted many members of the school. Among them was Wajsberg. To Wajsberg belongs the credit of showing that the logic of pure equivalence can be axiomatized with the help of single axiom. In 1925, still as a third-year student, he found the following two 15-letter single axioms (see WAJSBERG 1937, footnote 1):

$$EEEEpqrsEsEpEqr$$

and

$$EEEpEqrEErssEpq.$$

In 1930 five more 15-letter single axioms for pure equivalence were found by Łukasiewicz, Sobociński, and J. Bryman (see SOBOCIŃSKI 1932). Later all these results were sharpened by Łukasiewicz who in 1933 found the following three 11-letter single axioms for pure equivalence:

$$EEpqEErqEpr,\ EEpqEEprErq,\ \text{and}\ EEpqEErpEqr.$$

Łukasiewicz proved that each of these axioms is the shortest possible single axiom for pure equivalence, thus solving the problem of the length of single axioms for this logic (see ŁUKASIEWICZ 1939).

In 1963 C.A.Meredith found seven more shortest single axioms for pure equivalence (see MEREDITH 1963 and PETERSON 1976). One more such single axiom was added by J.A. Kalman in 1978 (see

KALMAN 1978). Continuing earlier efforts by Kalman and Peterson, L. Wos and S. Winker finally established that the number of all single axioms for pure equivalence is thirteen (WOS-WINKER 1980). More historical information concerning investigations into the logic of pure equivalence may be found in SURMA 1973b.

vii. Sheffer's Connective

As a fourth-year student Wajsberg made a contribution to the study of the Sheffer connective D (read as 'Not both'). He found the following axiom for D:

DDpDqrDDDsrDDpsDpsDpDpq

and he deduced from this axiom the following axiom:

DDpDqrDDtDttDDsqDDpsDpDps

which was found by J. Nicod as early as in 1917, and which became the first single axiom for propositional logic ever known (see NICOD 1917).

Wajsberg's axiom improves Nicod's one. First, it contains one less propositional variable. Besides, it is organic while Nicod's is not as it contains the tautology DtDtt as a subformula.

Wajsberg's own results on single axioms contributed to similar studies, already in their full swing, advanced considerably by Łukasiewicz, Tarski, and Sobociński, who found many single axioms for various fragments of propositional logic. For reference see ŁUKASIEWICZ-TARSKI 1930 and SOBOCIŃSKI 1932. All his early results were included into Wajsberg's master's thesis, entitled "Contribution to the Research on Mathematical Logic", which was written under Łukasiewicz's supervision. It is on the basis of this thesis that he was awarded his M.A. degree on October 2, 1928.

viii. Early Observation on Modal Logic

Still as a student Wajsberg became involved into the study of modal logic, the ancient subject which was revived by modern logicians, especially, by C.I. Lewis. He was the first to prove that none of Lewis's modal systems is equivalent to classical propositional logic. Following WAJSBERG 1937, footnote 7, his separating four-valued truth tables, used in the proof, were found by him as early as in 1926. He observed that formula 'La' (read as 'It is necessary that a') is already a theorem in Lewis's system S1, whenever 'a' itself is a classical tautology, an important fact pertaining to the so called Goedel-Lemmon-style formalization of modal logics. And for the first time in the history of modern modal logic he outlined an adequate semantic characterization of Lewis's system S5. A detailed description of the semantics was presented in his later paper WAJSBERG 1933a.

All these observations were communicated by Wajsberg to C.I. Lewis at least as early as in 1927, as it is acknowledged in Appendix ii of LEWIS-LANGFORD 1932. See also PARRY 1968.

ix. Wajsberg's Work on Many-Valued Logic

From August, 1929 to September, 1930 Wajsberg served in the army, first as a student in the cadet training unit, and then in the 4th Regiment of the Tatra Highland Gunners. In September, 1930 he qualified for Ph.D. studies at the Warsaw University. As a Ph.D. student he worked under Łukasiewicz's supervision. His research project centered on the three-valued logic of Łukasiewicz.

The three-valued logic of Łukasiewicz was discovered by Łukasiewicz in 1920, that is already a decade earlier (see ŁUKASIEWICZ 1919-1920 and 1921), and was described semantically with the help of his well-known three-valued truth tables, at that time referred to as the method of logical matrices (see ŁUKASIEWICZ 1930). In 1922 the three-valued logic was generalized by Łukasiewicz to n-valued logics, where n may be an arbitrary finite or even infinite number. Researches on Łukasiewicz's logics were carried out by a growing team of talented and devoted students and collaborators, which included not only Wajsberg but also Tarski, Lindenbaum, Sobociński, and, later, Słupecki and Jaśkowski.

Wajsberg accomplished his Ph.D. project in less than a year, entitled his manuscript "Axiomatization of the three-valued propositional logic", and submitted it oficially to the Warsaw University in fulfilment of the requirement for the degree of Doctor of Philosophy.

In his thesis Wajsberg found the following system of independent axioms for the three-valued logic of Łukasiewicz based on implication and negation as primitive connectives:

CpCqp, CCpqCCqrCpr, CCNpNqCqp, CCCpNppp

(Explanation of the symbolism: 'Cab' reads as 'if a, then b', as before; while 'Na' reads as 'it is not the case that a' so that 'N' stands for the connective of negation). He proved that each three-valued tautology and only such tautology can be deduced from the above axioms using the rules of detachment and substitution as the only rules of inference. This provided a solution to the completeness problem for the three-valued logic of Łukasiewicz, which was the first result of the kind in the history of many-valued logic.

In his thesis Wajsberg also proved that no subsystems of classical propositional logic can be axiomatized with the help of axioms built up of at most two propositional variables. An algebraic proof of this fact was given by A.H. Diamond and J.C.C. McKinsey in 1947 (see DIAMOND-McKINSEY 1947).

A paper based on the results contained in Wajsberg's thesis was presented by Łukasiewicz to the Warsaw Scientific Society for publication as early as January 19, 1931. It appeared in the Proceedings of the Society in the same year (see WAJSBERG 1931).

Formal defence of Wajsberg's Ph.D. thesis followed, with Łukasiewicz and S. Mazurkiewicz as referees, and the degree of Doctor of Philosophy was conferred upon him at the promotion ceremony on May 29, 1931.

Wajsberg's Ph.D. thesis did not contain all of his findings concerning Łukasiewicz's many-valued logics. At about the same time he proved axiomatizability of all those n-valued Łukasiewicz's logics, for which (n-1) is a prime number. This result was later extended by Lindenbaum to all natural n (see ŁUKASIEWICZ-TARSKI 1930).

Wajsberg also confirmed Łukasiewicz's conjecture on the axiomatizability of the infinite-valued Łukasiewicz's logics, namely, that the logic can be axiomatized by the detachments and substitution rules together with the following axioms:

CpCqp, CCpqCCqrCpr, CCCpqqCCqpp, CCCpqCqpCqp, CCNpNqCqp.

He announced in WAJSBERG 1936, p.240, that he had found proof for the conjecture but his proof has never been published (see ŁUKASIEWICZ-TARSKI 1930). The proof that the above axioms suffice for Łukasiewicz's infinite-valued logic was shown in print by A. Rose and J.B. Rosser only in 1958 (see ROSE-ROSSER 1958). C.A. Meredith and C.C. Chang then showed, independently, that axiom CCCpqCqpCqp is redundant and so can be omitted from above list (see MEREDITH 1958 and CHANG 1958).

Wajsberg also found a relatively simple axiomatization of the so called extended three-valued logic of Łukasiewicz. An extended propositional logic was defined in the school as a propositional logic admitting quantification over propositional variables (see ŁUKASIEWICZ-TARSKI 1930). As such it can be viewed as a particular case of Leśniewski's protothetic (see LEŚNIEWSKI 1929) in which also quantification over variable connectives is admissible.

x. Axiomatizability of Negation

In the year of 1931, apart from the paper containing his Ph.D. thesis, Wajsberg also prepared for publication his papers WAJSBERG 1932a and 1932b, which were published in the next year. In WAJSBERG 1932a he presented an axiomatizability criterion for the classical propositional logic based on implication and negation. According to this criterion, a set X of formulae built up of implication and negation in such a way that negation may only be followed by propositional variables, when added to the axioms for pure implication, axiomatizes the logic based on implication and negation if and only if each unary connective different from negation does not satisfy at least one formula from X. For reference see also ŻARNECKA-BIAŁY 1973.

In the paper WAJSBERG 1932b we find Wajsberg's organic axiom for the Sheffer connective along with his findings involving pure implication and pure equivalence which he found already as an undergraduate student.

xi. Wajsberg's Semantics for Lewis's Modal System S5

The year of 1932 Wajsberg also spent in Warsaw. In February and March he presented two papers to the Section of Logic of the Warsaw Philosophical Society, entitled "From the Research on the Theory of Deduction", and "Axiomatization of Predicate Logic", respectively. The precise contents of the papers is unknown. One may only guess that they were related to his papers WAJSBERG 1933a and 1933b which he prepared for publication at around that time.

In the paper WAJSBERG 1933a the author constructed an adequate semantic characterization of Lewis's system S5, the first example of an adequate semantics in the history of modal logic. As we mentioned in Section viii, this semantics was known to Wajsberg long before 1933 (see LEWIS-LANGFORD 1932, Appendix ii).

Using modern terminology and notation Wajsberg's semantics may be described as follows. Let A be a non-empty set, and let P(A) denote the set of all subsets of A. Let $-_A$ denote the set-complementation operation within A, i.e., if X is a subset of A, then $-_A(X)$ denotes the set of all those elements in A which are not members of X. Let $\cap$ and $\cup$ denote, as usual, the set-intersection and the set-union, respectively. Let I_A be a unary operation in P(A) defined, for every $X \subseteq A$, as shown below:

$$I_A(X) = \begin{cases} A & \text{if } X = A \\ \emptyset & \text{if } X \neq A \end{cases}$$

where, of course, $\emptyset$ denotes the empty set. Let $\underline{P(A)}$ denote the sequence:

$$(P(A), -_A, \cap, \cup, I_A)$$

Thus $\underline{P(A)}$ is a Boolean algebra of subsets of A with the additional unary operation I. A formula is defined as true in $\underline{P(A)}$ if and only if it takes on value A under every assignment of members of $\underline{P(A)}$ to its propositional variables, where the valuation function is defined in such a way that propositional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), and L (necessity) correspond to the operations: $-_A$, $\cap$, $\cup$, and I_A, respectively. Now, the main result of Wajsberg may be expressed as follows:

> An arbitrary formula is provable as a theorem of Lewis's system S5 if and only if the formula is true in the system $\underline{P(A)}$, for every non-empty set A.

For reference see also ZACHOROWSKI 1973.

To prove this theorem Wajsberg introduced a kind of normal form procedure. More specifically, he showed that every propositional formula of the form 'La' (read as 'It is necessary that a') is reducible in S5 to a kind of conjunctive normal form where each disjunct consist of 'L' or '$\neg$L' followed by a disjunction of variables (negated or un-negated). It should be noted, however, that this form cannot be used as a general normalization procedure for S5 because only formulae of the form 'La' and not all formulae are so reducible. G.F. Schumm has observed (see SCHUMM 1975) that a slight modification of Wajsberg's original form could do the normalization job. Namely, each formula is reducible in S5 to another conjunctive normal form where each disjunct consist of either 'L' or '$\neg$L' followed by a disjunction of variable or a negated variable.

Notice that Wajsberg's sequence $\underline{P(A)}$, as constructed above, appears to be a kind of the so called nowadays McKinsey-Tarski topological Boolean algebras (see McKINSEY-TARSKI 1944) which are widely used to construct algebraic-type semantics for modal logics.

An inspection of Wajsberg's proof also reveals that a formula containing precisely n propositional variables is a theorem of Lewis's system S5 if and only if it becomes true in Wajsberg's sequence P(A), for every set A consisting of 2^n elements. It follows here from that implicit in Wajsberg's proof is a decidability procedure for S5.

At the end of the paper WAJSBERG 1933a the author observed that the replacement of propositional variables: $p_1, p_2, p_3, \ldots$ by monadic formulae of predicate logic of one and the same variable x: $P_1x, P_2x, P_3x, \ldots$ and the replacement of the connective 'L' by the universal quantifier '$\forall$' binding 'x' we can get the (non-modal) monadic predicate logic in one individual variable 'x'. It should be added that similar relation between a modal system and a system of predicate logic has since been found also in respect to some other modal systems (see, for instance THOMAS 1962).

xii. Papers on Predicate Logic

Unlike previous papers, the paper WAJSBERG 1933b concerns the first order predicate logic. Let us call formula of predicate logic k-true if and only if it is true in any of its k-element models, and let us define a k-true formula, which is not (k+1)-true, as exactly k-true. In WAJSBERG 1933b the author constructed an exactly k-true formula from which every k-true formula is deducible,

$$(Ax_k) \qquad \bigvee^{i}_{i\leq k} F_i x_i \supset \bigvee^{m}_{i<m<k+1} F_i x_m$$

where the expression '$\bigvee^{i}_{i\leq k} B(x_i)$' abbreviates the disjunction '$B(x_1) \vee B(x_2) \vee \ldots \vee B(x_k)$', and where '$\supset$' stands for the connective of implication. To see better the syntactic structure of (Ax_k) we give below three particular cases, for k = 1, k = 2, and k = 3, respectively

(Ax_1) $F_1x_1 \supset F_1x_2$,

(Ax_2) $(F_1x_1 \supset F_1x_2 \vee F_1x_3) \vee (F_2x_2 \supset F_2x_3)$,

(Ax_3) $(F_1x_1 \supset F_1x_2 \vee F_1x_3 \vee F_1x_4) \vee (F_2x_2 \supset F_2x_3 \vee F_2x_4) \vee (F_3x_3 \supset F_3x_4)$

For reference see also WOLEŃSKI 1973.

The paper WAJSBERG 1933-1934 also deals with predicate logic. Applying Tarski's notion of the degree of completeness of a deductive system (see TARSKI 1930) Wajsberg provided in the paper a detailed proof that the degree of completeness, i.e., the number of all maximal consistent extensions of the first order logic is equal to the number of the continuum.

xiii. Wajsberg's Criterion of Axiomatizability of Finite Matrices

From Warsaw Wajsberg moved to Kowl in Volhynia where he worked as a teacher to the end of June, 1933. Then he returned to his native Łomża where he continued his teaching career and where the rest of his works were written.

The paper WAJSBERG 1935 included author's well-known result concerning the conditions of axiomatizability of finite logical matrices, including Łukasiewicz's matrices and the so called finite intermediate logics among others. According to his theorem if the formulae below:

CCpqCCqrCpr, CCqrCCpqCpr, CCpqCNqNp, CNqCCpqNp, CCqqCpp

are all satisfied in a finite logical matrix, then the matrix must be axiomatizable. The theorem, with formula CCqrCpp replacing formula CCqqCpp, was stated as Wajsberg's theorem without proof in ŁUKASIEWICZ-TARSKI 1930, i.e., as early as in 1930. Wajsberg's own proof of this theorem, included in his paper, is lengthy and rather difficult to comprehend. A detailed exposition of his proof, with only small changes in notation, can be found in ACKERMANN 1971. See also SZCZĘCH 1973.

xiv. General Approach to Logical Matrices

The general notion of logical matrix was introduced by Tarski (see ŁUKASIEWICZ-TARSKI 1930). The paper WAJSBERG 1936, written in 1934 and published two years later, was conceived as a contribution to the study of logical matrices. In this rather technical paper Wajsberg made an effort to classify logical matrices into types (distinguished in the paper are various special types of matrices such as congruence matrices; linear congruence matrices and sum-matrices as their special case; infinite linear matrices; and conditional matrices along with interval matrices as a special case of the latter). He also described some systematic methods for deciding which formulae built up of implication and negation are satisfied in which matrices of a given type. For reference see also SUCHOŃ 1973. It is rather striking that the discussed Wajsberg's paper has attracted almost no attention from the subsequent researches in the field. In particular, in Łoś's monograph ŁOŚ 1948 Wajsberg's paper is not even mentioned. Neither is it referred to in

J. Kalicki's works on logical matrices (see, for instance, ZYGMUNT 1981).

xv. Separability Property of Intuitionistic Connectives

In WAJSBERG 1938a the separability theorem for a system of intuitionistic propositional logic of axiomatic type was established to the effect that no intuitionistic theorem, from which any one of the four connectives:

(★) $\neg$ (negation), $\supset$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction)

is absent, requires for its proof any axiom in which the connective is present.

Wajsberg also added a number of interesting results concerning definability of propositional connectives to the effect that none of the mentioned in (★) can be expressed in intuitionistic logic in terms of the remaining three, the result which was also arrived at, independently, in a paper by J.C.C. McKinsey published one year later (see McKINSEY 1939).

In connection with the separability problem it may be worthwhile recalling that on A. Church's suspicion (see his errata to CHURCH 1956, footnote 211) Wajsberg's proof were to contain an error difficult to correct. Without further discussion of the nature of the alleged error Church seemed to suggest that the result should be, therefore, credited to H.B.Curry whose paper CURRY 1939, solving independently, the separability problem by a Gentzen's sequents' technique, appeared one year later. In his monograph CURRY 1963 the author confessed that though he had never examined Wajsberg's proof, he trusted others in considering it erroneous. The suspicion of error has since been repeated by many, among others, by A. Horn who provided the first modern-style algebraic proof of separability property of intuitionistic logic (see HORN 1962). Of the papers which have attempted a detailed reconstruction of Wajsberg's argument two are in order, KABZIŃSKI-PORĘBSKA 1975, and BEZHANISHVILI 1981. In the first paper it is shown that some of Wajsberg's preparatory lemmas admit, in fact slight strengthening which then implies the separability property without complications. In the second paper Wajsberg's Definition 2, #8 of an n-order thesis, claimed to be the source of the alleged error, was changed and so was the proof of Wajsberg's Theorem 14, #8. For reference see also KABZIŃSKI 1973b.

The formulation and the solution of the separability problem for intuitionistic logic did not come as a surprise. It was well motivated by the parallel investigations into the axiomatization of various fragments of the expressively complete ordinary, two-valued propositional logic. The latter can be viewed an investigations into the separation of properties of various classical connectives.

xvi. Miscellany on Propositional Logic

In WAJSBERG 1937 and 1938b the author included a rich crop of various 'incidental' results and remarks on different axiom systems of classical propositional logic and its fragments. Some of them come from his unpublished master's thesis. Various axiom systems for pure implication are listed in WAJSBERG 1937, #1 and #6, and in WAJSBERG 1938b, #1; axiom systems for implication and falsum are discussed in WAJSBERG 1937, #9, and in WAJSBERG 1938b, #2; paper WAJSBERG 1938, #2 also contains various axioms for implication and negation; paper WAJSBERG 1937, #7 is devoted to axiom systems for equivalence. The completeness property of each of the systems referred to above was established by syntactic means, i.e., by deducing from each of them another axiom system, already known to be complete. Paragraph 2 of WAJSBERG 1937 discussed the independence property of various axiom systems. For reference see also STĘPIEŃ 1973.

Paragraph 4 of paper WAJSBERG 1937, entitled "General Scheme of a Completeness Proof for the C-Pure", contains a schematic description of Wajsberg's method of proof of the completeness property for the propositional logic based on implication as the only primitive connective. To solve the completeness problem for the three-valued logic of Łukasiewicz as well as for some other many-valued logics, Wajsberg had to work out an original completeness argument. Later he adjusted the argument to provide a new proof of the completeness theorem for the two-valued logic of pure implication (see WAJSBERG 1937). It should be mentioned that the first proof of the completeness theorem for pure implication was found by Tarski but it was not published by the author (see ŁUKASIEWICZ-TARSKI 1930, and TARSKI 1934-1935).

Wajsberg's method can be characterized briefly as follows:

i. first one must prove that all tautologies built up of one propositional variable are formally deductible by the axioms and rules of an axiomatic system under consideration (a most laborious part of the completeness proof);oof);

ii. then, assuming that all tautologies built up of n different variables each are formally deducible, one must prove that all tautologies built up of (n+1) different variables each are also formally deducible in the axiomatic system (a comparatively easy part of the completeness proof).

Wajsberg's completeness argument has since been in frequent use. In 1938 W.V. Quine followed the plan sketched in WAJSBERG 1937 to solve the completeness problem for the logic based on implication and falsum as the only primitive connectives (see QUINE 1938). In 1943 K. Schroeter provided a Wajsberg-type completeness argument for the full classical logic based on all usual connectives as primitives (see SCHROETER 1943). Wajsberg's method has often been used in Poland (see, for instance, SADOWSKI 1961, and SURMA 1973a). A detailed discussion of Wajsberg's method is also contained in the monograph ASSER 1959.

xvi. Closing Remark

Since the outbreak of the second world war there has been no reliable information concerning Wajsberg's fate. The only fact known for certain is that he has perished prematurely and so all his unpublished manuscripts have been lost.

M. N. BEZHANISHVILI

NOTES ON WAJSBERG'S PROOF OF THE SEPARATION THEOREM

In his well-known paper M. Wajsberg (1938) stated some important results on the intuitionistic propositional logic. But in connection with Wajsberg's proof of the Separation Theorem (which was first formulated and proved by him), A. Church (1956) indicates that this paper of Wajsberg's contains an error which is difficult to correct (see the correction of footnote 211). Further Church notes that the correct Gentzen-style proof of this theorem for intuitionistic predicate logic was given by H. B. Curry (1939) which was in print when Wajsberg's paper appeared. Afterwards this proof was reproduced by S. C. Kleene (1952). Church writes also that for Curry's proof essential is the Gentzen's Cut Theorem but not the use of sequents which can be eliminated. Therefore the Cut Theorem can be applied in a suitable form to Hilbert type formulations (see H. B. Curry (1939) and K. Schütte (1950)). Concerning Wajsberg's proof of the Separation Theorem Curry (1963) writes that he has never examined this proof, but judging by Bernays and an errata sheet to the book of Church (1956) (see footnote 211), the proof contains an error (p.250). In early reviews (e.g., Heyting (1939), Rosser (1938)) the error of Wajsberg (1938) is not mentioned and in the preface to Wajsberg (1977) St. Surma refers to the same important information of Church. Repeating the same indication A. Horn (1962) gives the algebraic proof of the Separation Theorem for the intuitionistic propositional calculus. J. Kabziński and M. Porębska (1974) give its proof by the Wajsberg's method without indicating the error in Wajsberg's initial proof. They write that this error is not indicated in the literature and they think that all the objections to the Wajsberg's proof of separability are caused by his oversights rather than error and that the arguments presented by Wajsberg for establishing some preliminary results prove in fact slightly strengthened formulations of these results allowing to obtain the separability without any complications (see p.31). A. Tsitkin (1979) tried to find and correct all the errors. He thought that when stating the fact of separability Wajsberg (1938) erred in case 3 of the proof of Theorem 2 (§11). He also indicated an error in Wajsberg's proof of identity of the sets of all theses and all consequences of conjunction-free axioms, namely, in the proof of Theorem 14 (§8). Although Wajsberg's proof of the separability is independent of this theorem, some other interesting results of his depend on it. Tsitkin proposed the correction of the proof of Theorem 14 by introducing an additional parameter of induction and by strengthening the formulations of Wajsberg's Theorems 10 and 11 (§8).

The aim of the present paper, which mainly gives an account of the author's former results (see Literature), is also a correction of the same Wajsberg's work.

For his investigations Wajsberg chooses the formulation W of the Münster School axiom system of the intuitionistic propositional calculus modified by him. Formulae of W are called propositions and are presented in the symbolic notation of Łukasiewicz. We will use the usual notation.

Formulae are constructed from propositional variables p, q, r, s (with or without indices) and a propositional constant 0 (which means "false") by means of the connectives $\supset$, &, $\vee$ (implication, conjunction and disjunction). Propositional variables and the constant are called propositional signs. As metavariables for them are used a, b, c, d and for formulae – $\alpha, \beta, \gamma, \delta, \varepsilon$ (with or without indices). Capital Latin letters denote sets of formulae and i, j, k, l, m, n, u – natural numbers (including zero).

$(\alpha_1 \ldots \alpha_k \supset \beta)$ is the abbreviation for formula

$$(\alpha_1 \supset (\alpha_2 \supset \ldots \supset (\alpha_k \supset \beta) \ldots)).$$

When k = 0, this expression means the same as β. Each formula α can be uniquely presented in the form $(\alpha_1 \ldots \alpha_k \supset \beta)$, $k \geqslant 0$, where $\supset$ is not a principal sign of β. $\alpha_1, \ldots, \alpha_k$ are called the parts of α, and β is called the end of α. The length of a formula α is equal to the number of occurrences of the propositional signs in α (every propositional sign of α is counted as many times as it actually occurs in α).

The consequences of the set of formulae X are called X-consequences or X-derivable formulae. α and β are called X-equivalent if $((\alpha \supset \beta) \& (\beta \supset \alpha))$ is X-derivable. Two formulae should be called deductively X-equivalent if after adding one of them to X, the other is X-derivable.

For each logical sign (connective and the constant) the calculus W contains corresponding groups of axioms:

$\supset$: 1. $(p \supset (q \supset p))$,

2. $((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)$.

&: 1. $((p \,\&\, q) \supset p)$,

2. $((p \,\&\, q) \supset q)$,

3. $((p \supset q) \supset ((p \supset r) \supset (p \supset (q \,\&\, r))))$.

$\vee$: 1. $(p \supset (p \vee q))$,

2. $(q \supset (p \vee q))$,

3. $((p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r)))$.

0: 1. $(0 \supset p)$.

Rules of inference are: the rule of substitution for propositional variables and the rule of modus ponens.

P, V and U will denote the systems of 0-free, &-free and $\vee$-free axioms respectively.

SEPARATION THEOREM (Theorem 6, Wajsberg (1938), §11).
Each consequence α of W is derivable from the group of axioms for implication together with only those groups of axioms which contain logical signs actually appearing in α other than implication.

The plan of Wajsberg's proof is the following: for every J (J = & or $\vee$ or 0), Wajsberg gives the method by means of which to each formula α a formula β can be assigned (when J = &, then $\beta = \mathop{\&}_{i=1}^{m} \beta_i$, where all β_i are &-free) in such a way that if α is J-free, then $\alpha = \beta$ and, α is W-consequence iff β (accordingly each β_i) is derivable from the J-free axioms of W.

When J = 0, β is given according to the following:

DEFINITION OF 0-REDUCT (Definition 1, Wajsberg (1938), §7).
Let $b_1, \ldots, b_n$ be all different variables of α and α' results from replacing every occurence of the constant 0 in α by a variable a. We say that β is the reduct of α with respect to a ($\beta = R^0_a(\alpha)$) iff β has the form $((a \supset b_1) \ldots (a \supset b_n) \supset \alpha')$ when 0 occurs in α, or $\beta = \alpha$ otherwise. (The examples 1 – 3 of §7 confirm that this definition coincides with that of Wajsberg).

Proving the fact if α is a W-consequence, then each reduct of α with respect to any variable is P-derivable (Theorem 2, §7 in Wajsberg (1938)), Wajsberg errs in one point: in case 3, when $R^0_a(\beta \supset \gamma)$ and $R^0_a(\beta)$ are P-derivable, it can be proved that $((a \supset b_1) \dots (a \supset b_n) \supset \gamma')$ is also P-derivable, where b_i represents all different variables of $(\beta \supset \gamma)$ and a is a fixed one which does not occur in γ. But when β contains 0, γ is 0-free and more than one variable occurs in it, from the formula $((a \supset b_1) \dots (a \supset b_n) \supset \gamma')$ it is impossible to obtain $\gamma = R^0_a(\gamma)$ by the Wajsberg's substitution (a instead of those b_i which do not occur in γ). However, we can obtain it by the substitution ($\&_{i=1}^{m} b_i$ instead of a) used for this case in Kabziński, Porębska (1974), lemma 0, possibility (iii) (cf. Wajsberg's note 2, §7). The same authors require associating with each W-consequence the first (with respect to any well-ordering) W-derivation and replacing each occurence of the constant 0 in every reduct of W-consequence by a fixed variable not occurring in that W-derivation. But the requirement is not necessary, as Wajsberg's Theorem 2 of §7 holds for each 0-reduct of a formula with respect to any variable. In fact if γ and $(\delta \supset \varepsilon)$ are P-consequence, where γ is a variant of δ (in Church's (1956) sense, p.86), then δ and therefore ε are P-consequences too.

Tsitkin (1979) proposed also a certain reformulation of Theorem 2 of §11 without changing Wajsberg's initial proof of Theorem 2 of §7. But actually the correction of the first is superfluous when the latter holds. Wajsberg proves Theorem 2 of §11 by induction on modus ponens rule. Substitutions are done in axioms only, and the modus ponens rule is used for formulae $(\varphi_1 \supset \varphi_2)$ and φ_1 which satisfy the following condition: if φ_2 is 0-free, the same holds for φ_1. This requirement does not limit the generality of reasoning since if 0 occurs in φ_1, but does not occur in φ_2, we can eliminate it from φ_1 according to Theorem 2 of §7. Thus each $\vee$-reduct of W-consequence is U-derivable and so forth.

Wajsberg's (1938) proof of the Separation Theorem gives the original method by means of which every given W-derivation of α, having no separation property, can be transformed into the derivation of α, having this property. But it implicitly contains also the method of searching for W-derivations with separation property, founded on Fundamental Theorem of §8 and Theorem 3 of §6. This searching method involves idea of another, Gentzen-style proof of separability for the intuitionistic propositional logic, at least so far as the idea of Curry's proof of separability, by his own evidence, was involved in Gentzen's (1934) Main Theorem. However, Wajsberg did not realize that idea when chose his plan of proving separability without the use of Fundamental Theorem of §8.

From above mentioned Theorem 3 of §6 follows that to each formula α can be assigned in standard way a finite set of &-free formulae, such that α is W-consequence iff the elements of this set are V-consequences. Therefore, in Wajsberg (1938) the decision problem is solved for calculus V (see Theorem 1, §9, by proving which Wajsberg describes a decision procedure which is different from that one of Gentzen (1934).

A formula of the form $(\alpha_1 \dots \alpha_k \supset a)$, where a is a propositional sign $(k \geq 0)$, is called modified if each of its part α_i $(1 \leq i \leq k)$ is either v-free, or has the form $(\rho_1 \dots \rho_l \supset \rho \vee \gamma)$ where $l \geq 0$ and all ρ_j $(1 \leq j \leq l)$ are v-free, or has the form $(\rho \vee \gamma \supset \delta)$ where ρ, γ and δ are v-free. The parts of the last kind can be omitted, as by means of V-equivalent transformations such a part can be replaced by two new parts $(\rho \supset \delta)$ and $(\gamma \supset \delta)$.

It should be noted that using the methods of the proof of Wajsberg's (1938) Theorem 1 (§4) we can easily state that each formula (in particular, each &-free formula) is V-deductively equivalent to a certain modified formula.

One of the main Wajsberg's (1938) concepts is the notion of a thesis introduced as follows.

DEFINITION 1 (Wajsberg (1938), §8).

α is a thesis of the first order iff α has the form $(\alpha_1 \dots \alpha_k \supset (b \supset (\rho_1 \dots \rho_l \supset a)))$ where $b = a$ or $b = 0$ $(k, l \geq 0)$. The part b is called the proof part of α.

DEFINITION 2 (Wajsberg (1938), §8).

α is a thesis of the order n $(n > 1)$ iff n is the smallest natural number such that at least one of the cases I-III holds:

I. α has the form $(\alpha_1 \dots \alpha_k \supset \rho)$, $k > 0$ and the following three conditions are satisfied:

(a) ρ is a propositional sign or disjunction of formulae;

(b) for certain m $(1 \leq m \leq k)$ α_m has the form $(\rho_1 \dots \rho_l \supset \gamma)$, $l > 0$, where 1° if ρ is propositional sign then $\gamma = \rho$ or $\gamma = 0$, and 2° if ρ is a disjunction of formulae, then γ is also a disjunction;

(c) for every i ($1 \leqslant i \leqslant l$) formulae

(1i) $$(\alpha_1 \ldots \alpha_{m-1} \supset ((\beta_1 \ldots \beta_l \supset \gamma) \supset (\alpha_{m+1} \ldots \alpha_k \supset \beta_i)))$$

and the formula

(2) $$(\alpha_1 \ldots \alpha_{m-1} \supset (\gamma \supset (\alpha_{m+1} \ldots \alpha_k \supset \beta)))$$

are theses of order lower than n.

II. α has the form $(\alpha_1 \ldots \alpha_k \supset \gamma \vee \delta)$ where $k \geqslant 0$ and one of the formulae

(3) $$(\alpha_1 \ldots \alpha_k \supset \gamma)$$

or

(4) $$(\alpha_1 \ldots \alpha_k \supset \delta)$$

is a thesis of order (n - 1).

III. α has the form $(\alpha_1 \ldots \alpha_{j-1} \supset ((\gamma \vee \delta) \supset (\alpha_{j+1} \ldots \alpha_k \supset \beta)))$ and both the formulae

(5) $$(\alpha_1 \ldots \alpha_{j-1} \supset (\gamma \supset (\alpha_{j+1} \ldots \alpha_k \supset \beta)))$$

and

(6) $$(\alpha_1 \ldots \alpha_{j-1} \supset (\delta \supset (\alpha_{j+1} \ldots \alpha_k \supset \beta))).$$

are theses of order lower than n

DEFINITION 3 (Wajsberg (1938), §8).

a) If for α and a certain n holds the case I of Definition 2, then the part α_m is called a proof part of α and the formulae (1i) and (2) are called proof theses of α.

b) If for α and a certain n holds the case III of Definition 2, then that one of the formulae (3) and (4) which is a thesis of the order (n - 1), is called a proof thesis of α.

c) If for α and a certain n holds the case III of Definition 2, then the part $(\gamma \vee \delta)$ is called the proof part of α and the formulae (5) and (6) are called proof theses of α.

DEFINITION 4 (Wajsberg (1938), §8).

α is a thesis iff for certain positive natural number n α is a thesis of the order n.

A formula α is said to be T-decidable if for α we can decide in a finite number of steps whether α is or is not a thesis.

FUNDAMENTAL THEOREM (Wajsberg (1938), §8).

The set of all theses is identical with the set of all V-derivable formulae.

Wajsberg first proves that each thesis is V-derivable (Theorem 1, §8). To prove the converse statement, that each V-derivable formula is a thesis, he states that the result of any substitution in every axiom of V is a thesis (Theorem 8 of §8), and then shows that if $(\alpha \supset \beta)$ and α are theses, then β is also a thesis (Theorem 15 of §8). Wajsberg uses this Fundamental Theorem also for providing the decidability of calculus V. In §9 namely, he shows that each modified formula is T-derivable and, as every &-free formula is V-deductively equivalent to a certain modified formula, from the Fundamental Theorem follows the decidability of V.

But the counter example below shows that in fact the set of theses which satisfy the above definition of Wajsberg is smaller than the set of V-derivable formulae. The cause of this is an error, which was committed by Wajsberg while formulating the item (b) in case I of the above Definition 2. Actually, as is easily seen, the formula

(7) $\quad ((p \supset (q \supset r)) \supset (p \supset ((q \supset s) \supset ((r \supset s) \supset s))))$

is V-derivable, but with regard to the definition of Wajsberg it is not a thesis (note that (7) is a modified formula). In fact, (7) cannot be a thesis of the first order. Then if (7) is a thesis, its order is greater than 1. Therefore, (7) must satisfy one of the cases of Definition 2. (7) does not satisfy case II because it ends with a propositional sign, neither does it satisfy case III because no part of it has the form $(\gamma \vee \delta)$. Therefore, if the formula (7) is a thesis, it must satisfy case I. The condition (a) is fulfilled, as the end of (7) is the propositional sign s. Further, (7) has four parts. According to the item 1° of condition (b) the end of a proof part of (7) must be identical with the propositional variable s or the constant 0. Therefore, a proof part of (7) can be only its third or fourth part: $(q \supset s)$ or $(r \supset s)$. In the first case, according to condition (c), proof theses of (7) must be the formulae:

(8) $\quad ((p \supset (q \vee r)) \supset (p \supset ((q \supset s) \supset ((r \supset s) \supset q))))$

and $((p \supset (q \vee r)) \supset (p \supset (s \supset ((r \supset s) \supset s))))$. The latter actually is a thesis of the first order. But (8) is not a thesis, according to the item 1° of the condition (b), because all the ends of the implicational parts of (8) are distinct from q and 0. Thus, $(q \supset s)$ cannot be a proof part of (7). Similarly, we are convinced that $(r \supset s)$ also cannot be a proof part of (7). But the end of the first part of (7) is a disjunction of the propositional variables q and r, and again according to the item 1° of the condition (b) $(p \supset (q \vee r))$ cannot be a proof part of (7).

It is not difficult to point to many other similar and even more simple examples. The V-derivable formula $((p \supset (q \vee q)) \supset$ $\supset (p \supset q))$, for example, is not a thesis as its first part also does not satisfy the item 1°.

The fact that the course of Wajsberg's reasoning has not been misrepresented above can be seen when considering the correspondingly erroneous passage in his proof of Theorem 1 of §9 and his example 3 of §9. "As a modified proposition, α should end in a propositional sign, and, therefore, is of the type $(\alpha_1 ... \alpha_k \supset a)$ $(k = 1,2,...)$. The proof part of α should end in zero or α (cf. Def. 2). If no part of α ends in such a way, then α is not a thesis" (p.86 - 87; cf. Wajsberg (1977), p.160). This is confirmed also by the example 3 of §9 in which, to state that the modified formula

(9) $\quad ((r \supset p) \supset s) \supset (((r \supset q) \supset s) \supset ((r \supset (p \vee q)) \supset s)))$

is not a thesis, Wajsberg verifies only its first two parts (which end with a propositional sign).

In the formulation of the condition (b) in case I of Definition 2, Wajsberg does not take into account that the end of implicational proof part can be a disjunction of formulae even when the given formula is ended with a propositional sign.

The Fundamental Theorem of §8 will be valid, when instead of Wajsberg's Definition 2, we accept the Definition 2' in which the item 1° of the condition (b) of case I is transformed as follows:

1° if β is a propositional sign, then $\gamma = \beta$ or $\gamma = 0$ or γ is a disjunction of formulae.

Actually, in Wajsberg's proof of the statement that each thesis is V-derivable, the items 1° and 2° of the condition (b) are not used. Therefore, this statement holds also for theses in the sense of Definition 2'. As to the converse statement, its proof requires the following

CUT THEOREM (Theorem 14, Wajsberg (1938), §8; Fundamental Lemma, Tsitkin (1979)).

If a thesis α has the form

$$(\alpha_1 \dots \alpha_{j-1} \supset (\delta \supset (\alpha_{j+1} \dots \alpha_k \supset \rho))), \quad j = 1, 2, \dots, k$$

and the formula

$$(\alpha_1 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset \delta)) \tag{10}$$

is also a thesis, then the formula

$$(\alpha_1 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset \rho)) \tag{11}$$

is a thesis.

Wajsberg proves this statement by induction on the order n and the length l of α. But formulating explicitly the induction hypothesis, Tsitkin (1979) indicates that it is not sufficient to prove the Cut Theorem. He reconstructs Wajsberg's proof of Theorem 14 introducing the third additional parameter of induction on the length of the cut part δ of α. Tsitkin modifies Wajsberg's notion of a thesis, but he repeats Wajsberg's error (see, the conditions of rule R_1, p.244). This leads to the error in case III.a.1) of the reconstructed proof of Theorem 14 (as well as in corresponding case of the Wajsberg's initial proof).

In fact, in that case α has the form

$$(\alpha_1 \dots \alpha_{j-1} \supset (\delta \supset (\alpha_{j+1} \dots \alpha_k \supset \rho)))$$

and is a thesis according to case III of Definition 2. δ is its proof part and, therefore, it is a disjunction of formulae. Thus, (10) is a thesis of order higher than 1 and one of the cases I - III of Definition 2 occurs.

Suppose that (10) is a thesis according to the case I. Without limiting the generality of reasoning we can assume that α_1 is its proof part. Then α_1 has the form $(\rho_1 \dots \rho_m \supset \gamma)$. According to item 2° of the case I (b) of Definition 2, γ must be a disjunction as δ is a disjunction. But the end ρ of α may be a propositional sign.

The proof theses of (10) have the form

$$((\rho_1 \dots \rho_m \supset \gamma) \supset (\alpha_2 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset \rho_i))), \quad i = 1, \dots, m \tag{12i}$$

and

$$(\gamma \supset (\alpha_2 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset \delta))). \tag{13}$$

Now through the m-tuple application of strengthened Theorem 10 (cf. Wajsberg's §8 and Tsitkin's Lemma 2'. If in a thesis α we replace the part of the form $(\delta \supset \varepsilon)$ by ε, we again obtain a thesis the order of which is not greater than the order of α) we obtain from α the thesis

(14) $$(\gamma \supset (\alpha_2 \dots \alpha_{j-1} \supset (\delta \supset (\alpha_{j+1} \dots \alpha_k \supset \beta)))$$

which is shorter than α. Therefore, by induction hypothesis from (14) and (13) follows that the formula

(15) $$(\gamma \supset (\alpha_2 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset \beta)))$$

is a thesis. But in order to state that (11) is also a thesis in a general case we have no right to apply Definition 2 to (12i) and (15) because of item 1° of the case I, (b) (as γ is a disjunction of formulae and β may be a propositional sign). However, if we accept Definition 2' instead of Definition 2, the proof of this case will be obtained directly from the transformed item 1°.

REMARK 1.

In the reconstructed proof of Theorem 14 the case I(a) must be considered according to Wajsberg's plan. Because in this case the induction hypothesis (for uniqueness of presentation of formulae in the form $(\alpha_1 \dots \alpha_k \supset \beta)$ where $\supset$ is not a principal sign of β) must be applied not to the pair of theses

$$(\alpha_1 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset (\beta_1 \supset (\beta_2 \dots \beta_m \supset \gamma))))$$

and

(16) $$(\alpha_1 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset \beta_1)),$$

as it is done in Tsitkin **(1979)**, but to the first of these formulae and to the formula

$$(\alpha_1 \dots \alpha_{j-1} \supset (\alpha_{j+1} \dots \alpha_k \supset (\beta_2 \dots \beta_m \supset \beta_1))),$$

as it is done in Wajsberg, where the last formula is obtained from (16) according to the strengthened Theorem 12 (cf. Wajsberg **(1938)**, §8). If α is a thesis, then $(\gamma_1 \dots \gamma_u \supset \alpha)$, $u \geqslant 1$, is also a thesis the order of which is not greater then the order of α).

As it was shown above, by Definition 2' a proof part of the formula (7) may also be its first part $(p \supset (q \vee r))$. In this case the proof theses of (7) must be the formulae

$$((p \supset (q \vee r)) \supset (p \supset ((q \supset s) \supset ((r \supset s) \supset p)))),$$

which indeed is a thesis of the order 1, and

(16) $((q \vee r) \supset (p \supset ((q \supset s) \supset ((r \supset s) \supset s))))$

which by case III of Definition 2' is a thesis if both formulae

$$(q \supset (p \supset ((q \supset s) \supset ((r \supset s) \supset s))))$$

and

$$(r \supset (p \supset ((q \supset s) \supset ((r \supset s) \supset s))))$$

are theses of the lower order. But they indeed are theses of the order 2. $(q \supset s)$ is the proof part of the first formula and $(r \supset s)$ is the proof part of the second one. Therefore, the formula (17) is a thesis of the order 3 and (7) is a thesis of the order 4.

***REMARK* 2.**

In Bezhanishvili (1981) there are proposed also other alternative corrections of Definition 2 and proofs of theorems dependent on it. For example, the Fundamental Theorem of §8 will be also valid, if we accept Definition 2'' which is obtained from the initial one by the omission of the items 1° and 2° of the case I(b).

It must be emphasized that at the end of §8 of Wajsberg (1938) gives another (this time correct) definition of a thesis of the order n (see Definitions 1** and 2**), but when formulating theorems and proving them he bases only on the above mentioned incorrect Definition 2. Further, if we omit from Definition 2 the conditions concerning the formulae containing the disjunction sign, we obtain a correct definition of a thesis of the order n for $\vee$-free formulae (see Definition 2*, §8). If we omit the phrase "or 0" from the latter, we obtain also correct definition for theses which contain only an implication sign.

Now consider the following

DECISION THEOREM (Theorem 1, Wajsberg (1938), §9).
Each modified formula is T-decidable.

Incorrectness in the proof of this theorem, as it was indicated above, is connected with the use of the item 1° of Definition 2 (case I, condition (b)) and it can be removed quite simply. In particular, when α is a thesis according to the case I of Definition 2', it ends with a propositional sign as a modified formula. Therefore, the end of its proof part must coincide with the end of α or must be the constant 0 or a disjunction of formulae. In the third case, which Wajsberg does not take into account, the proof thesis of the form (2), as a shorter formula than α, is T-decidable according to the induction hypothesis: Wajsberg proves this theorem by induction on increasing length of α and (in case of equal length) on decreasing number of the parts of α. All the other cases in Wajsberg's proof of this theorem remain valid.

In accordance with this, in his example 3 (§9) Wajsberg (**1938**) does not bring to the end the solution of the question, that the formula (9) is not a thesis, because the proof part of (9) would have been its third part which ends with a disjunction.

REMARK 3.

Depending on acceptance of one of the alternative definitions of a thesis, we must correspondingly transform Wajsberg's decision procedure. Bezhanishvili (**1983**) considered the method which corresponds to Definition 2'' (see Remark 2).

Wajsberg (**1938**) also gives the first proof of the independence of logical signs of the intuitionistic propositional logic (see Theorems 1 - 6 of §10). In the proof of Theorem 2 of §10 Wajsberg states that in a certain case a certain φ cannot be a propositional sign because $(p \supset q) \supset \varphi$ cannot be a thesis. This assertion requires an additional explanation if we recall that Wajsberg's decision method is erroneous: the assertion holds because $(p \supset q) \supset \varphi$ does not contain a disjunction sign.

In connection with this result of Wajsberg McKinsey noticed that since writing his work of 1939 he has discovered that the same problem was solved by M. Wajsberg in 1938. Wajsberg's method of proving this result, writes further McKinsey, is quite unlike his, as it involves application of a decision method for the Heyting calculus (see McKinsey (**1939**), p. 158). Kabziński (**1973**) does not mention either this peculiarity of Wajsberg's proof, or the fact that Wajsberg's decision method needs a correction.

EDWARD NIEZNAŃSKI

LOGICAL ANALYSIS OF THOMISM
The Polish Programme that Originated in 1930's

1. Introduction

In the thirties Poland was dominated by a style in philosophy related in some respects to the Vienna Circle, and called by Kazimierz Ajdukiewicz the Polish Antiirrationalism. This was distinguished by three main characteristics: 1° antiirrationalism, i.e. the decision to accept only fully verifiable theses which can be demonstrated and verified; 2° the postulate of linguistic precision and exactness; 3° inclusion of logistical conceptual system, together with marked influence of symbolic logic. While, however, the Vienna Circle opted for the death of metaphysics and theology, the Polish philosophers were opposed to that, and postulated the revival of these disciplines by the means of improvement of linguistic clarity together with the application of formal logic.

This new way of philosophical thinking originated in three places: Lvov, Warsaw and Cracow. The precursors of this philosophy, later linked with the tradition of symbolic logic, were: Kazimierz Twardowski (1866 - 1938) in Lvov, a pupil of Franz Brentano, Władyslaw Weryho (1868 - 1916) in Warsaw, and Władyslaw Heinrich (1869 - 1957) in Cracow, a pupil of Richard Avenarius. The centres produced such illustrious philosophers as: Jan Łukasiewicz (1878 - 1956), Leon Chwistek (1884 - 1944), Stanisław Leśniewski (1886 - 1939), Alfred Tarski (1902 - 1983), Kazimierz Ajdukiewicz (1890 - 1963), and Tadeusz Kotarbiński (1886 - 1981). Rev. Jan Salamucha (1903 - 1944) was a pupil of Jan Łukasiewicz, father Jozef Bocheński (b. 1902), graduate of the universities of Lvov and Poznan, and Jan Franciszek Drewnowski (1896 - 1978), who was taught by Tadeusz Kotarbiński, were the main figures who applied the new symbolic logic also to Thomism.

2. A Programme of the Improvement of Thomism by Means of Symbolic Logic

In the climate of interest in the precise language of science and under the pressure of great discoveries in mathematical logic, a new programme was developing in Poland. It was the first to make use of symbolic logic in order to attain maximum precision in Thomist philosophy and theology. As far back as 1934, Jan Franciszek Drewnowski advocated in the paper **An Outline of a Philosophical Programme (1934)** , an improvement of the language of classical philosophy and theology by means of formal logic. He sketched this idea in definite form. However, the programme wasn't given it's final form till the Third Polish Conference of Philosophy (Cracow, 24 - 27 Sept. 1936). It was then presented in the book **Catholic Mind in Relation to Modern Logic (A 1937)**[1].

Members of the Conference first decided unanimously that "a follower of Christian philosophy cannot shut his eyes to the development of formal logic" (Michalski **(1937)**, p.10), and that "the development of science can neither be reversed nor is it allowed to reverse it, since the new intellectual tools cannot be denied. The thrust of the logistic criticism cannot be avoided by simply turning one's back on it" (Salamucha **(1937)**, p.152). Although the utilization of new logic "for the creation of a general world view exceeds the capacity of one generation" (Drewnowski **(1937)**, p.50), it has to be remembered that "the followers of a world view have at least three tasks to perform: to formulate that view, to defend it against objections, and to promote it. It is clear that we are far from accomplishing the first, the accomplishment of the other two is, in consequence, badly lacking" (Drewnowski **(1937)**, p.52). Therefore, the programme of the improvement of Thomism should start with the improvement of formulation and with the improvement of precision. J. Bocheński (**(1937)**, p.30) reminded the members of the Conference of the fact that "the Catholic thought has from the very beginning been characterized by a tendency to attempt maximum precision". He explained "what we mean exactly by "precision": as far as words are concerned, they must be unequivocal signs of simple things, features, experiences, etc.; they are to be clearly defined in relation to those simple signs, and in accordance with precisely stated rules. Where propositions are concerned, they cannot be accepted till we know exactly what they mean and why we assent to them. Sometimes we accept them as evident. Sometimes on the basis of faith or proof. A proof should be based on clearly formulated and verified logical rules" ((A 1937), p.28). Thus conceived precision implies deduction, "but in the field of scholastic philosophy there exist vast, strictly deductive domains. Those, if they are to represent a scientific value, must be transformed right away and developed by means of new tools" (Salamucha (1937), p.48). Mathematical logic doesn't remove the intuition, to which a philosopher often appeals, but - as J. Łukasiewicz points out ((1937), p.18) - it is necessary to enter philosophy with a logistic apparatus "in order to verify, regulate and rationalize the achievements of easily fallible intuition. Logistics strengthens our critical faculty and reveals an overabundance of error in philosophical speculation". In such circumstances Rev. Piotr Chojnacki ((1937), p.68) suggests: "First of all, it would be necessary to specify ontology as the essential philosophical discipline. I would consider it advisable to proceed with its axiomatization, according to the requirements of epistemology and modern logic".

Maximum precision is to be obtained only by the means of formalization. As far back as the Eighth International Conference of Philosophy (Prague, 2 - 7 Sept. 1934), Kazimierz Ajdukiewicz (1934) distinguished two general conceptions of formalization: the first was descriptive in relation to the natural language and followed the phenomenological method, the second was arbitrary and placed propositions among postulates. He claimed also that "one might well expect more from the second procedure than from the phenomenological method which should, however, be attempted just in case" (p.137). The members of the Cracow Conference, who opted for the formalization of deductive areas of both Thomist philosophy and theology, obviously intended only the phenomenological method. Ajdukiewicz's distinction evolved in time into two different formalization practices. For those who stuck to the phenomenological method formalization remained a sort of translation of a natural language into a symbolic one, and care was used to preserve the meaning of the translated texts. For the others, the formalization was an arbitrary procedure consisting in creating formalized theories for the use of given concepts or models, where the solution of the problems posed in the first place was the only desideratum. The texts or opinions constituted, in that case, nothing but inspiration, while, if there was translation, it was precisely reverse, from the symbolic language of a formalized theory into the natural one.

Jan Salamucha wrote: "Scholastics generally admit that philosophy is mostly a deductive science. Various simple deductions can be actually expressed in the syllogistic form. But in case of more complicated proofs, if one wants to formalize them precisely in accordance with the schemes taken from a textbook, it is necessary to do violence to the arguments, or to alter the schemes to such an extent that the incommensurability of the logical apparatus with the substantive material becomes obvious" ((A 1937), p.39). Therefore Salamucha maintains that it is necessary not only "to opt for the application of logistics to philosophy in order to preserve the traditional postulate of maximal precision of tools" ((A 1937), p.47), but "perhaps it will be necessary to develop the logic even more for the use of philosophy" ((A 1937), p.47). Rev. Konstanty Michalski claimed: "I, for my part, think that from the three parts of mathematical logic, the third, that is to say, the theory of relations, provides the most incentive for philosophic work" ((A 1937), p.10). He shared here the earlier view of Rev. Stanisław Kobylecki (1934). The latter suggested that "all knowledge, and, in the first place, all scientific and philosophical knowledge of the world, depends in fact on two conditions: on relating the things that constitute the world to each other and on ordering these connections" (p.348), where "the idea of relation is the most elementary, the most general and the most essential for both philosophy and mathematical logic" (p.353).

It is of interest that the creators of the programme of the improvement of Thomism by the means of symbolic logic, in the heat of passionate and fervent discussions, faced with numerous apparent problems advanced by their adversaries, were still aware of some difficulties presented by their own programme. Rev. P. Chojnacki warned: "Before the precise use of formal signs is introduced, it is necessary to determine exactly what is to be precisely stated, that is, it is necessary to determine exactly the meaning and sense. Logistics will be helpful here, though it won't do everything, as it cannot replace semantics" ((A 1937), p.67). Further, Michalski and Salamucha foresaw the inevitable contact of formalization with the area of analogy. They thought that "only the most precise formalization could protect us from the deformation of Thomism by involuntary sliding into other areas" (Salamucha (1937), p.147) and "block the way to a slide into arbitrariness" (Michalski (1937), p.11).

As far back as the thirties there occurred the formalization of two basic Thomist proofs, i.e. the proof of the existence of God (1934) and the proof of the immortality of the soul (1938).

3. Formalization of the Thomist proof of the immortality of the soul

Józef Bocheński's formalization of the proof of the immortality of the soul, i.e. the formalization of St. Thomas Aquinas' **Summa Theologica** I, 75, 6, is the only one so far attempted. It appeared in the second appendix to the book **Nove lezioni di logica simbolica** (Roma 1938, pp.147-155).

The essential part of Aquinas' argument is in the form: "every divisible being disintegrates *per se* or *per accidens*". But the human soul does not disintegrate *per se* or *per accidens*. Therefore the human soul is an indivisible, that is to say, an immortal being. Bocheński says that the argument has its logical form:

$$(p \rightarrow (q \vee r)) \,\&\, {\sim}q \,\&\, {\sim}r \rightarrow {\sim}p.$$

Its detailed expansion, accompanied by reasoning the premises, required formalisation in terms of predicate logic.

The formalisation begins with the introduction of 11 symbols of extra-logical constants. There are 3 individual constants:

a =: the human soul, *in suppositione simplici*[2],

s =: *per se*,

c =: *per accidens*;

and 8 extra-logical predicates (i.e. derived from outside logic):

Sx =: x is a substantial being (a substance),

Exy =: x exists in the way y does, where $y \in \{s,c\}$,

Dx =: x is a being subject to dissolution,

Bxy =: x disintegrates in the way y does, where $y \in \{s,c\}$,

Pxy =: y belongs to x *per se*,

Mxy =: x may exist apart from y,

Fx =: x is a pure form,

Ixy =: x is the existence of y.

Three kinds of numbering formulas are adopted: Roman numerals indicate logical axioms, Arabic numerals indicate extra-logical axioms and double Arabic numerals (e.g. 1.1, 1.2, ...) indicate the main theorem and all the lemmata.

The principal theorem states:

(1.3) $\sim Da$. The human soul is not a divisible being (it is immortal).

The thesis is proved from an extra-logical axiom (1) and two lemmata (1.1 and 1.2).

(1) $\bigwedge x\ (Dx \rightarrow (Bxs \vee Bxc))$. *Dupliciter enim aliquir corrumpitur. Uno modo per se, alio modo per accidens.*

(1.1) $\sim Bas$.

(1.2) $\sim Bac$.

I $1 \rightarrow (1.1 \rightarrow (1.2 \rightarrow 1.3))$, logical axiom.

One obtains theorem (1.3) from I, 1, 1.1, and 1.2 by means of the rule of detachment.

At the beginning of the text Aquinas reasons in lemma 1.2 that the human soul is a substantial being, therefore, as such, it is not a divisible being *per accidens*. Here is the formalization of the argument:

(2) $\bigwedge x\ (Sx \rightarrow Exs)$,

(3) $\bigwedge x \bigwedge y\ (Exy \rightarrow (Dx \rightarrow Bxy))$,

(4) $\bigwedge x\ (Bxs \rightarrow \sim Bxc)$,

(5) $\bigwedge x\ ((Sx \rightarrow (Dx \rightarrow \sim Bxc)) \rightarrow (Sx \rightarrow \sim Bxc))$,

(6) Sa.

(2.1) $\bigwedge x\ (Sx \rightarrow (Dx \rightarrow Bxs))$,

(2.2) $\bigwedge x\ (Sx \rightarrow (Dx \rightarrow \sim Bxc))$,

(2.3) $\bigwedge x\ (Sx \rightarrow \sim Bxc)$.

II $2 \rightarrow (3 \rightarrow 2.1)$,

III $4 \rightarrow (2.1 \rightarrow 2.2)$,

IV $2.2 \rightarrow (5 \rightarrow 2.3)$,

V $2.3 \rightarrow (6 \rightarrow 1.2)$.

Lemma (1.2) derives from logical axioms II - IV by means of detaching extra-logical axioms (2) - (6) and lemmata (2.1) - (2.3) derived on the way.

The main idea of Aquinas' proof of lemma (1.1), that the human soul does not disintegrate *per se* is as follows: no pure form is a divisible being, for existence is proper to the form *per se*. But the human soul is such a form, therefore it does not disintegrate *per se*. Here is the formalization of the proof:

(7) $\bigwedge x \bigwedge y\ (Pxy \rightarrow \sim Mxy)$,

(8) $\bigwedge x \bigwedge y\ (Fx \rightarrow (Iyx \rightarrow Pxy))$,

(9) $\bigwedge x\ (\bigwedge y\ (Iyx \rightarrow \sim Myx) \rightarrow \sim Bxs)$,

(10) Fa.

(3.1) $\bigwedge x \bigwedge y\ (Fx \rightarrow (Iyx \rightarrow \sim Myx))$

(3.2) $\bigwedge x\ (Fx \rightarrow \sim Bxs)$

VI $7 \rightarrow (8 \rightarrow 3.1)$

VII $3.1 \rightarrow (9 \rightarrow 3.2)$,

VIII $3.2 \rightarrow (10 \rightarrow 1.1)$.

If repeated detachment is used with respect to logical axioms VI - VIII, extra-logical axioms 7 - 10 and lemmata 3.1, 3.2, derived on the way, lemma 1.1 is obtained as a conclusion.

Since logical axioms are valid a priori, the proof of the immortality of the human soul, formalized by J. Bocheński, can be seen as based on ten metaphysical propositions assumed without proof, that is to say, axiomatically.

4. Formalized Thomist proofs of the existence of God

Where the formalized Thomist proofs of the existence of God are concerned, we have to take into consideration the five "ways" of St. Thomas Aquinas, presented in his **Summa Theologica** (I, q.2) and **Summa contra Gentiles** (I, 13). The following relations are taken into account:

(1) xR_1y =: x moves y,

(2) xR_2y =: x is an efficient cause of y,

(3) xR_3y =: x is a reason for the existence of y,

(4) xR_4y =: x is no more perfect than y; or, x is less perfect than y or equal to y,

(5) xR_5y =: x provides principles for intentional transformations of y.

To express ourselves in a formalized language of the first order, we assume that in its metalanguage there is a non-empty universum U, as a range of variability of individual variables. Set B of all real individual beings differs from the universum U. Then for any relation R we have:

(6) $DR \overset{\mathrm{df}}{=} \{x: \bigvee y\ xRy\}$, domain of relation R,

(7) $D'R \overset{\mathrm{df}}{=} \{y: \bigvee x\ xRy\}$, counter-domain of R,

(8) $FR \overset{\mathrm{df}}{=} DR \cup D'R$, field of relation R.

Now it is possible to show that in all formalizations of Aquinas' "ways" presented below, it is assumed implicitly that:

(9) $B \neq \emptyset$ (where $\emptyset$ is the empty set),

(10) $B \subset U$ and $B \subset FR_i$, $i = 1, 2, ..., 5$, i.e. the transcendentality of the universe and of the field of relation in question,

(11) $\breve{R}_i B \subset B$, $i = 1, 2, ..., 5$, i.e. the image of B with respect to the converse of R_i is included in the set B; or, B is closed with respect to R_i; or, R_i is real; or, the reality of the relation in question - the inheritance of real existence - is due to the converse of the relation R_i[3].

All the formalized Thomist proofs of the existence of God deal with the problem of extreme elements in the respective relations[4]. We are concerned with:

(a) the set of first elements of initial relation R (denoted as IR):

$$IR \overset{\mathrm{df}}{=} \{x \in FR: \bigwedge y\ (y \in FR \ \&\ x \neq y \rightarrow xRy)\},$$

or the set of last elements of R (marked by LR):

$$LR \overset{\mathrm{df}}{=} I\breve{R};$$

(b) or at most a one-element set of initial elements of R (symbolized by 1IR):

$$1IR \overset{\mathrm{df}}{=} \{x \in IR: \bigwedge y\ (y \in IR \rightarrow x = y)\},$$

or at most a one-element set of last elements of R (denoted by 1LR):

$$1LR \overset{\mathrm{df}}{=} 1I\breve{R};$$

(c) or the set of minimal elements of R (marked by MinR):

$$\mathrm{Min}R \overset{\mathrm{df}}{=} \{x \in FR: \bigwedge y\, (yRx \rightarrow x = y)\},$$

or the set of maximal elements of R (symbolized by MaxR):

$$\mathrm{Max}R \overset{\mathrm{df}}{=} \mathrm{Min}\breve{R};$$

(d) or the set of relatively initial elements of R (denoted as IR/y):

$$IR/y \overset{\mathrm{df}}{=} \{x \in \mathrm{Min}R: x = y \vee xR_{po}y\}, \quad \text{where}$$

$$xR_{po}y \overset{\mathrm{df}}{\longleftrightarrow} \bigwedge X\, (x \in X \;\&\; RX \subset X \rightarrow y \in RX), \quad \text{and}$$

$$y \in RX \overset{\mathrm{df}}{\longleftrightarrow} \bigvee x\, (x \in X \;\&\; xRy),$$

or the set of relatively last elements of R (marked by LR/y):

$$LR/y \overset{\mathrm{df}}{=} I\breve{R}/y.$$

The problems raised for the use of formalized theodicy can be put together into the following groups of questions:

1° for i = 1, 2, 3, 5:

a: Is $IR_i \neq \emptyset$ or $IR_{ipo} \neq \emptyset$ or $L\breve{R}_i \neq \emptyset$ or $L\breve{R}_{ipo} \neq \emptyset$?

b: Is $1IR_i \neq \emptyset$ or $1IR_{ipo} \neq \emptyset$ or $1L\breve{R}_i \neq \emptyset$ or $1L\breve{R}_{ipo} \neq \emptyset$?

c: Is $\mathrm{Min}R_i \neq \emptyset$ or $\mathrm{Max}\breve{R}_i \neq \emptyset$?

d: Is $IR_i/y \neq \emptyset$ or $L\breve{R}_i/y \neq \emptyset$ for any $y \in B$?

2° for i = 4:

a: Is $LR_i \neq \emptyset$ or $LR_{ipo} \neq \emptyset$ or $I\breve{R}_i \neq \emptyset$ or $I\breve{R}_{ipo} \neq \emptyset$?

b: Is $1LR_i \neq \emptyset$ or $1LR_{ipo} \neq \emptyset$ or $1I\breve{R}_i \neq \emptyset$ or $1I\breve{R}_{ipo} \neq \emptyset$?

c: Is $\mathrm{Max}R_i \neq \emptyset$ or $\mathrm{Min}\breve{R}_i \neq \emptyset$?

All the formalized Thomist proofs of the existence of God will be presented here in a metalanguage, that is to say, within a kind of "applied" set theory. Let's assume further the following indispensable set-theoretic notions:

(12) $$R \in irr \overset{\mathrm{df}}{\longleftrightarrow} \bigwedge x \bigwedge y\, (xRy \rightarrow x \neq y),$$
where irr =: the set of irreflexive relations,

(13) $$R \in refl \overset{\mathrm{df}}{\longleftrightarrow} \bigwedge x\, (x \in FR \rightarrow xRx),$$
where refl =: the set of reflexive relations,

(14) $$R \in as \overset{\mathrm{df}}{\longleftrightarrow} \bigwedge x \bigwedge y\, (xRy \rightarrow \sim yRx),$$
where as =: the set of asymmetrical relations,

(15) $R \in ants \overset{df}{\longleftrightarrow} \bigwedge x \bigwedge y\, (xRy \,\&\, yRx \rightarrow x = y)$,
where ants =: the set of antysymmetrical relations,

(16) $R \in trans \overset{df}{\longleftrightarrow} \bigwedge x \bigwedge y \bigwedge z\, (xRy \,\&\, yRz \rightarrow xRz)$,
where trans =: the set of transitive relations,

(17) $R \in con \overset{df}{\longleftrightarrow} \bigwedge x \bigwedge y\, (x,y \in FR \rightarrow (x = y \vee xRy \vee yRx))$,
where con =: the set of connected relations,

(18) $ord_s \overset{df}{=} irr \cap trans$,
where ord_s =: the set of strongly ordering relations,

(19) $ord_w \overset{df}{=} refl \cap ants \cap trans$,
where ord_w =: the set of weakly ordering relations,

(20) $chain_s \overset{df}{=} ord_s \cap con$,
where $chain_s$ =: the set of strong chains (strongly linear ordering relations),

(21) $chain_w \overset{df}{=} ord_w \cap con$,
where $chain_w$ =: the set of weak chains (weakly linear ordering relations),

(22) $R \in MQ \overset{df}{\longleftrightarrow}$
$\bigwedge x \bigwedge y\, (x,y \in FR \rightarrow (x = y \vee xRy \vee yRx \vee \bigvee z\, (zRx \,\&\, zRy)))$,
where MQ =: the set of multiplicative quasi-half-lattices.

4.1 Formalized versions of the argument *ex motu*

[A] Rev. Jan Salamucha was the first to formalize in 1934 Aquinas' argument *ex motu* of the existence of God, inserted in **Summa contra Gentiles** (I, 13). Józef Bocheński reviewed Salamucha's paper in 1935 and proposed some important modifications of his formalization. William Bryar ((**1951**), pp.211-219) presented his views on Salamucha's formalization and Bocheński's comments. Salamucha's paper was translated into English in 1958. A biographical note to this translation was written by Bolesław Sobociński.

Salamucha wrote in a tortuous style. Of his (numerous) assumptions only the 9 following are significant in the proof:

(A1) $R_1 \in trans$;

(A2) $R_1 \in con$;

(A3) $\bigwedge x\, (x \in D'R_1 \rightarrow \bigvee a \bigvee b\, (aPx \,\&\, bPx))$,
where aPx =: a is a proper part of x. (A3) indicates that if something is in motion, it consists of proper parts);

(A4) $$\bigwedge x\, (\bigvee a \bigvee b\, (aPx \,\&\, bPx) \rightarrow \sim xR_1x).$$

If an object x comprises of two proper parts, a and b, it is not true that object x makes itself move[5];

(A5) $$D'R_1 \subset C,$$

where $x \in C$ =: x is a body. If an object x is in motion, that very object x is a body;

(A6) $$C \cap D'R_1 \subset D'H,$$

where tHx =: t is the measure of the continuity of the movement of x. If a material object is in motion, a certain segment of time is the measure of the continuity of that movement;

(A7) $$\bigwedge x \bigwedge t\, (x \in C \,\&\, tHx \rightarrow t \in Fin),$$

where $t \in Fin$ =: t is a finite segment of time. According to (A7), if a material object is in motion, the measure of the continuity of the movement of that object is a finite segment of time;

(A8) $$\bigwedge x \bigwedge y \bigwedge t_1 \bigwedge t_2\, (xR_1y \,\&\, t_1Hx \,\&\, t_2Hy \rightarrow t_1 = t_2).$$

The measure of the continuity of the movement of a mover is equal to the continuity of the movement of the object moved;

(A9) An infinite body, or even an infinite class of bodies which seem to form a single body *per continuationem* or *per contiguationem* cannot be in motion in a finite segment of time. Salamucha didn't formalize the quoted assumption (A9).

In accordance with the assumptions, Salamucha first proves a few lemmata:

L1. $R_1 \in chain_s \,\&\, IR_1 \neq \emptyset \rightarrow IR_1 - D'R_1 \neq \emptyset.$

This lemma can be considered as a particular instance of the law of the calculus of relations:

$$\bigwedge R\, (R \in chain_s \,\&\, IR \neq \emptyset \rightarrow IR - D'R \neq \emptyset).$$

In **Summa contra Gentiles** Aquinas provides three proofs of the thesis *omne quod movetur ab alio movetur*, i.e.

$$\bigwedge y\, (y \in D'R_1 \rightarrow \bigvee x\, (xR_1y \,\&\, x \neq y)).$$

Salamucha formalized the first proof basing on assumptions (A3) and (A4). The second proof omitted, he formalized the third one. But the thesis in question turns out to be insignificant to the proof of the main theorem referring to the existence of the first mover.

Therefore, Salamucha demonstrates the following lemma separately:

L2. $R_1 \in \mathrm{irr}$.

Here is the proof of that lemma:

1. xR_1y, supp. (supposition of the proof)
2. $y \in D'R_1$, from the definition of $D'R_1$ and 1
3. $\bigvee a \bigvee b$ (aPy & bPy), from (A3) and 2
4. $\sim yR_1y$, from (A4) and 3
5. $x \neq y$, since 1 and 4.

From assumptions (A1) and (A2), and from lemma L2 follows the lemma:

L3. $R_1 \in \mathrm{chain}_s$.

The proof of the lemma referring to the non-occurrence of the so-called regress into infinity in relation R_1 occupies a major part of the argument:

L4. $DR_1 - D'R_1 \neq \emptyset$.

Apagogical suppositional proof of L4:

1. $DR_1 - D'R_1 = \emptyset$, s.a.p. (the supposition of the apagogical proof)
2. $IR_1 = DR_1 - D'R_1$, from $\bigwedge R$ ($R \in \mathrm{chain}_s \rightarrow IR = DR - D'R$) and L3
3. $IR_1 = \emptyset$, from 1 and 2
4. $\overline{\overline{FR}} \geqslant \aleph_0$, since $\bigwedge R$ ($R \in \mathrm{chain}_s$ & $\overline{\overline{FR}} < \aleph_0 \rightarrow IR \neq \emptyset$), L3, 3,
   where $\overline{\overline{X}}$ =: the cardinal number of the set X
5. $FR_1 = D'R_1$, since $\bigwedge R$ ($DR - D'R = \emptyset \rightarrow FR = D'R$) and 1
6. $FR_1 \subset C$, from (A5) and 5
7. $FR_1 \subset D'H$, from 5, 6 and (A6)
8. $FR_1 \subset H(Fin)$, from (A7), 6, 7, $R(X) \overset{df}{=} \{y: \bigvee x\, (x \in X \;\&\; xRy)\}$
9. $\bigwedge x \bigwedge y$ ($x,y \in FR_1 \rightarrow \bigwedge t_1 \bigwedge t_2$ ($x \neq y$ & t_1Hx & $t_2Hy \rightarrow t_1 = t_2$)),
   from (A8), (A2)
10. k $\overset{df}{=}$ the object being the mereologic sum of all elements of the set $D'R_1$
11. $\bigwedge a \bigwedge x$ ($aP^+x \overset{df}{\longleftrightarrow}$ (aPx $\vee$ a = x)),
    where aP^+x =: a is a non-proper part of x

12. $\bigwedge a\,(aP^{+}x \longleftrightarrow a \epsilon D'R_1)$, from 11, 10

13. $k \epsilon D'R_1$, since $kP^{+}k$ (from 11, $k = k$) and 12.

14. $k \epsilon C$, from (A5), 13

15. $\bar{x} \overset{\mathrm{df}}{=} \overline{\overline{\{a:\ aP^{+}x\}}}$, where $\bar{x}$ =: the mereologic power of x

16. $\bar{k} = \overline{\overline{\{a:\ aP^{+}k\}}} = \overline{\overline{D'R_1}} = \overline{\overline{FR_1}}$, from 15, 12, 5

17. $\bar{k} \geqslant \aleph_0$, from 16 and 4

18. $k \epsilon H(Fin)$, from 8, 13, 5

19. $\bigwedge x \bigwedge t\,(\bar{x} \geqslant \aleph_0\ \&\ tHx \rightarrow \sim t \epsilon Fin)$, i.e. (A9)

20. $\bigvee x \bigvee t\,(\bar{x} \geqslant \aleph_0\ \&\ tHx\ \&\ t \epsilon Fin)$, from 17, 18

contradiction: 19, 20.

Line 10 is undoubtedly the weakest item of the proof, where the mereologic sum of all real beings is assumed as an individual real being.

The last lemma and Salamucha's main theorems follow simply from the lemmata:

L5. $IR_1 \neq \emptyset$, since $\bigwedge R\,(R \epsilon chain_s \rightarrow IR = DR - D'R)$, L3, L4.

Th.1. $IR_1 - D'R_1 \neq \emptyset$, from L1, L2, L5.

Th.2. $1IR_1 - D'R_1 \neq \emptyset$, since Th.1, $\bigwedge R\,(R \epsilon chain_s \rightarrow IR = 1IR)$, L3.

Salamucha's formalization of the argument *ex motu* was rather a loose translation of Aquinas' text into symbolic language. Salamucha, as an author of the formalization, thought that he had managed to demonstrate the formal correctness of Aquinas' deduction or, in other words, its exemption from *non sequitur* errors. But still, he correctly doubted whether the proof was convincing. He was also wrong when he falsely assumed that the relations of moving R_1 were connected and that they ordered their own field in a linear fashion. He also misread Aquinas, attributing to him the view that the relation was a chain.

[B] Father J. Bocheński in his review **(1935)** opts for an interpretation of Aquinas' text, according to which the argument *ex motu* does not prove exactly the existence of a *primum movens immobile* but only the existence of *movens immobile*, i.e. the thesis: $MinR_1 \neq \emptyset$. Bocheński, however, assumes also $R_1 \epsilon chain_s$. Hence by the law $\bigwedge R\,(R \epsilon as \cap con \rightarrow IR = MinR)$, we obtain $IR_1 = MinR_1$..

Bocheński suggested also several important terminological modifications of Salamucha's symbolic language. Notably, he introduced the notion of the triadic relation of motion:

$$Mxyz =: x \text{ moves } y \text{ towards } z.$$

The connection of notions R_1 and M is expressed by the definition:

(23) $$xR_1y \overset{df}{\leftrightarrow} \bigvee z\, Mxyz$$

Bocheński presented the Thomist doctrine concerning relation M, included in the thesis *nihil enim movetur nisi secundum quod est in potentia ad illud ad quod movetur*, by the means of four axioms:

(B1) $$\bigwedge y\, (y \in D'R_1 \rightarrow \bigvee x \bigvee z\, Mxyz);$$

(B2) $$\bigwedge x \bigwedge y \bigwedge z\, (Mxyz \rightarrow xAz),$$
where $xAz =: x$ is in act with respect to z;

(B3) $$\bigwedge x \bigwedge y \bigwedge z\, (Mxyz \rightarrow yPz),$$
where $yPz =: x$ is in potency in respect to z;

(B4) $$\bigwedge x \bigwedge z\, (xPz \rightarrow \sim xAz).$$

[C] A German philosopher, Johannes Bendiek (1956) was the immediate continuator of Salamucha's work. He didn't so much formalize Aquinas, but rather presented three formalized proofs of the existence of the first unmoved mover.

Bendiek accepts 7 assumptions in system I (among others $R_1 \in chain_s$), but for the proof he uses only:

(C1) $$R_1 \in irr \cap con,$$

together with the assumption of the absence of an *regressus ad infinitum*:

(C2) $$\sim\bigwedge x \bigvee y\, (x \neq y \;\&\; yR_1x).$$

In accordance with the metalanguage compact: $FR_1 = U$, assumption (C2) means that $MinR_1 \neq \emptyset$. Since $\bigwedge R\, (R \in con \rightarrow MinR \subset IR)$, thus $MinR_1 \subset IR_1$, i.e. $IR_1 \cap MinR_1 = MinR_1$, Since $MinR_1 \neq \emptyset$, thus also $IR_1 \cap MinR_1 \neq \emptyset$, which is the final theorem of system I.

It is assumed in system II that $R_1 \in as$ and $IR_1 \neq \emptyset$, hence by the law $\bigwedge R\, (R \in as \rightarrow IR \subset 1IR)$ the theorem $1IR \neq \emptyset$ is obtained.

Bendiek assumes in system III that 1° $R_1 \in chain_s$, 2° $R_1 \neq \emptyset$, 3° $D'R_1 \subset D'S$, where $S =:$ *causa sufficiens*, and 4° $D'R_1 \cap DS = \emptyset$.

Here is the proof of the theorem of that system:[6]

1. $S \neq \emptyset$, from 3° and 2°
2. $\bigwedge x \bigwedge y \bigwedge z\, (xSz \rightarrow \sim yR_1x)$, from 4°
3. $\bigwedge x \bigwedge y \bigwedge z\, (xSz \rightarrow (xR_1y \vee x = y))$, from 2, 1°, $FR_1 = U$
4. $\bigvee x \bigvee z\, xSz \rightarrow \bigvee x \bigwedge y\, (x \neq y \rightarrow xR_1y)$, from 3
5. $IR_1 \neq \emptyset$, from df.IR, 4, 1, $FR_1 = U$.

[D] The Italian Francesca Rivetti Barbò did a lot towards the formalization of Aquinas' text from **Summa Theologica** (I, q.2,a.3) inclusive of the argument *ex motu*. She investigated the problem in the papers **(1960)**, **(1962)**, **(1966)** and **(1967)**. While in the **(1960)** paper Barbò formalizes the whole of the "first way", in subsequent ones she limits formalization to the thesis *omne quod movetur ab alio movetur*. With respect to the remaining proof of the existence of an unmoved mover, she maintains that it can be done only intuitively. Barbò's parallel development of each of the conceptual contents of the propositions, in relation to their sense in the premises of every one of Aquinas' "ways", makes formalization all but impossible in F. Rivetti's opinion.

Barbò introduced many innovations with respect to the formalization of the argument *ex motu* **(1960)**. First of all she rejected Salamucha's, Bocheński's and Bendiek's idea that $R_1 \in chain_s$, opposing especially the assumption that $R_1 \in con$. Adopting Bocheński's symbolic notation she yet formulated the problem of Thomist deduction in a completely different way. She attempted to demonstrate that $\bigvee x\, (IR_1/x \neq \emptyset)$.

In order to prove the thesis *omne quod movetur ab alio movetur* Barbò assumes four axioms (where the meaning of the symbols "A", "M", "P" is the same as for Bocheński):

(D1) $$\bigwedge x \bigwedge z\, (xGz \rightarrow xPz),$$
where $xGz =: x$ is in motion to z;

(D2) $$\bigwedge x \bigwedge z\, (\bigvee y\, Mxyz \rightarrow xAz);$$

(D3) $$\bigwedge x \bigwedge z\, (xPz \rightarrow \sim xPz);$$

(D4) $$\bigwedge x \bigwedge z\, (xGz \longleftrightarrow \bigvee y\, Myxz).$$

It follows from the definition (23) and from the above axioms that $R_1 \in irr$. If it is assumed that for an x: xR_1x, then by (D1), (D2) and (D3) it would be valid that $\bigvee z\, (xAz \;\&\; xPz)$, which contradicts (D3).

In order to prove the theorem of the existence of a relatively first mover, Barbò uses three additional assumptions:

(D5) $$R_1 \in \mathrm{trans};$$

(D6) $$\bigwedge x\,(x \in D'R_1 \rightarrow \sim\bigwedge y\,(yR_{1po}x \rightarrow y \in D'R_1);$$

(D7) $$D'R_1 \neq \emptyset.$$

Here is the proof of the theorem $\bigvee x\,(IR_1/x \neq \emptyset)$:

1. $R_1 \in \mathrm{ord}_s$, since $R_1 \in \mathrm{irr}$ and (D5)
2. $R_{1po} \in \mathrm{irr}$, since $\bigwedge R\,(R \in \mathrm{trans} \rightarrow R_{po} = R)$ and 1
3. $\bigwedge y\,(IR_1/y = \{x \in \mathrm{Min}R_1\colon xR_{1po}y\})$, from df.IR/y and 2
4. $\bigwedge x\,(x \in D'R_1 \rightarrow \bigvee y\,(yR_{1po}x \;\&\; \sim y \in D'R_1))$ from (D6)
5. $\bigwedge x\,(x \in D'R_1 \rightarrow \bigvee y\,(y \in FR_1 - D'R_1 \;\&\; yR_{1po}x))$, from 4
6. $\bigwedge x\,(x \in D'R_1 \rightarrow \bigvee y\,(y \in \mathrm{Min}R_1 \;\&\; yR_{1po}x))$,
   from 5, 2 and $\bigwedge R\,(R \in \mathrm{irr} \rightarrow \mathrm{Min}R = FR - D'R)$
7. $\bigwedge x\,(x \in D'R_1 \rightarrow IR_1/x \neq \emptyset)$, from 6, 3
8. $\bigvee x\,(IR_1/x \neq \emptyset)$, from 7 and (D7).

The above deduction is nothing but a simple transition "from the general to the particular". For axiom (D6) is strong enough to assume that there exists a relatively first mover for each moved object. While the conclusion only that there exists a relatively first mover. But what the axiom assumes is evident neither a priori nor a posteriori.

[E] Ivo Thomas **(1960)** points out that assumption (D5) and lines 1 - 3 are idle in Barbò's formalized proof. Assumption (D4) is a mere definition of an extra-logical constant "G", and since $\bigwedge R\,(R \subset R_{po})$ it is possible to reduce the assumption (D6) to the following form: $\bigwedge x\,(X \in D'R_1 \rightarrow \breve{R}_1\{x\} - D'R_1 \neq \emptyset)$. The proof should stop at line 7 and, in consequence, assumption (D7) should be eliminated.

However, it should be said in Barbò's defence, that her formalization was constructed with a different purpose than Thomas'. While Barbò formalized Aquinas' specific text in order to demonstrate its final correctness, Thomas tried to create a clear formalized proof of the existence of a relatively first mover.

[F] In 1975 Fr. Korneliusz Policki (Academy of Catholic Theology in Warsaw), also constructed a formalized proof of the existence of the unique *primum movens immobile,* without formalizing any Thomist text. If one disregards Bowman L. Clarke's remark (in his book **(1966)**), vis à vis the applicability of Zorn's lemma to the first "way", Policki was the first to present a formalized argument *ex motu,* using the mentioned lemma. To begin with, he supposes that

(F1) $\qquad FR_1 \neq \emptyset$ and

(F2) $\qquad \breve{R}_1 \in ord_s$.

Then he proves that relation $\breve{R}_1 \cup I \in ord_w$, where $xIy =: x = y$. Finally he assumes

(F3) for every two weak chains included in the relation $\breve{R}_1 \cup I$ there exists a common upper bound.

Hence Zorn-Kuratowski's lemma yields the following theorem:

(TF) $\qquad 1L(\breve{R}_1 \cup I) \cap Max(\breve{R}_1 \cup I) \neq \emptyset$.

Assumption (F3) and conclusion (TF) are equivalent inferentially in Policki's formalization, while the whole argument cannot be written down in elementary language.

[G] Edward Nieznański suggests (**(1980)**, p.107) the weakening of assumption (F3) by reduction to two assumptions:

(F3.1) $\qquad MinR_1 \neq \emptyset$, and

(F3.2) $\qquad R_1 \in MQ$.

It will be possible to write them both down in elementary language:

(F3.1') $\qquad \sim\bigwedge x \bigvee y\, yR_1x$;

(F3.2') $\qquad \bigwedge x \bigwedge y\, (x = y \vee xR_1y \vee yR_1x \vee \bigvee z\, (zR_1x \,\&\, zR_1y))$.

From those assumptions follows the thesis $1IR_1 \neq \emptyset$, since $\bigwedge R\, (MinR \neq \emptyset \,\&\, R \in MQ \rightarrow 1IR \neq \emptyset)$.

[H] The argument *ex motu* for the existence of God, presented by Leibniz in **Demonstratio Existentiae Dei ad Mathematicam Certitudinem Exacta** is worth attention as compared to Aquinas' text **Summa contra Gentiles** (I, 13). This argument was formalized by Krystyna Błachowicz **(1982)**.

The formalized argument of the existence of a *primum movens immobile* is based on three primary theorems:

(H1) $$\bigvee y \bigwedge y\,(xPy \longleftrightarrow \Phi(x)).$$

This is Leibniz's postulate, where xPy =: x is a non-proper part of y, while $\Phi(x)$ is a metalanguage variable for propositional formulas comprising free variable x (and not comprising free variable y);

(H2) $$P \in \mathrm{refl};$$

(H3) $$\bigwedge x \bigwedge y\,(xR_1y \rightarrow x \neq y\ \&\ {\sim}xPy)^7.$$

Here is the proof of the main theorem: $IR_1/a \neq \emptyset$, where a is any individual constant and $a \in B$:

1. $IR_1/a = \emptyset$, s.a.p. (the supposition of the apagogical proof)
2. $\bigvee y \bigwedge x\,(xPy \longleftrightarrow x \in D'R_1\ \&\ xR_{1*}a)$,
   from (H1) and $xR_*y \overset{df}{\longleftrightarrow} (xR_{po}y \vee x = y)$
3. $\bigwedge x\,(xPk \longleftrightarrow x \in D'R_1\ \&\ xR_{1*}a)$, from 2, k is a constant
4. $k \in D'R_1$, from 3, (H2)
5. $kR_{1*}a$, from 3, (H2)
6. $\bigwedge x\,(xR_{1*}a \rightarrow {\sim}x \in MinR_1)$, from 1 and df.IR/y
7. ${\sim}k \in MinR_1$, from 6, 5
8. $\bigvee x\,(x \neq k\ \&\ xR_1k)$, from 7, 4, df.MinR
9. $b \neq k\ \&\ bR_1k$, from 8, b is a constant
10. $bR_{1*}a$, from 9, 5
11. ${\sim}bPk$, from (H3) and 9
12. ${\sim}b \in D'R_1$, from 3, 11, 10
13. $b \in MinR_1$, from 12, 9, df.MinR
14. $b \in IR_1/a$, from df.IR/y, 13, 10
15. $IR_1/a \neq \emptyset$, from 14

   contradiction: 1, 15.

[I] Lastly let's mention some contributions to the formalization of the Thomist theory of motion, movement and change, that is to say, two papers by Witold Marciszewski **(1959)** and **(1960)**, a review of L. Larouche's dissertation **(1964)** by Rev. Stanisław Kamiński **(1967)** and a book by Stanisław Kiczuk **(1984)**.

4.2. Formalized versions of the argument *ex ratione causae efficientis*

[A] The first formalized argument *ex causae efficientis,* which in its keynote refers to Aquinas' second "way" (**Summa Theologica** I, q.2,a.3), derives from an Austrian professor Wilhelm K. Essler (**1969**).

It is assumed

(A1) $$R_2 \neq \emptyset,$$

(A2) $$D'R_2 \subset R_2(\mathrm{Min}R_2),$$

(A3) $$R_2 \in \mathrm{irr}$$

(A4) $$R_2 \in MQ$$

and (implicitly) $U = FR_2$. Proof of two theorems is provided:

(AT1) $\mathrm{Min}R_2 \neq \emptyset$, from (A1) and (A2), and

(AT2) $$1IR_2 \neq \emptyset$$
from the law $\bigwedge R\,(R \in MQ \,\&\, \mathrm{Min}R \neq \emptyset \rightarrow 1IR \neq \emptyset)$, (A4) and (AT1).

With respect to the argument it should be pointed out that an equivalence occurs in it: (A2) $\leftrightarrow$ (AT1). Here is the proof of the implication (AT1) $\rightarrow$ (A2):

1. $y \in D'R_2$, supp.
2. $\sim y \in R_2(\mathrm{Min}R_2)$, s.a.p.
3. $\bigwedge x\,(x \in \mathrm{Min}R_2 \rightarrow \sim xR_2y)$, from df.RX and 2
4. $a \in \mathrm{Min}R_2$, from (AT1), a is a constant
5. $a \in IR_2$, since $\bigwedge R\,(R \in MQ \rightarrow \mathrm{Min}R \subset IR)$, 4, (A4)
6. $\sim aR_2y$, from 3, 4
7. $a = y \vee aR_2y$, from df.IR, 1, 5
8. $a = y$, from 7, 6
9. $a \in D'R_2$, from 1, 8
10. bR_2a, from 9, df.D'R, b is a constant
11. $b \neq a$, from (A3), 10
12. $b = a$, from 4, 10, df.MinR

contradiction: 11, 12.

[B] Nieznański develops two formalizations **(1982)** and **(1983/84)**, the first for Aquinas' "second way" and the other for the modern argument *ex ratione causae efficientis*. The latter comes about under Kazimierz Kłósak's, a distinguished Polish Thomist, who wrote (**(1973)**, p.205): "the thesis about the impossibility of the regress into infinity within the efficient causes should be replaced by the principle of sufficient reason".

Here are the assumptions of the formalized argument:

(B1) $$U = FR_2;$$

(B2) $$R_2 \subset T,$$
where xTy =: x outpaces y in existence;

(B3) $$T \in irr;$$

(B4) $$R_2 \subset R_3;$$

(B5) $$U = D'S_3,$$
where xS_3y =: x is a sufficient reason for the existence of y and $S_3 \overset{\text{df}}{=} (MinR_3)/R_3$ (where $x(X)/Ry \overset{\text{df}}{\longleftrightarrow} xRy \;\&\; x \in X$);

(B6) $$U = B \neq \emptyset.$$

From assumptions (B2) and (B3) by

$$\bigwedge R \bigwedge S\,(S \subset R \;\&\; R \in irr \rightarrow S \in irr)$$

we obtain the lemma

(b7) $$R_2 \in irr.$$

From (B6) and (B5) follows the next lemma

(b8) $$D'S_3 \neq \emptyset.$$

The proof of the main theorem

(b9) $$MinR_2 \neq \emptyset$$

is as follows:

1. $D'S_3 = D'R_3$, from df.S_3
2. $D'R_3 = U$, from (B5) and 1
3. $DR_3 \subset D'R_3$, since $DR_3 \subset U = D'R_3$, 2
4. $FR_3 = U$, since 3, $D'R_3 = DR_3 \cup D'R_3 = FR_3$, 2
5. $FR_2 = FR_3$, from 4, (B1)
6. $MinR_2 \neq \emptyset$, since $\bigwedge R \bigwedge S\,(S \subset R \;\&\; FR = FS \rightarrow MinR \subset MinS)$, (B4), 5, (b8), df.$S_3$.

4.3 Formalized versions of the argument *ex contingentia*

Kłósak sees (1957) that modern Thomist proofs of the existence of a necessary being as having their origin not in Aquinas' but in Leibniz.

[A] Nieznański (1977) attempted the logical analysis of the notion of "the essence, to which existence belongs". He supposes two separate universes: B (the set of beings) and T (the set of all intervals of time). He then assumes that $B \neq \emptyset$, $T \neq \emptyset$ and $B \cap T = \emptyset$. The set of all beings present at moment t is

$$A_t \overset{df}{=} \{x\colon xAt\},$$

where A_t =: x is a present being at moment t.

Hence $A \subset B \times T$, and $B = \bigcup_{t \in T} A_t$. The definitions of various notions of essence are obtained in formalized elementary language. Here are their denotations:

(1) for the present essence: $F_t(x) \overset{df}{=} \{x\} \cap A_t$;

(2) for the real essence: $F(x) \overset{df}{=} \bigcup_{t \in T} F_t(x)$, hence $F(x) = \{x\} \cap B = \{x\}$;

(3) for the essence of the species: $G(X) = Y \overset{df}{\longleftrightarrow} X = Y$, and

(4) for the universal essence: $H(x) \overset{df}{=} \bigcap_{t \in T} F_t(x)$,

hence $H(x) = \bigcap_{t \in T}(\{x\} \cap A_t) = \{x\} \cap \bigcap_{t \in T} A_t$.

With the help of the notion of H, there were also defined some closely related notions of necessary existence. Here are their denotations:

(5) $N \overset{df}{=} \{x\colon H(x) \neq \emptyset\}$, hence $N = \{x\colon x \in \bigcap_{t \in T} A_t\}$;

(6) $N' \overset{df}{=} \{X \subset B\colon \bigwedge_{t \in T}(X \cap A_t \neq \emptyset)\}$, and

(7) $N'' \overset{df}{=} \{X \subset B\colon \bigwedge_{t \in T} \bigvee_{z \in B}(X \cap A_t = \{z\})\}$.

The main question of the theodicy, whether $N \neq \emptyset$, i.e. $\bigcap_{t \in T} A_t \neq \emptyset$ is regarded as an open problem. It wants non-arbitrary solutions.

[B] Nieznański gives several versions of the formalized proofs of the existence of a necessary existence **(1979)**, **(1981)** and **(1982b)**. The simplest system gets the following form:

1) assumptions:

(B1) $U = B \neq \emptyset$; (B2) $R_3 \subset B \times B$; and (B3) $B \subset D'S_3$, where $S_3 \overset{df}{=} (\mathrm{Min}R_3)/R_3$;

2) conclusions:

(b1) $DS_3 \neq \emptyset$, from (B1), (B3) by $\bigwedge R\ (D'R \neq \emptyset \rightarrow DR \neq \emptyset)$;

(b2) $\bigwedge x\ (X \in DS_3 \rightarrow xS_3x)$, since:

1. $x \in DS_3$, supp.
2. $x \in MinR_3$, from df.S_3, df.(X)/R, 1
3. $\bigwedge z\ (zR_3x \rightarrow x = z)$, from df.MinR, 2
4. $x \in B$, from 1, df.S_3, (B2)
5. $x \in D'S_3$, from (B3), 4
6. aR_3x, from df.D'R, 5, a is a constant
7. $a = x$, from 3, 6
8. xR_3x, from 6, 7
9. xS_3x, from df.S_3, df.(X)/R, 2, 8.

(b3) $N \neq \emptyset$, since $N \stackrel{df}{=} \{x: xS_3x\}$, (b1), (b2).

The formalism in question is developed further on by means of some defined secondary notions:

$$I \stackrel{df}{=} \{x: B \subset S_3\{x\}\},$$

where I =: the set of first beings,

$$1I \stackrel{df}{=} \{x \in I:\ I \subset \{x\}\},$$

where 1I =: at most a one-element set of necessary first existences.

The supplementary assumption is as follows:

(B4) $$S_3 \in MQ.$$

We obtain the following conclusions:

(b4) $N = MinR_3$, since $\bigwedge R\ (MinR \subset DR)$, df.N, (b2), df.$S_3$;

(b5) $MinR_3 \neq \emptyset$, from (b4) and (b3);

(b6) $FR_3 = FS_3$, since:

1. $FR_3 \subset B$, from (B2)
2. $B \subset D'S_3 = D'R_3 \subset FR_3$, from (B3), df.$S_3$, df.FR
3. $FR_3 = B$, from 1 and 2
4. $B \subset FS_3$, from (B3), df.FR
5. $S_3 \subset R_3 \subset B \times B$, from df.$S_3$, (B2)

6. $FS_3 \subset B$, from 5
7. $FS_3 = B$, from 4 and 6
8. $FR_3 = FS_3$, from 3 and 7.

(b7) $MinS_3 \neq \emptyset$, since $\bigwedge R \bigwedge S (S \subset R \ \& \ FR = FS \rightarrow MinR \subset MinS)$, df.$S_3$, (b6), (b5);

(b8) $1I \neq \emptyset$, since $\bigwedge R (R \in MQ \ \& \ MinR \neq \emptyset \rightarrow 1IR \neq \emptyset)$, (B4), (b7);

(b9) $1NI \neq \emptyset$, since $\bigwedge R (1IR \neq \emptyset \ \& \ MinR \neq \emptyset \rightarrow 1IR \cap MinR \neq \emptyset)$, (b7), (b8), df.1NI.

[C] Nieznański **(1983b)** and **(1984)** presented the logical analysis of three concepts of necessary beings. They are:

1) substantial necessary beings: $N \overset{df}{=} \{x \in MinR_3 : xR_3x\}$;

2) distributive necessary totality of beings:
$$N' \overset{df}{=} \{X \subset FR_3 : \breve{R}_3X \subset X \neq \emptyset\};$$

3) collective necessary totality of beings:
$$N^+ \overset{df}{=} \{x : \breve{R}_3\{x\} \subset \breve{P}\{x\}\},$$
where P denotes the relation of non-proper parts.

The theorems: $N \neq \emptyset$, $N' \neq \emptyset$ and $N^+ \neq \emptyset$ are obtained in the three simple formalized systems, based on Leibniz's principle of a sufficient reason of existence.

[D] We have a kind of combination of Aquinas' third and fourth "ways" in Charles Hartshorne's **(1961, 1962)** formalization of St. Anselm's ontological proof as found in **Proslogion 3**.

Let's assume a few definitional abbreviations: $E!X \overset{df}{\longleftrightarrow} X \neq \emptyset$ (where $E!X$ =: X exists), $C_1X \overset{df}{\longleftrightarrow} E!X \ \& \ \Diamond{\sim}E!X$, $C_2X \overset{df}{\longleftrightarrow} \Diamond E!X \ \& \ \Diamond{\sim}E!X$[8] (where C_iX =: X is contingent in the sense i, for i = 1,2). Hartshorne **(1962)** and A.G. Nasser **(1971)** confirm that in terms of modal logic S5 there is the equipollence: $\bigwedge X (C_1X \longleftrightarrow C_2X)$. Let us assume two more abbreviations: $G_1 \overset{df}{=} MaxR_4$ (according to St. Anselm) and $G_2 \overset{df}{=} LR_4$ (according to St. Thomas Aquinas), where $x \in G_i$ =: x is God in the sense i, for i = 1,2.

Hartshorne's proof of the thesis $E!G_1$ is based on the two assumptions (D1) ${\sim}C_1G_1$ and (D2) $\Diamond E!G_1$. The conclusions are as follows:

(d1) ${\sim}C_2G_1$, from (D1) and $\bigwedge X (C_1X \longleftrightarrow C_2X)$;

(d2) ${\sim}\Diamond E!G_1 \vee {\sim}\Diamond{\sim}E!G_1$, from (d1) and df.$C_2$;

(d3) $\sim\Diamond\sim E!G_1$, from (d2) and (D2);

(d4) $E!G_1$, since (d3) and $\sim\Diamond\sim p \rightarrow p$.

It is possible to obtain a reduction of the proof done by C.G. Vaught **(1972)**, who assumed initially that $\sim C_2G_1$. If G_1 is replaced by G_2 in the proof, the proofs of Aquinas' thesis are obtained automatically. Theodor G. Bucher **(1984)**, however, was right saying that the arguments were of questionable value, since it was enough to replace assumption (D2) with (D2') $\Diamond\sim E!G_1$ in order to obtain the unexpected conclusion: (d4') $\sim E!G_1$.

The formalization of the ontological arguments for the existence of God wasn't popular among Poles. Only Nieznański mentions them.

4.4 Formalizations of generalized arguments and a general logical theory of extreme elements of relations for the use of theodicy

Bendiek **(1956)** regards his formalized arguments to be calculi suitable for various interpretations. Nieznański **(1980)** created a formalized system for the relations R_1, R_2 and R_3 taken together. He undertook the problems of Thomist theodicy within that system. The generalizations of the above mentioned arguments refer to some particular relations and are based on extra-logical assumptions. However, it is also possible and necessary to develop a purely logical theory or relations for the use of theodicy. Anthony Kenny **(1969)** noted that numerous proofs of the existence of the absolute are based on the following law of logic

$$\bigwedge R\,(R \neq \emptyset \;\&\; R \in ord_s \;\&\; \overline{\overline{FR}} < \aleph_0 \rightarrow MinR \neq \emptyset).$$

Dozens of other logical theorems, useful for the theodicy, were proved by Nieznański **(1980)**, who constructed a general logical theory of the extreme elements of relations.

5. Logical analysis of some Thomist notions

Several important Thomist notions, viz. "existence" (4.1), "analogy" (4.2), "the omnipotence and omniscience of God" (4.3), "authority and faith" (4.4) went through logical analyses in Poland.

5.1. An extensive review of the symbolic demonstration of the concepts referring to the notions of existence is given by Nieznański (**(1980)**, pp.118-125). The same author also suggests

(1983) some kind of semantics of the scholastic theory *de modis essendi*. The definition of an arithmetical scheme for 2^k-element Boolean algebras, the universes of which are Cartesian products $\{0,1\}^k$, is followed by the denotations of sixteen *mod essendi* and by the description of the system establishing logical connections among the *mod*.

5.2. The traditional knowledge of the analogy of notions and beings derives mainly from Aquinas', and particularly from his question *utrum Deus nominari possit*. Bocheński (1948) was the first in the history of that field of research, who carried out a thorough logical analysis of the notion of analogy.

To begin with, Bocheński states that "the meaning of the name" (denoted by symbol S) is a four-argument relation: S(a,l,f,x) =: in the language l the name a means the property f of the thing x. Bocheński states further on that the analogy is a relation of two, possibly even isomorphic names and it is a form of ambiguity. Thus, he defines first the relation of the ambiguity of names Am(a,b,l,f,g,x,y) =: names a and b are ambiguous in language l referring to properties f and g, and things x and y.

Here is the definition:

$$Am(a,b,l,f,g,x,y) \overset{df}{\leftrightarrow} S(a,l,f,x) \mathbin{\&} S(b,l,g,y) \mathbin{\&} I(a,b) \mathbin{\&} f \neq g \mathbin{\&} x \neq y,$$

where I(a,b) =: names a and b are of the same shape.

The analogy of attribution (denoted At) has the following definition:

$$At(a,b,l,f,g,x,y) \overset{df}{\leftrightarrow} Am(a,b,l,f,g,x,y) \mathbin{\&} (C(x,y) \vee C(y,x)),$$

where C(x,y) =: x is a cause of y.

The notion of the analogy of proportionality (Apl) is defined by Bocheński by means of the theory of relational isomorphism

$$Apl(a,b,l,f,g,x,y) \overset{df}{\leftrightarrow} Am(a,b,l,f,g,x,y) \mathbin{\&} \bigvee P \bigvee Q\, (fPx \mathbin{\&} gQy\ P \text{ smor } Q)^{9},$$

where P smor Q =: relations P and Q are isomorphic.

5.3. The Viennese professor Curt Christian presented (1957) a logical analysis of the notions: omnipotence (AM), omniscience (AW) and God (G). He defined them as follows:[10]

(1) $$AM_i x \overset{df}{\leftrightarrow} \bigwedge p\, (WLxp \rightarrow p),$$

where $AM_i x$ =: x is omnipotent in the sense i; and WLxp =: x wants p to;

(2) $$AWx \overset{df}{\leftrightarrow} \bigwedge p\, (p \rightarrow WSxp),$$

where AWx =: x is omniscient; and WSxp =: x knows that p;

(3) $$G_1x \overset{df}{\longleftrightarrow} AM_1x \,\&\, AWx,$$
where G_ix =: x is God in the meaning i.

Nieznański (1976) assumes that:

(4) $$AM_2x \overset{df}{\longleftrightarrow} AM_1x \,\&\, \bigvee p\, WLxp \quad \text{and}$$

(5) $$G_2x \overset{df}{\longleftrightarrow} AM_2x \,\&\, AWx.$$

The Salzburger professor Paul Weingartner (1974) defined the notions "onmipotence" and "omniscience" in another way:

(6) $$AM_3x \overset{df}{\longleftrightarrow} \bigwedge p\,(p \longleftrightarrow Kpx),$$
where Kpx =: x can do that p, and

(7) $$AW_2x \overset{df}{\longleftrightarrow} \bigwedge p\,(p \longleftrightarrow WSxp).$$

Rev. Czesław Oleksy (Academy of Catholic Theology in Warsaw) holds (1984) that Weingartner's definition 7) determines both omniscience (AW) and infallibility (IN):

(8) $$INx \overset{df}{\longleftrightarrow} \bigwedge p\,(WSxp \rightarrow p),$$
where INx =: x is infallible.

In that case he obtains a new definition of God:

(9) $$G_3x \overset{df}{\longleftrightarrow} AM_1x \,\&\, AWx \,\&\, INx, \quad \text{and}$$

(10) $$G_4x \overset{df}{\longleftrightarrow} AM_2x \,\&\, AWx \,\&\, INx.$$

Oleksy suggests still other specifications of the notions "omnipotence" and "God":

(11) $$AM_4x \overset{df}{\longleftrightarrow} \bigwedge p \bigwedge t \bigwedge s\,(\sim A_1pt \,\&\, WL'xtp \,\&\, t \neq s \rightarrow A_1ps),$$
where A_1pt =: occurrence p is actual at the time t; and $WL'xtp$ =: at time t x desires p;

(12) $$AM_5x \overset{df}{\longleftrightarrow} \bigwedge a \bigwedge t \bigwedge s\,(\sim A_2at \,\&\, WL^+xta \,\&\, t \neq s \rightarrow A_2as),$$
where A_2at =: the being a is present at the time t; and WL^+xta =: at time t x wants object a to become an actual being;

(13) $$G_ix \overset{df}{\longleftrightarrow} AM_kx \,\&\, AWx \,\&\, INx, \quad \text{for } 4 \leqslant i < 9 \text{ and } 1 \leqslant k < 5.$$

Several formalized theorems referring to the notions of omnipotence, omniscience and God were proved by Christian (1957) and by Nieznański (1976). Weingartner (1974) refuted the thesis of the so-called religious fatalism:

(14) $$\bigwedge p\,(p \rightarrow WLgp),$$
where g =: God; and $g \overset{df}{=} (\iota x)\,(AM_3x \,\&\, AW_2x)$,

while Oleksy (1984) rejected the thesis of negative predestination (*reprobatio*).

5.4. The notion of authority is essential for the definition of the theological concept of faith. Bocheński (1965) and (1974) was the first to carry out a logical analysis of that notion. He distinguished and determined the notions of epistemic and deontic authority. Nieznański (1985) unfolded the concept of authority and faith on the basis of the logical theory of belief. Bocheński's book (1965) is the first to be concerned entirely with the problems of faith and religion considered from the logical point of view.

6. Final remarks

The Polish programme of the logical analysis of Thomism was accomplished to a considerable extent, and especially with respect to the study of the formal correctness of the Thomist deductions. Formalized proofs of the existence of God acquired a desired standard of precision. However, the objection that the proofs in question were too formal was not, so far, either overcome or refuted. Even if the conclusions follow from the assumptions without *non sequitur* errors, the assumptions remain empirically undecidable. Proofs of that kind used to be seen in traditional logic as subject to the error *ignoratio elenchi*, i.e. they didn't demonstrate what needed demonstration, since they proved demonstratively nothing, being limited to implications from the conjunction of premises to the conclusions. And, e.g., H. Scholz (1969), who maintained that "Ein Beweis ist verbindlich für jedermann, oder es ist überhaupt kein Beweis" (p.64), is a contemporary logician who shares this opinion. It would seem that the future of the application of logic to Thomism lies more in the area of semantics of the language of this philosophy. Thomism is a doctrine with a future, as long as it accepts the postulate of maximum precision of proof and method.[11]

Notes

[1] Cf. the bibliographical list at the end of this book.

[2] Here (and everywhere below) the symbol =: is a sign of abbreviating.

[3] The definition of the converse of relation R:

$$\breve{R} \overset{df}{=} \{\langle y,x\rangle: xRy\}.$$

The definition of R-image of set X:

$$RX \overset{df}{=} \{y: \bigvee x \,(x \in X \;\&\; xRy\}$$

[4] Cf. E. Nieznański **(1981b)**.

[5] It is possible to reduce Salamucha's axioms (A3) and (A4) replacing the expression: $\bigvee a \bigvee b\,(aPx \;\&\; bPx)$ by the expression: $\bigvee a\; aPx$.

[6] Assumption 1° was used in the proof only as $R_1 \in con$.

[7] Segment "$x \neq y$" is superfluous in axiom (H3), since it follows from segment "$\sim xPy$" and axiom (H2), already assumed.

[8] *Contingeas est quod potest esse et non esse* (**Summa Theologica** I, q.86,a.3).

[9] Smor =: *simili ordine*.

[10] All indices below denote the succeeding meanings of the extra-logical constants and "p" is a propositional variable.

[11] Translated by Stefania Szczurkowska.

LEON KOJ

ON JUSTIFICATION OF QUESTIONS

Introduction

1. It is not unnatural to suppose, in view of the existence of several kinds of questions, that different questions are justified in different ways, and if so, then the theory of justification of questions must depend heavily on the type of questions considered. But one is tempted to formulate general conditions under which asking questions can, in the majority or in all cases, be justified. Thus we have to enumerate the types of questions which will be taken into account in this paper and have to see whether general rules of justification of questions can be found.

The most interesting category of questions are cognitive questions posited to obtain the information which the question concerns. On the whole, the majority of theories of justification, or theories of arising of questions or their evocation, are concerned with cognitive questions or, to be exact, with their subset. This paper departs from this body of practice, taking into account a wider class of questions. It includes: examination questions, asked about things very well known to the questioner who is interested in whether the questioned person can give the right answer; deliberative questions we ask ourselves without expecting an answer; rhetorical questions which presuppose common knowledge on the part of the questioner and the questioned person alike: they both know the answer and know that they know it. Sometimes a distinction is made between questions which are concerned with analytical matter, as in: 'Is 2 = 2?' or empirical matter as in: 'Who was Shakespeare?'. Sometimes open questions, e.g. 'Why do you smoke?' are distinguished from closed questions, e.g. 'What is your name?' (J. Giedymin (1966), p.16). Open questions do not determine the structure of the answer as is the case with closed questions. The questions just enumerated will be discussed in some detail below.

2. In order to describe the justification of questions we have to ponder a little about the language in which this could be done. First, this language must be more inclusive than the languages of most theories of questions. If so, we should be able to tackle problems which are not describable in poorer languages. But still there will remain some problems which in our language are inexpressible.

The relatively simplest language in which questions can be described is purely syntactical. The most important concept - apart from the terminology which serves to formulate the grammar (rules for constructing sentences) - is one referring to proof. In such languages a question is taken to be justified when some sentences have no proof. Theories which use this kind of languages can deal, at most, with questions which arise only in deductive sciences where lack of empirical corroboration is inexpressible. But even in deductive sciences only very few questions can be taken to be justified, namely only those arising from sentences which are independent of the axioms of a given system. In these syntactical languages there is no possibility to introduce the notion of proof that is actually arrived at. In the majority of questions posited with regard to analytical matters there may exist a proof (in the abstract sense of existence used in formal logic) of the sentences under discussion but either the proof can not be discovered by questioning person or simply, it may be unknown to him. And it is precisely in those cases that we ask questions very often. Thus the justification of the majority of questions cannot be described; A. Wiśniewski ((**1986**), p.17) claims that analytical sentences cannot generate any questions. The great hypothesis of Fermat - provided it has a proof which has not been discovered yet - cannot be claimed to generate any questions. As we want to consider a much wider range of questions, clearly a purely syntactical language is too poor for our purposes.

3. Similar problems are encountered in the case of a semantic terminological basis. Here, the concepts of truth and entailment are relevant to the explanation of generation of questions. Thus Wiśniewski defines generation of questions in the following way:

> A set of sentences X generates a question Q if and only if (iff) the set X does not entail any direct answer to Q and entails the presupposition of the question Q. The set of sentences which are entailed by all consistent answers to the question Q forms the presupposition of Q (Wiśniewski (**1986**)).

Now, it looks as if this definition did not allow for generation of a number of questions which in fact are asked seriously. Let us assume that X does not entail the presupposition of Q and that we do not know the answers to Q. According to the definition of generation cited above we cannot generate the question Q; but on all probability we shall ask the questions. The lack of knowledge of the answers, it seems, is decisive here. For instance, take an open question Q and a set of consistent answers to it. What does this set entail?, i.e. what is the presupposition? The answers are not determined by the structure of Q nor can their structure be derived from the meaning of the question. It appears, therefore, that almost anything can be entailed by the answers and it is very difficult to say whether the presupposition of Q is sufficient for generating the question. As we shall see a little later the answers to an open question are determined by their relation to the sentence included in the question. This relation, as a rule, is not the relation of entailment in either direction.

Let us assume that X entails the presupposition of Q and does not entail the answers, and that we know the answers. In this case, although the question can be legitimately generated, we shall not ask it. These examples show convincingly that there is a gap between our habits of asking questions and generation of questions as defined above. We also point to the fact that this definition of generating questions does not allow forming questions on the basis of any set of tautological sentences or on the basis of inconsistent set of sentences. This last thesis goes counter to the rather widespread conviction that contradictions bring forth problems of various kinds.

The sentences $2^5 = 32$ and $2 \cdot 16 = 32$ are logically equivalent, and they entail each other. If we ask questions (a) 'Is $2^5 = 32$?' and (b) 'Is $2 \cdot 16 = 32$?' the positive answers to them are equivalent, and so are the negative answers. A consistent set of answers (let them be positive) to (a) is then logically equivalent to the set of similar answers to (b). Both sets then entail the same presupposition which generates the same question or two questions which are equivalent in some derivative sense. But (a) may be asked by a boy who attends the second class of a primary school while (b) will not be asked by him at all: he knows the result of the multiplication. Our habits of asking questions then are outside the domain of erotetic logic, or the logical tools used in this kind of erotetic logic are too clumsy.

4. The definition of generation of questions establishes a set of sentences X which generates a question by pointing to the answers of the question which is to be generated by X. This kind of definition is formally quite correct: all these things - sets of sentences X, questions generated by them and their answers are given by rules of formation without any reference to their succession in time. But from a methodological point of view and from the point of view of our practice this definition of generation is strange: to know the set of sentences which generates a question (unknown so far) we have to consider the answers to the question - and they are known at the very end of the whole procedure of forming questions and solving the problem. The real procedure starts with a set of sentences which generates the question; we get to know the question later and still later we find the answers. It seems - and we take it as a postulate - that a theory of justification of questions which is to be methodologically useful should provide us with rules describing the actual procedure.

Terminological framework

1. The examples we cited above purported to show that questions are asked in case some information, namely that expressed with the help of answers, is not known to us. However, this is not always so; there are questions the answers to which are known to us. Rhetorical questions are a case in point. What is important is that to describe the examples we used the notion of knowledge, i.e. an epistemic concept. There are more epistemic notions, which eventually can enter our discourse: belief, assumption, doubt, certainty, etc. Possibly all these notions can be useful in characterizing the conditions which justify asking cognitive questions. The problem is that we have to choose the notion(s) which suits best our ends. Thus far we have mentioned knowledge. Very often knowledge is conceived of as a complex notion - as a true belief well justified. Thus if one happens not to know something then

he does not believe in it or

it is not true or

it is not sufficiently corroborated.

As already remarked it is not the proof or, generally - corroboration, which makes people ask questions. A sentence may be well proved or corroborated and somebody who never encountered the proof may ask the question. He may even have seen the proof and asked the question if he doubted the correctness of the proof. As to the truth, people ask questions regarding both true and false propositions if only they do not exhibit any cognitive relation to the values of the propositions taken into account. Thus only lack of belief is a constant factor present in the case of serious cognitive questions. As we shall see later even in the case of non-cognitive questions this notion plays some role.

2. We do not ask questions concerning things we have not heard about. We do not ask them although we have no belief about those things. In this case we do not feel any lack of knowledge. As a rule, we ask a question when some answer is expected, required, necessary, ordered or requested. We also ask questions if we want to get the answer. These attitudes are reactions to our awareness of lack of knowledge or belief. We have to choose that of the cited attitudes which is always present when we ask questions.

Let us begin with the concept of necessity which was used by Aqvist (1965). If we took this notion into the description of justification we would have to introduce something like this: a question is justified int. al. when it is necessary to believe that.... This condition is a very strong one and given the principle: *ab necesse ad posse valat illation*, it precludes to ask questions which have no believable answers. The principle implies: it is possible to believe that.... But we never know in advance whether a question has believable answers at all. If the opinion prevailed at the very beginning of the XIX century that there were no meteors, one could not ask what the chemical compositions of the meteors are, as no believable answer could be offered from the point of view of the accepted theory. If we were to ask only questions which have believable answers it is quite probable that no accepted false theory would be rejected. Therefore we rather have to exclude from considerations the concept of necessity and such like which imply that only questions with believable answers can be asked in a justified way. As far as the notion of ordering is concerned, we are confronted with a concept which refers to two persons: to the speaker and to the addressee. A question does not exhibit any person in its meaning. But more importantly, an order presupposes that the action which is ordered can be performed, i.e. an answer, and especially a true answer can be given. Once more we face the same problem as before: seemingly, when we ask questions we have to assume that answers can be given. It is this assumption that is taken by

Ajdukiewicz (1938) as a condition of correctness of questions. The views of Positivists in this respect are also well known: no value, they claimed, could be given to questions to which no answer (empirically meaningful) could be given. Such questions – they declared – are pseudo-questions. If we took this assumption as a rule of justification of questions not only would we encounter the difficulties already mentioned but we would be forced to reject all deliberative questions which very often do not presuppose any answers – just as many philosophical questions do. Thus the notion of order put forward by R. M. Hare (1949) as a general requirement on all questions is of no use to us.

Very similar objections can be raised in the case of the notion of expectation, of requirement and request. Thus only the notion of wish survives our objections. It is this notion which was propounded by B. Bolzano (1929). Nowadays it is used by B. Bogusławski (1977). As N. Rescher (1968) has shown a kind of logic called optative logic is possible where the notion of wish is considered. Thus the foundation of the theory of justification of questions is, as for now, not outside logic: it includes the two specific primitive concepts considered so far – belief and wish.

3. The introduction of the concept of wish necessarily implies the following: what one hopes to get by putting forward a question. People who ask questions want to achieve very different things and information. Not all of these things are of interest to us. We are interested only in those pieces of information which from the point of view of the meaning of the question are to be treated as desirable. We may e.g. wonder whether by asking questions we want to know the truth of the answer, or at least to grasp some possibilities or verisimilitudes. We may also strive to believe the answer. It is possible that it is not truth and the like, but certainty that we are trying to establish. To make the choice between these possible objects of our wishes let us begin with certainty. Sometimes we get the answer: possibly.... It is quite a good answer, though perhaps not always fully satisfying. Sometimes we do not arrive at certainty as perhaps sometimes certainty is not necessary. If somebody does not do something in case of risk an answer which begins with "possibly" is good enough to restrain him from the action he planned. Thus it appears that not always we are looking for certainty.

Also, when questions are asked, not always truth is wanted. In the case of examination questions, for example, we already know the true answer and we want not so much the true answer; rather we want to know whether the person questioned knows the true answer. It happens many times that we ask questions without any wish to get a true answer. We ask to know something else, to know the opinion of the questioned person. We, ourselves may not know the true answer. But after questioning many people we can find out what the popular belief is.

It can also be doubted whether we want to hear an answer at all. For example in the case of rhetorical questions nobody is requested to answer. Sometimes we do answer, though we do not utter a word. When one is asked: 'Where is Hotel Eden?' he can simply point to the nearby building.

The latest example gives us a hint where to look for a solution to our problem: what do we always want when we do ask questions. In all these examples the person asked is at least to present a piece of information. In the case of rhetorical questions and deliberative questions everyone has to present the information himself - not to the person who asks, in the case of rhetorical questions; in the case of deliberative questions the speaker and the addressee are identical. In the case of pointing the addressee presents the information by pointing to the source of information. In the case we utter the answer we present the information indirectly with the help of an expression which, in turn, refers to the information. Sometimes pointing to an object, showing a picture or drawing is a better means of answering then uttering an expression. This fact is sometimes forgotten. The notion of presenting is necessary, it seems, to describe some components of the overall concept of justification. But there is a difficulty here: it seems there are no logical considerations so far which tackle the notion of presenting information. To some degree this notion is similar to the notion of perceiving. Unfortunately, the analogy is rather superficial.

4. Besides these three primitive concepts of our attempted theory of justification of questions we shall use usual logical, syntactical and semantical terminology without going into any explanation. We want to concentrate on the problems which can be solved with the notions of belief, wish and presentation. It is obvious that a still more inclusive theory of questions has to include additional methodological terminology, at least making possible the definition of corroboration, testing etc.

So far the term "justification" has been left unexplained and undefined. For the time being let us assume that it is a very complex notion including admissibility of questions, motivation of questions and their well-foundedness.

Admissibility of cognitive questions

1. Most of us do not know who discovered Grenada. Let us then ask the following question:

(1) Who discovered Grenada?

as a model example. We ask this question because we do not know the discoverer. This means that there is nobody we know that discovered Grenada. This lack of knowledge is part of the admissibility (and therefore also of the justification) of (1). Let the variables z, z', z'' range over the set of people. "K" may be used as the symbol for "knows". "K" is therefore a functor with one individual category and one propositional category argument. The sentences built with the help of "K" are of the form $K(z,p)$. Now, the situation in which we ask (1) can be described as (2) or, equivalently, as (3)

(2) $\bigwedge_{z'} \sim K(z, z' \text{ discovered Grenada})$

(3) $\sim\bigvee_{z'} K(z, z' \text{ discovered Grenada})$

It has already been mentioned many times that the notion of knowledge is inapplicable. This notion must be weakened and adjusted to suit those cases in which the notion of knowledge is not an adequate one. We have to pass therefore to the notion of belief. In (4) and (5) "B" stands for "believes" and replaces "knows".

(4) $\bigwedge_{z'} \sim B(z, z' \text{ discovered Grenada})$

(5) $\sim\bigvee_{z'} B(z, z' \text{ discovered Grenada})$

This precondition of asking questions is a very weak one. Sometimes it is satisfied when a person does not ask any question. Suppose that person z has never heard anything about Grenada and simply does not know about its existence or that it has been ever discovered. To ask a question, z has to assume at least that Grenada is something that is discoverable and that it has been discovered by someone. Thus the second condition of asking questions is:

(6) $B(z, \bigvee_{z'} (z' \text{ discovered Grenada}))$

Stipulation (6) seems to be similar to the postulates that every question has at least one true answer (Ajdukiewicz **(1938)**). In fact, the stipulation made by Ajdukiewicz and expressed in metalanguage is much stronger than (6). From the existence of an object that has been discovered does not follow that we know anything concerning the discovery and can formulate an answer to question (1). We are far from claiming that we can always know about all things that exist or existed.

Let us compare (4) and (6) with respect of the place of "B" relative to the quantifier. In (4) "B" is in the scope of the quantifier "$\bigwedge$". In (6) the quantifier "$\bigvee$" is in the scope of "B". The meaning of (4) is that there is nobody about whom person z thinks with conviction that he discovered Grenada or, simply, z does not know who discovered Grenada. The difference between a belief functor appearing within the scope of quantifier and quantifier put in the scope of belief functor is made more clear if we compare two formulae containing the same quantifier:

(α) $$B(z, \bigvee_{z'} (z' \text{ discovered Grenada}))$$

(β) $$\bigvee_{z'} B(z, z' \text{ discovered Grenada}))$$

In (α) we say that z believes that somebody discovered Grenada. In (β) we say there is somebody about whom z thinks with conviction that he discovered Grenada. Plainly (β) is the stronger formula: (α) follows from (β), but not vice versa (Kutschera (1976), p.92, TG_{1h}). Prima facie it may look as if (5) and (6) contradicted each other. In fact, contradiction could arise only on the condition that (6) implied (5):

(7) $$B(z, \bigvee_{z'} (z' \text{ discovered Grenada})) \Rightarrow \bigvee_{z'} B(z, z' \text{ discovered Grenada}))$$

The consequent of (7) is a blatant contradiction of (5). Fortunately - as already remarked - (7) is not true and so the contradiction does not arise.

(7) and its negation points to the necessity of a good characterization of the notion of belief. Outside logic it is taken as a purely psychological concept which defies any logical attempts at making it more precise. In logic this notion is given an over-rational definition. According to R. M. Martin (1959) men believe in all logical theses. If they believe in some sentence p and if from p logically follows the sentence q, then they always believe in q. Human being that do so are more logical then a computer. We wish to assume a much weaker notion of belief (which will make sense of our well-foundedness of question) which is less rationalistic (Kutschera (1979), p.80). We assume that a person z believes in some logical theses; if a person z believes that p and q is a logical consequence of p (or p entails q) then person z believes q on the additional provision that it had been presented to him that q follows from p or p entails q. The additional condition includes the notion of presenting. This

notion implies, among others, the possibility that person z has made the proof of q relative to p and thus has shown to himself that q follows from p.

2. Conditions (5) and (6) must be refined a little. We simply assumed that z' ranged over the set of people. This stipulation is an unfortunate one because in different questions we may ask about different kind of objects. To avoid new variables each time we ask about new kind of entities, it is more convenient to let the variables range over a very vast set of objects and each time provide a short notice what kind of object is under consideration. But let the letters z, z' etc. range over the set of people. The more inclusive set may be represented by the variables x, x', x''. More exactly, the last variables will be typically ambiguous - more about it a little later. With this convention in mind instead of (6)

(6) $$B(z, \bigvee_{z'} (z' \text{ discovered Grenada}))$$

we introduce

(8) $$B\left[z, (\bigvee_{x} (x \text{ discovered Grenada}) \wedge \wedge \bigwedge_{x} (x \text{ discovered Grenada} \Rightarrow x \text{ is human}))\right]$$

What (8) says is that person z believes that an object discovered Grenada and whatever did it is human. This last conviction is expressed in (1) by the word "who".

We have to modify (5) in the same way:

(9) $$\sim\bigvee_{x} B(z, (x \text{ is human} \wedge x \text{ discovered Grenada}))$$

There is no need to state explicitly that z is human, first of all, because of the range of the variable "z" and because "B" is understood as a two-place predicate (see above). Secondly, we take the formula below to be a thesis of our logic of belief:

(10) $$\bigwedge_{z} (B(z,p) \Rightarrow z \text{ is human})$$

To include questions asked in fairy tales by their non-human heroes we can generalize (10) by the extension of the set z is ranging over so that it could include all human-like objects.

3. Let us call the consequent of the second part of (8) the confinement of the range of question. The antecedent of this part may be called *datum questionis*. The convention is to be understood as fairly general. Every time we have sentences of the form:

(11) $$B(z,(\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)))$$

(12) $$\sim\bigvee_x B(z,(gx \wedge fx))$$

the sentence symbolized by *gx* is the confinement of the range of a question and the sentence symbolized by *fx* is a datum questionis. Now, we can formulate the rule which describes admissibility of cognitive questions. First let us say something about the question mark of a question. It will be put at the very beginning of question - just as quantifiers are put. Similarly to quantifiers, the question mark will be endowed with a variable which will reappear inside the question. This variable is bound by the question mark. According to the ideas of T. Kubiński (1971) - to whom I owe very much - the question mark is conceived of as an operator. We read the question operator "$?_x$" as "which all x's are such that...". For the reasons for such treatment of the question operator see (Koj (1971)). Questions are admissible for person z if and only if (11) and (12) are true. We can express this idea in:

(R) The question $\ulcorner ?_x\,_{(gx)}\,[fx]\urcorner$ is admissible for person z iff (11) and (12).

The round parentheses and square brackets are signals that the enclosed expressions are not interchangeable - their relative succession is fixed by (11). On the basis of (11) and (12) the confinement of the range is put to the right of the question operator. The datum questionis is still farther to the right.

4. This rule - contrary to appearance - is very general and applies in the case of cognitive questions of all possible structures. Since the structures of questions were considered in another paper of mine (Koj (1972)), here I confine myself merely to a brief presentation of the results of my considerations. Each type of questions is presented here in its natural-language form, then two paraphrases are given which bring us closer to the logical form.

A a Did <u>Columbus</u> discover America? (the underlined word is stressed).
Was it Columbus who discovered America?

b Did Columbus or somebody else discover America?

c Who of the two: Columbus or somebody else discover America?

d $?_x{}_{(x\in\{\text{Columbus, Non-Columbus}\})}$ [x discovered America]

Non-Columbus is understood as indefinite description: $x = \text{Non-}y \equiv \bigvee_{x'}(x' \neq y \wedge x = x')$. The same pertains to all Non-y phrases; they differ only as to their syntactical category.

B a Did Columbus discover <u>America</u>?

b Did Columbus discover America or something else?

c What did Columbus discover: America or Non-America?

d $?_x{}_{(x\in\{\text{America, Non-America}\})}$ [Columbus discovered x]

C a Did Gödel <u>get</u> the Nobel Prize?

b Did Gödel get or did he not get the Nobel Prize?

c Did Gödel get or non-get the Nobel Prize?

d $?_x{}_{(x\in\{\text{get, non-get}\})}$ [Gödel x the Nobel Prize]

D a Is the hat yellow, orange or violet?

b What is the hat like: yellow, orange or violet?

c Which out of {yellow, orange, violet} is the property of the hat?

d $?_x{}_{(x\in\{\text{yellow, orange, violet}\})}$ [hat $\in$ x]

E a Who discovered Grenada?

b Which human being discovered Grenada?

c Which object out of the set of humans discovered Grenada?

d $?_x{}_{(x\in\text{human})}$ [x discovered Grenada]

F a Which two pupils from the fourth class threw stones at the window?

b Which all pupils of the fourth class, members of a two-element set threw stones at the window?

c Which all x-es of a two element set included in the fourth class threw stones at the window?

d $?_x{}_{(x\in\text{fourth class} \wedge \bigvee_u(x\in u \wedge \bar{\bar{u}}=2))}$ [x threw stones at the window]

In F.d. we use the set-theoretic notion of cardinality of sets. Thus the set u has the cardinality two. When we talked about the language of our consideration it was assumed that the whole logical terminology is included. Among others the notions of elementhood and identity and all definitional derivatives are at our disposal.

G a Is $\bigwedge_{p,q}(q \Rightarrow (p \Rightarrow q))$?

b Is it that $\bigwedge_{p,q}(q \Rightarrow (p \Rightarrow q))$ or the other way round?

c Is it $\bigwedge_{p,q}(q \Rightarrow (p \Rightarrow q))$ or $\sim\bigwedge_{p,q}(q \Rightarrow (p \Rightarrow q))$?

d $?_{f\ (f \in \{\vdash, \sim\})}\ [f\ (q \Rightarrow (p \Rightarrow q))]$

H a Why does he study mathematics?

b What caused him to study mathematics?

c What is the non-analytic state of affairs that stands in the causal relation to his studying mathematics?

d $?_{x\ (x \in \text{Non-analytic})}$ [x C He studies mathematics]

In H.d. the letter C means the causal relation, whatever it might be.

The adduced examples show that many questions fall under the general schema $\ulcorner ?_{x\ (gx)}\ [fx] \urcorner$ where x is typically ambiguous: one time it is an individual variable, another time a predicate variable, etc. The schema can be also applied in the case of questions with more than one question operator. This kind of questions is illustrated with the help of: 'When and how did he escape from prison?' After Conrad Rudi's (1978) overview of different theories concerning the structure of questions it seems that the one expounded here is adequate.

5. To test whether our R-rule is applicable in all cases of belief as described in (11) and (12) and whether we get the questions we intuitively expect let us take new examples covering all types of questions from A to H. We shall test the rule as schematically as in the case of A and H. First a short story is presented (a), then the respective counterparts of (11) and (12) are given (b). The application of R to them yields a question (c) which is then translated into normal natural language (d), to find out whether it fits our intuitions aroused by (a). Point e is the evaluation of this last question.

A' a Mary (as person z) believes that somebody has stolen her necklace and she (i.e. z) suspects that it was Jane.

b $B\Big[z, (\bigvee_x (x = \text{Jane} \lor x = \text{somebody else}) \land$

$\land \bigwedge_x (x \text{ stole the necklace} \Rightarrow (x = \text{Jane} \lor x \text{ is someone else}))\Big]$

$\sim\bigvee_x B(z, (x \text{ stole the necklace} \land (\text{Jane} = x \lor x = \text{Non-Jane})))$

c $?_{x\ (x \in \{\text{Jane, Non-Jane}\})}$ [x stole the necklace]

d Did <u>Jane</u> steal the necklace?

e Question d. was to be expected on the basis of a.

B' a A simple-minded pupil has doubts as to whether Copernicus invented the heliocentric system, discovered it or simply described it, etc.; he cannot decide by himself which possibility is true.

b $B\Big[z, (\bigvee_x (\text{Copernicus } x \text{ heliocentric system or}$

$\text{or Copernicus } \bar{x} \text{ heliocentric system}) \land$

$\land \bigwedge_x (\text{Copernicus } x \text{ heliocentric system} \Rightarrow$

$\Rightarrow (x = \text{invention} \lor x = \text{non-invention}))\Big]$

$\sim\bigvee_x B(z, (\text{Copernicus } x \text{ heliocentric system} \land$

$\land (x = \text{invention} \lor x = \text{non-invention})))$

c $?_{x\ (x \in \{\text{invention, non-invention}\})}$ [Copernicus x heliocentric system]

d Did Copernicus <u>invent</u> the heliocentric system?

e In fact, the pupil who has these doubts may be justified to ask d.

C' a Barens was a famous discoverer and a southern European boy knows it. But he has no opinion as to Barens' discovery of Spitzbergen. Perhaps Barens discovered Spitzbergen but it may have been something else.

b $B\Big[z, (\bigvee_x (\text{Barens discovered } x) \land$

$\land \bigwedge_x (\text{Barens discovered } x \Rightarrow$

$\Rightarrow (x = \text{Spitzbergen} \lor x = \text{Non-Spitzbergen})))\Big]$

$\sim\bigvee_x B(z, (\text{Barens discovered } x \land$

$\land (x = \text{Spitzbergen} \lor x = \text{Non-Spitzbergen})))$

c $?_{x\ (x \in \{\text{Spitzbergen, Non-Spitzbergen}\})}$ [Barens discovered x] d Did Barens discover Spitzbergen?

e This question is also natural enough when we take into account the boy's doubts.

D' a John does not remember the type of Timothy's car; he remembers only that it is Japanese and he can remember only three Japanese types: Honda, Mazda and Toyota.

b $$B\left[z, \left(\bigvee_x (\text{Timothy's car is an } x) \wedge \bigwedge_x (\text{Timothy's car is an } x \Rightarrow (x = \text{Honda} \vee x = \text{Mazda} \vee x = \text{Toyota}))\right)\right]$$

$$\sim\bigvee_x B(z, (\text{Timothy's car is an } x \wedge (x = \text{Honda} \vee x = \text{Mazda} \vee x = \text{Toyota}))$$

c $?_{x\ (x \in \{\text{Honda, Mazda, Toyota}\})}$ [Timothy's car is an x]

d Is Timothy's car a Honda, a Mazda or a Toyota?

e d. is the question we can expect in view of all what John (he is the person z) knows and remembers.

E' a Somebody took Peter's hat. He believes it was one of his friends who came to a meeting with him.

b $$B\left[z, \left(\bigvee_x (x \text{ took Peter's hat}) \wedge \bigwedge_x (x \text{ took Peter's hat} \Rightarrow x \text{ was Peter's friend})\right)\right]$$

$$\sim\bigvee_x B(z, (x \text{ took Peter's hat} \wedge x \text{ was Peter's friend}))$$

c $?_{x\ (x \text{ Peter's friend})}$ [x took Peter's hat]

d Who from among Peter's friends took Peter's hat?

e The R-rule once more generated the expected question.

F' a The policeman was told that three persons had beaten Richard on the train. The policeman had no choice but to assume that they were travelling on the train. Of course he did not know the culprits.

b $$B\left[z, \left(\bigvee_x (\text{Richard had been beaten by } x) \wedge \bigwedge_x (\text{Richard had been beaten by } x \Rightarrow \bigvee_u (x \in u \wedge \bar{\bar{u}} = 3 \wedge u \subset \text{set of travellers}))\right)\right]$$

$$\sim\bigvee_x B(z, (\text{Richard had been beaten by } x \wedge \bigvee_u (x \in u \wedge \bar{\bar{u}} = 3 \wedge u \subset \text{set of travellers}))$$

c $?_{x\ (\bigvee_u (x \in u\ \wedge\ \bar{\bar{u}}=3\ \wedge\ u\ \subset\ \text{set of travellers}))}$ [Richard had been beaten by x]

d Which three travellers had beaten Richard?

e Indeed this is the question the policeman had to ask.

G' a Some students cannot agree as to whether the following $\bigvee_y (fy \Rightarrow gy) \Rightarrow (\bigvee_y (fy) \Rightarrow \bigvee_y (gy))$ is true or false.

b $B\Big(z, \bigvee_x \Big(x\ (\bigvee_y (fy \Rightarrow gy) \Rightarrow (\bigvee_y (fy) \Rightarrow \bigvee_y (gy)))\Big)\ \wedge$
$\wedge\ \bigwedge_x \Big(x\ (\bigvee_y (fy \Rightarrow gy) \Rightarrow (\bigvee_y (fy) \Rightarrow \bigvee_y (gy))) \Rightarrow x \in \{\vdash, \sim\}\Big)\Big)$
$\sim\bigvee_x B\Big(z, x\ (\bigvee_y (fy \Rightarrow gy) \Rightarrow (\bigvee_y (fy) \Rightarrow \bigvee_y (gy)))\ \wedge\ x \in \{\vdash, \sim\}\Big)$

c $?_{x\ (x \in \{\vdash, \sim\})}$ [$x\ (\bigvee_y (fy \Rightarrow gy) \Rightarrow (\bigvee_y (fy) \Rightarrow \bigvee_y (gy)))$]

d Is $\Big(\bigvee_y (fy \Rightarrow gy) \Rightarrow (\bigvee_y (fy) \Rightarrow (gy))\Big)$?

e Just as A, B, C so G is a general question. But here we do not ask about the subject or the predicate; rather we ask about the assertion or negation of the whole sentence.

H' a Philosopher A thought that every action has a *causa finalis* and treated life as a kind of activity. Unfortunately, he did not know the causa finalis of his life. (Take F as "is causa finalis of").

b $B(z, (\bigvee_x (x\ F\ A\text{'s life})\ \wedge\ \bigwedge_x (x\ F\ A\text{'s life} \Rightarrow x\ \text{non-analytical})))$
$\sim\bigvee_x B(z, (x\ F\ A\text{'s life}\ \wedge\ x\ \text{non-analytical}))$

c $?_{x\ (x \in \text{non-analytical})}$ [x F A's life]

d What is A living for? What is the causa finalis of A's life?

e The philosopher may ask himself what is he living for, especially when he is tired and unhappy.

The adduced examples seem to show that the R-rule generates the expected questions. We can take it as a partial solution to the problem of justification of questions and turn to another part of the problem: to the motivation of questions. Obviously, the R-rule may be tested in the case of still new kinds of questions. The reader is asked to do it himself, e.g. he may try to find out whether the R-rule is adequate in the case of questions with two or more question operators.

Motivation of questions

It is obvious that some admissible questions are not asked. This happens when the person who believes in all that (11) and (12) say is not interested in finding out what in fact is true. The person is not motivated to ask questions. Worse, non-cognitive questions which are not admissible in the way described above are, in fact, asked. The questioner is strongly motivated and this is enough to ask questions. Motivation is therefore the second part of the problem of the justification of questions, admissibility was the first. Usually, when one wants to know something, one asks a serious cognitive question. In this case the questioner's eagerness to know is the motivation which triggers the uttering of the question. Very often, however, somebody's wish to know the answer is not enough to be taken for motivation. It is sufficient that the answer or rather the information carried by the answer is simply presented. Knowledge as true information, or reliable information or believable information is not aimed at. Generally, it is the presentation of information which is a necessary object of one's wish. All that exceeds presentation of information is characteristic only of some questions. As will be shown later the desire for a presentation of information is present in all possible questions. Thus to describe motivation we have to introduce two additional predicates (mentioned above): wants and presents. Let "W" be the symbol for "wants" and "P" for "presents".

Let us assume that a question of the form $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is admissible for person z. Person z is then cognitively motivated to ask this question when he wants the information carried by the answer to be presented to him by someone z'. These remarks give way to:

(13) $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is cognitively motivated for z iff

$\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is admissible for z $\wedge$ $W[z, \bigvee_{z'} P(z', z, fx)]$

The signs $\ulcorner\ \urcorner$ are quasi-quotation marks (Quine (1955), §66)

The motivation of an examination question is different. The examiner believes that the question $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ may be admissible for the questioned person. The examiner simply assumes that the questioned person may not know the answer or he may be in doubt of it. The examiner wants the questioned person to present to him the answer. This wish is caused by another wish, which may not interest us here: to know the knowledge of the questioned person. To avoid further extension of our language, the word "may" will be substituted by an existential quantifier (the affinity of possibility to the existential quantifier is well known). Instead of saying: the question is possibly admissible for the person z, we shall say: there exists some z' for whom the question is admissible. This substitution alters a little the original formulation and its meaning. But from our point of view the change is permissible. The motivation of an examination question is as follows:

(14) $B\left[z, \bigvee_{z'}\left[(\ulcorner ?_{x\,(gx)}\,[fx]\urcorner \text{ is admissible for } z') \wedge W(z,P(z',z,fx))\right]\right]$
iff $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is for z a motivated examination question.

(14) is an equivalence and its parts may be reversed. A stronger form of (14) is (14') where there is a particular addressee of the question. The parts of the equivalence are given now in the reverse order:

(14') $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is for z an addressed motivated examination question
iff $\bigvee_{z'} B\left[z, (\ulcorner ?_{x\,(gx)}\,[fx]\urcorner \text{ is admissible for } z') \wedge W(z,P(z',z,fx))\right]$

Similar motivations hold in the case of rhetorical and deliberative questions. The person z who asks the question takes it as possibly admissible. This person wants the answer to be presented to somebody. While in the case of the rhetorical questions person z wants everybody to present to himself the answer (or rather the respective information), in deliberative questions person z wants that somebody (it may be himself) presents the answer to z. The assumption of admissibility seems to be doubtfull at first inspection. In fact, the doubts can be dispersed rather easily. When somebody asks the rhetorical question: 'Is it possible to deny that 2 = 2?' he assumes that all normal people will not deny that 2 = 2; only an idiot, he thinks, can deny it. Thus idiots are those people for whom the question is admissible. People who have no doubts in regard to 2 = 2 are normal. Everybody who understands the questions and sees the admissibility assumption can treat himself as normal only when he has no doubts in regard to 2 = 2. Thus asking this question helps people to see their extraordinary mental abilities. Many times this is aimed at by asking rhetorical questions.

The motivation of a rhetorical question is then as follows:

(15) $\ulcorner ?_x \,_{(gx)}\, [fx] \urcorner$ is for z a motivated rhetorical question iff
$$B\left[z, \bigvee_{z'} (\ulcorner ?_x \,_{(gx)}\, [fx]\urcorner \text{ is admissible for } z') \wedge W(z, \bigwedge_{z'} P(z',z',fx))\right]$$

The clause $W(z, \bigwedge_{z'} P(z',z',fx))$ is to the effect that z wants z' to present to himself the preposition (information) *fx*.

(16) $\ulcorner ?_x \,_{(gx)}\, [fx] \urcorner$ is for z a motivated deliberative question iff
$$B\left[z, \bigvee_{z'} (\ulcorner ?_x \,_{(gx)}\, [fx]\urcorner \text{ is admissible for } z') \wedge W(z, \bigvee_{z'} P(z',z,fx))\right]$$

In this case z wants someone to present to him the information *fx*. Person z is not sure that someone will do it in fact.

While considering general questions we cannot fail to notice that the clauses (11) and (12) include statements of the form $\ulcorner x = a \;\vee\; x = \text{non-}a \urcorner$. In this alternative the parts can be interchanged. We get then $\ulcorner x = \text{non-}a \;\vee\; x = a \urcorner$. When we apply the R-rule to (11) and (12) with the changed alternative we get questions of another form, e.g.:

A'' Did Non-Jane steal the necklace?

B'' Did Copernicus non-discover the heliocentric system?

C'' Did Barens discover Non-Spitzbergen?

All these questions sound very unnatural and, bacause of the symmetry of alternative, we cannot help such a change. This problem can be solved only when some new condition is added to the R-rule, namely:

$\ulcorner ?_x \,_{(gx)}\, [fx] \urcorner$ is admissible for z iff (11) and (12) and $W(z, \bigvee_{z'} P(z',z,fx))$ and *fx* is the shortest form of the information.

The obstacle is that our language is too poor to formulate the clause: *fx* is the shortest form of information. Thus at least one problem remains unsolved.

Well-foundedness of questions

1. Even when questions are admissible and motivated they may be in some sense very stupid. We may, for example, be astonished when somebody seriously (not rhetorically) asks if 2 + 2 = 4.

Most often when the problem of generating questions is raised, only its scientific status is considered: do questions only express subjective lack of knowledge or - and this is important - is there a scientific reason for asking them. Let us call this issue the problem of well-foundedness of questions.

The basis of all questions is their admissibility which in the case of questions of the form $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ reduces to (11) and (12). As we know, these two clauses, in turn, state that the person z who asks the question: 1) believes that *fx* and that the objects which are *f* are also *g*; 2) the person z does not know the exact objects which are *f*. The problem of well-foundedness is reduced to the question whether these beliefs are reasonable. In the framework of our poor language it is not possible to describe reasonableness of corroboration of hypotheses, etc. There, in order to point to ways of arriving at a solution, we have to resort to the notion of proof. This notion (notions) is well defined and well known. But we ought to bear in mind that a more realistic solution will be possible only if instead of the notion of proof the more general notion of corroboration or argument is introduced.

2. In order to find out whether the questions posited by person z are well-founded we must examine the beliefs of this person in regard to their reasonableness. In the case of questions of the form $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ person z believes that

X. $\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)$

and cannot tell what object a, b, c, ... satisfies *fx*. There is uncertainty as to

Y. fa, fb, fc, fd,...

The question asked by z is well-founded if X has some proof (corroboration) and no member of the series Y has a proof. Unfortunately, this general ascertainment has many ramifications. First of all, we have to settle the problem of the bases of the proofs for X. Secondly, so far we have deliberately not explained what is meant by "X has a proof". Is it to be understood in the sense it is used in methodological proof-theory? Or does it mean that person z knows such a proof and X has proof from the standpoint and knowledge of person z? Or can this phrase be taken to mean that the proof of X has actually been performed by somebody? Or, finally, does it point to the fact that the proof is actually performed and generally known?

Similarly, different views are connected with well-foundedness of Y. As was already mentioned, lack of belief in Y is well-founded if Y has no proof. Once more, we can ask what was taken into account here. It is possible that on the basis of one set of sentences (and one set of rules of inference) Y cannot be proved, while it can be proved on the basis of another set of sentences? The set of sentences which is the basis of the lack of proof is to be fixed and it is to be identical with the set which makes the proof of X possible. Possibly, there is no proof of Y (in the sense of proof theory) on the basis of the whole set of true sentences, which are expressible in the language of the question. But there may be such a proof but it has not been performed by anybody, including z who asks the question. It may be that the proof has been performed but it has not been presented to person z; it is also possible that the proof has been performed but it has not been presented to a group of people including z who is a member of this group. Perhaps z knows the proof but his scientific environment does not know it.

It is obvious that we get a whole range of well-foundedness concepts if we consider all these possibilities. Only some of them can be presented here.

3. I. Let us assume that

1. The question (α) $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is cognitively motivated for z. Automatically, the question is admissible for z and if so, then (11) and (12) are true. This being the case the question is well-founded in the sense I if:
2. T is the set of true sentences which are expressible in the language which is couched.
3. There is a proof of $\bigvee_{x} (fx) \wedge \bigwedge_{x} (fx \Rightarrow gx)$ on the basis of T.
4. For all a: there is no proof of fa on the basis of T.
5. For all a: fa is couched in the language of sentences T.

The sentences fa are independent from the set of true sentences. The sentences fa are obviously false and are possible answers to (α). Thus questions without possible true answers are not well-founded in sense I.

II. Once more let us assume that

1. $\ulcorner ?_{x\,(gx)}\,[fx]\urcorner$ is cognitively motivated for z. Then the question is well-founded in the sense II if:

2. S is the set of principles of an empirical theory which is believed in the community C.

3. There is a proof of $\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)$ on the basis of S.

4. For all *a*: there is no proof of *fa* on the basis of S.

5. The sentences *fa* are worded in the language of the set S.

It seems that sentences *fa*, which possibly can be true, are falsifying instances of the theory.

III. In case 1. is true the question is well-founded in the sense III if:

2. S is a set of principles of a theory (not necessarily an empirical one)

3. There is a proof of $\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)$ on the basis of S.

4. For all *a*: there is a proof of *fa* on the basis of S but the proof is not presented to anybody.

5. The sentences of *a* are worded in the language of S.

Possibly the great problem of Fermat is well-founded in this sense.

IV. In case 1. is true and similarly 2. and 3. of III and

4. For some *a* there is a proof of *fa* on the basis of S but the proof is not presented to anybody with the exception of the person who discovered the proof.

5. The sentences *fa* are worded in the language of S.

If Fermat had succeeded in finding the proof of his great hypothesis, it in sense IV in which his problem is well-founded.

From among the variety of further senses of well-foundedness let us cite only two without any comments.

V. 1. Question (α) is motivated for z.

2. S is the set of principles of a theory accepted in the community C (the members of the community believe in S).

3. There is a proof of $\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)$ on the basis of S and nobody is presented with this proof – it is not discovered.

4. For some *a* there is a proof of *fa* on the basis of S but the proof is not presented to anybody – it is unknown.

5. As above in I – IV.

VI 1. Question (α) is motivated for z.

2. As above in V.

3. There is a proof of $\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)$ on the basis of S and this proof is presented by members of community C to person z.

4. For some *a*: there is a proof of *fa* on the basis of S but the proof is not presented to anybody of the community C.

The adduced examples show the way how to construct successive concepts of well-foundedness. The basic sentences may be given logically, presented or not presented, believed or not by z or by a community. The proof of $\bigvee_x (fx) \wedge \bigwedge_x (fx \Rightarrow gx)$ may be given logically and presented or not to z or to a community. Similarly with the fourth clause. A new numerous set of notions of well-foundedness comes into existence when the notion(s) of proof is substituted by different concepts of corroboration or argument. So far we have considered only well-foundedness of cognitive questions. Well-foundedness of rhetorical, examination and deliberative questions reduces to the problem of the foundation of the belief that they are or can be admissible. In the case of the rhetorical question: 'Is it possible to deny that 2+2=4?' all arguments show that for some people 'Is 2+2=4?' is admissible.

4. The considerations put forward are rather sketchy. I would like to call attention to three important points which call for further study. The variable x is typically ambiguous – it ranges over sets of different syntactical categories. In this paper the problem of the grammar of the language in which the questions are worded was carefully omitted. As a consequence nothing was said about how many and what kinds of syntactical categories are possible in language of the questions. In fact, we had the natural language in mind and its categories and grammar are not well discerned.

The notion of presentation is very important in these consideration. It deserves a thorough analysis. The notion of belief - the weaker one which was introduced here - depends heavily on the notion of presentation and is also worth a detailed study, as only the stronger concepts of belief were in the focus of interest.

WOJCIECH BUSZKOWSKI

THE LOGIC OF TYPES

0. INTRODUCTION AND PRELIMINARIES

By the logic of types (TL) we mean systems of type transformation. The formulae of these systems are simply types or type transformation rules. For example, the system introduced by Ajdukiewicz **(1935)** (under the influence of Leśniewski's doctrine of semantic categories) employs the schema:

(A.1) $(ab)a \rightarrow b$,

which can be interpreted as a law or rule of type reduction. The much stronger system of Lambek **(1958)** also admits the schemata:

(1) $(ab)(ca) \rightarrow (cb)$ (rediscovered by Geach **1968**),

(2) $(ab) \rightarrow ((ca)(cb))$,

and many others. The system of van Benthem **(1983a, 1985)** affixes to the latter:

(3) $a \rightarrow ((ab)b)$ (implicit in Montague **1973**).

Notice that, according to (2), each functor of type (ss) is also of type ((ns)(ns)). For instance, negation is a sentence-forming functor, but it can be regarded as a predicate-forming functor as well. The basic type of a quantifier is ((ns)s) (as in $\forall xP(x)$). By (2), it expands to ((n(ns))(ns)) (as in $\forall xP(x,y)$), to ((n(n(ns)))(n(ns))) (as in $\forall xP(x,y,z)$), and so on (see Levin **1982** who uses (1) instead of (2)). Due to (3), each individual name (type n) can be lifted up to the type of nominal phrases ((ns)s).

Systems of TL play a fundamental role in theory of categorial grammars, a logically oriented branch of mathematical linguistics. From the linguistical point of view, a natural semantics for them is an algebraic semantics, based on residuated semigroups (Buszkowski **1982, 1985, 1985a**). A long proof-theoretic tradition suggests another semantics, involving typed lambda calculus. For example, van Benthem (**1983a**) proves that the type transformations derivable in his system are precisely those which can be defined by means of a limited class of typed lambda terms. In fact, the very idea of this correspondence was implicit in Cresswell (**1973**). As is well known, typed lambda calculus is naturally modelled by Cartesian closed categories (Scott **1980**); in a sense, systems of typed lambda calculus are simply equivalent to Cartesian closed categories (Lambek, Scott **1984**). Accordingly, TL may also be viewed as a logic of these categories, which deals with some universal arrows and transformations.

Beyond doubt, the subject-matters of TL are of great significance for the foundations of logic and linguistics. Their significance follows from the obvious role of types in logical syntax and semantics. On the other hand, not many studies of genuine logical character have been devoted to the matters in question. Furthermore, they mainly focus on the linguistics side of TL (categorial grammars), just ignoring finer logical aspects. In the author's opinion, TL deserves a profound research from the stand-point of logic (for its position in logic see also van Benthem **1983**, **1984**, Buszkowski **1986a**). Methodologically, it is rather close to abstract propositional logics (cf. Rasiowa, Sikorski **1963**, Wójcicki **1984**), but it calls for some special methods, for instance, of linguistic flavour.

In this paper we consider several systems of TL, all related to that of Lambek (**1958**). The Lambek calculus will be denoted by L. We distinguish some interesting supersystems and subsystems of L, among them the commutative L (CL), which amounts to the calculus of van Benthem (**1983a**). The paper penetrates into the correspondences between TL and typed lambda calculus, different axiomatizations of systems of TL, and matrix semantics for TL.

In section 1 we establish a number of correspondences between systems of TL and classes of typed lambda terms. There are regarded non-directional, unidirectional, as well as bidirectional systems. However, we confine ourselves to product-free systems of TL, which correspond to the traditional versions of types and typed lambda calculus.

Section 2 provides different axiomatizations of the systems distinguished in section 1. We consider Gentzen-style axiomatizations (which yield decidability results), Hilbert-style ones, and linear ones. In particular, we show that CL admits no finite Hilbert-style axiomatization, though such an axiomatization exists for the closely related system $CL_{\bullet}$. These results possess a nice connection with lambda calculus. For instance, the class of terms corresponding to CL cannot be generated by any finite family of term schemata (with application as the only operation).

In section 3 we examine matrix semantics for TL. Precisely, the so-called e-free systems (as L, CL, etc.) require modified matrices, where instead of a distinguished subset one uses a distinguished binary relation. We prove a number of representation, completeness, and adequacy theorems. For example, such systems, as L, L_e, CL, CL_e are shown to admit no finite adequate matrix, though L and CL possess the finite model property.

Many results of this paper are less or more akin to earlier ones. Those from section 1 generalize the afore-mentioned theorem of van Benthem **(1983a)**. Section 2 refers to the Gentzen-style axiomatization of L given by Lambek **(1958)** and the axiomatization of L by "cancellation schemata" considered by Cohen **(1967)**, and Zielonka **(1981)**. In section 3 we widely employ the author's earlier results on algebraic semantics for L.

Below we recapitulate some basic notions of TL. Our mathematical terminology and notation is rather standard and need not be explained.

We fix a denumerable set Pr, of primitive types. The set Tp, of types, is defined by the inductive clauses:

(i) $Pr \subseteq Tp$,

(ii) if $a,b \in Tp$ then $(ab) \in Tp$.

The letters a, b, c (p, q, r) are to denote (primitive) types, and X, Y, Z finite strings of types (e stands for the empty string). Expressions of the form $X \rightarrow a$ are called arrows. An arrow of the form $a \rightarrow b$ is said to be simple. By the complexity of $a \in Tp$ ($c(a)$) or an arrow $X \rightarrow a$ ($c(X \rightarrow a)$) we mean the total number of primitive types occuring in it. The order of $a \in Tp$ ($o(a)$) is a non-negative integer, defined by the following induction on $c(a)$:

(o.1) $$o(p) = 0, \text{ for } p \in Pr,$$

(o.2) $$o((ab)) = \max(o(b), o(a)+1),$$

and we set:

$$o(e) = -1,$$

(4) $$o(a_1 \ldots a_n) = \max_{1 \leq j \leq n} o(a_j),$$

$$o(X \rightarrow a) = \max(o(X), o(a)).$$

For all a and X, type (X,a) is defined by induction on the length of X as follows:

(5)
$$(e,a) = a,$$
$$(Xb,a) = (X,(ba)).$$

Notice that for each type a there are unique $p \in Pr$ and X, such that a = (X,p) (p is called the head of a). Also observe that:

(6) $$o((X,p)) = o(X)+1.$$

The Ajdukiewicz calculus admits (A.1) and:

(A.0) $$a \to a,$$

as its axioms, and:

(CUT) $$XaZ \to b \text{ and } Y \to a \text{ yield } XYZ \to b,$$

as its only inference rule. We denote this system by A^r (the reason for the superscript r will be provided later on). We write $\vdash_{A^r} X \to a$ if $X \to a$ is derivable in A^r, and similarly for other systems. By affixing to A^r the rule:

(R.1) $$Xa \to b \text{ yields } X \to (ab) \quad (X \neq e),$$

we obtain the right-directional fragment of L (L^r). Verify that (1) and (2) (but not (3)) are derivable in L^r. CL equals L^r + (3). CL is commutative, which means that it admits the rule:

(COM) $$XabY \to c \text{ yields } XbaY \to c,$$

and, consequently, $a_1 \ldots a_n \to b \vdash_{CL} a_{i_1} \ldots a_{i_n} \to b$, for every permutation $i_1, \ldots, i_n$ of $1, \ldots, n$. (Given a calculus C and a set of arrows R, $R \vdash_C X \to a$ means, as usual, that $X \to a$ is derivable from R in C.) The commutativity of CL has been shown by van Benthem **(1983a)**, but his axiomatization of CL is richer than ours; so, we give new proof.

First, we show that:

(7) $$(a(bc)) \to (b(ac)),$$

is derivable in CL. Both L^r and CL admit the rules:

(EXP.1) $$a \to b \text{ yields } (ca) \to (cb),$$

(EXP.2) $$a \to b \text{ yields } (bc) \to (ac).$$

Using these rules together with (2) and (3), we can get in CL:

(8) $$(a(bc)) \to (((a(bc))(ac))(ac)) \to (((bc)c)(ac)) \to (b(ac)),$$

hence $\vdash_{CL}$ (7), by (CUT).

To derive (COM) assume $XabY \to c$. By (R.1), we get $Xab \to (Y,c) = c'$. Suppose $X \neq e$. Then, (R.1) yields $X \to (ab,c')$, hence we obtain $X \to (ba,c')$, by (7) and (CUT). Using (A.1) and (CUT) we come to $Xba \to c'$, and finally, to $XbaY \to c$. Suppose $X = e$. Then, $ab \to c'$ yields $a \to (bc')$. By (3), (A.1), and (CUT), the schema:

(A.1') $$a(ab) \to b,$$

is derivable in CL, hence we get $b(bc') \to c'$, which yields $ba \to c'$, by (CUT). Finally, using (A.1) and (CUT), we come to $baY \to c$.

Notice that CL also amounts to (A.0) + (A.1') + (CUT) + (R.1). For, (R.1) transforms (A.1') into (3), and (3), (A.1') and (CUT) yield (A.1).

By $(R.1_e)$ we denote the rule (R.1) with the constraint $X \neq e$ dropped. Systems L^r_e and CL_e result from substituting $(R.1_e)$ for (R.1) in L and CL, respectively. Both L^r_e and CL_e but neither L^r nor CL produce derivable arrows of the form $e \to x$, which we write $\to x$. Furthermore, L^r_e (CL_e) is a non-conservative extension of L^r (CL), since, for instance, $((qq)p) \to p$ is derivable in L^r_e but not CL. (If we claim underivability, we employ decision methods for these systems, which will be considered in section 2.)

The systems described above are restricted to non-directional types. The set Tp^*, of bidirectional types, is defined as Tp with (ii) replaced by:

(ii*) if $a,b \in Tp^*$ then $(ab),(ab)^* \in Tp^*$.

All the technical notions defined above preserve their sense for bidirectional types. Only (5) must be supplemented by:

(5*) $$(e,a)^* = a,$$ $$(Xb,a)^* = (b(X,a)^*)^*.$$

The bidirectional version of A^r (A) uses bidirectional types and the additional schema:

(A.1*) $$a(ab)^* \to b.$$

Actually, this version of A^r is due to Bar-Hillel **(1953)**, Bar-Hillel *et.al.* **(1960)**. Similarly, by affixing to A (R.1) and:

(R.1*) $$aX \to b \text{ yields } X \to (ab)^* \quad (X \neq e),$$

we get the Lambek calculus (L) (L amounts to the product-free fragment of the system of Lambek **1958**). L_e arises from L in the same way as L^r_e from L^r. For $a \in Tp^*$, a^0 denotes the non-directional type which results from dropping all stars in a, and we set:

$$(9) \quad \begin{aligned} e^o &= e, \\ (a_1 \ldots a_n)^o &= a_1^o \ldots a_n^o, \quad \text{for } a_j \in Tp^*,\ 1 \leqslant j \leqslant n. \end{aligned}$$

L (L_e) can be treated as a system intermediate between L^r (L_e^r) and CL (CL_e), since the following conditional holds true:

(10) if $\vdash_L X \rightarrow a$ ($\vdash_{L_e} X \rightarrow a$) then $\vdash_{CL} X^o \rightarrow a^o$ ($\vdash_{CL_e} X^o \rightarrow a^o$).

If should be noticed that A^r and L^r coincide in the scope of arrows of the form $X \rightarrow p$, where $o(X) \leqslant 1$ and $p \in Pr$, and similarly A and L (Buszkowski **1982**). In this scope, CL yields precisely the arrows $Y \rightarrow p$ such that, for some string X, $\vdash_{A^r} X \rightarrow p$ and Y is a permutation of X. Consequently, the generative capacity of CL reaches all commutative closures of context-free languages, hence it allows some non-context-free languages (Buszkowski **1984**, van Benthem **1985**).

Types can be translated into purely implicational propositional formulae. Precisely, for $a \in Tp$, we define a formula F(a) by the inductive clauses:

(i) $F(p) = p$, for $p \in Pr$,

(ii) $F((ab)) = (F(a) \rightarrow F(b))$.

We also set: $F(a) = F(a^o)$, for $a \in Tp^*$. One easily verifies that all the systems distinguished above are in fact subsystems of positive intuitionistic logic. Precisely, if $\vdash_C X \rightarrow a$, where C is any of those systems of TL, then $F(X) \vdash F(a)$, where F(X) stands for the set of all $F(\bar{a})$, for $\bar{a}$ appearing in X, becomes a valid inference pattern of this logic. Clearly, (A.1), (A.1'), (A.1*) correspond to Modus Ponens, (1) is the rule of transitivity, (R.1) and (R.1*) (also (R.1_e) and (R.1_e^*)) represent some forms of the deduction theorem, and so on. So, identifying types with formulae and arrows with rules, we see that systems of TL, as presented above, are systems of rules rather than formulae. Such an approach to TL appears to be much expedient for various purposes.

1. TL VERSUS TYPED LAMBDA CALCULUS

To each type $a \in Tp$ we ascribe a denumerable set VAR_a, of variables of type a, to be denoted by x_a, y_a, z_a, etc. The set TER, of typed lambda terms (shortly: terms), is the union of pairwise disjoint sets TER_a, of terms of type a, being defined by the following induction:

(TER.1) $VAR_a \subseteq TER_a$,

(TER.2) if $t \in TER_{(ab)}$ and $u \in TER_a$ then $(tu) \in TER_b$,

(TER.3) if $x_a \in VAR_a$ and $t \in TER_b$ then $\lambda x_a.t \in TER_{(ab)}$.

With each term t we associate a string var(t), containing all the free occurences of variables in t in the order of their appearance in t. We give an inductive definition:

(var.1) $var(x_a) = x_a$,

(var.2) $var((tu)) = var(t)var(u)$,

(var.3) $var(\lambda x_a.t) =$ the string that results from dropping x_a in var(t).

For $t \in TER$, by typ(t) we denote the string of types which results from replacing in var(t) each variable by its type. The only type a such that $t \in TER_a$ will be denoted by Typ(t). Finally, Ar(t), $t \in TER$, stands for arrow $typ(t) \rightarrow Typ(t)$. Given a set $T \subseteq TER$, we set:

(1) $$Ar(T) = \{Ar(t): t \in T\}.$$

Let C be a system of TL (whose formulae are arrows), and let $T \subseteq TER$. We say that C is complete with respect to T (T-complete) or T corresponds to C if the arrows derivable in C are precisely those from Ar(T). As observed as far back as Curry *et. al.* **(1958)**, the full class TER corresponds to the purely implicational intuitionistic logic. Van Benthem **(1983a)** proves that CL is complete with respect to the class TER_{CL} of all terms, fulfilling the following constraints:

(C.1) each subterm contains a free variable,

(C.2) no subterm contains more then one free occurence of the same variable,

(C.3) each occurence of the lambda abstractor binds some variable within its scope.

We consider classes TER_G, $G \subseteq \{1,2,3\}$, of all the terms fulfilling the constraints (C.i), for $i \in G$. So, $TER_\emptyset = TER$ and $TER_{123} = TER_{CL}$ (we write 123 for {1,2,3}, and similarly for other cases). TER_3 is the typed version of Church's first concept of lambda terms (Church **1941**).

Our goal is to find systems of TL, complete with respect to these classes of terms. The most natural way seems to proceed as follows. First, we look for systems whose derivation trees strictly harmonize with the structure of terms from the corresponding classes. We omit standard definitions of a derivation tree of arrow $X \to a$ in a system C and the tree of subterms of a term. Now, a system C is said to be compatible with a class $T \subseteq TER$ if the following conditions hold true:

(i) for any $t \in T$, one obtains a derivation tree of Ar(t) in C, after he has replaced in the tree of subterms of t each node u by the arrow Ar(u),

(ii) if $\vdash_C X \to a$ then there exists a derivation tree of $X \to a$ in C, such that, for some $t \in T$, fulfilling $Ar(t) = X \to a$, this derivation tree results from the tree of subterms of t in the way indicated in (i).

It immediately follows from this definition that:

1.1 ***Lemma.***
If C is compatible with T, and T is closed under subterms, then C is T-complete.

All the classes of terms we consider in this paper are closed under subterms. So, to find a system complete with respect to such a class T it suffices to formulate axioms and rules, strictly mirroring the principles of construction for T, and next, to look for some equivalent axiomatizations.

Consider the rules:

(R.0) $\quad X \to (ab)$ and $Y \to a$ yield $XY \to b$,

(R.1') $\quad X_1aX_2a \ldots X_naX_{n+1} \to b$ yields $X_1X_2 \ldots X_{n+1} \to (ab)$.

The scope of (R.1') will be limited by the constraints:

(c.1) $\quad X_1X_2 \ldots X_{n+1} \neq e$,

(c.2) $\quad n \leqslant 1$,

(c.3) $\quad n \geqslant 1$ (i.e. one excludes: $X_1 \to b$ yields $X_1 \to (ab)$).

For each $G \subseteq \{1,2,3\}$, by $(R.1'_G)$ we denote the rule (R.1') restricted by all the constraints (c.i), for $i \in G$. Thus, for instance, $(R.1'_\emptyset)$ is simply (R.1'), $(R.1'_{123})$ amounts to:

(2) $\quad X_1aX_2 \to b$ yields $X_1X_2 \to (ab)$ $\quad (X_1X_2 \neq e)$,

and so on. We also define C_G = (A.0) + (R.0) + $(R.1'_G)$, for all $G \subseteq \{1,2,3\}$.

1.2 ***Theorem.***
For each $G \subseteq \{1,2,3\}$, the system C_G is compatible with the class TER_G.

Proof. Fix $G \subseteq \{1,2,3\}$. Let $t \in TER_G$. If t is a variable, then $Ar(t)$ is an axiom (A.0). If $t = (u_1u_2)$, then $Ar(t)$ results from $Ar(u_1)$ and $Ar(u_2)$ by (R.0). If $t = \lambda x_a.u$, then $Ar(t)$ results from $Ar(u)$, by (R.1'$_G$). This yields the clause (i) of the definition of compatibility. To prove (ii) we proceed by induction on derivations of $X \rightarrow a$ in C_G. For an axiom $a \rightarrow a$, we get $a \rightarrow a = Ar(x_a)$, where x_a is an arbitrary variable of type a. Assume that $X \rightarrow a$ results from $X_1 \rightarrow (ba)$ and $X_2 \rightarrow b$ by (R.0). By induction, we find terms $u_1, u_2 \in TER_G$, such that $Ar(u_1) = X_1 \rightarrow (ba)$, $Ar(u_2) = X_2 \rightarrow b$, and the trees of subterms of u_1 and u_2 fulfil (ii). We can change the free variables in u_1 and u_2 so to obtain $(u_1u_2) \in TER_G$. Consequently, the term $t = (u_1u_2)$ fulfils (ii) with respect to $X \rightarrow a$. Finally, assume that $X \rightarrow a$ results from $Y \rightarrow c$ by (R.1'$_G$). Then, $a = (bc)$ and $Y = Y_1bY_2b \ldots Y_nbY_{n+1} \rightarrow c$, where Y satisfies the constraints imposed on (R.1'$_G$). Again, by induction, we find a term $u \in TER_G$ whose tree of subterms fulfils (ii) with respect to $Y \rightarrow c$. After identifying, if necessary, some free variables of type b in u, we get $t = \lambda x_b.u \in TER_G$, $typ(t) = Y_1Y_2 \ldots Y_{n+1} = X$, and the tree of subterms of t yields a derivation of $X \rightarrow a$ in C_G. The proof is finished.

We wish to compare C_G's with the systems of TL introduced in the preceding section. For $a \in Tp$, by $\vec{a}$ we denote a string of a's, and by $l(\vec{a})$ the length of $\vec{a}$. Consider the rule:

(R.1°) $X\vec{a} \rightarrow b$ yields $X \rightarrow (ab)$,

and the constraints:

(c.1°) $X \neq e$,

(c.2°) $l(\vec{a}) \leqslant 1$,

(c.3°) $l(\vec{a}) \geqslant 1$.

(R.1°$_G$), where $G \subseteq \{1,2,3\}$, stands for the rule (R.1°) restricted by all the constraints (c.i°), for $i \in G$, and by CL_G we denote the system A^r + (3) + (R.1°$_G$) ((3) refers to section 0). Clearly, $CL_{123} = CL$ and $CL_{23} = CL_\bullet$. We prove:

1.3 ***Lemma.***
For all $G \subseteq \{1,2,3\}$, and for all arrows $X \rightarrow a$, $\vdash_{CL_G} X \rightarrow a$ iff $\vdash_{C_G} X \rightarrow a$.

Proof. Fix $G \subseteq \{1,2,3\}$. By induction on derivations, one easily proves that C_G is closed under (CUT). Using (R.0) and (A.0), we derive (A.1) in C_G, hence also (3) (from section 0), by (R.1'$_G$). Since (R.1°$_G$) is an instance of (R.1'$_G$), we infer that $\vdash_{CL_G} X \to a$ entails $\vdash_{C_G} X \to a$. To show the converse observe, first, that CL_G admits (R.0)^G (use (A.1) and (CUT)). In the same way as for CL one proves that CL_G also admits (COM) (in fact, if $1 \notin G$ then the proof goes more smoothly!), and consequently, it must admit (R.1'$_G$), which finishes the proof.

From 1.1 - 1.3 we infer:

1.4 ***Theorem.***
For all $G \subseteq \{1,2,3\}$, CL_G is TER_G-complete.

In particular, we have given a new proof of van Benthem's result for CL. Notice, furthermore, that $CL_\emptyset$ provides some axiomatization of purely implicational intuitionistic logic. Accordingly, the commutative systems of TL are certain subsystems of this logic, corresponding to some natural constraints on the structure of terms.

The extensions of CL we have described above may also find some interesting applications in categorial grammar. Consider, for instance, a grammar which admits the following type assignment:

(3) Joan$\to$n, runs$\to$(ns), springs$\to$(ns), and$\to$(ss,s).

If based on A^r + (3) (from section 0), this grammar assigns type *s* to the sentence:

(4) Joan runs and Joan springs,

but even CL does not allow to accept:

(5) Joan runs and springs.

The latter sentence is nonetheless accepted by a grammar, based on CL_{13}. For, the arrow:

(6) (aa,b)$\to$((ca)(ca),(cb)),

is derivable in CL_{13} (use (R.1'$_{13}$) to the A^r-derivable arrow (aa,b)(ca)c(ca)c$\to$b), hence (ss,*s*) can be expanded to ((ns)(ns),(ns)) (see also Cresswell 1973 who points out this fact in the terminology of lambda terms).

For any G, the system CL_G, though TER_G-complete, is however not compatible with TER_G. Of course, derivations in CL_G do not reflect trees of subterms. In particular, (CUT) corresponds to the substitution of a term for a free variable in a term. It would certainly be interesting to examine the operations on terms corresponding to the rules of these systems in detail, but we avoid this matter here.

We shall briefly discuss other systems of TL. Clearly, A^r is complete with respect to the class of lambda-free terms (show that A^r yields the same theorems as the system (A.0) + (R.0), and the latter is compatible with this class). L^r is complete with respect to the class TER_L, consisting of all the terms which fulfil (C.1), (C.2) and:

(C.3') each occurence of the lambda abstractor binds the right-most occurence of a free variable within its scope.

For, L^r is (weakly) equivalent to the system (A.0) + (R.0)+ (R.1) and the latter is compatible with TER_L. Similarly, L^r_e is complete with respect to the class TER_{L_e}, of all the terms fulfilling (C.2) and (C.3').

To manage bidirectional systems we need a bidirectional version of typed lambda terms. The set TER^*, of directional terms, is again the union of pairwise disjoint sets TER^*_a, for $a \in Tp^*$, which are defined by the following induction:

(TER^*.1) $VAR_a \subseteq TER^*_a$,

(TER^*.2) if $t \in TER^*_{(ab)}$ $(t \in TER^*_a)$ and $u \in TER^*_a$ $(u \in TER^*_{(ab)^*})$ then $(tu) \in TER^*_b$ $((tu)^* \in TER^*_b)$,

(TER^*.3) if $x_a \in VAR_a$ and $t \in TER^*_b$ then $\lambda x_a.t \in TER^*_{(ab)}$ and $\lambda^*.t \in TER^*_{(ab)^*}$.

As a natural semantics for TER consists of Cartesian closed categories, TER^* can be interpreted by means of biclosed monoidal categories (Lambek 1958). In particular, we can employ a standard hierarchy of ontological categories (see e.g. Suszko 1958–60), but distinguish between a function $f \in CAT_{(ab)}$ and its "copy" $f^* \in CAT_{(ab)^*}$. Linguistically, that means that we relativize possible designata to the actual syntactic roles of the expressions which denote them.

All the notions introduced above for TER admit an obvious extension for TER^*. This also holds for compatibility, T-completeness, and so on. Let TER_L^* ($TER_{L_e}^*$) denote the class of all the directional terms fulfilling (C.1), (C.2), and:

(C.3*) each occurence of λ (λ^*) binds the right-most (left-most) occurence of a free variable within its scope.

We prove:

1.5 ***Theorem.***
L (L_e) is $TER_{L_e}^*$-complete ($TER_{L_e}^*$-complete).

Proof. One shows that L is (weakly) equivalent to the system (A.0) (for $a \in Tp^*$) + (R.1) + (R.1*) + (R.0) + the rule:

(R.0*) $X \rightarrow a$ and $Y \rightarrow (ab)^*$ yield $XY \rightarrow b$,

which is compatible with TER_L^*. The case of L_e can be treated in a similar way.

Clearly, A is complete with respect to the class of all lambda-free bidirectional terms. As for non-directional systems, we could consider extensions of L, corresponding to some wider classes of bidirectional terms. Since they behave quite analogously to the previous ones, we omit all details.

Till now we have considered the language of typed lambda calculus, but not the very calculus (e.g. reducibility, equalities, etc.). Of course, by regarding these matters we come to a more advanced level of TL. In this paper we however neglect this direction of research in TL except for some simple observations. Notice that the commutativity of the systems corresponding to such classes, as TER_C's, is a consequence of the fact these classes contain terms which are not in normal form. For instance, (A.1'), which results from (A.1) by (COM), equals Ar(t), where t is the term:

(7) $$(\lambda x_{(ab)}.(x_{(ab)}y_a)x_{(ab)}),$$

whose normal form is $u = x_{(ab)}y_a$, and $Ar(u) = (A.1)$. As in semantics equal terms posses equal designata, we may suppose that just terms in normal form provide a semantically relevant variety of type transformations. Consequently, we should look for systems of TL, being complete with respect to some classes of terms in normal form.

Consider, for instance, the class $NTER_{CL}$, of all the terms from TER_{CL} which are in the weak normal form. One easily shows that the system NCL = (A.0) + (2) + the rule:

(NR.0) $X_1 \rightarrow a_1, \ldots, X_n \rightarrow a_n$ yield $(a_1 \ldots a_n, b)X_1 \ldots X_n \rightarrow b$, is compatible with $NTER_{CL}$, hence it is $NTER_{CL}$-complete. NCL is closed under neither (COM), nor (CUT), but, interestingly, it completely determines CL, due to the equivalence:

(8) $\vdash_{CL}$ $a_1 \dots a_n \to b$ iff, for some permutation $i_1, \dots, i_n$ of the sequence $1, \dots, n$, $\vdash_{NCL}$ $a_{i_1} \dots a_{i_n} \to b$.

The proof of (8) uses the fact that each typed term has a weak normal form. Since commutative systems seem rather strange from the point of viev of linguistics, as they enforce us to accept every permutation of an accepted string of words, we suggest that systems like NCL, which lack that failure, may deserve a serious attention.

2. AXIOMATIZABILITY PROBLEMS

In this section we consider various axiomatizations of systems of TL. We begin from the Gentzen-style ones, following that given by Lambek (1958).

We introduce a new rule:

(R.2) $XbZ \to c$ and $Y \to a$ yield $X(ab)YZ \to c$.

Denote GL^r = (A.0) + (R.1) + (R.2). We prove:

2.1 ***Lemma.***
GL^r is closed under (CUT).

Proof. We must show:

(1) if $\vdash_{GL^r}$ $XaZ \to b$ and $\vdash_{GL^r}$ $Y \to a$ then $\vdash_{GL^r}$ $XYZ \to b$.

We use induction on c(a). Let $a \in Pr$. Then, (1) holds by a straightforward induction on GL^r-derivations of $XaZ \to b$. Let $a = (a_1a_2)$. Again, we proceed by induction on GL^r-derivations of $XaZ \to b$. The only non-trivial case is if $XaZ \to b$ results from $Xa_2Z_2 \to b$ and $Z_1 \to a_1$ by (R.2) (so, $Z = Z_1Z_2$). We use induction on GL^r-derivations of $Y \to a$. Again, the only non-trivial case is if $Y \to a$ results from $Ya_1 \to a_2$ by (R.1). Then, from $Xa_2Z_2 \to b$ and $Ya_1 \to a_2$ we obtain $XYa_1Z_2 \to b$, hence with applying $Z_1 \to a_1$ we come to $XYZ_1Z_2 \to b$, as desired (the final steps employ the first induction on c(a)).

Although the above lemma was essentially proved by Lambek (1958), we have given another proof which can easily be adapted for different systems to be considered in what follows.

2.2 ***Corollary.***
GL^r is weakly equivalent to L^r.

Proof. Clearly, L^r admits (R.2) (use (A.1) and (CUT)), hence it is supersystem of GL^r. Since (A.1) follows from (A.0) and (R.2), then – in the presence of 2.1 – we infer that $\vdash_{L^r} X \rightarrow a$ entails $\vdash_{GL^r} X \rightarrow a$. Consequently, L^r and GL^r yield the same theorems.

2.3 ***Theorem*** (essentially Lambek 1958).
L^r is decidable.

Proof. Observe that in both (R.1) and (R.2) the conclusion has a greater complexity than the premiss(es). Then, GL^r admits a standard "proof-search" decision method.

By $L^r(R)$, where R is a set of arrows, we denote the system resulting from L^r after one has affixed to it all the arrows from R as new axioms, and similarly for other systems. We shall describe Gentzen-style systems equivalent to $L^r(R)$'s. First, observe that due to the equivalence:

(2) $$\vdash_{L^r(R)} X \rightarrow (Y,a) \text{ iff } \vdash_{L^r(R)} XY \rightarrow a,$$

every arrow is equivalent in $L^r(R)$ to an arrow of the form $X \rightarrow p$, $p \in Pr$. Accordingly, with no loss of generality we may assume that the arrows in R are in the latter form. To each arrow $a_1 \dots a_n \rightarrow p$ we ascribe a new rule:

(3) $$X_1 \rightarrow a_1, \dots, X_n \rightarrow a_n \text{ yield } X_1 \dots X_n \rightarrow p.$$

Then, by $GL^r(R)$ we denote the system arising from GL^r after one has affixed to it all rules ascribed to the arrows from R. The argument given for 2.1 also yields:

2.4 ***Lemma.***
$GL^r(R)$ is closed under (CUT).

As a result, $GL^r(R)$ and $L^r(R)$ are equivalent. For the case of a finite set R, consisting of arrows of one of the forms:

(4) $$p_1 \cdots p_n \rightarrow p,$$

(5) $$p/q \rightarrow r,$$

$GL^r(R)$, hence also $L^r(R)$, is decidable. Arrows (4) correspond to production rules of phrase structure grammars. Consequently, L^r can be treated as a decidable transformation system over phrase structure grammars (Buszkowski 1987). On the other hand, (5) give rise to the rules:

(6) $Xq \to p$ yields $X \to r$,

being some forms of deletion. Accordingly, sets R, containing both (4) and (5), correspond to so called generalized phrase structure grammars (Gazdar *et. al.* 1985). As proved in Buszkowski (1982a), for such sets R, the system $L^r(R)$ is, in general, undecidable. Precisely, every recursively enumerable language can be generated by some system of that form.

Quite similar results can be obtained for other systems of TL. GCL (Gentzen-style form of CL) results from GL^r by affixing (COM) (Buszkowski 1984). GL^r_e and GCL_e are almost equal to GL^r and GCL, respectively, except for dropping $X \neq e$ in (R.1). To get GL and GL_e one has to expand GL^r and GL^r_e, respectively, on bidirectional types and to affix the rule:

(R.2*) $XbZ \to c$ and $Y \to a$ yield $XY(ab)^*Z \to c$.

For systems CL_G from section 1, the corresponding Gentzen-style systems result from GCL after one has replaced (R.1) by (R.1^o_G). Now, verify that 2.1 - 2.3 hold for each of these systems with essentially analogous proofs.

In section 0 systems of TL have been interpreted as subsystems of positive intuitionistic logic. Following this line we shall look for Hilbert-style axiomatizations of these systems. Such axiomatizations will be exemplified for CL and Cl_e.

We begin from CL_e. Since $X \to a \vdash_{CL_e} \to (X,a)$ and conversely, we can identify in CL_e the arrow $X \to a$ with the type (X,a) (then, several arrows are represented by the same type). By HCL_e we denote the following system:

(a.1) (aa),

(a.2) ((a(bc))(b(ac))),

(a.3) ((ab)((ca)(cb))),

(MP) (ab) and a yield b.

Each type derivable in HCL_e is also derivable in CL_e. For, $\vdash_{CL_e}$ (a.1), by (A.0) and $(R.1_e) = (R.1^o_{23})$, $\vdash_{CL_e}$ (a.2), by (7) (from section 0) and $(R.1_e)$, $\vdash_{CL_e}$ (a.3), by (2) (from section 0) and $(R.1_e)$, and finally, (MP) is an instance of (R.0). We show the converse. First, $\vdash_{HCL_e}$ (A.0), (A.1), by (a.1), and $\vdash_{HCL_e}$ (3)

(from section 0), by (a.1), (a.2), and (MP). Obviously, HCL_e admits ($R.1_e$) (both $Xa \rightarrow b$ and $X \rightarrow (ab)$ are represented by (Xa,b)). It suffices to prove that HCL_e admits (CUT). By (a.2) and (MP), we get:

(7) $$(ab,c) \vdash_{HCL_e} (ba,c).$$

Since HCL_e admits (EXP.1) (use (a.3) and (MP)), (7) can be generalized to:

(8) $$(Xab,c) \vdash_{HCL_e} (Xba,c),$$

which means the same as:

(9) $$(XabY,c) \vdash_{HCL_e} (XbaY,c).$$

Consequently, HCL_e admits (COM). Now, assume that $XaZ \rightarrow b$, and $Y \rightarrow a$ hold true. By (COM), we get $aXZ \rightarrow b$, which amounts to $(a(XZ,b))$. By (EXP.1), we infer $((Y,a)(YXZ,b))$, which together with (Y,a) yields (YXZ,b), by (MP). Finally, (XYZ,b), which amounts to $XYZ \rightarrow b$, holds by (COM). We have proved:

2.4 ***Theorem.***
CL_e and HCL_e are strongly equivalent, that means, they provide the same consequence relations.

In a similar way we find Hilbert-style forms of systems CL_2, CL_3, and $CL_\emptyset$ (notice that $CL_e = CL_{23}$). Precisely, to obtain HCL_2 one affixes to HCL_e the axiom-schema:

(10) $$(b(ab)),$$

and HCL_3 employs:

(11) $$((aa,b)(ab));$$

finally, $HCL_\emptyset$ requires both (10) and (11). The corresponding equivalence results can easily be obtained in the way sketched above.

A different situation arises for systems which do not allow derivable arrows of the form $\rightarrow a$. Then, we cannot identify arrow $X \rightarrow a$ with type (X,a). Instead, we look for Hilbert-style systems, employing arrows $X \rightarrow a$, that means, operating on inference schemata rather than formulae. A system of this form will be provided for CL.

By HCL we denote the system axiomatized by (A.0), (A.1) (now standing for (MP)), (2) and (3) (from section 0), together with rules (CUT), (EXP.1) and (EXP.3).

2.5 ***Lemma.***
HCL is closed under (R.1).

Proof. By induction on derivations in HCL we prove that:

(12) if $\vdash_{HCL}$ $Xa \to b$ then $\vdash_{HCL}$ $X \to (ab)$, provided $X \neq e$.

If $Xa \to b$ amounts to (A.1) then $X \to (ab)$ equals (A.0). For the remaining axioms of HCL, we have $X = e$, and similarly for the case if $Xa \to b$ arises by (EXP.1) or (EXP.2). Assume that $Xa \to b$ arises by (CUT). We consider three cases:

(I) $X = X_1X_2X_3$ and the premisses of (CUT) are $X_1cX_3a \to b$ and $X_2 \to c$. Then, $\vdash_{HCL}$ $X_1cX_3 \to (ab)$, by induction, hence $\vdash_{HCL}$ $X \to (ab)$, by (CUT).

(II) $X = X_1X_2$, where $X_2 \neq e$, and the premisses are $X_1c \to b$ and $X_2a \to c$. Then, $\vdash_{HCL}$ $X_2 \to (ac)$, by induction. If $X_1 \neq e$ then also $\vdash_{HCL}$ $X_1 \to (cb)$, by induction, hence $\vdash_{HCL}$ $X_1 \to ((ac)(ab))$, by the axioms and (CUT), which yields $\vdash_{HCL}$ $X_1(ac) \to (ab)$, by (A.1) and (CUT). If $X_1 = e$, we also get $\vdash_{HCL}$ $X_1(ac) \to (ab)$, by (EXP.1). From $\vdash_{HCL}$ $X_1(ac) \to (\bar{ab})$ and $\vdash_{HCL}$ $X_2 \to (ac)$ we infer $\vdash_{HCL}$ $X \to (ab)$.

(III) The premisses of (CUT) are $Xc \to b$ and $a \to c$. Since $X \neq e$, we get $\vdash_{HCL}$ $X \to (cb)$, by induction, and $\vdash_{HCL}$ $(cb) \to (ab)$, by (EXP.2). Consequently, $\vdash_{HCL}$ $X \to (ab)$ holds by (CUT).

2.6 ***Corollary.***
For all arrows $X \to a$, $\vdash_{HCL}$ $X \to a$ iff $\vdash_{CL}$ $X \to a$

As a matter of fact, the role of (EXP.1) and (EXP.2) can be reduced to that of axiom-forming rules. We define a sequence C_n, $n \geq 0$, of sets of arrows, by the following recursion:

(i) C_0 consists of all arrows (2) and (3) (from section 0),

(ii) C_{n+1} consists of all arrows $(ca) \to (cb)$ and $(bc) \to (ac)$, such that $a \to b \in C_n$.

We also define $\bar{C}$ (resp. $\bar{C}_n$) as the union of all C_m (resp. all C_m, such that $m \leq n$). Clearly, the arrows in $\bar{C}$ are derivable in CL. By $\overline{HCL}$ (resp. $\overline{HCL}_n$) we denote the system resulting from affixing to A^r all the arrows from $\bar{C}$ (resp. $\bar{C}_n$), as new axioms. One easily checks that $\overline{HCL}$ is closed under (EXP.1) and (EXP.2), hence it is (weakly) equivalent to HCL. Accordingly, HCL is equivalent to an axiomatic extension of A^r. Since $\bar{C}$ contains infinitely many axiom-schemata, $\overline{HCL}$ is an infinite extension of A^r. We shall prove that HCL is equivalent to no finite axiomatic extension of A^r, i.e. no extension of A^r by a finite number of axiom-schemata (or: a finite number of axioms and the rule of substitution). Our argument follows a similar one, given by Zielonka (1981) for L, but involves some new details.

It suffices to show that each system $\overline{HCL}_n$, for $n \geqslant 0$, is properly weaker then $\overline{HCL}$. For, assume that the above holds true but HCL is finitely axiomatizable over A^r by, say, the axiom-schemata $S_1, ..., S_k$. Clearly, each of these schemata must be derivable in $\overline{HCL}_n$, for some $n \geqslant 0$, hence they all must be so. Consequently, HCL is equivalent to $\overline{HCL}_n$, against our assumption.

Observe that $a \to b$ is derivable in $\overline{HCL}_n$ iff there are types $a = a_0, a_1, ..., a_m = b$ $(m \geqslant 0)$, such that $a_{i-1} \to a_i \in \overline{C}_n$, for all $1 \leqslant i \leqslant m$; the sequence $a_0, a_1, ..., a_m$ is called a linear derivation of $a \to b$ in $\overline{HCL}_n$. For $a \to b \in \overline{C}$, the only $n \geqslant 0$ such that $a \to b \in C_n$ is called the rank of $a \to b$, and by the rank of a linear derivation we mean the sum of all ranks of the (occurences of) axioms involved in this derivation.

2.7 ***Lemma.***
Let $a_0, a_1, ..., a_m$ be a linear derivation of minimal rank of $a_0 \to a_m$ in $\overline{HCL}_n$, such that $a_m = (ap)$, where $p \in Pr$, and $a_{m-1} \to a_m \notin C_0$. Then, this derivation uses no axiom from C_0 at all.

Proof. We proceed by induction on m. For $m = 0$ and $m = 1$, the thesis is obvious. Take $m \geqslant 2$. Since $a_{m-1} \to a_m \notin C_0$ and no axiom in $\overline{C}$ has the form $c \to p$ (!), then $a_{m-1} = (bp)$ with $a \to b \in \overline{C}_n$. We show that $a_{m-2} \to a_{m-1} \notin C_0$. Assume the contrary. Then, $b = (a_{m-2}p)$ and the latter arrow is an instance of (3) (from section 0). We consider two cases:

(I) $a \to b \in C_0$. Then, $a_{m-2} = (ap) = a_m$, against the assumption of minimality.

(II) $a \to b \notin C_0$. Then, $a = (cp)$ with $a_{m-2} \to c \in \overline{C}$, hence we get:

(13) $$a_0, a_1, ..., a_{m-2}, c, ((cp)p) = a_m,$$

being a linear derivation of $a_0 \to a_m$ in $\overline{HCL}_n$ with a rank less than that of the initial one, against the assumption of minimality.

Consequently, the sequence $a_0, a_1, ..., a_{m-1}$ fulfils the assumptions of our lemma, hence it employs no axiom from C_0. Evidently, the same must hold for $a_0, a_1, ..., a_m$.

We define types a_n, b_n, $n \geqslant 0$, by the following recursion:

(14) $$a_0 = p,\ b_0 = ((pq)q),\ \text{where } p,q \in Pr,\ p \neq q,$$

(15) $$a_{n+1} = (b_n p),\ b_{n+1} = (a_n p),\quad \text{if n is even,}$$
$$a_{n+1} = (b_n q),\ b_{n+1} = (a_n q),\quad \text{if n is odd.}$$

We set $\overline{HCL}_{-1} = A^r$. There holds:

2.8 ***Lemma.***
For all $n \geq 0$, $a_n \rightarrow b_n$ is derivable in $\overline{HCL}_n$ but not $\overline{HCL}_{n-1}$.

Proof. Obviously, $a_n \rightarrow b_n \in C_n$, for all $n \geq 0$. By induction on $n \geq 0$, we show that $a_n \rightarrow b_n$ is not derivable in $\overline{HCL}_{n-1}$. It is obvious for $n = 0$. Assume $n > 0$, and suppose that $a_n \rightarrow b_n$ is derivable in $\overline{HCL}_{n-1}$. Let $a_n = c_0, c_1, \ldots, c_m = b_n$ be a linear derivation of $a_n \rightarrow b_n$ in $\overline{HCL}_{n-1}$, having the minimal rank. By (14), (15), b_n cannot be the right type of any axiom from C_0, hence $c_0, c_1, \ldots, c_m$ fulfils the assumptions of 2.7. Therefore, no axiom of rank 0 appears in this derivation, and consequently, $a_{n-1} \rightarrow b_{n-1}$ is derivable in $\overline{HCL}_{n-2}$, against the inductive hypothesis, which finishes the proof.

2.9 ***Theorem.***
HCL cannot be axiomatized over A^r by any finite collection of axiom-schemata.

By a term-schema we mean the totality of all typed lambda-terms that result from a single term from TER by means of substitution of types for primitive types in this term. As a consequence of 2.9, we obtain:

2.10 ***Theorem.***
The type transformations definable by the terms from TER_{CL} can be generated by application from no finite family of term-schemata.

Proof. Assume the contrary. Then, the arrows derivable in CL are provided by a system, based on (R.0) and a finite number of CL-derivable axiom-schemata. Since HCL admits (R.0), it follows that HCL allows an axiomatization by a finite collection of axiom-schemata, which contradicts 2.9.

On the other hand, the axiom-schemata of $HCL_\bullet$ are precisely the principal type-schemata of combinators I, B, C and (Curry and Feys 1958), hence $TER_{2,3}$ is generated by these combinators (in the above sense).

As concerns non-commutative systems, neither L^r nor L admit a finite Hilbert-style axiomatization (Zielonka 1981 and modification of the above argument). This problem remains unsolved for $L_{\bullet}$ (recently W. Zielonka communicated a negative solution for $L^r_{\bullet}$).

On the basis of (A.1), (CUT), and (R.1), the arrow $aX \to b$ is equivalent to $a \to (X,b)$. Consequently, all the contents of such systems, as L^r, L, $L_{\bullet}$, CL, $CL_{\bullet}$, etc. can be expressed by arrows of the form $a \to b$. By a linear system we mean a system which operates on simple arrows and uses the only rule:

(CUT_0) $\qquad a \to b$ and $b \to c$ yield $a \to c$.

Clearly, $a \to b$ is derivable in a linear system C if and only if there is a linear derivation in the sense introduced above which employs the axioms of C. One easily checks that $\bar{C}$ + (CUT_0) provides a linear system, equivalent to CL. A linear axiomatization of L^r consists of all the arrows arising from (2) (from section 0) by means of (EXP.1) and (EXP.2). Linear systems, corresponding to $L^r_{\bullet}$, $CL_{\bullet}$, etc., require additional arrows of the form $\to a$. For instance, to obtain a linear axiomatization of $CL_{\bullet}$ it suffices to add the schema $\to (aa)$ to $\bar{C}$. We leave to the reader further exercises in this matter.

3. MATRIX SEMANTICS

A (logical) matrix is an algebra with a distinguished subset (the set of designated elements). As is well known, matrices form a fundamental semantics for sentential logics which has been thoroughly investigated by Łukasiewicz and Tarski **(1930)**, Łoś **(1949)**, Kalicki **(1950)**, Suszko **(1957)**, and others (especially in the group conducted by Professor Ryszard Wójcicki, see Wójcicki **1984**).

In this section we establish some basic properties of matrices corresponding to various systems of TL. It is impossible to regard all interesting systems and variants of semantics. Therefore we shall merely illustrate the matters by typical examples.

First of all, we observe that the above notion of matrix is not adequate for such systems, as e.g. L^r, L, CL, which employ formulae of the form $X \to a$, but not $\to a$. These systems need a concept of matrix which allows a direct interpretation of $\to$, so, instead of a set of designated elements one considers a distinguished binary relation on the underlying algebra. Matrices of this form provide, in a sense, an interpretation of entailment rather than validity. On the other hand, systems $L^r_{\bullet}$, $L_{\bullet}$, $CL_{\bullet}$, etc. can be referred to the usual matrices. To avoid confusions we use the term relational matrices (R-matrices) for the modified matrices.

We focus on R-matrices $M = (U_M, /, \leqslant)$, such that U_M (the universe of M) is a nonempty set, / is a binary operation, and $\leqslant$ is a binary relation on U_M. Given an assignment $f: Pr \longrightarrow U_M$, the value of f for type a (f(a)) is defined according to the inductive clause: $f((ab)) = f(b) / f(a)$. We say that an arrow $a \to b$ is satisfied by an assignment f if $f(a) \leqslant f(b)$. Given a class K of R-matrices, a set R of arrows, and an arrow $\to b$, we say that R K-entails $a \to b$ if, for every $M \in K$, and all assignments $f: Pr \longrightarrow U_M$, $a \to b$ is satisfied by f whenever all the arrows from R are so (sometimes, we use more general notion, where K is a class of assignments: if $\emptyset$ K-entails $a \to b$ then $a \to b$ is said to be K-valid).

We shall describe a class of R-matrices which is strongly adequate for L^r, that means, the entailment with respect to this class coincides with the consequence relation for L^r. It is expedient to identify arrows $aX \to b$ and $a \to (X,b)$. By K_0 we denote the class of all R-matrices $M = (U_M, /, \leqslant)$, such that $\leqslant$ is a partial ordering on U_M, and, for all $x,y,z \in U_M$, the following conditions hold true:

(1) $$x/y \leqslant (x/z)/(y/z),$$

(2) $$\text{if } x \leqslant y \text{ then } x/z \leqslant y/z \text{ and } z/y \leqslant z/x.$$

3.1 ***Theorem.***
$R \vdash_{L^r} X \to a$ iff R K_0-entails $X \to a$.

Proof. To prove the "only if" direction observe that (A.0), (A.1) are K_0-valid, and K_0 admits (R.1) (due to the above convention). We show that K_0 admits (CUT). Let f satisfy $XaZ \to b$ and $Y \to a$. We consider several cases:

(I) $X = e$, $Y = c$. Then, $f(c) \leqslant f(a)$ and $f(a) \leqslant f((Z,b))$ yield what desired, by the transitivity of $\leqslant$.

(II) $X = e$, $Y = cY'$ with $Y' \neq e$. Then, $f(c) \leqslant f((Y',a))$ and $f(a) \leqslant f((Z,b))$ by the assumption, hence $f((Y',a)) \leqslant f((Y'Z,b))$, by (2), and consequently, $f(c) \leqslant f((Y'Z,b))$.

(III) $X = a'X'$, $Y = c$. Then, from $f(a') \leqslant f((X'aZ,b))$ and $f(c) \leqslant$ $\leqslant f(a)$ it follows that $f(a') \leqslant f((X'cZ,b))$, by (2).

(IV) $X = a'X'$, $Y = cY'$ with $Y' \neq e$. Then, $f(a') \leqslant f((X'aZ,b))$ and $f(c) \leqslant f((Y',a))$, by the assumption. Observe that $(X'aZ,b) =$

$= (X',(a(Z,b)))$. Consequently, (1), (2), and the transitivity of $\leqslant$ yield $f((X'aZ,b)) \leqslant f((X'(Y',a)Y'Z,b)) \leqslant f((X'cY'Z,b))$, hence $f(a') \leqslant f((X'cY'Z,b))$, as desired.

The "if" direction is proved by the method of Lindenbaum matrices. We write $a \approx_R b$ if both $a \to b$ and $b \to a$ are derivable in $L^r(R)$. Clearly, $\approx_R$ is a congruence on the absolutely free algebra of types. By $[a]_R$ we denote the equivalence class of $\approx_R$ that contains a. Let U consist of all equivalence classes of $\approx_R$. For $a,b \in Tp$, we define:

(3) $$[a]_R/[b]_R = [(ba)]_R,$$

(4) $$[a]_R \leqslant [b]_R \text{ iff } R \vdash_{L^r} a \to b.$$

The matrix $M = (U,/,\leqslant)$ belongs to K_0, and we consider the assignment $f\colon Pr \longrightarrow U$ given by: $f(p) = [p]_R$, for $p \in Pr$. Clearly, f satisfies $a \to b$ iff $[a]_R \leqslant [b]_R$ iff $R \vdash_{L^r} a \to b$, which finishes the proof.

As an immediate consequence, we infer that the class K_1 of all $M \in K_0$ which validate (3) (from section 0) is strongly adequate for CL.

The theory of categorial grammars is especially interested in R-matrices of the following kind. Let V denote a nonempty set, which we refer to as a vocabulary. V^+ denotes the set of nonempty strings over V. A string $\alpha \in V^+$ is called a (right-directional) functor from a set $B \subseteq V^+$ into a set $C \subseteq V^+$ if, for every $\beta \in B$, we have $\alpha\beta \in C$, where $\alpha\beta$ stands for concatenation of α and β, by C/B we denote the set of all functors from B into C. One easily checks that the R-matrix $M(V) = (P(V^+),/,\subseteq)$ (P(W) symbolizes the power-set of W) belongs to K_0. R-matrices of the form M(V) will be referred to as standard R-matrices. In Buszkowski **(1982)** it has been proved that L^r is strongly complete with respect to the class of standard R-matrices. Actually, this claim follows from 3.1 and:

3.2 ***Theorem.***
Each R-matrix from K_0 can be embedded into a standard R-matrix.

Proof. Fix an R-matrix $M \in K_0$. Take $V = U_M$. For $\alpha \in V^+$, $x \in V$, we define x/α in the same way as (X,a) in section 0. We write $\alpha \leqslant x$ if $y \leqslant x/\alpha'$, where $y\alpha' = \alpha$. Now, for, $x \in U_M$, by f(x) we denote the set of all $\alpha \in V^+$, such that $\alpha \leqslant x$. We show that f is monomorphism of M into M(V). Clearly, $x \leqslant y$ iff $f(x) \subseteq f(y)$. As a result, $f(x) = f(y)$ entails $x = y$. We have to prove that $f(x/y) = f(x)/f(y)$. Let $\alpha \in f(x/y)$ and $\beta \in f(y)$. Then, $\alpha y \in f(x)$, hence

$\alpha\beta \in f(x)$, which can be obtained in a way similar to that applied in the "only if" part of 3.1. This yields $f(x/y) \subseteq f(x)/f(y)$. To prove the converse inclusion take $\alpha \in f(x)/f(y)$. Since $y \in f(y)$, then $\alpha y \in f(x)$, which yields $\alpha \in f(x/y)$. The proof is finished.

A pair (M,f) such that M is an R-matrix and $f\colon Pr \longrightarrow U_M$ is an assignment is called a model. A model (M,f) is said to be standard if M is a standard R-matrix. A standard model (M,f) is said to be commutative if, for all $p \in Pr$, f(p) is closed under arbitrary permutations of strings. Obviously, if (M,f) is commutative then also f(a), for all $a \in Tp$, is invariant under permutations. The class of all standard and commutative models will be denoted by SC.

3.3 ***Theorem.***
$R \vdash_{CL} X \to a$ iff R SC-entails $X \to a$.

Proof. We have already mentioned that the above equivalence holds with K_1 in the place of SC. Now, the embedding f constructed in the proof of 3.2, if applied to an R-matrix $M \in K_1$, sends each $x \in U_M$ into a permutation-closed set $f(x) \subseteq U_M^+$, which immediately yields the thesis.

Structures $(P(V^+),\cdot,/,\subseteq)$, where $\cdot$ stands for concatenation, in the following sense:

(5) $$B \cdot C = \{\alpha\beta\colon \alpha \in B,\ \beta \in C\}, \quad \text{for } B,C \subseteq V^+,$$

fulfil the axioms of so-called right-residuated semigroups (cf. Fuchs 1963). Consequently, standard R-matrices result from right-residuated semigroups by dropping concatenation. We can replace V^+ by an arbitrary semigroup G, just getting the structure $(P(G),\cdot,/,\subseteq)$ which is also a right-residuated semigroup. What we have called standard and commutative models can equivalently be characterized by means of structures $(P(G),\cdot,/,\subseteq)$, where G is a free Abelian semigroup. The reader is referred to Buszkowski (1982), (1985), (1986), where Lambek-style systems are modelled by various classes of residuated semigroups. It should be noticed that, due to affixing concatenation, one can define the relation "$X \to a$ is satisfied by f", where $X = a_1 \dots a_n$, by the simple formula:

(6) $$f(a_1) \cdot \dots \cdot f(a_n) \subseteq f(a).$$

Standard R-matrices are infinite structures. We shall define a closely related class of R-matrices, which contains finite structures. By a quasi-standard R-matrix we mean an R-matrix of the form $(P(U),/,\subseteq)$, where $\emptyset \neq U \subseteq V^+$, for some nonempty set V, U is closed under nonempty substrings, that means:

(7) $$\text{if } \alpha\beta\gamma \in U,\ \beta \neq e, \text{ then } \beta \in U,$$

and / is defined by setting:

(8) $$B/C = \{\alpha \in U\text{: for all } \beta \in C, \text{ if } \alpha\beta \in U \text{ then } \alpha\beta \in B\}.$$

Again, quasi-standard R-matrices belong to K_0, and each standard R-matrix is quasi-standard. Consequently, quasi-standard R-matrices form a strongly adequate semantics for L^r. By the methods of Buszkowski (1982a) it can be proved that L^r is complete with respect to the class of finite quasi-standard R-matrices, that means, the arrows derivable in L^r are precisely those valid in this class. One does not obtain the strong completeness, however. Accordingly, L^r possesses the finite model property. Below we establish an analogous result for CL.

By FSC we denote the class of finite quasi-standard commutative models, i.e. models (M,f) such that M is a finite quasi-standard R-matrix, and f(p) is permutation-invariant, for all $p \in Pr$. We aim to show that CL is complete with respect to FSC. The proof uses some auxiliary notions, introduced in Buszkowski (1982a).

Let T be a finite set of types, closed under subtypes. By a norm on T we mean a pair (m_1, m_2) of functions from T into the set of positive integers which satisfy following conditions:

(9) $$m_1((ab)) = \max(1, m_1(b)-m_1(a), m_2(b)-m_2(a)),$$

(10) $$m_2((ab)) = m_2(b)-m_1(a),$$

(11) $$m_1(a) < m_2(b), \quad \text{for all } a,b \in T.$$

If R is a set of arrows, by Tp(R) we denote the set of all types occuring in the arrows from R. For $T \subseteq Tp$, sub(T) denotes the set of all subtypes of the types from T. We consider sets R, such that every arrow from R is of the form $a \to b$. R is said to be normable if there exists a norm (m_1, m_2) on sub(Tp(R)), which fulfils:

(12) $$m_1(a) \geqslant m_1(b),\ m_2(a) \geqslant m_2(b), \quad \text{for all } a \to b \in R.$$

Observe that the problem of whether a finite set R is normable is effectively solvable. For, the sentence "R is normable" (for a fixed R) can be expressed in the language of

Presburger arithmetic, and the latter is a decidable theory. We have to prove:

3.4 ***Theorem.***
If R is a finite and normable set of arrows then CL(R) is complete with respect to FSC, that means, $R \vdash_{CL} X \to a$ *iff* R FSC-entails $X \to a$.

Proof. Since the argument is similar to that given for theorem 2 in Buszkowski (1982a), we only sketch the main lines. Of course, FSC is contained in K_1, hence R FSC-entails $X \to a$ whenever $R \vdash_{CL} X \to a$. To prove the converse fix a finite and normable set R, and take an arbitrary finite set $T \subseteq Tp$, such that $Tp(R) \subseteq T$ and T is closed under subtypes. It suffices to find a model (M,f) in FSC, fulfilling the equivalence:

(13) $f(a) \subseteq f(b)$ iff $R \vdash_{CL} a \to b$, for all $a,b \in T$.

Choose a primitive type $q \notin T$. We set $V = T \cup \{q\}$. It can be shown that there exists a norm (m_1,m_2) on V, satisfying (12) and such that $m_1(q) = 1$. Let N denote the maximal integer $m_2(a)$, for $a \in V$. We define $U \subseteq V^+$, as the set of all strings $X \in V^+$, such that $l(X) \leqslant N$. Consider the R-matrix $M = (P(U),/,\subseteq)$. We define an assignment $f\colon Pr \longrightarrow P(U)$, by setting:

(14) $f(p) = U$, if $p \notin T$,

(15) $f(p) = \{X \in U\colon m(X) > m_2(p) \text{ or } R \vdash_{CL} X \to p\}$, for $p \in T$,

where $m(X) = m_1(a_1) + \ldots + m_1(a_n)$ if $X = a_1 \ldots a_n$. Clearly, $(M,f) \in FSC$. By the simultaneous induction on c(a) we can prove the following claims: for all $X \in U$, $a \in T$,

(16) if $m(X) < m_1(a)$ then $X \notin f(a)$,

(17) if $m_1(a) \leqslant m(X) \leqslant m_2(a)$ then, $X \in f(a)$ iff $R \vdash_{CL} X \to a$,

(18) if $m(X) > m_2(a)$ then $X \in f(a)$.

We show (13)($\Leftarrow$). It suffices to verify that f satisfies each arrow from R. Take $a \to b \in R$. If $X \in f(a)$ then $m_1(a) \leqslant m(X)$, by (16), hence $m_1(b) \leqslant m(X)$, by (12). If $m(X) > m_2(a)$ then also $m(X) > m_2(b)$, again by (12), hence $X \in f(b)$, by (18). Otherwise, from $X \in f(a)$ we infer $R \vdash_{CL} X \to a$, by (17), and consequently $R \vdash_{CL} X \to b$, by (CUT), which yields $X \in f(b)$. To show (13)($\Rightarrow$) assume that $a \to b$ is not derivable in CL(R), where $a,b \in T$. Since $m_1(a) < m_2(b)$ then, according to (16), (17), $a \notin f(b)$. On the other hand, $a \in f(a)$, and consequently, f(a) is not contained in f(b). The proof is finished.

Since the empty set is evidently normable, we get:

3.5 ***Corollary.***
$\vdash_{CL} X \rightarrow a$ iff $X \rightarrow a$ is FSC-valid.

Clearly, 3.5 provides a new proof of decidability of CL. The set $\{(aa) \rightarrow a\}$ is neither normable, nor FSC-complete (i.e. CL(R) is not so), though it still remains decidable. It would be much interesting to verify whether normability amounts to the finite model property (our conjecture is: no).

We have shown that CL possesses the finite model property. There arises the question of whether it admits a finite adequate R-matrix, i.e. a finite R-matrix M, such that the arrows derivable in CL are precisely those valid in M. The negative answer to this question follows from some results concerning the generative capacity of CL. Given a finite set $T \subseteq Tp$, and $p \in Pr$, by CL(T,p) we denote the set of all $X \in T^+$, such that $\vdash_{CL} X \rightarrow p$. It follows from some results of Buszkowski (1984), van Benthem (1985) (see also Buszkowski 1987) that CL(T,p) may be even non-context-free. On the other hand, if CL admitted a finite adequate R-matrix then CL(T,p) would be a regular language, for all finite T and $p \in Pr$. Consequently we obtain:

3.6 ***Theorem.***
For some finite $T \subseteq Tp$ and $p \in Pr$, the equivalence:

(19) $\vdash_{CL} X \rightarrow p$ iff $X \rightarrow p$ is valid in M, for all $X \in T^+$,

holds for no finite R-matrix M.

All the theorems given in this section possess their analogues for many others systems of TL. Without essential changes they can be proved for L^r, L (in the case of bidirectional systems one must employ R-matrices with two operations: / and \), $L^r_\bullet$, $L_\bullet$, $CL_\bullet$, etc. For the latter systems, one needs R-matrices with a distinguished element 1, fulfilling $x/1 = x$, for all elements x. Instead we can consider matrices $(U_M,/,1,D)$, where D consists of all $x \in U_M$, such that $1 \leqslant x$. Then, an assignment $f\colon Pr \longrightarrow U_M$ satisfies an arrow $a \rightarrow b$ if $f((ab)) \in D$. Accordingly, we come to this form of semantics which is well known from the theory of propositional logics.

The matrix semantics for TL deserves a further intensive investigation. Besides its evident logical importance we should emphasize its intuitive foundations, anchored in the theory of categorial grammars. As a matter of fact, a model (M,f), where M is standard, corresponds to the intuitive concept of the family of syntactic categories in a language. That syntactic categories constitute an algebra like a residuated semigroup was first observed by Lambek (1958). Accordingly, the algebra of residuated semigroups, and especially standard R-matrices and their variants, may be identified with the syntactic theory of types, as the semantic theory of types refers to Cartesian closed categories, typed lambda calculus and so on.

WITOLD MARCISZEWSKI

SYSTEMS OF COMPUTER-AIDED REASONING FOR MATHEMATICS AND NATURAL LANGUAGE

1. INTRODUCTORY REMARKS

1.1. Every device created by humans is intended either for dealing with energy or for dealing with information. As for energy, some devices save it through facilitating the work of humans or animals, as do wheels, or through conserving it, as do clothes, while others produce new work due to their abilities of transforming energy (as do engines). As for information, it can be either recorded, or transformed, or else increased. One records a message, e.g. with pencil, one can transform it, e.g. with the help of calculating device, moreover one can increase a piece of knowledge, as when using a microscope, or measuring a quantity. Those devices which are to be more advanced, more sophisticated, need a theoretical science for their construction; physics when dealing with energy, and informatics when dealing with information (the latter is also dealt with by genetics, but this is another story). Thus physics and informatics, both rooted in mathematics, prove to be crucial for civilizational development. The import of physics has been acknowledged at least since the last century; the import of informatics is starting to be recognized only recently, since only now, in our age, we have created such advanced and involved information tools that they need a theoretical science for their construction and development (once upon a time also energetical devices, e.g. those for producing fire, had managed without any resort to theoretical physics). Let it be observed, by the way, that the recent appearance of informatics as a new discipline, both theoretical and applied, may raise a new and fundamental problem, viz. that of mutual relationship between energy and information; possibly, however, before we face such an involved question, we should do our best to develop information sciences, including the oldest, logic, but seen in the most recent informational perspective.

This recalling of so general a law of civilization progress is meant to give a perspective in which the import of this essay's subject might be appreciated. Reasoning, like calculating, is a fundamental human activity of processing (i.e. transforming) information, hence any device that makes it either more economical or more efficient (efficiency including reliability) deserves our attention. One should expect that such devices would involve computer programs checking logical and linguistic correctness of human reasoning and to assist humans in their most vital activity (as reasoning *must* be essential for what *ex definitione* is a reasonable being). In fact, such programs, appropriately called CHECKERs, started to appear since the sixties. The checker's task consists in showing and describing errors, if any, committed in reasoning, thus assisting a human in producing a correct proof (i.e. a piece of deductive reasoning; note that so-called inductive reasonings exceed the checker's competence as defined in this discussion).

1.2. Such a procedure can be adopted in at least four fields of human activities:

(i) scientific research, at least in its more routine stages;

(ii) the teaching of logic, mathematics, philosophy, etc., inasmuch it includes a training in reasoning;

(iii) the evaluation of arguments before the text in question gets approved for publication, which is the main task of editors and reviewers;

(iv) the checking of correctness of computer programs (such programs being similar to proofs of conditionals of the following form: if such and such operations are performed, then so and so is obtained).

It is the second of the above applications that was, in fact, practised while the others remained rather in the realm of projects and theoretical considerations. This may be explained by the fact that didactic applications provide us with the most convenient field for gathering experience.

A word is to be said about computer-aided reasoning in scientific research. Obviously, a checking program cannot envisage all the methods of reasoning that may be intended by creative human minds, since every computer program has to be based on the methods already known. But, let us note that in every research there are some routine components, and in this respect computer's services may be welcome.

On the other hand, the situation in teaching is quite the reverse: clever students may have some creative insights, but nobody can be taught such insights (at the most they may be inspired by the teacher's example and personality). Every-day instruction consists mainly in imparting some routine procedures of research, and this can be done better by a computer than by human being.

The aim of the present paper is to discuss some steps already made; they consist in producing software devices, based on classical logic alone, meant to assist logical and mathematical instruction. In this context we are to consider certain possibilities of extending the use of these devices to include natural language reasonings, as those being performed in every-day life as well as in philosophy and social sciences.

2. A COMMENT ON ALGORITHMIC PROCEDURES FOR ARGUMENT CHECKING (APACs)

2.1. There are several places in the world where people started to work on algorithmic procedures for argument checking; for short, let us call them APACs. The most successful ones have proved to be those produced in the following centers: (a) Stanford University, where a number of APAC projects, related to P. Suppes' teaching activities, came into use; (b) Eindhoven Institute of Technology in the Netherlands working on the system called AUTOMATH (abbreviation for "automatized mathematics"); (c) Warsaw University, where a family of systems called MIZAR was created for use in teaching logic and mathematics.

Projects of this kind should be mentioned at the very beginning of the present discussion to indicate that genuine results, not just plans or intentions will be discussed. However, what is said in this section is not an empirical generalization based on the results of some concrete projects. A concrete exemplification is to be given later, while in the present section we consider some a priori possible approaches to the construction of APAC systems. Owing to such a discussion, we shall be better equipped to appreciate those approaches which have been adopted in such projects as the above mentioned.

2.2. Each logician is familiar with certain algorithms for checking the validity of proofs, namely those regarding formalized proofs. Such an algorithm is a set of instructions which makes possible for anybody to solve the subsequent task (in a finite number of steps): to decide whether the final conclusion does follow from the assumptions as listed at the start. The phrase "for anybody" means all the people who are able to attentively use their eyes to trace the transformations of the physical shapes of symbol strings, according to the instructions put at the proof margin. Each instruction involves an inference rule and a reference to some former lines of the proof in question. The set of such instructions forms the algorithm for checking the given proof. Owing to the fact that the perceptual apparatus alone is sufficient to follow the algorithm, the correctness of a formalized mathematical proof can be checked even by a person without any mathematical competence, who do not grasp the content of the proof. Even the understanding of logical consequence is not required, the only thing necessary is to follow the instructions which describe physical transformations.

Owing to these features, a formalized proof can be reproduced in a computer, with the only difference that the optical mechanism of recognizing objects is replaced by another mechanism that operates on computer states instead of graphical symbols. Thus, the whole structure of a proof text is reproduced in the isomorphic structure of the sequence of machine states.

2.3. Discussed above was a theoretically possible approach to the construction of APACs, viz. the approach imitating the technique of formalization elaborated by Hilbert, Tarski, etc. for metamathematical investigations. One can imagine adopting this technique for new purposes, that is for checking proofs by a computer. However, it proves to be a theoretical option rather than a practical and feasible solution. This mode of formalization is too cumbersome for a human reasoner; at the same time it is too complicated for an automatic checker. This checker would be obliged to reproduce step by step the whole structure of thus formalized proof, putting it in the sequence of machine states.

Let us imagine a different approach. Instead of reproducing the whole structure, we adopt the method that can be called *source-and-target-reproduction*. This means that only the assumptions and the final conclusion have to be reproduced in the machine representation, while the rest, i.e. the way leading from the assumptions to the conclusion may be entirely different from the way chosen by a human reasoner. To illustrate this, let us take the analogy with arithmetical operations. One who is adding, say, 123 and 299 may obtain the result 422 either by successively adding 3 to 9, 2 to 9, and 1 to 2; or using the trick of defining 299 as 300 minus 1, and then, by adding 300 to (123 minus 1), one obtains the result in the easiest and quickest way. But whatever way is chosen by the human, the computer will perform addition always in the same way, namely that of successive additions step by step.

This example can illustrate a new approach to the construction of APACs, which does not depend on the exact and total reproduction of human-made proof in the sequence of computer states. In this new approach, instead of the total reproduction just a partial reproduction takes place, comprising only the assumptions (the source) and the final inference (the target), while the path leading from the source to the target in computer behavior may be quite different from that in human behavior. Thus, for a human reasoner a wide spectrum of options is left. Among these options may appear arguments as readable and concise as those appearing in the practice of scientific and every-day reasoning.

For these reasons, the constructors of various APACs adopted the source-and-target approach. Such a construction includes three components:

(i) a logical system of natural deduction involving an appropriate syntax and inference rules, both addressed to human reasoners;

(ii) a checker, i.e. a computer algorithm (a piece of software) to check the syntactic and logical validity of human reasoning;

(iii) a list of commands to set the algorithm in motion.

The last component can be constructed in either of two ways. Such commands as "start checking", "check the next line", "stop checking" etc. can either be added to the text of proof as separate units, or built into the text as its integral parts, serving both human communication and automatic text processing. Obviously, the latter solution is more economical; we shall see how it works in the system to be discussed below.

3. APAC AS REALIZED IN MIZAR SYSTEMS

3.1. There are three ways in which computers can assist human activities. Either a human activity is wholly replaced by computer operations (as in numerical computing, in automatic theorem proving etc.), or a computer provides us with a half-finished product to be evaluated and finished by a human (e.g. half-automatic text translation), or else a human produces something to be evaluated or checked by a computer. Computer-aided reasoning, that is an APAC implemented on a computer, falls in the last category.

Do we really need computer aid in reasoning while there is a stronger tool, namely full automatization of reasoning? It should be answered that even if the complete automatization of reasoning were a feasible solution, we would still need the procedure of computer-aided reasoning. For it is a fundamental property of human reason, closely related to the process of understanding; to understand a proposition is somehow equivalent to being able to prove it. Hence, we may need technical devices to make our reasoning more efficient.

3.2. Before introducing the APAC system being the main subject of this discussion, it is worth while to hint at some other systems of the same kind; this should provide us with a suitable comparative context.

The story of automatic proof checking goes back to McCarthy's (1961) work. Soon, first implementations appeared at Stanford University, esp. at the Institute for Mathematical Studies in Social Sciences, not without the influence and collaboration of P. Suppes whose two books have been used in the project: one on logic (1957), the other one on set theory (1972). The inference rules of predicate logic were taken from the former, while the latter provided the project with the basis for the APAC system called QUIP, constructed for teaching set theory. The system, containing both the prover and checker, has been used since 1974; the program was written in LISP. A checker for predicate logic was created in the same Institute ten years earlier.

A project based on lambda calculus with types, called AUTOMATH, started in the 1966 in the Eindhoven Institute of Technology in the Netherlands. It was initiated by N. G. Bruijn who explains the point as follows (see de Bruijn (1983), p.86).

> "The idea was to develop a system of writing entire mathematical theories in such a precise fashion that verification of correctness can be carried out automatically, yet keeping, step by step, contact with ordinary mathematical presentation. A similar idea possibly existed in the mind of Leibniz but not develop at that time."

To support this important historical conjecture, let me add the following quotation taken from Leibniz himself: "let the truth be perceived like a picture printed out in a chart with the aid of a machine" (*ut veritas quasi picta, velut machinae ope in charta* *expressa deprehendatur;* letter to Oldenburg of 28th Dec. 1675, quoted in Couturat (1901), p.99).

Similar ideas were proposed in the Soviet Union by at least two authors: L. A. Kaluznin (1964), and V. M. Glushkov (1972). Since Kaluznin's paper was published in Poland in a widely read mathematical journal, his terminological suggestion to call the languages in question "information languages" has been accepted in Poland. It is worth noting that Kaluznin's proposal coincides in time with the start of Stanford and Eindhoven projects; this may be seen as one more confirmation that when the time is ripe the same ideas appear independently at different and distant places.

3.3. The Polish project started in the middle 70ties, initiated and led by A. Trybulec, at Warsaw University, Białystok Branch; the research is mainly carried on in the Section of Computer Science in the Institute of Mathematics, while didactic experiments are realized at the Department of Logic of the same University. The name MIZAR denoting the Polish family of APAC systems has been randomly chosen from among the collection of star names. Its purpose and main idea are like those expressed in de Bruijn's comment quoted above.

MIZAR's logical basis is a system of inferential predicate logic (natural deduction) somehow similar to that of Słupecki and Borkowski (1967) which, in turn, is an improved version of the system of natural deduction created by S. Jaśkowski (1934), simultaneously with, but independently of, the systems of G. Gentzen.

All the MIZAR systems have been programmed in PASCAL and implemented on very many machines, recently on personal computers Apple 2, IBM PC/XT, Amstrad PCW 8256. Students are trained in MIZAR mainly with the help of SM-4. There are collaborators who carry on either research or instruction in university centers abroad, e.g. in Connecticut (USA), Alberta (Canada), Gent and Louvain la Neuve (Belgium), Kopenhagen, Stockholm. There was an exchange of visits and experiences between the Polish research team and the Dutch Eindhoven group, there were also contacts with a representative of the Stanford group.

MIZAR is meant to be universal language of mathematics; however, it is applicable to non-mathematical discourses as well, if the logical means required do not exceed the scope of classical logic. Extended predicate logic, that is involving identity and functions, is used at more advanced levels of teaching or research. An easier and simpler system called MIZAR MSE, is used at more popular levels. In the sequel we shall concentrate upon this simplified version.

3.4. The affix MSE means Multi-Sorted predicate logic with Equality. The fact of being multi-sorted belongs to the main syntactic features of the language in question. This feature is fairly rare in the current logical systems. An instructive example of its application is found in Hilbert's axiomatization of geometry as accomplished in "Grundlagen der Geometrie" (1899). In this system there are three universes and three respective sorts of individual variables: for points, lines, and planes.

Multi-sortedeness is a useful property from the viewpoint of the economy of formalization. It makes formulas shorter and more readable then those resulting from the use of respective predicates. It should be noted that MIZAR multi-sortedness is not quite like the multi-sortedness of Hilbert, as in the former the declaration of sorts does not hold for ever, that is for the whole theory in question; instead, it is given locally, that is in the preface to a particular proof or a cluster of proofs. This is somehow similar to restricted quantification, but the symbols for sorts restricting the range of variable are not predicates; rather they are names standing for sorts, like those referring to types in programing languages (e.g. types "integer", "boolean").

Here is an example. The first Euclidean postulate in the Hilbert formalization reads as follows. Let R denote the relation *...may be drawn from...to...*, let A, B be variable ranging over points, and a, b ranging over lines. Then we have:

(1) $$(\forall A)(\forall B)(\exists a)\ R(a,A,B).$$

Let us put the same proposition in a unisorted language which has predicates P (for point), and L (for lines) and whose variables range over the whole domain of geometric objects. Then we obtain a less economical formulation, viz.:

(2) $$(\forall x)(\forall y)(\exists z)\ (P(x)\ \&\ P(y)\ \&\ L(z)) \rightarrow R(z,x,y)).$$

Now, remaining in the same unisorted language we make use of the restricted quantification:

(3) $$\forall(Px)\ \forall(Py)\ \forall(Lz)\ \ R(z,x,y).$$

Let us compare (3) with the corresponding MIZAR formulation, where <u>for</u> is the universal quantifier, <u>ex</u> ... <u>st</u> (satisfying) is the existential quantifier, while POINT and LINE are sort names (not predicates as are P and L, respectively, in (3)).

(4) <u>for</u> x,y <u>being</u> POINT <u>ex</u> z <u>being</u> LINE <u>st</u> R(z,x,y).

3.5. Besides multisortedeness MIZAR MSE possesses other means to increase conciseness and readability, and to approximate mathematical practice. One of them consists in the suitable selection of inference rules, another one in the selection of directives regarding the structure of proof texts.

As for the rules of inference, the following solutions have been adopted in MIZAR MSE. Otherwise than in some current systems, there are infinitely many rules of propositional logic, viz. as many as the number of tautologies which provide us with inference rules. Hence, it can be said that all the propositional tautologies are obvious for the checker.

For both quantifiers hold the rules of introduction and of elimination, like in the mentioned system of Słupecki and Borkowski. However, an important novelty is introduced to the use of elimination rules: each is applied in the context of a construction characteristic for the rule in question.

To eliminate the universal quantifier from a formula like

<u>for</u> x <u>being</u> INTEGER <u>holds</u> F(x),

we make the assumption: <u>let</u> x' <u>be</u> INTEGER, where x' designates a fixed but random object. Thus, in the context "let..be.." the letter x' appears as a constant, in the sense that it designates a fixed object.

The existential quantifier can be also eliminated, provided that previously the existence of the object in question has been proven. In MIZAR this fact is recorded with the help of a construction which has behind it, as its rationale, the choice rule. This rule tells us following: if it is known that a certain existential statement is satisfied, then it is possible to introduce an named object, i.e. the object satisfying the conditions listed in the construction proposing an object to be considered, and thus allowing to drop the existential quantifier. We can eliminate this quantifier from a sentence

<u>ex</u> x <u>st</u> A(x)

provided that we know an object x' satisfying the condition A. Then we can put down:

<u>consider</u> x' <u>such</u> <u>that</u> A(x').

There are conceptual analogies (in spite of notational differences) between these MIZAR constructions and Hilbert's methods of dealing with quantifiers. Hilbert and Bernays in their "Grundlagen der Mathematik" introduce special symbols to replace bounded variables in the procedure of quantifier elimination. But, unlike more recent approaches, the symbols which replace existentially bounded variables are different from those which replace universally bounded variables. This distinction, though usually disregarded, has a good theoretical motivation, since in the operation of eliminating the existential quantifier it is obligatory to reference an existential assertion, while the analogous operation on the other quantifier is free of this obligation. In MIZAR the same distinction is expressed not with the help of different kinds of symbols, but with a different context, which is closer to mathematical practice.

3.6. In MIZAR there is an original contribution to the structuralization of proof texts. Let us discuss this in a comparative context.

A traditional mathematical proof, as structured e.g. with Euclid, starts from the proposition to be proven, then there is a sequence of propositions being ordered by the relation of consequence, possibly with references to previously accepted assertions; at the very end we put the proven assertion, sometimes distinguished through letters "q.e.d.".

In a formalized proof the structure is more rigorously defined. It is required that each step be justified by referencing both the relevant previous lines and the rules of inference being used. It is this requirement, as not admitting any shortcuts, that makes the usual formalized proofs so long and cumbersome.

MIZAR systems attempt to combine the flexibility and conciseness of natural proofs, with the rigour and the algorithmic style of formalized proofs; the latter feature is necessary for automatic text processing, the former enables addressing MIZAR to human reasoners. A detailed description of how this goal has been achieved would exceed the limits of this essay, hence the exposition has to be restricted to chosen examples.

A solution that must be mentioned consists in the mode of justification by referencing only the premisses without any reference to the deductive rules being used. This allows for considerable shortcuts. E.g. if a reasoner makes use of five rules of propositional logic at once instead of doing it in five steps, it is up to checker to decide whether the formula in question does, or does not (semantically) follow from the referenced premiss. If it does, the shortened text obtains the checker's O.K.

Another device is a schematic frame of proof construction performing two duties: for human users it yields a readable proof construction, for a computer it expresses commands corresponding to the given stage of processing. For instance, the word "environ" indicates that in the text section contained between it and the word "begin" there are collected the syntactic stipulations (e.g. symbols to stand for introduced types) as well as the axioms to be referred to in the proof. The word "proof" hints that the checking process should start, while "end" means stopping this process. Within these two basic commands more specific ones may appear, for instance "hence" meaning that the formula following this word should be checked with respect to the immediately preceding formula.

Note that such items are addressed both to human readers and to the checking system; for the former they determine the proof structure, for the latter they are commands concerning the text processing. There is also the possibility of inserting comments that are addressed to humans alone, while being disregarded by the computer. Such devices built into the MIZAR systems provide the desired economy and flexibility that can still grow, given extra programming effort.

4. HOW TO EXPRESS INTENSIONALITY IN CLASSICAL LOGIC esp. IN MIZAR SYSTEMS?

4.1. MIZAR MSE and other MIZAR systems are suited for mathematical proofs. Obviously, the same predicate logic which provides mathematicians with logical means for their reasoning is valid also for natural languages; to this extent the classical predicate logic as contained in the system MIZAR MSE, hence the system itself, can be applied to natural language arguments.

However, in natural language there are means of reasoning that exceed the scope of classical logic possibilities; at least, there is a widespread belief that this is the case. This belief motivated the creation of so many and so various non-classical logics, with the exception of intuitionistic logic that grew from some genuine problems of mathematics itself. In the sequel I shall concentrate on the first kind of non-classicality - that related to the natural language (it is also closely related to philo- sophy, requiring natural language for its arguments, hence the term "philosophical logic" is often applied to non-classical logic, but these relations will not be discussed here).

There are two features of the real world, as described by natural languages, which distinguish it from the abstract mathematical world; temporality and intensionality. the latter means that the products of human mind, such as concepts, propositions, problems and theories, as well as mind's relations to these products (believing, proving, doubting, etc.) can be dealt with in the language in question; we call it **intensional**, for the mentioned entities are characterized as having contents, i.e. (in Latin) *intensiones*. Both features are logically relevant, in the natural language there are corresponding rules of inference, even if they are disguised in the form of rules of grammar, such as *consecutio temporum* rules (giving rise to modern systems of temporal logic). As for intensionality, the rules in question are, e.g. those which govern reported speech, in which we speak of somebody's thoughts, beliefs, etc.

Both temporal and intensional reasonings have a common core, namely the idea of **modality**. In the temporal discourse we somehow feel a relation between, eg. future end possibility, or between past and necessity (if one has, e.g. fatalistic feeling that what had happened, had necessarily happened). In the intensional discourse we distinguish, e.g. between knowledge and belief with the help of the modal notions of necessity and possibility respectively (cf. Hintikka (1962), Marciszewski (1972)). Therefore, even if the modal notions do not suffice to render all the varieties of temporal and intensional arguments, they provide us with a typical example of logical peculiarities of natural languages. It is why I shall focus on them in the subsequent discussion.

4.2. In natural languages there are at least three grammatical devices to express modal ideas. The same modal concept, e.g. that of possibility, can be expressed either with an adverb (*modalitas de re*, according to the ancient terminology), or with a predicate, or else with a phrase prefixing a sentence (*modalitas de dicto*). Here are examples.

John possibly agrees	modality *de re*
It is possible that John agrees	modality *de dicto*
That John agrees is possible	predicative modality

In the last case the sentence "John agrees" is prefixed by "that" to transform the sentence into a nominal phrase which is equivalent (roughly) with "John's agreeing", "John's agreement" etc.

This role of words like "that" was clearly noticed by Frege who even found a symbolic representation for it in his "Begrifsschrift"; it was a horizontal line without a stroke, as opposed to the horizontal line with the vertical stroke (⊢), the latter expressing the assertion operation. Frege read this horizontal line as "der Umstand, daß", or "der Satz, daß". Following Frege's terminological suggestion, we can call the horizontal line the content operator, as it produces the mere content of a proposition, devoid of acceptance (assertion). Of the content in question it may be predicated that it is the case, is true, possible, necessary, obligatory, permitted, expected, likely, beautiful, etc. Obviously, that list includes modal predicates.

These observations are to show that there are content operators in natural languages which can be used to eliminate modality *de re*, likewise modality *de dicto*, in favour of modal predicates. These are predicated of contents, in this way forming sentences which fit into the scheme of classical logic: neither do they contain non-classical constants, nor do they receive any logical values apart from the true and the false.

4.3. If the state of affairs appearing in natural languages is to be reproduced in formal logic, we should introduce a symbolic content operator. This move was made by Frege, and developed by A. Church in his logic of sense and denotation. However, this line of developing logic did not receive a wider acceptance. Possibly the reason lies in the fear of platonism, as felt by many modern logicians who suspect contents (as obtained with the content operator) to be too much like Platonian ideas. A more convincing argument is that we did not succeed in finding ways to deal with contents as efficient as those of dealing with sets.

In any case, the predicative mode of treating modalities gave way to the mode deriving from the idea of modality *de dicto*; but the situation changed again with the appearance of computers and programming languages. The philosophical objections such as those concerning the existence of contents can be given up in the case when we obtain technical tools to handle such entities without any harm to argument preciseness.

Such a tool is involved in the notion of a type (mode, sort), that is a set of objects for which certain operations are defined. In particular, we may introduce the type PROPOSITION for which (among others) holds the operation of prediction, e.g. in the context: PROPOSITION 2 + 2 = 4 is necessary. According to MIZAR rules for proof structuration, the types to be used in the proof in question should be introduced in the section called "environ" by means of the following declaration:

<u>let</u> --- <u>denote</u> ... ;

where in blanks there first appears a variable or variables (---) and then the name of a type (...), for instance:

<u>let</u> s <u>denote</u> PROPOSITION.

In the same section the declared type is implicitly defined through suitable axioms, for instance:

A1: <u>for</u> s <u>holds</u> false[s] <u>iff</u> <u>not</u> true[s];

Now it is known that the type PROPOSITION includes objects of which truth and falsity can be predicated. On the same footing modal predicates can be applied to objects of the type PROPOSITION, thus contributing to the further explanation of its nature, e.g.

A2: true[s] <u>implies</u> possible[s];

which renders the old maxim: *de esse ad posse valet illatio.*

If we need to deal with an individual object of type PROPOSITION, we can introduce it as a constant by using the special construction for this purpose, expressed with <u>given</u>, for instance:

<u>given</u> it_rains <u>being</u> PROPOSITION;

Obviously predicates which are applicable to type variables (like s in the above examples) can be applied to type constants (like it_rains).

To put the suggested method to a rather demanding test, let us apply it to the Barcan formula (cf. Marciszewski (1981)), viz.

$$\mathbf{P}((\exists x)A) \rightarrow (\exists x)\mathbf{P}(A)$$

where **P** stands for the modal operator to be read "it is possible that" (the bold face to distinguish it from the modal predicate *P*, read "is possible", to be used in the proposed transformation). Now we stipulate:

<u>given</u> $(\exists x)A$ <u>being</u> PROPOSITION;

<u>given</u> A <u>being</u> PROPOSITION;

Then the Barcan formula obtains the following form:

$$P((\exists x)A) \rightarrow (\exists x)P(A)$$

where the above stipulation is realized due to the context *P* (as being an expression of syntactic category *s/n)* instead of the previous **P** (category *s/s)*.

Natural languages like English and German possess a suitable syntactic device for such a transformation, namely the English "that" and the German "daß", both functioning either as a part of a modal operator (e.g. "it is possible that", "es is möglich daß") or as a functor forming a name (of a proposition) out of a proposition ("that A is possible", "daß A ist möglich"). When transforming the Barcan formula with the help of these natural language devices (from modality *de dicto* to predicative modality), we obtain something like the following:

<u>from</u>: if it is possible that there is x who is immortal, then there is x such that it is possible (about him) that he is immortal;

<u>to</u>: if THAT THERE IS x WHO IS IMMORTAL is possible, then there is x which satisfies (the formula): THAT x IS IMMORTAL is possible.

In this example capital letters indicate the object of the type PROPOSITION of which property of being possible is predicated.

4.4. Let me summarize the foregoing discussion. The problem of analysing reasonings in a natural language has been restricted to checking logical correctness and this, in turn, has been restricted to checking with the use of computer program (called a checker).

When resorting to such technical devices, we got bound by economy more than in other methods of analysis, since the lack of economy brings about financial losses (e.g. computer time is measured also in terms of money). From this view-point the analysis of arguments carried out in terms of classical logic may prove more convenient than that in terms of modal logic, if there are ready and simple programs prepared on the classical basis. Thus restricted, and disregarding more theoretical considerations, when we analyse natural language reasonings that involve modal logical constants, it is worth while to try to eliminate them in favour of modal extralogical predicates. After such a translation a natural language argument can be expressed in a standard logical computer communication language based on classical logic, like the MIZAR language discussed above.

The strategy used in the suggested translation consists in the nominalization of a sentence in order to apply a predicate to the name thus obtained, instead of applying a corresponding sentential operator to the sentence in question. There may be philosophical objections against this strategy but they can be answered with the following argument: any systematic procedure that brings about success, like a correct decision concerning argument validity, obtains the verdict of being right from the high court of practice. There remains the problem how this practical rightness is related to the genuine truth, as looked for by philosophers; for instance, whether the utility of the nominalization of a sentence does confirm the view of the existence of sentential contents. This however, is a new and different question. It cannot be disregarded in the totality of our theoretical inquires, but it could have been disregarded within the limited contentions of this essay.

ANNA ZALEWSKA

AN APPLICATION OF MIZAR MSE IN A COURSE IN LOGIC

I would like to present some educational experiences collected during computer-aided courses in logic with the assistance of Mizar MSE[1]. I limit myself only to my own experiences and the experiences of my colleagues, because I did not get any detailed data about the courses delivered by others[2].

The principal goal of logic training is to develop the skill in deductive reasoning. Students achieve this skill, in general, by exercises in a properly chosen domain; their self-reliance and activity are critical.

Traditional teaching of logic comprises of two parts: firstly, the introduction of necessary information in a specific domain; secondly, exercises both in class and at home. There are no problems when we work with students of more or less equal abilities, actively participating in classes, but it rarely happens. The teaching usually concerns a group of learners who have different capabilities and work at a different pace. The teacher has no opportunity to adjust to individual needs. In order to enable all student students to master a subject it is necessary to present and explain the same matter repeatedly, boring the more enterprising members of the class. The introduction of non-trivial exercises and more sophisticated examples, interesting for better students would be too difficult for the less capable ones. It is also difficult for the teacher to encourage greater activity of ordinary students and to encourage self-reliance and independence.

Let us then have a look at computer aided instruction of logic with the aid of Mizar MSE. The teaching of logic based on this system starts with a short introduction of the Mizar MSE language. Students are given necessary information about how to build sentences in this language, how to construct Mizar MSE proofs, how to justify statements, and so on.

After this preliminary presentation students begin to work individually. Each of them receives a separate exercise.

The exercise can be prepared in two ways. The first one consists in presenting the exercise in natural language. The student's task is to write correctly the theorem and its proof in Mizar MSE language together with all the definitions, axioms and remaining facts used in the proof. Since the *Environment* is verified by the system only syntactically, the teacher must watch carefully the logical correctness of an *Environment* written by a student.

Another way of preparing exercises is as follows. All students receive the same exercises concerning the same domain (e.g. the elementary theory of sets) together with a proper *Environment*. It is helpful to add several examples of correctly proved theorems in the prepared *Environment*. The student's task is to write a correct proof of the given theorem in Mizar MSE language. As far as the contents of the theorem is concerned it is sometimes better to present the students with the exercise in natural language. In particular, when the exercise involves functions, they should be replaced by predicates. If students do it themselves, the theorems become clearer for them. An example of such an exercise is given in the appendix.

From the very beginning of the work with the Mizar MSE system the student must be active. First of all, the system requires the learner to possess his own conception of proof construction. Mizar MSE only decides whether a given step of the presented proof is correct, but does not help to construct this proof. The proof text prepared by the student is checked by the system. If this text is not accepted, the student must correct the indicated mistakes and again verify his text with Mizar MSE. If he is not able to fill the reasoning gaps correctly and to improve the text in this manner, he can apply to the teacher for additional explanations. Generally, after several attempts (4–5 if the proof is not difficult, 8–9 if it is more complicated), the student can get his proof accepted by Mizar MSE.

Some remarks about Mizar MSE as a tool of teaching logic

1. Mizar MSE, in comparison to the traditional methods, is more attractive for students. The system is characterized by the following features:

- it decides at once whether the analysed text is correct, so there is no loss of time;
- it marks individual errors and provides explanations;
- the language includes some of the typical natural language constructions found in mathematical proofs, e.g. "let ... be ... such that ...", "assume that ...", "thus ...", thus students can build proofs resembling genuine mathematical practice;
- it defines a notion of obviousness consistent with our intuitions, so it is possible for students to construct proofs without needless details.

2. Individual work with the Mizar MSE system allows the student to work at his own pace, and the teacher to follow the progress of each student separately. There is then a better opportunity for intensive learning independent of variations in student capability.

3. We learned that the difference between good students and poorer ones is more apparent than usually. Weaker students need much more time to correct their proofs.

4. When beginning a computer aided course in logic we ought to have at our disposal several groups of exercises. At the very beginning really simple exercises are often needed, especially with students lacking mathematical training. In such cases it might be necessary to acquaint students with inference rules as the first step. With more experienced candidates non-trivial and more interesting exercises can be offered.

5. A formalized language (also Mizar MSE), in contradistinction to a natural one, requires rather strict discipline. Most of the students, who have never used such a language, experienced some difficulty being used to informal reasonings, often not at all precise. This leads to errors; students should be advised to write their proof in the following way:

- first construct a skeleton of the proof;
- then fill in the reasoning gaps until the proof is detailed enough to be accepted by the Mizar MSE checker.

Appendix

```
MIZAR.MSE                                    AMSTRAD VERSION 3.02
WARSAW UNIVERSITY BIALYSTOK CAMPUS

Example.MSE

:: This Environment concerns binary relations

environ

reserve x,y,y',z for member;
reserve P,Q,R,RR for relation;

Extensionality: for P,Q holds
          (for x,y holds Rel[x,P,y] iff Rel[x,Q,y] implies P = Q;

Inclusion: for P,Q holds
          Incl[P,Q] iff for x,y st Rel[x,P,y] holds Rel[x,Q,y];

Union: for P,Q,R holds
     Union [P,Q,R] iff (for x,y holds Rel[x,R,y] iff
                                   (Rel[x,P,y] or Rel[x,Q,y]));

Intersection: for P,Q,R holds
     Inter[P,Q,R] iff (for x,y holds Rel[x,R,y] iff
                                   (Rel[x,P,y] & Rel[x,Q,y]));

Complement: for P,Q holds
     Comp[P,Q] iff (for x,y holds Rel[x,P,y] iff not Rel[x,Q,y]);

Converse: for P,Q holds
          Conv[P,Q] iff (for x,y holds Rel[x,P,y] iff Rel[y,Q,x]);

Composition: for P,Q,R holds
     Comp[P,Q,R] iff (for x,z holds Rel[x,R,z] iff
                              (ex y st Rel[x,P,y] & Rel[y,Q,z]));

Transitivity: for R holds
Tr[R] iff for x,y,z st Rel[x,R,y] & Rel[y,R,z] holds Rel[x,R,z]);

Symmetry: for R holds Sym[R] iff (for x,y st Rel[x,R,y] holds Rel[y,R,x]);

Reflexivity: for R holds Refl[R] iff (for x holds Rel[x,R,x]);

given I being relation;

Identity: for x,y holds Rel[x,I,y] iff x = y;
```

```
begin

:: Two examples of correctly solved theorems are given below:

Ex1: for P st Tr[P] & Sym[P] & (for x ex y st Rel[x,P,y]) holds Refl[P]
     proof let P be relation;
      assume Tr[P]; then 1: for x,y,z st Rel[x,P,y] &  Rel[y,P,z]
                                            holds Rel[x,P,z] by Transitivity;
      assume Sym[P]; then 2: for x,y st Rel[x,P,y]
                                              holds Rel[y,P,x] by Symmetry;
      assume 3: for x ex y st Rel[x,P,y];
      now let x be member;
       consider z such that 4: Rel[x,P,z] by 3;
       Rel[z,P,x] by 4,2;
       hence Rel[x,P,x] by 4,1;
      end;
      hence Refl[P] by Reflexivity;
      end;

:: The contents of the second example is:
:: For any relation R holds that
::   R is transitive if and only if R∘R ⊂ R

Ex2: for R, RR st Comp[R,R,RR] holds Tr[R] iff Incl[RR,R]
     proof let R, RR be relation such that 1: Comp[R,R,RR];
      2: for x,z holds Rel[x,RR,z] iff
                    (ex y st Rel[x,R,y] & Rel[y,R,z] by 1,Composition;
      N1: now assume Tr[R]; then
          3: for x,y,z st Rel[x,R,y] & Rel[y,R,z] holds
                                                  Rel[x,R,z] by Transitivity;
           now let x,y be member such that 4: Rel[x,RR,y];
            consider y' such that 5: Rel[x,R,y'] & Rel[y',R,y] by 4,2;
            thus Rel[x,R,y] by 5,3;
           end;
           hence Incl[RR,R] by Inclusion;
          end;
      now assume Incl[RR,R]; then
       6: for x,y st Rel[x,RR,y] holds Rel[x,R,y] by Inclusion;
       now let x,y',z be member such that 7: Rel[x,R,y'] & Rel[y',R,z];
        ex y st Rel[x,R,y] & Rel[y,R,z] by 7;
        then Rel[x,RR,z] by 2;
        hence Rel[x,R,z] by 6;
       end;
```

```
  hence Tr[R] by Transitivity;
 end;
 hence thesis by N1;
 end;
```

:: *A theorem to prove is the following:*
:: *For any relations R,S such that R is transitive and R∘S = I*
:: *both R = I and S = I are true.*

THANKS OK

Notes

[1] Other publications on computer aided instruction using Mizar MSE include:

[1] Cours avancé de Mizar MSE donné par A. Trybulec, solutionné par St. Żukowski, Summer Mizar Workshop, Louvain-la-Neuve, 16.06 - 15.09.1985, Rapport Technique, Cahiers de Mathématiques Appliquées aux Sciences Humaines, pp.53-72, 1985.

[2] M. Mostowski, Textbook of Logic based on Mizar MSE (manuscript), 1985.

[3] M. Mostowski and A. Trybulec, A certain experimental computer aided course of logic in Poland, Proc. of World Conference on Computers in Education, Norfolk, Va, July - August 1985, North Holland.

[4] M. Mostowski and A. Zalewska, Logical exercises in the Mizar MSE language (manuscript), 1986.

[5] K. Prażmowski and P. Rudnicki, Mizar MSE - a course in the monthly DELTA, Nos 9-12/1983, 1-6/1984.

[2] Since the Mizar MSE system is in the public domain we don't exactly know how it is used and where. Up to now we have gathered the following information:

1. **Warsaw University, Poland:**
   - course in logic: the Department of Philosophy (first year) and Information and Library,
   - course in foundations of geometry (fourth and fifth year): the Department of Mathematics.

2. **Warsaw University - Białystok Campus, Poland:**
   - course in logic: the Department of Mathematics (first year), Pedagogics (first year) (cf. [2], [4] above) and Technical Training (fourth year),
   - course in methodology of physics (fifth year): The Department of Physics,
   - a correspondence course of logic taught with the aid of Mizar MSE in the Polish monthly DELTA, completed two years ago (cf. [5], [3]).
3. **University of Connecticut, USA:**
   - course in discrete mathematics for computer science, (cf. [1]).
4. **Washington State University, USA:**
   - course in discrete mathematics for computer science.
5. **University of Alberta, Canada:**
   - course in formal systems in computer science.
6. **University of Tokyo, Japan:**
   - course in logic.

LESŁAW W. SZCZERBA

THE USE OF MIZAR MSE IN A COURSE IN FOUNDATIONS OF GEOMETRY

Anna Zalewska (in this volume) has described some educational experiments in computer aided courses involving Mizar. Among others, she mentioned a course in foundations of geometry held at the Department of Mathematics of the University of Warsaw in the spring semester of 1985.

I used Mizar MSE installed on the Riad 60 in the batch system. The return time was, for technical reasons, quite substantial - sometimes as long as one week which proved to be disastrous to the experiment. In fact it is one of the reasons for which, in my opinion, it is pointless to give precise results of the experiment, therefore I will restrict myself only to some general remarks and observations.

The course was based upon the textbook (M. Kordos, W. Szczerba 1976) containing the formal exposition of axiomatic Euclidean geometry. Since Mizar MSE does not support function symbols, the use of Mizar MSE during the course had to be limited to the material contained in the first half of chapter 1, all of chapter 2, half of chapter 3 of part I and first three sections of chapter 1 of part II. Students attended a course during which the main idea was developed, which consisted mainly in proving simple geometrical theorems. Some theorems, together with the proper environment, were assigned to students for proving. The objective was to prepare the proof in Mizar language and obtain the 'OK' from the computer. To get credit for the course the students were required to get three such theorems 'OK-ed'.

The class consisted of ten students in the senior year who were majors in Education of Mathematics. They represented low to average level. The main difficulty in the experiment was to get students to learn how to use Mizar; at first there has been considerable resistance on their part. One of the students restricted his proof to the mere statement of the theorem, and to my surprise got it accepted. So later on I was more careful in assigning problems. After the initial difficulties were overcome the students started to like it. I understand that they were treating the computer as a part of reality and therefore getting an 'OK' was an objective success in contrast with their attitude to the teacher who is susceptible to all kinds of psychological pressures. This gave them a feeling of accomplishment.

There occurred a substantial change in my role as a teacher. Earlier, when assigning homework tests, I was treated as an enemy who has to be forced to accept the solution, sometimes in not an exactly honest way. Now the enemy to be defeated was the computer and I was turned into an ally helping to fight this horrible device. This small fact has seriously influenced my contacts with the students. They were much more eager to approach me with their problems, to report on their difficulties, and ask for help.

Using Mizar in such a course has, for a teacher, an additional advantage: it lifts off the burden of checking homeworks. It has to be paid for by increased help required by the students at the beginning. However, the change in the nature of contacts between student and teacher is an obvious gain. Another gain was visible during the final tests, namely the students understood the notion of proof much better than their peers from other classes. Of course the size of the class rules out any use of statistical methods, but for an experienced teacher such an outcome was evident.

Thus Mizar seems to be very well suited to be a help in teaching mathematics on any course where proving constitutes an essential part. Mizar MSE, however, has several drawbacks when applied to teaching of mathematics:

1. Lack of support for functional symbols. This limits considerably the scope of applicability.

2. Lack of user defined characters, infix and postfix notations and generalised notations, such as $a < b < c$.

Of course, theoretically Mizar MSE is an universal language, and anything expressible in mathematics is expressible in Mizar MSE. Still, sometimes it turns out to be impractical. The reservations mentioned above are of unequal importance - the lack of functional symbols is much more serious. A version of Mizar supporting functional symbols will lead to much more successful experiments.

LITERATURE

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The list starts from bibliographic abbreviations, mostly those used for Leibniz editions.

A = **Sämtliche Schriften und Briefe,** *edited by* Preußische (later Deutsche) Akademie der Wissenschaften. *[Lenzen, Note 1]*.

AV = **Voraus - edition to the series VI of A.** *[Lenzen, Note 1]*.

C = **G. W. Leibniz Opuscules et Fragments inédits,** Paris 1903 *ed.* L. Couturat. *Reprinted* Hildesheim 1961. *[Juniewicz]*, *[Lenzen, Note 1]*.

GI = **Generales Inquisitiones de Analysi Notionum et Veritatum** *ed.* F. Schupp. *[Lenzen, Note 1]*.

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